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A RELAXATION RESULT FOR A CLASS OF DEGENERATE HAMILTON-JACOBI EQUATIONS

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Abstract. In this paper we prove a relaxation result for a class of degenerate Hamilton–Jacobi equations; i.e., equations which do not admit a strict subsolution, extending in some directions the results in [6] and [7].

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be an open set, $H : \mathbb{R}^N \to \mathbb{R}$ continuous and positively homogeneous and $H(p) > 0$ for $p \neq 0$. Denote by $\widehat{H}$ the convex envelope of H. In [6], the following result, called by the authors a relaxation property, was proved:

The maximal almost everywhere subsolution (the unique viscosity solution) u of

$$\begin{cases} \widehat{H}(Du) = 1 & x \in \Omega \\ u = 0 & x \in \partial\Omega \end{cases}$$

satisfies

$$\begin{aligned} u(x) = \max\{w(x) :\ & w \in W^{1,\infty}(\Omega) \cap C^0(\overline{\Omega}) \\ & \text{s.t. } w = 0 \text{ on } \partial\Omega,\ H(Dw) \leq 1 \text{ almost everywhere in } \Omega\}. \end{aligned} \tag{1.1}$$

In other words, the maximal almost everywhere subsolution of the convex problem achieves the supremum of the almost everywhere subsolutions of the nonconvex problem.

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In [7] the result was extended to Hamilton–Jacobi equations depending also on x. Besides some condition relating $Z(x) = \{p \in \mathbb{R}^N : H(x,p) \leq 0\}$ to its convex hull, it is assumed that there exists $\gamma > 0$ such that

$$B(0,\gamma) \subset Z(x) \quad \text{for any } x \in \Omega \tag{1.2}$$

or, equivalently, that $w \equiv 0$ is a strict subsolution of the problem (note that w is an almost everywhere subsolution of $H(x, Du) = 0$ if and only if $Dw(x) \in Z(x)$ almost everywhere).

Our aim is to relax assumption (1.2) for a class of coercive Hamilton-Jacobi equations. We prove that if the set $K = \{x \in \Omega : 0 \in \partial Z(x)\}$; i.e., the set where the null function fails to be a strict subsolution, is composed of a finite number of connected components, the relaxation property still holds.

Even if in this paper, as in [6] and [7], viscosity solutions play no role, they are clearly connected with the problem at hand. We recall that (1.2) guarantees the uniqueness of the viscosity solution of the Dirichlet problem [5]. If this condition is violated even at a single point, in general uniqueness does not hold. In this case, the object of interest is the maximal almost everywhere subsolution. In [3], [4], [8], and [9] suitable modifications of the notion of viscosity solution were introduced in order to characterize this function (see also [10] for a control theoretic approach). In particular we will use some results contained in [3] and [4], where a distance function associated to the equation was studied.

We proceed as follows. In Section 2, we state the assumptions and describe some classes of HJ equations satisfying them. In Section 3 we prove the relaxation result splitting the proof in some steps.

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2. Assumptions and preliminaries

Let Ω be an open, bounded subset of $\mathbb{R}^N$ with a locally Lipschitz continuous boundary. We consider a set–valued map $Z : \overline{\Omega} \to P(\mathbb{R}^N)$ continuous with respect to the Hausdorff metric. We assume that there exists $\psi \in C^1(\overline{\Omega})$ such that for any $x \in \Omega$

> $Z(x)$ is compact, contains $D\psi(x)$ and is strictly star-shaped with respect to $D\psi(x)$; i.e., if $p \in Z(x)$, $p \neq D\psi(x)$, and $\lambda \in (0,1)$, then $\lambda p + (1-\lambda)D\psi(x) \in int(Z(x))$.

(2.1)

For every $T > 0$ we denote by $AC_{xy}(T)$ the set of Lipschitz continuous curves $\xi : [0, T] \to \overline{\Omega}$ such that $\xi(0) = x$ and $\xi(T) = y$; i.e., the curves contained in $\overline{\Omega}$ joining x to y in a time T. Define

$$K = \{x \in \overline{\Omega} : \, D\psi(x) \in \partial Z(x)\},$$

i.e., the set of points $x \in \overline{\Omega}$ where $Z(x)$ is degenerate ,with respect to $D\psi(x)$. Note that, by the assumptions on $Z(x)$ and ψ, K is closed. We assume that

$$\begin{array}{l} K \subset \Omega \text{ with } K = \cup_{i=1}^{M} K_i,\ K_i \cap K_j = \emptyset \text{ for } i \neq j \\ \text{and if } x,\ y \in K_i \text{ there exists } \xi \in AC_{xy}(T), \text{ for some } T > 0, \\ \text{such that } \xi(t) \in K_i \text{ a.e. in } [0, T]; \end{array} \tag{2.2}$$

$$Z(x) = \{D\psi(x)\} \qquad \text{if } x \in K; \tag{2.3}$$

$$\begin{array}{c} \text{there exists a neighborhood } K_0 \text{ of } K \text{ and a nonnegative} \\ \text{continuous function } s(x) \text{ with } s^{-1}(0) = K \text{ such that} \\ \text{for any } x \in K_0 \setminus K,\ p \in \partial Z(x),\ q \in N_{Z(x)}(p) \setminus \{0\} \end{array} \tag{2.4}$$

$$(p - D\psi(x))\, \tfrac{q}{|q|} \geq s(x)$$

($N_{Z(x)}(p)$ is the normal cone to $Z(x)$ at p; see [2]).

We introduce a (nonsymmetric) distance associated to the map Z. We set

$$\begin{aligned} \sigma(x, q) \quad &= \sup\{(p - D\psi(x))q : \, p \in Z(x)\} \\ &= \sup\{(p - D\psi(x))q : \, p \in \overline{co}(Z(x))\} \end{aligned} \tag{2.5}$$

(here $\overline{co}$ stands for the closed convex hull). σ is a continuous function in $\overline{\Omega} \times \mathbb{R}^N$, convex and positive homogeneous in q. Moreover $\sigma(x, q) \geq 0$ for any (x, q), and, if $x \in K$, $\sigma(x, q) = 0$ for any $q \in \mathbb{R}^N$. For the proof of the next proposition we refer to [3].

Proposition 2.1. *Let $S : \overline{\Omega} \times \overline{\Omega} \to \mathbb{R}^+$ be defined by*

$$S(x, y) = \inf\left\{\int_0^T \sigma(\xi(t), -\dot{\xi}(t))dt : \xi \in AC_{xy}(T),\ T > 0\right\}. \tag{2.6}$$

Then

i) *$S(x, y) \geq 0$, $S(x, y) \leq S(x, z) + S(z, y)$ for any x, y, $z \in \overline{\Omega}$.*

ii) *If M is such that $Z(x) \subset B(D\psi(x), M)$ for any $x \in \overline{\Omega}$, then*

$$S(x, y) \leq M d_\Omega(x, y) \qquad \textit{for any } x,\ y \in \overline{\Omega}$$

where d_Ω is the Euclidean geodesic distance in $\overline{\Omega}$ (i.e., the distance function defined as in (2.6) *with $\sigma(x, q) = |q|$).*

iii) *If* $x,\ y \in K_i$, *for some* $i = 1, \dots, M$, *then* $S(x,y) = S(y,x) = 0$.

iv) *If* w *is a Lipschitz continuous function, then* $Dw(x) \in \overline{co}(Z(x))$ *almost everywhere in* $\overline{\Omega}$ *if and only if* $w(x) - w(y) \le S(x,y) + \psi(x) - \psi(y)$ *for any* $x,\ y \in \Omega$.

Note that in the definition (2.6) the homogeneity of σ with respect to q makes the integral $\int_0^T \sigma(\xi(t), -\dot{\xi}(t))dt$ independent of rescaling of the trajectory in the time variable.

We set $B_S(K,r) = \{x \in \overline{\Omega} : S(K,x) \vee S(x,K) \le r\}$ where $S(K,x) = \inf\{S(y,x) : y \in K\}$ and $S(x,K)$ is defined similarly.

We illustrate the previous assumptions with some examples. Assumption (2.1) is satisfied if

$$Z(x) = \{p \in \mathbb{R}^N : H(x,p) \le 0\}, \tag{2.7}$$

where $H : \overline{\Omega} \times \mathbb{R}^N \to \mathbb{R}$ is continuous in (x,p), $\liminf_{|p|\to\infty} H(x,p) = +\infty$, there exists $\psi \in C^1(\overline{\Omega})$ such that $H(x, D\psi(x)) \le 0$ for any $x \in \Omega$ and

$$\begin{gathered}\text{if } \lambda \in (0,1),\ p \ne D\psi(x) \text{ such that } H(x,p) \le 0, \\ \text{then } H(x, \lambda p + (1-\lambda)D\psi(x)) < 0.\end{gathered}$$

In this case $K = \{x \in \Omega : H(x, D\psi(x)) = 0\}$.

If K is empty then we are in the nondegenerate case considered in [7]. Assumptions (2.2) and (2.3) imply that two points on the same connected component of K have 0 distance S (see (iii) in Proposition 2.1).

Assumption (2.4) was introduced in [4]. It guarantees that if we move out of $Z(x)$ in the direction of a normal to $\partial Z(x)$ we get far from $Z(x)$. Moreover (2.4) implies the following property which guarantees that the relaxation result in [7] holds in $\Omega \setminus B_S(K,\varepsilon)$ for any $\varepsilon > 0$

$$\begin{aligned}&\text{For any } \varepsilon, \text{ there exists } \theta_\varepsilon > 1 \text{ and for each } \theta \in (1, \theta_\varepsilon) \\ &\text{a constant } \delta \text{ such that if } x, y \in \Omega \setminus B_S(K,\varepsilon) \text{ and } |x-y| \le \delta, \\ &\text{then } Z(y) \cap \partial(\overline{co}Z(y)) \subset \theta Z(x) + (1-\theta)\{D\psi(x)\}.\end{aligned} \tag{2.8}$$

To show that (2.4) implies (2.8), we recall that in [4], Proposition 5.2, it was proved that for any $\theta > 1$ and $\varepsilon > 0$, there exists $\overline{\eta} > 0$ such that for any $x \in \Omega \setminus B_S(K,\varepsilon)$ and $\eta < \overline{\eta}$, the set $Z^\eta(x) = \{q : d(q, Z(x)) \le \eta\}$, d being the Euclidean distance in $\mathbb{R}^N$, is contained in $\theta Z(x) + (1-\theta)\{D\psi(x)\}$. Since $Z(x)$ is continuous with respect to the Hausdorff metric, we can find δ sufficiently small such that if $x,\ y \in \Omega \setminus B_S(K,\varepsilon)$ and $|x-y| < \delta$, then $Z(y) \subset Z^\eta(x) \subset \theta Z(x) + (1-\theta)\{D\psi(x)\}$. From this (2.8) easily follows.

Hypothesis (2.4) is for example satisfied if the Hamiltonian expressing $Z(x)$ is Lipschitz continuous in p and satisfies (see [1] and [4])

> there exist δ, β positive and a neighborhood K_0 of K such that for any $x \in K_0 \setminus K$ and for almost every p satisfying $H(x,p) < \delta$
> $$D_pH(x,p)(p - D\psi(x)) \geq \beta|p - D\psi(x)|.$$

Another example of a problem satisfying (2.1)–(2.4) is given by the class of equations considered in [8]. In this case $Z(x)$ is expressed by a Hamiltonian $H(x,p) = F(x,p) - f(x)$ where f is nonnegative and F satisfies

$$\begin{aligned}
&F(x,0) = 0, \qquad F(x,p) > 0 \qquad \text{if } |p| > 0\\
&F(x,\lambda p) \leq \lambda^\beta F(x,p) \qquad \text{for any } \lambda \in (0,1),\ (x,p) \in \Omega \times \mathbb{R}^N\\
&|F(x,p_1) - F(x,p_2)| \leq L|p_1 - p_2| \quad x \in \overline{\Omega},\ p_1,\ p_2 \in \mathbb{R}^N
\end{aligned}$$

for some $\beta > 0$. The set K is given by $\{x \in \Omega : f(x) = 0\}$ and must satisfy (2.2).

Note that all the conditions we have assumed depend only on $Z(x)$. In general, there are several Hamiltonians representing the same set–valued map $Z(x)$ in the sense of (2.7); for example, the signed distance $d^*(p, Z(x)) := d(p, Z(x)) - d(p, \mathbb{R}^N \setminus Z(x))$.

3. The relaxation result

We consider a continuous function $g : \partial\Omega \to \mathbb{R}$ satisfying the compatibility condition

$$g(x) - g(y) \leq S(x,y) + \psi(x) - \psi(y) \qquad x, y \in \partial\Omega \tag{3.1}$$

and the function

$$u(x) = \psi(x) + \min\{g(y) + S(x,y) - \psi(y) :\ y \in \partial\Omega\} \qquad x \in \overline{\Omega}. \tag{3.2}$$

It is shown in [3] and [4] that u is the maximal almost everywhere subsolution of the problem

$$\begin{cases} \widehat{H}(x, Du) = 0 & x \in \Omega,\\ u = g & x \in \partial\Omega, \end{cases}$$

where $\widehat{H}$ is a Hamiltonian representing $\overline{co}(Z(x))$ in the sense of formula (2.7). Therefore,

$$\begin{aligned} u(x) = \max\{w(x) :\ & w \in W^{1,\infty}(\Omega) \cap C^0(\overline{\Omega}) \text{ s.t.}\\ & w \leq g \text{ on } \partial\Omega,\ Dw(x) \in \overline{co}(Z(x)) \text{ a.e. in } \Omega\}. \end{aligned} \tag{3.3}$$

Our relaxation result is

Theorem 3.1.

$$u(x) = \sup\{w(x) :\ w \in W^{1,\infty}(\Omega) \cap C^0(\overline{\Omega})\ \text{s.t.}\ w \le g \text{ on } \partial\Omega,\ Dw(x) \in Z(x) \text{ a.e. in } \Omega\}. \tag{3.4}$$

Proof. To simplify the proof we assume that $\psi \equiv 0$, $g \equiv 0$, moreover, that K is composed of only one connected component (for the general case see Remark 3.2 at the end of the proof).

Since $Z(x) \subset \overline{co}(Z(x))$, by (3.3)

$$u(x) \ge \sup\{w(x) :\ w \in W^{1,\infty}(\Omega) \cap C^0(\overline{\Omega})\ \text{s.t.}\ w \le g \text{ on } \partial\Omega,\ Dw(x) \in Z(x) \text{ a.e. in } \Omega\}.$$

We denote by $\mathcal{S}$ the class of function in the right-hand side of (3.4). Since $g \equiv 0$ and $\psi \equiv 0$, this class contains the function $w \equiv 0$, hence $u \ge 0$ in $\overline{\Omega}$. Our aim is to show that for any $\eta > 0$, there exists a function $w_\eta \in \mathcal{S}$ such that

$$w_\eta(x) \le u(x) \le w_\eta(x) + \eta \qquad x \in \overline{\Omega}. \tag{3.5}$$

The proof is split into some steps.

Step 1: In the first step we show that u can be approximated by the maximal subsolution of another problem whose $Z(x)$ are convex in a neighborhood of K.

We define a function $r : \Omega \to \mathbb{R}^+$ by

$$r(x) = \sup\{r :\ B(0,r) \subset Z(x)\}. \tag{3.6}$$

The function $r(\cdot)$ is continuous in Ω, $r(x) > 0$ if $x \notin K$ and $r(x) = 0$ if $x \in K$; moreover $B(0, r(x)) \subset Z(x)$ for any $x \in \overline{\Omega}$ (see [4], Lemma 3.1).

For any $n \in \mathbb{N}$, let $\eta_n : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth function such that $\eta_n(t) = 0$ for $t \le 1/n$, $\eta_n(t) = 1$ for $t \ge 2/n$ and $0 \le \eta_n \le 1$. In view of Assumption (2.2) we can take n such that $B_S(K, 2/n) \subset \Omega$ and we set

$$Z_n(x) = [\eta_n\big(S(K,x) \vee S(x,K)\big)Z(x)] \cup B(0, r(x)). \tag{3.7}$$

Observe that

i) $Z_n(x) \subset Z(x)$ for any $x \in \Omega$, $Z_n(x) = B(0, r(x))$ if $x \in B_S(K, 1/n)$, $Z_n(x) = Z(x)$ if $x \in \Omega \setminus B_S(K, 2/n)$;

ii) $Z_n : \overline{\Omega} \to P(\mathbb{R}^N)$ is continuous with respect to the Hausdorff metric and for any $x \in \Omega$, $Z_n(x)$ is compact, contains 0, and is strictly star-shaped with respect to 0;

iii) the set $K_n = \{x \in \overline{\Omega} :\ 0 \in \partial Z_n(x)\}$ coincides with K;

iv) Assumption (2.8) is satisfied with $Z_n(x)$ in place of $Z(x)$.

Properties i)-iii) are easy to verify. For property iv), observe that if $0 < S(K,x) \vee S(x,K) \leq 1/n$, then $Z_n(x) = B(0,r(x))$ and (2.4) is satisfied with $s(x) = r(x)$. For $1/n < S(K,x) \vee S(x,K) \leq 2/n$, either $Z_n(x) = B(0,r(x))$, and therefore (2.4) is satisfied, or $\eta_n\big(S(K,x) \vee S(x,K)\big)Z(x) \setminus B(0,r(x))$ is not empty. Since, by continuity, $r(x) \geq \delta$ for $x \notin B_S(K,1/n)$, it follows that in the latter case there exists $\epsilon_n > 0$, independent of x, such that $\eta_n\big(S(K,x) \vee S(x,K)\big) \geq \epsilon_n$. Hence (2.4) follows since $\eta_n(S(K,x) \vee S(x,K))Z(x)$ and $B(0,r(x))$ satisfy this assumption with $(\eta_n(S(K,x) \vee S(x,K))s(x)$ and $r(x)$ in place of $s(x)$, respectively. Finally, if $2/n < S(K,x) \vee S(x,K)$, $Z_n(x) = Z(x)$ hence (2.4) is satisfied by assumption.

Define σ_n and S_n as in (2.5) and (2.6) with Z_n in place of Z and set

$$\begin{aligned} u_n(x) = \max\{w(x) :\ & w \in W^{1,\infty}(\Omega) \cap C^0(\overline{\Omega}) \text{ s.t.} \\ & w \leq 0 \text{ on } \partial\Omega,\ Dw(x) \in \overline{co}(Z_n(x)) \text{ a.e. in } \Omega\}. \end{aligned}$$

We want to show that u_n converges uniformly to u in $\overline{\Omega}$. Recalling the representation formula (3.2) (with $g \equiv 0$ and $\psi \equiv 0$) and the corresponding one for u_n; i.e., $u_n(x) = \min\{S_n(x,y) : y \in \partial\Omega\}$, it is sufficient to prove that $S_n(x,y) \to S(x,y)$ uniformly for $y \in \partial\Omega$, $x \in \overline{\Omega}$.

Since $Z_n(x) \subset Z(x)$ for any $x \in \overline{\Omega}$, we have that $S_n(x,y) \leq S(x,y)$ in $\overline{\Omega} \times \overline{\Omega}$. Given $\varepsilon > 0$, let $x \in \overline{\Omega}$, $y \in \partial\Omega$ and $\xi \in AC_{xy}(T)$, for some $T > 0$, be such that

$$S_n(x,y) \geq \int_0^T \sigma_n(\xi, -\dot{\xi})dt - \varepsilon. \tag{3.8}$$

If $\xi([0,T]) \cap B_S(K,2/n) = \emptyset$, then $\int_0^T \sigma_n(\xi,-\dot{\xi})dt = \int_0^T \sigma(\xi,-\dot{\xi})dt$ and therefore $S_n(x,y) \geq S(x,y) - \varepsilon$. Otherwise, set

$$\begin{aligned} t_- &= \inf\{t \in [0,T] : \xi(t) \in B_S(K,2/n)\} \\ t_+ &= \sup\{t \in [0,T] : \xi(t) \in B_S(K,2/n)\}. \end{aligned}$$

Let $\overline{\xi} \in AC_{\xi(t_-)\xi(t_+)}(t_+ - t_-)$ be such that

$$S(\overline{\xi}(t_-), \overline{\xi}(t_+)) \geq \int_0^{t_+ - t_-} \sigma(\overline{\xi}, -\dot{\overline{\xi}})dt - \varepsilon$$

and define a trajectory $\gamma \in AC_{xy}(T)$ by

$$\gamma(t) = \begin{cases} \xi(t) & t \in [0,t_-] \\ \overline{\xi}(t - t_-) & t \in [t_-, t_+] \\ \xi(t) & t \in [t_+, T]. \end{cases}$$

By (3.8)

$$\begin{aligned} S_n(x,y) &\geq \int_0^T \sigma_n(\gamma, -\dot\gamma)dt - \int_{t_-}^{t_+} \sigma(\overline{\xi}, -\dot{\overline{\xi}})dt - \varepsilon \\ &\geq S(x,y) - S(\xi(t_-), \xi(t_+)) - 2\varepsilon. \end{aligned} \tag{3.9}$$

To estimate $S(\xi(t_-), \xi(t_+))$, let z_1, z_2 be such that $S(K, \xi(t_+)) = S(z_1, \xi(t_+))$ and $S(\xi(t_-), K) = S(\xi(t_-), z_2)$ and use iii) of Proposition 2.1 to estimate

$$S(\xi(t_-), \xi(t_+)) \leq S(\xi(t_-), z_2) + S(z_2, z_1) + S(z_1, \xi(t_+)) \leq \frac{4}{n}. \tag{3.10}$$

Substituting the previous inequality in (3.9), we get

$$S_n(x,y) \geq S(x,y) - \frac{4}{n},$$

since ε is arbitrary. Therefore, $S_n(x,y) \leq S(x,y) \leq S_n(x,y) + 4/n$ if $x \in \overline{\Omega}$, $y \in \partial\Omega$. We now fix n in such a way that

$$u_n(x) \leq u(x) \leq u_n(x) + \eta \qquad x \in \overline{\Omega}. \tag{3.11}$$

Moreover, by Proposition 2.1.iv) we get

$$|u_n(x) - u_n(y)| \leq S_n(x,y) \vee S_n(y,x) \leq S(x,y) \vee S(y,x),$$

so we can take n such that for any x, $y \in B_S(K, 1/n)$, $|u_n(x) - u_n(y)| \leq \eta$. Hence,

$$\min_{B_S(K,1/n)} u_n \leq u_n(x) \leq \min_{B_S(K,1/n)} u_n + \eta \qquad \text{for } x \in B_S(K, 1/n). \tag{3.12}$$

Step 2: In this step we apply the relaxation result in [7] in a set where Z_n is nondegenerate.

For $\delta < 1/2n$, where n is fixed from the previous step, we set $\Omega_\delta = \Omega \setminus B_S(K, \delta)$ and we consider the boundary datum

$$g_{n\delta}(x) = \begin{cases} 0 & x \in \partial\Omega \\ \min_{B_S(K,1/n)} u_n & x \in \partial B_S(K, \delta). \end{cases} \tag{3.13}$$

Define the function

$$u_{n\delta}(x) = \min\{g_{n\delta}(y) + S_{n\delta}(x,y) : \, y \in \partial\Omega_\delta\}, \qquad x \in \overline{\Omega}_\delta, \tag{3.14}$$

where

$$S_{n\delta}(x,y) = \inf\left\{ \int_0^T \sigma_n(\xi(t), -\dot\xi(t))dt : \xi \in AC^\delta_{xy}(T), \ T > 0 \right\}$$

and $AC^\delta_{xy}(T)$ is the set of Lipschitz continuous curves connecting x to y in a time T and contained in Ω_δ. We want to show that $u_{n\delta}$ takes the datum

$g_{n\delta}$ on $\partial\Omega_\delta$. Since $u_n \geq 0$ in $\overline{\Omega}$, it is sufficient to show that if $x \in \partial B_S(K,\delta)$, $y \in \partial\Omega$, then

$$g_{n\delta}(x) - g_{n\delta}(y) \leq S_{n\delta}(x,y).$$

Let $\overline{x} \in B_S(K,1/n)$ be such that $u_n(\overline{x}) = \min_{B_S(K,1/n)} u_n$. Hence,

$$g_{n\delta}(x) - g_{n\delta}(y) = u_n(\overline{x}) \leq u_n(x) \leq S_n(x,y) \leq S_{n\delta}(x,y).$$

It follows that $u_{n\delta}$ is a solution of the problem

$$\begin{cases} \widehat{H}_n(x,Du) = 0 & x \in \Omega_\delta \\ u = g_{n\delta} & x \in \partial\Omega_\delta, \end{cases}$$

where $\widehat{H}_n$ is a Hamiltonian representing Z_n in the sense of formula (2.7).

From a comparison theorem for the previous problem (or directly from (3.14) and the corresponding formula for u_n) and recalling (3.12) we get

$$0 \leq \max_{\Omega_\delta}[u_n - u_{n\delta}] \leq \max_{\partial\Omega_\delta}[u_n - u_{n\delta}] \leq \eta. \tag{3.15}$$

Applying the relaxation theorem of [7] in Ω_δ, where Z_n is nondegenerate and satisfies hypothesis (2.8), we can find a function $w_{n\delta} \in W^{1,\infty}(\Omega_\delta) \cap C^0(\overline{\Omega}_\delta)$, $w_{n\delta} \geq 0$, such that

$$w_{n\delta} \leq u_{n\delta} \leq w_{n\delta} + \eta \qquad \text{in } \Omega_\delta, \tag{3.16}$$

$$Dw_{n\delta}(x) \in Z_n(x) \qquad \text{a.e. in } \Omega_\delta. \tag{3.17}$$

By (3.17), $w_{n\delta}$ satisfies

$$w_{n\delta}(x) - w_{n\delta}(y) \leq S_{n\delta}(x,y) \qquad x,y \in \Omega_\delta. \tag{3.18}$$

Step 3: In this last step, we extend $w_{n\delta}$ from $\overline{\Omega} \setminus B_S(K,1/n)$ to all $\overline{\Omega}$ preserving the subsolution property with respect to Z_n.

We first estimate $w_{n\delta}(x) - w_{n\delta}(y)$ for x, $y \in \partial B_S(K,1/n)$. We distinguish two cases

$$i) \qquad S_n(x,\partial B_S(K,\delta)) + S_n(\partial B_S(K,\delta),y) > S_n(x,y).$$

In this case, for ε sufficiently small, we can find a curve $\xi \in AC^\delta_{xy}(T)$, for some $T > 0$, such that

$$S_n(x,y) \geq \int_0^T \sigma_n(\xi,-\dot{\xi})dt - \varepsilon.$$

Hence

$$S_{n\delta}(x,y) \geq S_n(x,y) \geq \int_0^T \sigma_n(\xi,-\dot{\xi})dt - \varepsilon \geq S_{n\delta}(x,y) - \varepsilon$$

and, since ε is arbitrary,

$$w_{n\delta}(x) - w_{n\delta}(y) \le S_n(x,y), \qquad x,y \in \partial B_S(K,1/n). \tag{3.19}$$

ii) $\quad S_n(x, \partial B_S(K,\delta)) + S_n(\partial B_S(K,\delta), y) \le S_n(x,y).$

Let $\xi \in AC_{x,y}(T)$ be such that

$$S_n(x,y) \ge \int_0^T \sigma_n(\xi, -\dot{\xi})dt - \varepsilon$$

and set, if they exist,

$$t_- = \inf\{t \in [0,T] : \xi \in B_S(K,\delta)\}, \quad t_+ = \sup\{t \in [0,T] : \xi \in B_S(K,\delta)\}.$$

Take $\overline{\xi} \in AC^\delta_{\xi(t_-)\xi(t_+)}(t_+ - t_-)$ such that

$$S_{n\delta}(\overline{\xi}(t_-), \overline{\xi}(t_+)) \ge \int_0^{t_+ - t_-} \sigma_n(\overline{\xi}, -\dot{\overline{\xi}})dt - \varepsilon$$

and define a trajectory $\gamma \in AC^\delta_{xy}(T)$ by

$$\gamma(t) = \begin{cases} \xi(t) & t \in [0, t_-] \\ \overline{\xi}(t - t_-) & t \in [t_-, t_+] \\ \xi(t) & t \in [t_+, T]. \end{cases}$$

Estimate

$$\begin{aligned} S_n(x,y) &\ge \int_0^T \sigma_n(\gamma, -\dot{\gamma})dt - \int_{t_-}^{t_+} \sigma(\overline{\xi}, -\dot{\overline{\xi}})dt - \varepsilon \\ &\ge S_{n\delta}(x,y) - S_{n\delta}(\xi(t_-), \xi(t_+)) - 2\varepsilon. \end{aligned} \tag{3.20}$$

Now, recalling that we are in the case *ii)*, that $\delta < 1/2n$ and that $K \subset B_S(K,\delta)$, we have

$$S_n(x,y) \ge S_n(x, \partial B_S(K,\delta)) + S_n(\partial B_S(K,\delta), y) \ge \omega_{n\delta} \ge \omega_0 > 0$$

for any δ sufficiently small. Take a curve $\tilde{\xi}$ connecting $\xi(t_-)$ to $\xi(t_+)$ and contained in $B_S(K,2\delta) \setminus B_S(K,\delta)$. Since $B_S(K,2\delta) \subset B_S(K,1/n)$, then $Z_n(\tilde{\xi}(t)) = B(0, r(\tilde{\xi}(t))$, hence we have

$$\begin{aligned} S_{n\delta}(\xi(t_-), \xi(t_+)) &\le \int_0^T r(\tilde{\xi}(t))|\dot{\tilde{\xi}}(t)|dt \\ &\le \text{diam } (\Omega) \max_{B_S(K,2\delta)\setminus B_S(K,\delta)} r(x) := C_\delta \end{aligned} \tag{3.21}$$

and $C_\delta \to 0$ for $\delta \to 0$. By (3.18), (3.20), and (3.21) and the arbitrariness of ε we obtain

$$w_{n\delta}(x) - w_{n\delta}(y) \le S_{n\delta}(x,y) \le \Big(1 + \frac{C_\delta}{\omega_0}\Big) S_n(x,y), \quad x,y \in \partial B_S(K,1/n). \tag{3.22}$$

From (3.19) and (3.22), it follows that, if we define

$$\overline{w}_{n\delta}(x) = \Big(1 + \frac{C_\delta}{\omega_0}\Big)^{-1} w_{n\delta}(x),$$

then for any $x,\, y \in \partial B_S(K, 1/n)$,

$$\overline{w}_{n\delta}(x) - \overline{w}_{n\delta}(y) \le S_n(x, y). \tag{3.23}$$

Moreover, by (3.17) and since $Z_n(x)$ is star-shaped with respect to 0

$$D\overline{w}_{n\delta}(x) \in Z_n(x) \quad \text{a.e. in } \Omega \setminus B_S(K, \delta) \tag{3.24}$$

and, taking δ sufficiently small,

$$\overline{w}_{n\delta}(x) \le w_{n\delta}(x) \le \overline{w}_{n\delta}(x) + \eta \qquad x \in \overline{\Omega} \setminus B_S(K, \delta). \tag{3.25}$$

Let $v_{n\delta} : B_S(K, 1/n) \to \mathbb{R}$ be defined by

$$v_{n\delta}(x) = \min\{\overline{w}_{n\delta}(y) + \overline{S}_n(x, y) :\, y \in \partial B_S(K, 1/n)\},$$

where

$$\begin{aligned}\overline{S}_n(x, y) = \quad \inf \Big\{ \int_0^T \sigma_n(\xi(t), -\dot{\xi}(t))dt :\, \xi \in AC_{xy}(T) \\ \text{with } \xi(t) \in B_S(K, 1/n) \text{ a.e. },\ T > 0 \Big\}.\end{aligned}$$

Note that, by (3.23), $v_{n\delta} = \overline{w}_{n\delta}$ on $\partial B_S(K, 1/n)$. Moreover,

$$Dv_{n\delta}(x) \in Z_n(x) \qquad \text{a.e. in } B_S(K, 1/n). \tag{3.26}$$

Now define $w_\eta : \overline{\Omega} \to \mathbb{R}$ by

$$w_\eta(x) = \begin{cases} \overline{w}_{n\delta}(x) & x \in \Omega \setminus B_S(K, 1/n) \\ v_{n\delta}(x) & x \in B_S(K, 1/n). \end{cases} \tag{3.27}$$

Therefore, w_η is continuous on $\overline{\Omega}$, nonnegative, and $w_\eta = \overline{w}_{n\delta} = 0$ on $\partial\Omega$. Observe that, since S is locally equivalent to the euclidean distance outside K, then $S(K, x) \vee S(x, K)$ is Lipschitz continuous and therefore the set $\partial B_S(K, 1/n)$ has null Lebesgue measure. Hence, by (3.24) and (3.26),

$$Dw_\eta \in Z(x) \qquad \text{a.e. in } \Omega.$$

Finally w_η satisfies

$$w_n(x) \le u(x) \le w_n(x) + 5\eta,$$

in $\Omega \setminus B_S(K, 1/n)$, by (3.16) and (3.25), and in $B_S(K, 1/n)$, observing that, by (3.25), (3.16), (3.15), (3.12), and (3.11)

$$\begin{aligned} v_{n\delta}(x) &\ge \inf_{\partial B_S(K,1/n)} \overline{w}_{n\delta} \ge \inf_{\partial B_S(K,1/n)} w_{n\delta} - \eta \\ &\ge \inf_{\partial B_S(K,1/n)} u_{n\delta} - 2\eta \ge \inf_{\partial B_S(K,1/n)} u_n - 3\eta \end{aligned}$$

$$\geq u_n(x) - 4\eta \geq u(x) - 5\eta.$$

□

Remark 3.2. If K is composed of a finite number of connected components, we modify Step 1 in the following way. We take n sufficiently small in such a way that $B_S(K_i, 2/n) \cap B_S(K_j, 2/n) = \emptyset$ for $i \neq j$ and we set $t_0 = 0$ and for $i = 1, .., M$

$$t_-^i = \inf\{t \in [t_{i-1}, T] : \xi \in B_S(K_i, 2/n)\}$$
$$t_+^i = \sup\{t \in [t_{i-1}, T] : \xi \in B_S(K_i, 2/n)\}.$$

We proceed as above defining a new trajectory γ given by ξ as in (3.8) in $[0, T] \setminus \cup_{i=1}^M [t_-^i, t_+^i]$ and by a trajectory $\overline{\xi}_i$ satisfying

$$S(\overline{\xi}_i(t_-), \overline{\xi}_i(t_+)) \geq \int_0^{t_+^i - t_-^i} \sigma(\overline{\xi}_i, -\dot{\overline{\xi}}_i) dt - \varepsilon.$$

It is easy to see that $\gamma(t)$ satisfies estimates similar to (3.9) and (3.10) giving in this case

$$S_n(x, y) \geq S(x, y) - \frac{4M}{n}.$$

A similar modification is also needed in Step 3.

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