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VORTEX SOLITONS FOR 2D FOCUSING NONLINEAR SCHRÖDINGER EQUATION

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Abstract. We study standing wave solutions of the form $e^{i(\omega t+m\theta)}\phi_\omega(r)$ to the nonlinear Schrödinger equation

$$iu_t + \Delta u + |u|^{p-1}u = 0 \quad \text{for } x \in \mathbb{R}^2 \text{ and } t > 0,$$

where (r, θ) are polar coordinates and $m \in \mathbb{N} \cup \{0\}$. We prove that standing waves which have no node are unique for each m and that they are unstable if $p > 3$.

1. Introduction

In this paper, we study standing wave solutions with a vortex for 2-dimensional nonlinear Schrödinger equations

$$\begin{cases} iu_t + \Delta u + f(u) = 0 & \text{for } (t, x) \in \mathbb{R} \times \mathbb{R}^2, \\ u(x, 0) = u_0(x) & \text{for } x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $f(u) = |u|^{p-1}u$. Let $\omega > 0$, $m \in \mathbb{N} \cup \{0\}$, and $u(x, t) = e^{i(\omega t+m\theta)}\phi_\omega(r)$ be a standing wave solution of (1.1), where $e^{im\theta}\phi_\omega(r) \in H^1(\mathbb{R}^2)$ and r and θ are polar coordinates in $\mathbb{R}^2$. Then $\phi_\omega = \phi_\omega(|x|)$ is a solution to

$$\Delta\phi_\omega - \Big(\omega + \frac{m^2}{|x|^2}\Big)\phi_\omega + f(\phi_\omega) = 0 \quad \text{for } x \in \mathbb{R}^2. \tag{1.2}$$

A standing wave solution of the form $e^{i(\omega t+m\theta)}\phi_\omega(r)$ appears in the study of nonlinear optics (see references in [12]).

If $m = 0$, (1.2) has a ground state ϕ_ω, which is positive and radially symmetric (see [2]). Uniqueness of positive radially symmetric solutions to (1.2) was proved by Kwong [10] in the case where $m = 0$.

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In the case where $m \neq 0$, existence of standing waves (vortex solitons) was studied by Iaia and Warchall [9] (see also [4]). One of our goals in the present paper is to show uniqueness of positive radially symmetric solutions to (1.2) in the case where $m \neq 0$.

Theorem 1.1. *Let m be an integer and $1 < p < \infty$. Then there exists a unique positive radially symmetric solution ϕ_ω to* (1.2) *that belongs to $H^1(\mathbb{R}^2)$.*

Another goal is to investigate stability and instability of vortex solitons. If φ_ω is a ground state, the standing wave solution $e^{i\omega t}\varphi_\omega$ is stable if $dN[\varphi_\omega]/d\omega > 0$ and unstable if $dN[\varphi_\omega]/d\omega < 0$, where $N[u] := \int_{\mathbb{R}^2} |u(x)|^2 dx$ (see [1], [5], [7], [13], [14], [15] and references therein).

Recently, Pego and Warchall [12] proved linear stability of vortex solitons to a nonlinear Schrödinger equation with cubic-quintic nonlinearity by numerical computation. In this paper, we study stability and instability of vortex solitons $e^{i(\omega t+m\theta)}\phi_\omega(r)$ to perturbations of the form $e^{im\theta}v(r)$.

Let $u_0(x) = e^{im\theta}(\phi_\omega(r) + v_0(r))$. In the present paper, we say a standing wave solution $e^{i(\omega t+m\theta)}\phi_\omega(r)$ is *stable* if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if $\inf_{\gamma\in\mathbb{R}} \|u_0 - e^{i(m\theta+\gamma)}\phi_\omega\|_{H^1(\mathbb{R}^2)} < \delta$, the solution of (1.1) satisfies

$$\sup_{t\geq 0} \inf_{\gamma\in\mathbb{R}} \|u(\cdot,t) - e^{i(m\theta+\gamma)}\phi_\omega\|_{H^1(\mathbb{R}^2)} < \varepsilon.$$

Otherwise the standing wave solution is called *unstable*.

Theorem 1.2. *Let $m \in \mathbb{N} \cup \{0\}$ and ϕ_ω be a positive radially symmetric solution of* (1.2) *that belongs to $H^1(\mathbb{R}^2)$. Then the standing wave solution $e^{i(\omega t+m\theta)}\phi_\omega$ of* (1.1) *is unstable if $p > 3$ and stable if $1 < p < 3$.*

Remark 1.1. In Theorem 1.2, stability of vortex solitons is proved for a limited class of perturbations. It remains open whether vortex solitons are stable to small perturbations in H^1.

Remark 1.2. If $p = 3$, (1.1) has conformal invariance. Thus for any $\delta > 0$, there exists a $u_0 \in H^1(\mathbb{R}^2)$ with $\|u_0 - e^{im\theta}\phi_\omega\|_{H^1(\mathbb{R}^2)} < \delta$ such that the solution u to (1.1) blows up in a finite time.

Our plan for the present paper is as follows. In Section 2, we show uniqueness of positive radially symmetric solutions to (1.2). To prove the result, we use the classification theorem developed by Yanagida and Yotsutani ([17] and [18]). Recently their method was applied to a nonlinear Schrödinger equation with harmonic potential (see [8]). In Section 3, we investigate stability and instability of standing wave solitons with a vortex by applying

Grillakis-Shatah-Strauss [7]. To apply their theory, we need to prove that ϕ_ω is nondegenerate. Since our problem has a singular term, we cannot prove the non-degeneracy following the argument of [10] and [11]. Instead, we use local manifold theory and the Pohozaev identity to prove the non-degeneracy.

Finally, we introduce several notations. We use the notation $f(r) \sim g(r)$ if there exist some positive constants c_1 and c_2 satisfying $c_1 f(r) \le g(r) \le c_2 f(r)$. We denote by $\sigma(A)$ the spectrum of an operator A. Various constants will be simply denoted by C and C_i $(i \in \mathbb{N})$ in the course of calculations.

2. Uniqueness of positive radial solutions to (1.2)

Let $m \in \mathbb{N}$ and $\varphi \in H^1(\mathbb{R}^2)$ be a solution to

$$\Delta\varphi - \omega\varphi + |\varphi|^{p-1}\varphi = 0 \quad \text{for } x \in \mathbb{R}^2,$$

which can be written in the form $\varphi(x) = e^{im\theta}\phi_\omega(r)$. Then $\phi_\omega(r)$ is a C^2-solution to

$$\phi'' + \frac{1}{r}\phi' - \Big(\omega + \frac{m^2}{r^2}\Big)\phi + |\phi|^{p-1}\phi = 0, \tag{2.1}$$

$$\lim_{r\to 0}\frac{1}{r^m}\phi(r) = \alpha, \quad \lim_{r\to 0}\frac{1}{r^{m-1}}\phi'(r) = m\alpha, \tag{2.2}$$

for an $\alpha \in \mathbb{R}$ and satisfies

$$\lim_{r\to\infty}\phi(r) = 0. \tag{2.3}$$

When the amplitude is small, the behavior of solutions to (2.1) is governed by the linear equation

$$\eta_{rr} + \frac{1}{r}\eta_r - \Big(\sigma^2 + \frac{m^2}{r^2}\Big)\eta = 0, \tag{2.4}$$

where $\sigma = \sqrt{\omega}$. Linearly independent solutions of this equation are the modified Bessel functions $I_m(\sigma r)$ and $K_m(\sigma r)$ satisfying the following:

$$\begin{aligned} &I_m(\sigma r) \sim \frac{1}{2^m m!}(\sigma r)^m, \quad K_m(\sigma r) \sim 2^{m-1}(m-1)!(\sigma r)^{-m} \quad (r \to 0),\\ &I_m(\sigma r) \sim \frac{1}{\sqrt{2\pi\sigma r}}e^{\sigma r}, \quad K_m(\sigma r) \sim \sqrt{\frac{\pi}{2\sigma r}}e^{-\sigma r} \quad (r \to \infty), \end{aligned} \tag{2.5}$$

where $m \in \mathbb{N}$. Hence $e^{im\theta}\phi_\omega(r) \in H^1(\mathbb{R}^2)$ if and only if $\phi_\omega(r) \sim I_m(\sigma r)$ as $r \to 0$ and $\phi_\omega(r) \sim K_m(\sigma r)$ as $r \to \infty$.

We remark that every regular solution at $r=0$ satisfies (2.2). By the change of variables $x=\log r$, (2.1) is transformed into

$$\phi_{xx}-m^2\phi=e^{2x}(\omega\phi-|\phi|^{p-1}\phi).$$

Let $\Phi={}^t(\phi,\phi_x)$ and $G(u)={}^t(0,\omega u-|u|^{p-1}u)$. Note that $r\phi_r(r)$ is bounded as $r\to 0$. Indeed, let $a=\lim_{r\downarrow 0}\phi(r)$. Suppose, to the contrary, $a>0$. Then by

$$(r\phi_r)_r=\left(\omega r+\frac{m^2}{r}\right)\phi-r|\phi|^{p-1}, \tag{2.6}$$

it follows that $\phi_r(r)\sim\frac{\log r}{r}$ as $r\to 0$ and $\lim_{r\to 0}\phi(r)=\infty$. But this leads to a contradiction because $\phi(r)$ is continuous at $r=0$. It follows from (2.6) and the fact that $a=0$ that $r\phi_r$ is monotone increasing and bounded around $r=0$. Otherwise, we have $\lim_{r\to 0}r\phi_r(r)=-\infty$ and $\lim_{r\to 0}\phi(r)=-\infty$ as $r\to 0$.

Using the variation of constants formula and the fact that ϕ and $\phi_x=r\phi_r$ are bounded as $x\to-\infty$, we have

$$\Phi(x)=e^{mx}P_+\Phi(R)+T(\Phi)(x), \tag{2.7}$$

where R is a negative number and

$$T(\Phi)(x)=\int_R^x P_+e^{B(x-y)}e^{2y}G(\phi(y))dy+\int_{-\infty}^x P_-e^{B(x-y)}e^{2y}G(\phi(y))dy,$$

$$B=\begin{pmatrix}0&1\\m^2&0\end{pmatrix},\quad P_+=\begin{pmatrix}\frac12&\frac{1}{2m}\\\frac{m}{2}&\frac12\end{pmatrix},\quad P_-=\begin{pmatrix}\frac12&-\frac{1}{2m}\\-\frac{m}{2}&\frac12\end{pmatrix}.$$

Let Y and $\widetilde{Y}$ be Banach spaces defined as

$$Y=\left\{u\in C((-\infty,R];\mathbb{R}^2):\|u\|_Y<\infty\right\},$$
$$\widetilde{Y}=\left\{u\in C((-\infty,R];\mathbb{R}^2):\|u\|_{\widetilde{Y}}<\infty\right\},$$

where $\|\Phi\|_Y=\sup_{x\le R}e^{-mx}(|\phi(x)|+|\phi_x(x)|)$ and $\|\Phi\|_{\widetilde{Y}}:=\sup_{x\le R}(|\phi(x)|+|\phi_x(x)|)$. If R is sufficiently small, the mapping T defines a contraction mapping on both Y and $\widetilde{Y}$. That is, (2.7) has a unique solution in $\widetilde{Y}$ satisfying $(\phi,\phi_x)\sim(e^{mx},me^{mx})$ as $x\to-\infty$. Thus we prove $(\phi,\phi_r)\sim(r^m,mr^{m-1})$ as $r\downarrow 0$.

For any $\alpha\in\mathbb{R}$, there exists a unique global solution to (2.1) and (2.2) in the class $C([0,\infty))\cap C^2((0,\infty))$ (see [9]).

To show uniqueness of positive solutions to (2.1)–(2.3), we apply a classification theorem ([17]) to (2.1) and (2.2). Let $\phi(r,\alpha)=\eta(r)u(r,\alpha)$, where

$\eta(r)$ is a solution of (2.4) satisfying

$$\lim_{r\to 0} r^{-m}\eta(r) = 1 \quad \text{and} \quad \lim_{r\to 0} r^{-m+1}\eta'(r) = m. \tag{2.8}$$

Then $u(r,\alpha)$ satisfies

$$\begin{cases} (g(r)u_r)_r + g(r)K(r)|u|^{p-1}u = 0, \\ u(0,\alpha) = \alpha, \quad u_r(0,\alpha) = 0, \end{cases} \tag{2.9}$$

where $g(r) = r\eta(r)^2$ and $K(r) = \eta(r)^{p-1}$. In fact, we see that

$$\lim_{r\to 0} u(r,\alpha) = \lim_{r\to 0} \frac{\phi(r)}{\eta(r)} = \alpha,$$

and

$$\lim_{r\to 0} u_r(r,\alpha) = \lim_{r\to 0} \frac{\phi_r\eta - \phi\eta_r}{\eta^2} = \lim_{r\to 0} \frac{(r\phi_r\eta - r\phi\eta_r)_r}{(r\eta^2)_r} = 0.$$

Using (2.5) and the fact that $\eta(r) = cI_m(\sigma r)$ for some $c > 0$, one can easily check that $g(r)$ and $K(r)$ satisfy the following conditions:

$$\begin{cases} g(r) \in C^1([0,\infty)), \\ g(r) > 0 \quad \text{on } (0,\infty), \\ g(r)^{-1} \notin L^1(0,1), \quad g(r)^{-1} \in L^1(1,\infty), \end{cases} \tag{g}$$

and

$$\begin{cases} K(r) \in C((0,\infty)), \\ K(r) \geq 0 \text{ and } K(r) \not\equiv 0 \text{ on } (0,\infty), \\ h(r)K(r) \in L^1(0,1), \quad g(r)^{1-p}h(r)^pK(r) \in L^1(1,\infty), \end{cases} \tag{K}$$

where $h(r) = g(r)\int_r^\infty g(s)^{-1}ds$. Let

$$G(r) = \frac{2}{p+1}g(r)h(r)K(r) - \int_0^r g(s)K(s)ds,$$
$$H(r) = \frac{2}{p+1}h(r)^2\left(\frac{h(r)}{g(r)}\right)^p K(r) - \int_r^\infty h(s)\left(\frac{h(s)}{g(s)}\right)^p K(s)ds,$$

and

$$r_G = \inf\{r \in (0,\infty) \mid G(r) < 0\}, \quad r_H = \sup\{r \in (0,\infty) \mid H(r) < 0\}.$$

The classification theorem of Yanagida and Yotsutani (see Section 2 in [17]) tells us the following:

Proposition 2.1. *Assume that* $g(r)$ *and* $K(r)$ *satisfy the conditions* (g) *and* (K). *Let* $u(r,\alpha)$ *be the solution of* (2.9) *and suppose that* $G(r)\not\equiv 0$ *on* $(0,\infty)$. *If*

$$0 < r_H \le r_G < \infty, \tag{2.10}$$

there exists a positive number α_0 *satisfying the following:*

(A) *For every* $\alpha\in(0,\alpha_0)$, *the solution* $u(r,\alpha)$ *is positive on* $(0,\infty)$ *and satisfies*

$$\lim_{r\to\infty} g(r)u(r,\alpha)/h(r)=\infty.$$

(B) *For* $\alpha=\alpha_0$, *the solution* $u(r,\alpha)$ *is positive on* $(0,\infty)$ *and satisfies*

$$0<\lim_{r\to\infty} g(r)u(r,\alpha)/h(r)<\infty. \tag{2.11}$$

(C) *For every* $\alpha>\alpha_0$, *the solution* $u(r,\alpha)$ *has a zero in* $(0,\infty)$.

A solution of (2.9) has zeros in $(0,\infty)$ if and only if the corresponding solution of (2.1) and (2.2) has zeros in $(0,\infty)$. A solution of (2.9) satisfying (2.11) corresponds to an exponentially decaying solution.

Proposition 2.2. *Let* $1<p<\infty$, $m\in\mathbb{N}$ *and let* $\phi(r,\alpha)$ *and* $u(r,\alpha)$ *be continuous solutions of* (2.1)–(2.2) *and* (2.9) *that are positive on* $(0,\infty)$, *respectively. Then the following three conditions are equivalent:*

(1) $\phi(r,\alpha)$ *decays to* 0 *as* $r\to\infty$.
(2) $\phi(r,\alpha)\sim K_m(\sigma r)$ *as* $r\to\infty$.
(3) $u(r,\alpha)$ *satisfies* (2.11).

Proof. By (2.5), we have $g(r)\sim e^{2\sigma r}$ and $\int_r^\infty g(s)^{-1}ds\sim e^{-2\sigma r}$ as $r\to\infty$. Hence (2.11) is equivalent to $\lim_{r\to\infty} e^{2\sigma r}u(r,\alpha)\in(0,\infty)$. This is equivalent to $\phi(r,\alpha)\sim r^{-\frac12}e^{-\sigma r}$ since $\phi(r,\alpha)=\eta(r)u(r,\alpha)$. Thus we show the equivalence of (2) and (3).

If $\phi(r,\alpha)\to 0$ as $r\to\infty$, then $\phi(r,\alpha)$ decays at the same rate as $K_m(\sigma r)$. Hence (1) and (2) are equivalent. □

Propositions 2.1 and 2.2 tell us that it suffices to show (2.10) in order to obtain Theorem 1.1.

Lemma 2.3. *Let* $1<p<\infty$ *and* $m\in\mathbb{N}$. *Then* (2.10) *holds.*

To prove Lemma 2.3, we investigate the shape of $G(r)$ and $H(r)$ as in [8]. Differentiating $G(r)$ and $H(r)$, we have

$$G'(r)=\Big(\int_r^\infty g(s)^{-1}ds\Big)^{-p-1}H'(r)=\frac{2}{p+1}g(r)K(r)\Big(\Phi(r)-\frac{p+3}{2}\Big), \tag{2.12}$$

where

$$\begin{aligned}\Phi(r) &= \left(2g'(r) + \frac{g(r)K'(r)}{K(r)}\right)\int_r^\infty g(s)^{-1}ds \\ &= \left\{2\eta(r)^2 + (p+3)r\eta'(r)\eta(r)\right\}\int_r^\infty g(s)^{-1}ds.\end{aligned} \tag{2.13}$$

Since $g(r)$ and $K(r)$ are positive on $(0,\infty)$, $G(r)$ and $H(r)$ have the same critical points, which satisfy $\Phi(r) = (p+3)/2$.

First, we investigate the asymptotic behavior of $G(r)$ and $H(r)$ as $r \to 0$ and $r \to \infty$.

Lemma 2.4. *Let* $1 < p < \infty$ *and* $m \in \mathbb{N}$. *Then*

$$\lim_{r\to 0} G(r) = 0, \qquad \lim_{r\to\infty} G(r) = -\infty,$$
$$\lim_{r\to 0} H(r) = -\infty, \qquad \lim_{r\to\infty} H(r) = 0.$$

Proof. By the definition of $\eta(r)$ and $g(r)$, we have

$$\eta(r) = r^m + O(r^{m+1}) \quad \text{as } r \to 0, \tag{2.14}$$

and $g(r) = r^{2m+1} + O(r^{2m+2})$ as $r \to 0$. Using these and the fact that $g(r)^{-1} \in L^1(1,\infty)$, we obtain

$$r^{2m}\int_r^\infty g(s)^{-1} = \frac{1}{2m} + O(r) \quad \text{as } r \to 0. \tag{2.15}$$

Thus the definitions of $h(r)$ and $K(r)$, (2.14) and (2.15) imply $\lim_{r\to 0} G(r) = 0$. Furthermore, as $r \to 0$,

$$\begin{aligned}h(r)^2\left(\frac{h(r)}{g(r)}\right)^p K(r) &= r^2\eta(r)^{p+3}\left(\int_r^\infty g(s)^{-1}ds\right)^{p+2} \\ &= (2m)^{-p-2}r^{-m(p+1)+2}(1+O(r)),\end{aligned}$$

$$\begin{aligned}h(r)\left(\frac{h(r)}{g(r)}\right)^p K(r) &= r\eta(r)^{p+1}\left(\int_r^\infty g(s)^{-1}ds\right)^{p+1} \\ &= (2m)^{-p-1}r^{-m(p+1)+1}(1+O(r)).\end{aligned}$$

By the fact that $h(h/g)^pK \in L^1(1,\infty)$ and the above,

$$\begin{aligned}H(r) = {}& \frac{2}{(p+1)(2m)^{p+2}}r^{-m(p+1)+2} \\ & - \frac{1}{(2m)^{p+1}}\int_r^\infty s^{-m(p+1)+1}ds + O(r^{-m(p+1)+3})\end{aligned}$$

$$= \frac{-4}{(p+1)(2m)^{p+2}\{m(p+1)-2\}} r^{-m(p+1)+2}(1+O(r)) \to -\infty$$

as $r \to 0$.

Next, we prove $\lim_{r\to\infty} G(r) = -\infty$. Since

$$\begin{aligned} \eta(r) &= cr^{-\frac{1}{2}}e^{\sigma r}\left\{1 - \frac{3}{8\sigma r} + O(r^{-2})\right\} \quad \text{as } r \to \infty, \\ \eta'(r) &= c\sigma r^{-\frac{1}{2}}e^{\sigma r}\left\{1 - \frac{7}{8\sigma r} + O(r^{-2})\right\} \quad \text{as } r \to \infty \end{aligned} \tag{2.16}$$

for a $c > 0$ (see [16]), we have

$$g(r) = c^2 e^{2\sigma r}\left(1 - \frac{3}{4\sigma r} + O(r^{-2})\right), \tag{2.17}$$

$$\begin{aligned} \int_r^\infty g(s)^{-1}ds &= c^{-2}\int_r^\infty e^{-2\sigma s}\left(1 + \frac{3}{4\sigma s} + O(s^{-2})\right)ds \\ &= (2\sigma c^2)^{-1}e^{-2\sigma r}\left(1 + \frac{3}{4\sigma r} + O(r^{-2})\right), \end{aligned} \tag{2.18}$$

and

$$\begin{aligned} &2\eta^2 + (p+3)r\eta\eta' \\ =&c^2e^{2\sigma r}\left\{\frac{2}{r} + (p+3)\sigma\Big(1 - \frac{3}{8\sigma r}\Big)\Big(1 - \frac{7}{8\sigma r}\Big) + O(r^{-2})\right\} \\ =&c^2e^{2\sigma r}\left\{\sigma(p+3) - \frac{5p+7}{4r} + O(r^{-2})\right\} \end{aligned} \tag{2.19}$$

as $r \to \infty$. Combination of (2.18) and (2.19) yields

$$\begin{aligned} \Phi(r) =& \frac{1}{2\sigma}\left\{1 + \frac{3}{4\sigma r} + O(r^{-2})\right\}\left\{\sigma(p+3) - \frac{5p+7}{4r} + O(r^{-2})\right\} \\ =& \frac{p+3}{2} - \frac{p-1}{4\sigma r} + O(r^{-2}). \end{aligned}$$

Thus by (2.12) and the above, we have

$$G'(r) = \left\{-\frac{p-1}{2(p+1)\sigma} + O(r^{-1})\right\}\eta^{p+1} \to -\infty \quad \text{as } r \to \infty,$$

which implies $\lim_{r\to\infty} G(r) = -\infty$.

By (2.17) and (2.18), we have $h(r) = O(1)$ and

$$h^2(h/g)^pK = O(r^{-\frac{p-1}{2}}e^{-(p+1)\sigma r}) \quad \text{as } r \to \infty.$$

Since $h(h/g)^p K \in L^1(1,\infty)$ by (K) and (2.18), we have $\lim_{r\to\infty} H(r) = 0$. Thus we have completed the proof. □

Next, we investigate increase and decrease of $G(r)$ and $H(r)$.

Lemma 2.5. *Let $1 < p < \infty$ and $m \in \mathbb{N}$. Then there exists a unique positive number r_* such that $\Phi(r_*) = (p+3)/2$ and*

$$\begin{cases} \Phi(r) > (p+3)/2 & \text{for } r \in (0, r_*), \\ \Phi(r) \le (p+3)/2 & \text{for } r \in (r_*, \infty). \end{cases}$$

Furthermore, $\Phi(r) = (p+3)/2$ at most at one point in $r \in (r_*, \infty)$.

For the sake of the readers' convenience, we recall that the modified Bessel function $I_m(r)$ is monotone increasing on $(0,\infty)$.

Lemma 2.6. *Let $\eta(r) \in C([0,\infty)) \cap C^2((0,\infty))$ be the solution of (2.4) satisfying (2.8). Then $\eta'(r) > 0$ for every $r \in (0,\infty)$.*

Proof. In view of (2.8) and

$$r\eta'(r) = \int_0^r (\sigma^2 s + m^2 s^{-1})\eta(s)ds,$$

we have $\eta'(r) > 0$ on $(0,\infty)$. □

Proof of Lemma 2.5. Let $X(r) = r\eta'(r)/\eta(r)$. Then

$$\Phi(r) = \{2 + (p+3)X(r)\}\,\eta(r)^2 \int_r^\infty g(s)^{-1}ds. \tag{2.20}$$

By the definition of $\eta(r)$ and (2.15), we have

$$\lim_{r\to 0} X(r) = m \quad \text{and} \quad \lim_{r\to 0} \Phi(r) = (p+3)/2 + m^{-1}.$$

So $G(r)$ and $H(r)$ are increasing around $r = 0$. In view of Lemma 2.4, there exists $r > 0$ such that $G'(r) < 0$. Thus there exists a zero of $\Phi(r) - (p+3)/2$ in $(0,\infty)$.

Now, we will show that $\Phi(r) - (p+3)/2$ changes sign only once. For this purpose, we investigate the sign of $\Phi'(r)$ at zeros of $\Phi(r) - (p+3)/2$.

Differentiating $X(r)$ and $\Phi(r)$, we have

$$X'(r) = \sigma^2 r + m^2/r - X(r)^2/r, \tag{2.21}$$

$$\Phi' = \frac{\eta^2}{r}\left\{(p+3)(X^2 + \sigma^2 r^2 + m^2) + 4X\right\}\int_r^\infty g(s)^{-1}ds - \frac{1}{r}\left\{2 + (p+3)X\right\}. \tag{2.22}$$

If r is a zero of $\Phi(r) - (p+3)/2$, it follows from (2.20) that

$$\eta(r)^2 \int_r^\infty g(s)^{-1} ds = \frac{p+3}{2(p+3)X+4}.$$

Substituting the above into (2.22), we have

$$\begin{aligned}\Phi'(r) =& \frac{p+3}{2r\{(p+3)X+2\}} \left\{(p+3)(X^2+\sigma^2 r^2+m^2) + 4X\right\} - \frac{1}{r}\{2+(p+3)X\} \\ =& \frac{1}{2r\{(p+3)X+2\}} \Psi(r),\end{aligned}$$

where

$$\Psi(r) = -(p+3)^2 X(r)^2 - 4(p+3)X(r) + (p+3)^2(\sigma^2 r^2 + m^2) - 8.$$

Lemma 2.6 implies that $X(r)$ is positive on $(0,\infty)$ and that $\Psi(r)$ and $\Phi'(r)$ have the same sign if $\Phi(r) = (p+3)/2$.

Suppose that $\Psi(r)$ has a unique zero $\hat{r}$ on $(0,\infty)$ and that $\Psi(r) < 0$ for $r \in (0,\hat{r})$ and $\Psi(r) > 0$ on $r \in (\hat{r},\infty)$. Then

$$\Phi(r) < \frac{p+3}{2} \quad \text{for } r > \hat{r}. \tag{2.23}$$

In fact, if there exists a zero $\bar{r}$ of $\Phi(r) - (p+3)/2$ with $\bar{r} > \hat{r}$, it follows that $\Phi'(\bar{r}) > 0$ and $\Phi(r) > (p+3)/2$ for every $r \in [\bar{r},\infty)$ because $\Psi(r) > 0$ for every $r > \hat{r}$. Hence $G(r)$ is monotone increasing on $r \in (\bar{r},\infty)$, which contradicts the fact that $\lim_{r\to\infty} G(r) = -\infty$. This shows that $\Phi(r) - (p+3)/2$ has the same sign for every $r \in (\hat{r},\infty)$. Since $\lim_{r\to\infty} G(r) = -\infty$, there exists a sequence $\{r_k\}_{k=1}^\infty$ such that $\lim_{k\to\infty} r_k = \infty$ and $G'(r_k) < 0$. Hence (2.23) follows.

Let r_* be the first zero of $\Phi(r) - (p+3)/2$ on $(0,\infty)$. Since $\Phi(r)$ is continuously differentiable and $\Phi(r) > (p+3)/2$ around $r = 0$, we have $\Phi'(r_*) \le 0$. This shows that $r_* \le \hat{r}$. Suppose that $r_* < \hat{r}$. Then $\Phi'(r_*) < 0$ and there exists an $\varepsilon > 0$ satisfying $\Phi(r) < (p+3)/2$ for $r \in (r_*, r_*+\varepsilon)$. Since $\Psi(r)$ and $\Phi'(r)$ are negative for a zero of $\Phi(r) - (p+3)/2$ with $r \in (0,\hat{r})$, $\Phi(r) - (p+3)/2$ does not have the second zero on $(r_*,\hat{r})$. Combining this with (2.23), we conclude that $\Phi(r) > (p+3)/2$ for $r \in (0, r_*)$ and $\Phi(r) < (p+3)/2$ for $r \in (r_*,\infty)\setminus\{\hat{r}\}$. If $r_* = \hat{r}$ is the case, we can prove that $\Phi(r) > (p+3)/2$ for $r \in (0,r_*)$ and $\Phi(r) < (p+3)/2$ for $r \in (r_*,\infty)$ in the same way.

Thus to complete the proof, it suffices to show the following lemma. □

Lemma 2.7. *There exists a positive number $\hat{r}$ such that $\Psi(r) < 0$ for $r \in (0,\hat{r})$ and $\Psi(r) > 0$ for $r \in (\hat{r},\infty)$.*

Proof. By (2.8), $\lim_{r\to 0} X(r) = m$ and

$$\lim_{r\to 0} \Psi(r) = -4(p+3)m - 8 < 0.$$

As $r \to \infty$, it follows from (2.16) that

$$\frac{r\eta'}{\eta} = \frac{r\sigma r^{-\frac{1}{2}} e^{\sigma r}\left(1 - \frac{7}{8\sigma r} + O(r^{-2})\right)}{r^{-\frac{1}{2}} e^{\sigma r}\left(1 - \frac{3}{8\sigma r} + O(r^{-2})\right)} = \sigma r - \frac{1}{2} + O(r^{-1}).$$

Thus we have

$$\begin{aligned}\Psi(r) =& -(p+3)^2\Big(\sigma r - \frac{1}{2} + O(r^{-1})\Big)^2 - 4(p+3)\Big(\sigma r - \frac{1}{2} + O(r^{-1})\Big)\\ &+ (p+3)^2(\sigma^2 r^2 + m^2) - 8\\ =&(p+3)(p-1)\sigma r + O(1) \to \infty \quad \text{as } r \to \infty.\end{aligned}$$

By the intermediate value theorem, $\Psi(r)$ has a zero on $(0, \infty)$.

Secondly, we will prove uniqueness of an $r_* > 0$ with $\Psi(r_*) = 0$. Using (2.21), we have

$$\begin{aligned}\Psi'(r) =& -2(p+3)^2 XX' - 4(p+3)X' + 2(p+3)^2\sigma^2 r\\ =&\frac{2(p+3)^2}{r}X^3 + \frac{4(p+3)}{r}X^2 - 2(p+3)^2\Big(\sigma^2 r + \frac{m^2}{r}\Big)X\\ &- 4(p+3)\Big(\sigma^2 r + \frac{m^2}{r}\Big) + 2(p+3)^2\sigma^2 r.\end{aligned}$$

For an r with $\Psi(r) = 0$, we have

$$X(r)^2 = -\frac{4}{p+3}X(r) + \sigma^2 r^2 + m^2 - \frac{8}{(p+3)^2}, \tag{2.24}$$

$$\begin{aligned}X(r)^3 =& -\frac{4}{p+3}X(r)^2 + \Big(\sigma^2 r^2 + m^2 - \frac{8}{(p+3)^2}\Big)X(r)\\ =&\left\{\sigma^2 r^2 + m^2 + \frac{8}{(p+3)^2}\right\}X(r) - \frac{4}{p+3}\left\{\sigma^2 r^2 + m^2 - \frac{8}{(p+3)^2}\right\}.\end{aligned} \tag{2.25}$$

Then by (2.24) and (2.25),

$$\Psi'(r) = \frac{2(p+3)}{r}\left\{(p-1)\sigma^2 r^2 - 4m^2 + \frac{16}{(p+3)^2}\right\}$$

for an r with $\Psi(r) = 0$. Let r_0 be the positive root of

$$(p-1)\sigma^2 r_0^2 = 4m^2 - 16(p+3)^{-2} \tag{2.26}$$

and let $r_* = \inf\{r \mid \Psi(r) = 0\}$. Then it follows that $\Psi'(r_*) \geq 0$, which implies $r_* \geq r_0$.

Suppose $r_* > r_0$; then every zero of $\Psi(r)$ with $r \geq r_*$ satisfies $\Psi'(r) > 0$. Thus a zero of $\Psi(r)$ on $(0, \infty)$ is unique.

Next we prove $r_* > r_0$ by contradiction. If $r_* = r_0$, the definitions of r_0 and r_* yield that

$$\Psi(r_*) = \Psi'(r_*) = 0 \quad \text{and} \quad \Psi''(r_*) \leq 0.$$

Making use of (2.21), we rewrite $\Psi''(r)$ as

$$\begin{aligned}
&\Psi''(r) = -\frac{2(p+3)^2}{r^2}X^3 + \frac{6(p+3)^2}{r}X^2X' - \frac{4(p+3)}{r^2}X^2 + \frac{8(p+3)}{r}XX' \\
&- 2(p+3)^2\left(\sigma^2 r + \frac{m^2}{r}\right)X' - 2(p+3)^2\left(\sigma^2 - \frac{m^2}{r^2}\right)X \\
&- 4(p+3)\left(\sigma^2 - \frac{m^2}{r^2}\right) + 2\sigma^2(p+3)^2 \\
= &-\frac{6(p+3)^2}{r^2}X^4 - \frac{2(p+3)(p+7)}{r^2}X^3 + \left\{8(p+3)^2(\sigma^2 + \frac{m^2}{r^2}) - \frac{4(p+3)}{r^2}\right\}X^2 \\
&+ 2(p+3)\left\{-(p-1)\sigma^2 + (p+7)\frac{m^2}{r^2}\right\}X - 2(p+3)^2\left(\sigma^2 r + \frac{m^2}{r}\right)^2 \\
&- 4(p+3)\left(\sigma^2 - \frac{m^2}{r^2}\right) + 2(p+3)^2\sigma^2.
\end{aligned}$$

By (2.24) and (2.25),

$$\begin{aligned}
X(r)^4 = &\left\{\sigma^2 r^2 + m^2 + \frac{8}{(p+3)^2}\right\}X^2(r) - \frac{4}{p+3}\left\{\sigma^2 r^2 + m^2 - \frac{8}{(p+3)^2}\right\}X(r) \\
= &-\frac{8}{p+3}(\sigma^2 r^2 + m^2)X(r) + (\sigma^2 r^2 + m^2)^2 - \frac{64}{(p+3)^4}.
\end{aligned} \tag{2.27}$$

Substituting (2.24), (2.25) and (2.27) into the above, we have

$$\begin{aligned}
\Psi''(r) = &\frac{4(p+3)}{r^2}\left\{-(p-1)(\sigma^2 r^2 + m^2) + (p+3)m^2 - \frac{16}{(p+3)^2}\right\}X(r) \\
&+ \frac{2(p-1)}{r^2}\left\{(p+7)\sigma^2 r^2 + 4m^2 - \frac{16}{(p+3)^2}\right\}
\end{aligned}$$

for an r with $\Psi(r)$. Since $r_* = r_0$, it follows from (2.26) that

$$\Psi''(r_*) = 4(p-1)(p+3)\sigma^2 > 0,$$

which is a contradiction. Thus we have completed the proof. □

Now, we are in position to prove Lemma 2.3.

Proof of Lemma 2.3. By Lemma 2.5, there exists an $r_* > 0$ such that $G(r)$ and $H(r)$ are both increasing on $(0, r_*)$ and decreasing on (r_*, ∞) and take their maximum values at $r = r_*$, respectively. In view of this and Lemma 2.4, we see that $G(r_*)$ and $H(r_*)$ are positive and that $0 < r_H \leq r_G < \infty$. □

Combining Propositions 2.1 and 2.2 and Lemma 2.3, we obtain the following:

Proposition 2.8. *Let $1 < p < \infty$, $m \in \mathbb{N}$ and let $\phi(r, \alpha)$ be the solution to (2.1) and (2.2). Then there exists a positive number α_0 satisfying the following:*

(A) *For every $\alpha \in (0, \alpha_0)$, $\phi(r, \alpha)$ is positive on $(0, \infty)$ and satisfies*

$$\limsup_{r\to\infty} \phi(r, \alpha) > 0.$$

(B) *For $\alpha = \alpha_0$, $\phi(r, \alpha)$ is positive on $(0, \infty)$ and decays exponentially as $r \to \infty$.*

(C) *For every $\alpha > \alpha_0$, the solution $\phi(r, \alpha)$ has a zero in $(0, \infty)$.*

Proof of Theorem 1.1. Theorem 1.1 follows immediately from Proposition 2.8. □

3. STABILITY AND INSTABILITY OF STANDING WAVES

In this section, we study stability and instability of vortex solitons of the form $e^{i(\omega t + m\theta)}\phi_\omega$. Let

$$L_1 = -\Delta_r + \omega + \frac{m^2}{r^2} - p\phi_\omega^{p-1}, \quad L_2 = -\Delta_r + \omega + \frac{m^2}{r^2} - \phi_\omega^{p-1}.$$

To apply [7], we will prove the following:

Proposition 3.1. *The operator L_1 has exactly one negative simple eigenvalue and the kernel of L_2 is spanned by ϕ_ω. Furthermore, there exists a positive number c such that $\sigma(L_1) \setminus \{-\lambda\} \subset [c, \infty)$, $\sigma(L_2) \setminus \{0\} \subset [c, \infty)$.*

In order to prove Proposition 3.1, we will investigate the behavior of a solution to

$$\begin{cases} \psi_{rr} + \frac{1}{r}\psi_r - \left(\omega + \frac{m^2}{r^2}\right)\psi + p|\phi(r,\alpha)|^{p-1}\psi = 0 \quad \text{for } r \in (0,\infty), \\ \lim_{r\to 0} \frac{\psi(r,\alpha)}{r^m} = 1, \quad \lim_{r\to 0} \frac{\psi_r(r,\alpha)}{r^{m-1}} = m, \end{cases} \tag{3.1}$$

as $r \to \infty$.

Lemma 3.2. *Let α_0 be a positive number given in Proposition 2.8. Then there exists an $R \in (0,\infty)$ such that $\psi(R,\alpha_0) = 0$.*

Proof. Let λ be a positive number with $\lambda > 2/(p-1)$, $w(r) = r\partial_r\phi_\omega + \lambda\phi_\omega$ and let

$$R = \sup\left\{r_0 \;\middle|\; w(r) > 0 \text{ for } r \in (0, r_0)\right\}.$$

By (2.2), $\phi_\omega(r)$ is positive and monotone increasing for r close to 0. Thus we have $R > 0$. Since

$$\lim_{r\to\infty} \frac{\phi'_\omega(r)}{\phi_\omega(r)} = -\sqrt{\omega}$$

(see e.g. [4] for the proof),

$$w(r) = \phi_\omega(r)\left(\lambda + \frac{r\partial_r\phi_\omega}{\phi_\omega}\right)$$

becomes negative for large r and $R \in (0,\infty)$.

By a simple computation, we have

$$w_{rr} + \frac{1}{r}w_r - \left(\omega + \frac{m^2}{r^2}\right)w + p\phi_\omega^{p-1}w = 2\omega\phi_\omega + \{\lambda(p-1) - 2\}\,\phi_\omega^p > 0 \quad \text{on } (0,R).$$

Since $w(r)$ oscillates more slowly than $\psi(r,\alpha_0)$ and $w(0) = w(R) = 0$, the Sturm comparison theorem (see Lemma 1 in [10]) implies that $\psi(R_1) = 0$ for an $R_1 \in (0,R)$. □

Lemma 3.3. *Let α_0 be a positive number given in Proposition 2.8 and let $\psi(r,\alpha_0)$ be the solution to* (3.1). *Then $\psi(r,\alpha_0)$ has exactly one zero in $(0,\infty)$.*

Now, we use local manifold theory (see e.g. [3]) to investigate behavior of solutions to (2.1) and (2.2) for large r and α close to 0. Before we prove the lemma above, we introduce several notations. Let $\rho(r)$ be a nonnegative smooth function on $[0,\infty)$ satisfying

$$\rho(r) = \begin{cases} 1 & \text{for } 0 \le r \le 1, \\ 0 & \text{for } r \ge 2, \end{cases}$$

and let $\rho_\varepsilon(r) = \rho(r/\varepsilon)$. Let (x, y) be a solution to

$$\begin{cases} x_r = y, \\ y_r = \omega x + F(x, y, r, r_0), \\ x(0) = x_0, \quad y(0) = y_0, \end{cases} \tag{3.2}$$

where ε is a positive number and

$$F(x, y, r, r_0) = -\frac{1}{r + r_0} y + \frac{m^2}{(r + r_0)^2} x - \rho_\varepsilon |x|^{p-1} x.$$

Let $(x(r), y(r))$ be a solution to (3.2) with $|x(r)| < \varepsilon$. Then $x(r + r_0)$ is a solution to (2.1). We recall that the set of initial data such that solutions to (3.2) tends to $(0, 0)$ exponentially as $r \to \infty$ is a graph over $X_s := \{(x, y) \in \mathbb{R}^2 \,|\, y = -\sigma x\}$ and ‘almost ’tangent to X_s at $(0, 0)$.

The function F is Lipschitz continuous in x and y and satisfies

$$|F(x_2, y_2, r, r_0) - F(x_1, y_1, r, r_0)| \le L_F(|x_2 - x_1| + |y_2 - y_1|),$$

where $L_F = 1/r_0 + m^2/r_0^2 + (p + \sup|\rho'(r)|)\varepsilon^{p-1}$. Note that L_F tends to 0 as $\varepsilon \to 0$ and $r_0 \to \infty$. Let

$$W^s(r_0) = \{(x_0, y_0) : \text{the solution to (3.2) satisfies} \sup_{r \ge 0} e^{\frac{\sigma}{2} r}(|x(r)| + |y(r)|) < \infty\}.$$

By the use of the contraction mapping principle, we see that $W^s(r_0)$ is a graph over X_s if L_F is sufficiently small (see e.g. [3, 6]).

Lemma 3.4. *Let $\bar{\varepsilon} > 0$ be a sufficiently small number and $\bar{r} > 0$ be a sufficiently large number. Suppose that $\varepsilon \in (0, \bar{\varepsilon})$ and $r_0 \ge \bar{r}$. Then there exists a C^1-function $h(x, r_0)$ such that*

$$W^s(r_0) = \{(x, y) \,|\, y = -\sigma x + h(x, r_0)\}, \tag{3.3}$$

and satisfies

$$|h(x_2, r_0) - h(x_1, r_0)| \le C L_F |x_2 - x_1|, \tag{3.4}$$

where C is a positive number depending only on $\bar{\varepsilon}$ and $\bar{r}$.

Remark 3.1. Suppose that $(x_0, y_0) \in W^s(r_0)$ and that the solution $(x(r), y(r))$ to (3.2) remains in $U := \{(x, y) \,|\, |x| \le \varepsilon\}$ for every $r \ge 0$. Then $x(r + r_0)$ is a solution to (2.1) decaying exponentially as $r \to \infty$. In particular, $(\phi(r_0, \alpha_0), \phi_r(r_0, \alpha_0)) \in W^s(r_0)$ for sufficiently large r_0.

Proof of Lemma 3.3. First, we remark that $\psi(r,\alpha) = \frac{\partial\phi}{\partial\alpha}(r,\alpha)$. Suppose that $\psi(r,\alpha_0)$ has two or more zeros in $(0,\infty)$ and that the first two zeros of $\psi(r,\alpha_0)$ are at r_1 and r_2. Then $\frac{\partial\phi}{\partial\alpha}(r,\alpha_0) > 0$ on $(0,r_1)$, $\frac{\partial\phi}{\partial\alpha}(r,\alpha_0) < 0$ on (r_1,r_2) and $\frac{\partial\phi}{\partial\alpha}(r,\alpha_0) > 0$ on some interval to the right of r_2. Then for $\alpha < \alpha_0$ but close enough to α_0, the solutions $\phi(r,\alpha)$ intersect $\phi(r,\alpha_0)$ at least twice. Let the first two intersections be at $R_1(\alpha)$ and $R_2(\alpha)$. Note that

$$\phi(R_2(\alpha),\alpha) = \phi(R_2(\alpha),\alpha_0) \quad \text{and} \quad \phi_r(R_2(\alpha),\alpha) < \phi_r(R_2(\alpha),\alpha_0). \tag{3.5}$$

Now decrease α continuously to 0. The intersection points $R_1(\alpha)$ and $R_2(\alpha)$ depend continuously on α and cannot coalesce. Otherwise $\phi(r,\alpha)$ and $\phi(r,\alpha_0)$ would become tangent, which contradicts uniqueness of solutions to (2.1) and (2.2).

Since $\phi(r,\alpha)$ depends continuously on α and $\phi(r,0)\equiv 0$, $R_1(\alpha)\to\infty$ as $\alpha\to 0$.

Since $\phi_\omega(r)$ and $\phi'_\omega(r)$ decay exponentially as $r\to\infty$, it follows from (2.1) that there exists an $r_1 > 0$

$$r\phi'_\omega(r) = -\int_r^\infty \left\{ s(\omega\phi_\omega(s) - \phi_\omega(s)^p) + \frac{m^2}{s} \right\} ds < 0$$

for $r \ge r_1$. Hence if $\alpha > 0$ is sufficiently small, then $\phi_r(R_2(\alpha),\alpha_0) < 0$.

Let $\vec{\phi}(r,\alpha) = (\phi(r,\alpha),\phi_r(r,\alpha))$. Let $\bar{\varepsilon}$ and $\bar{r}$ be positive numbers in Lemma 3.4 and $U = \{(x,y) \,|\, |x| \le \varepsilon\}$. Since $\vec{\phi}(r,\alpha_0) = O(K_m(\sigma r))$ as $r\to\infty$, there exists $r_3 > \bar{r}$ such that $\vec{\phi}(r,\alpha_0) \in W^s(r)\cap U$ for $r \ge r_3$. For sufficiently small α, $R_2(\alpha) \ge r_3$. By Lemma 3.4, the Lipschitz coefficient of $h(x,r_0)$ with respect to x tends to 0 as $r_0\to\infty$ and $\varepsilon\to 0$. Combining this fact with (3.3)–(3.5), we see that $\vec{\phi}(R_2(\alpha),\alpha)\in U$ and lies under $W^s(R_2(\alpha))\cap U$. As long as $\vec{\phi}(r,\alpha)$ remains in the fourth quadrant $\{(x,y)\,|\, x>0, y<0\}$ for $r\ge R_2(\alpha)$, both $\phi(r,\alpha)$ and $-\phi_r(r,\alpha)$ are positive and decreasing because

$$\phi_{rr}(r,\alpha) = -\frac{1}{r}\phi_r(r,\alpha) + \left(\omega + \frac{m^2}{r^2} - \phi(r,\alpha)^{p-1}\right)\phi(r,\alpha) > 0,$$

and $\vec{\phi}(r,\alpha)$ remains in U. Since $W^s(r_0)\cap U$ and $\vec{\phi}(r_0,\alpha)$ depend continuously on r_0, $\vec{\phi}(r_0,\alpha)$ lies under $W^s(r_0)\cap U$ for $r_0 \ge R_2(\alpha)$ as long as $\vec{\phi}(r_0,\alpha)$ remains in the fourth quadrant. Indeed if $\vec{\phi}(r,\alpha)$ goes across $W^s(r)$ at $r = r_4 > R_2(\alpha)$, it follows that $\vec{\phi}(r_4,\alpha)\in W^s(r_4)\cap U$ and $\phi(r,\alpha)$ decays to 0 exponentially as $r\to\infty$. But this contradicts Proposition 2.8. Thus

$\vec{\phi}(r,\alpha)$ moves to the third quadrant $\{(x,y)\,|\,x<0, y<0\}$ in finite time or tends to $(0,0)$ as $r\to\infty$, which also contradicts Proposition 2.8. $\square$

Next, we will show that $\psi(r,\alpha_0)\to-\infty$ as $r\to\infty$. For this purpose, we make use of the Pohozaev identity. Let $u(r,\alpha)$ be the solution to (2.9) and $v(r,\alpha)=\frac{\partial u}{\partial\alpha}(r,\alpha)$. Then $v(r,\alpha)$ is a solution to

$$(gv_r)_r + pgK|u(r)|^{p-1}v(r)=0,\quad v(0)=1,\quad v_r(0)=0. \tag{3.6}$$

Let

$$P(r)=gu_r(hu_r+u)+\frac{2}{p+1}ghK|u|^{p+1},$$
$$Q(r)=2ghu_rv_r+g(u_rv+uv_r)+2ghK|u|^{p-1}uv.$$

Lemma 3.5. *Let $u(r)$ be a positive solution to* (2.9) *satisfying* (2.11) *in Proposition 2.1. Then* $\lim_{r\to\infty}P(r)=0$ *and* $P(r)>0$ *for every* $r\in(0,\infty)$.

Proof. Since

$$\frac{dP}{dr}(r)=\frac{2}{p+1}\left(\Phi(r)-\frac{p+3}{2}\right)gK|u|^{p+1}, \tag{3.7}$$

it follows from Lemma 2.5 that $P(r)$ is increasing on $(0,r_*)$ and decreasing on (r_*,∞). By (2.9), (2.14)–(2.18) and the definitions of g and h, we have $P(0)=0$. and $\lim_{r\to\infty}P(r)=0$. Thus we have proved the lemma. $\square$

Lemma 3.6. *Let $u(r)=u(r,\alpha_0)$ and $v(r)=v(r,\alpha_0)$. Then*

$$Q(r)=(p+1)\frac{v}{u}P(r)-(p+1)\int_0^r P(s)\left(\frac{v}{u}\right)_r ds. \tag{3.8}$$

Proof. By (2.9), (3.6), (2.14), (2.15) and the definitions of g and h, $Q(0)=0$. Differentiating (3.7) with respect to α, we have

$$\frac{dQ}{dr}(r)=(2\Phi-p-3)\,gK|u|^{p-1}uv,$$

which equals the derivative of the right hand side of (3.8). $\square$

Lemma 3.7. *Let $1<p<\infty$. Then*

$$\psi(r,\alpha_0)\sim -I_m(\sigma r)\quad \text{as } r\to\infty. \tag{3.9}$$

Proof. We abbreviate $u(r,\alpha_0)$ and $v(r,\alpha_0)$ as $u(r)$ and $v(r)$, respectively. Since $\phi(r,\alpha_0)$ decays exponentially as $r\to\infty$, a solution to (3.1) satisfies one of the following:

(a) $\lim_{r\to\infty}\psi(r,\alpha_0)=\infty$, that is,

$$\lim_{r\to\infty}\frac{\psi(r,\alpha_0)}{I_m(\sigma r)}=\lim_{r\to\infty}\frac{\psi_r(r,\alpha_0)}{\sigma I_m'(\sigma r)}\in(0,\infty).$$

(b) $\lim_{r\to\infty}\psi(r,\alpha_0)=0$, that is,

$$\lim_{r\to\infty}\frac{\psi(r,\alpha_0)}{K_m(\sigma r)}=\lim_{r\to\infty}\frac{\psi_r(r,\alpha_0)}{\sigma K_m'(\sigma r)}\in(-\infty,\infty).$$

(c) $\lim_{r\to\infty}\psi(r,\alpha_0)=-\infty$, that is,

$$\lim_{r\to\infty}\frac{\psi(r,\alpha_0)}{I_m(\sigma r)}=\lim_{r\to\infty}\frac{\psi_r(r,\alpha_0)}{\sigma I_m'(\sigma r)}\in(-\infty,0).$$

Suppose to the contrary that (b) holds. Since $\psi(r,\alpha_0)=\eta(r)v(r)$, we have $-v(r)$, $v_r\sim e^{-2\sigma r}$ as $r\to\infty$. Furthermore, u, $-u_r\sim e^{-2\sigma r}$. Combining this with (2.11) and (2.16)–(2.18), we obtain $\lim_{r\to\infty}Q(r)=0$.

Now, let $R(r)=g(uv_r-u_rv)$. By (2.9) and (3.6),

$$R_r=-(p-1)gK|u|^{p-1}uv. \tag{3.10}$$

Lemma 3.3 and (3.10) imply that $R(r)$ is decreasing on $(0,\rho)$ and increasing on (ρ,∞), where ρ is the zero of $v(r)$. Note that $v(r)$ and $\psi(r,\alpha_0)$ have the same sign. On the other hand, we have $R(0)=\lim_{r\to\infty}R(r)=0$. Hence it follows that $R(r)<0$ on $(0,\infty)$.

By Lemma 3.6,

$$\frac{1}{p+1}Q(r)=\frac{v}{u}P(r)-\int_0^r\frac{P(s)R(s)}{g(s)u(s)^2}ds. \tag{3.11}$$

Since $v(r)\sim e^{-2\sigma r}$ and $u(r)$ satisfies (2.11), $\limsup_{r\to\infty}|v(r)/u(r)|<\infty$. Lemma 3.5 implies that the first term of (3.11) tends to 0 as $r\to\infty$. On the other hand, Lemma 3.5 and the fact that $R(r)<0$ on $(0,\infty)$ imply that the integrand of the second term of (3.11) is negative. Thus there exists positive numbers r_1 and δ such that $Q(r)\geq\delta$ for $r\in(r_1,\infty)$. But this contradicts $\lim_{r\to\infty}Q(r)=0$. Hence (b) does not hold.

Lemma 3.3 shows that (a) does not hold. As a result, we have (c). Thus we have completed the proof. □

Now, we are in position to prove Proposition 3.1.

Proof of Proposition 3.1. Since $L_2\phi_\omega=0$ and $\phi_\omega(r)$ is positive on $(0,\infty)$, it follows from the Sturm comparison theorem that 0 is a simple eigenvalue of L_2 and $\sigma(L_2)\setminus\{0\}\subset[c,\infty)$ for a $c>0$. Furthermore, Lemmas 3.3 and 3.7

yield that L_1 has a simple negative eigenvalue $-\lambda$ and that $\sigma(L_1) \setminus \{-\lambda\} \subset [c, \infty)$. □

Finally, we prove Theorem 1.2.

Proof of Theorem 1.2. Let X be a Hilbert space equipped with the norm

$$\|u\|_X^2 := \int_0^\infty \left(|u_r(r)|^2 + \frac{|u(r)|^2}{r^2} + |u(r)|^2 \right) r dr.$$

We define the functional E, $Q\colon X \to \mathbb{R}$ by

$$E(w) = \int_0^\infty \left(|w_r|^2 + \frac{m^2}{r^2}|w|^2 - \frac{2}{p+1}|w|^{p+1} \right) r dr,$$

$$Q(w) = \int_0^\infty |w(r)|^2 r dr.$$

Suppose that $u_0(x) = e^{im\theta} w_0(r)$, where u_0 is the initial data of (1.1). Then the solution to (1.1) can be written as $u(x,t) = e^{im\theta} w(r,t)$, and satisfies the conservation laws

$$E(w(t)) = E(w_0) \quad \text{and} \quad Q(w(t)) = Q(w_0).$$

As can be easily seen, $E'(\phi_\omega) + \omega Q'(\phi_\omega) = 0$. Thus we see that Assumptions 1 and 2 in [7] are satisfied.

Set $d(\omega) = E(\phi_\omega) + \omega Q(\phi_\omega)$. Since $\phi_\omega(r) = \omega^{\frac{1}{p-1}} \phi_1(\omega^{\frac{1}{2}} r)$, it follows that $d''(\omega) > 0$ if $1 < p < 3$ and $d''(\omega) < 0$ if $p > 3$.

In view of Proposition 3.1 and the above, we conclude from Theorems 2 and 4.7 in [7] that a standing wave solution $e^{i(\omega t + m\theta)}\phi_\omega$ is stable if $1 < p < 3$ and unstable if $p > 3$ to the perturbation of the form $e^{im\theta} v(r)$. Thus we have completed the proof. □

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