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COEXISTENCE OF SIMULTANEOUS AND NONSIMULTANEOUS BLOW-UP IN A SEMILINEAR PARABOLIC SYSTEM

Julio D. Rossi
Departamento de Matemática, FCEyN
UBA (1428) Buenos Aires, Argentina

Philippe Souplet
Département de Mathématiques, INSSET
Université de Picardie, 02109 St-Quentin, France
and
Laboratoire de Mathématiques Appliquées, UMR 7641
Université de Versailles, 78035 Versailles, France

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Abstract. We study simultaneous and nonsimultaneous blow-up for solutions of the following system

$$\begin{cases} u_t = \Delta u + u^r + v^p, \\ v_t = \Delta v + v^s + u^q, \end{cases} \qquad \text{in } \Omega \times (0,T),$$

with Dirichlet boundary conditions. We show that, in the range of exponents where either component may blow up alone, there also exist initial data for which both components blow up simultaneously. The proof is based on a continuity argument, which requires upper and lower blow-up estimates, independent of initial data, and continuous dependence of the existence time. In turn, we prove a result of continuous dependence of the existence time, under the assumption of uniform upper blow-up estimates, in the framework of general abstract semiflows.

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1. INTRODUCTION.

We study the blow-up problem for the following system

$$\begin{cases} u_t = \Delta u + u^r + v^p, & \text{in } \Omega \times (0,T), \\ v_t = \Delta v + u^q + v^s, & \text{in } \Omega \times (0,T), \\ u = v = 0, & \text{on } \partial\Omega \times (0,T). \end{cases} \tag{1.1}$$

Here $p, q, r, s > 1$ and Ω stands for a C^2 smooth, bounded domain in $\mathbb{R}^N$.

In what follows, we fix some $\varepsilon \in (0,1)$ and we shall always assume that the initial data satisfy $u_0, v_0 \in C_0(\Omega)$, $u_0, v_0 \geq 0$, $(u_0, v_0) \not\equiv (0,0)$, along with the following assumptions:

$$\begin{cases} u_0, v_0 \in C^2(\overline{\Omega}), \quad \Delta u_0 = \Delta v_0 = 0 \ \text{ on } \partial\Omega, \\ \Delta u_0 + (1-\varepsilon)(u_0^r + v_0^p) \geq 0 \ \text{ in } \Omega, \\ \Delta v_0 + (1-\varepsilon)(v_0^s + u_0^q) \geq 0 \ \text{ in } \Omega \end{cases} \tag{1.2}$$

(such initial data can be constructed easily – see Remark 4.2). Problem (1.1) admits a unique, maximal, classical solution $u, v \geq 0$. We shall denote by $T = T(u_0, v_0) \in (0, \infty]$ its maximal existence time.

For parabolic systems like (1.1), it can happen that solutions become unbounded in finite time; that is,

$$T < \infty \quad \text{and} \quad \limsup_{t \to T} \big(\|u(t)\|_\infty + \|v(t)\|_\infty\big) = \infty.$$

This is, for instance, the case under assumption (1.2) (see Proposition 2.1 below). This phenomenon is known as blow-up and has been widely studied; see for example [6, 11, 14, 19] and references therein. The system (1.1) has been considered from the point of view of blow-up and global existence in [20, 21] and a similar system with reaction and absorption terms was studied in [3].

In what follows, we shall say that u (respectively, v) blows up if

$$T < \infty \ \text{ and } \ \limsup_{t \to T} \|u(t)\|_\infty = \infty \quad \text{(respectively,} \quad \limsup_{t \to T} \|v(t)\|_\infty = \infty).$$

We note that a priori there is no reason for both components of the system to blow up simultaneously. In fact it could happen that one of the components blows up as $t \to T$, while the other remains bounded on $[0, T)$. This phenomenon is called *nonsimultaneous blow-up*. For parabolic systems, this phenomenon seems to have been first suggested in [19] performing some numerical experiments. Further mathematical study was carried out in [16, 17, 21], where the range of exponents for which nonsimultaneous blow-up is possible was characterized in several parabolic systems.

For the particular problem (1.1), we know the following:

(1) if $r < q + 1$ (respectively, $s < p + 1$), then v (respectively, u) blows up whenever $T < \infty$. Consequently, if $r < q + 1$ and $s < p + 1$, then blow-up is always simultaneous (see [21, Proposition 2.1] for $\Omega = \mathbb{R}^N$; the argument there can be modified to cover the case of bounded domains);

(2) if $r > q+1$ (respectively, $s > p+1$), then there exist initial data such that u (respectively, v) blows up in finite time and v (respectively, u) remains bounded. Consequently, in the range of exponents

$$r > q + 1 \quad \text{and} \quad s > p + 1, \tag{1.3}$$

either component may have nonsimultaneous blow-up (see [21, Proposition 2.2] and Proposition 4.1 below).

None of the previous works deal with the question whether, for fixed values of the parameters p, q, r, s, it is possible to have both simultaneous and nonsimultaneous blow-up, depending only on the initial data. In particular, in the range (1.3), a natural question is to determine whether simultaneous blow-up may occur. The aim of this paper is to answer this question positively.

Our main result is the following (see Theorem 4.1 for a more detailed statement).

Theorem 1.1. *Assume* (1.3). *Then there exist initial data* (u_0, v_0) *such that both components of the solution of system* (1.1) *blow up simultaneously.*

We believe that simultaneous blow-up in the nonsimultaneous range (1.3) occurs for very exceptional initial data. Namely, for each fixed u_0, this should happen with the initial data $(u_0, \mu v_0)$ for at most one value of $\mu > 0$. However we are presently unable to prove this.

On the other hand, the possibility of simultaneous blow-up in the "half-simultaneous" range $r > q + 1$ and $s \leq p + 1$ does not seem to be known. This and related questions will be investigated elsewhere.

Next, we focus on the blow-up rates for system (1.1). We have obtained the following (see Propositions 2.1 and 2.2 for more precise results).

Theorem 1.2. *Assume* (1.2). *Suppose that blow-up is nonsimultaneous, or that blow-up is simultaneous and* (1.3) *holds. Then there exist constants*

$C_1, C_2 > 0$ *such that*

$$\begin{aligned} &C_1 \le (T-t)^{1/(r-1)} \|u(t)\|_\infty \le C_2, \quad 0 < t < T, \qquad \textit{if } u \textit{ blows up,} \\ &C_1 \le (T-t)^{1/(s-1)} \|v(t)\|_\infty \le C_2, \quad 0 < t < T, \qquad \textit{if } v \textit{ blows up.} \end{aligned} \tag{1.4}$$

Estimates (1.4) in the nonsimultaneous case were proved in [21] for $\Omega = \mathbb{R}^N$ and $r, s < (N+2)/(N-2)$ if $N \ge 3$. As for the simultaneous case, no blow-up rates were given in [21] in the range (1.3), whereas in the (disjoint) range $r < p(q+1)/(p+1)$, $s < q(p+1)/(q+1)$, the *slower* rates

$$\begin{aligned} &C_1 \le (T-t)^{(p+1)/(pq-1)} \|u(t)\|_\infty \le C_2, \quad 0 < t < T, \\ &C_1 \le (T-t)^{(q+1)/(pq-1)} \|v(t)\|_\infty \le C_2, \quad 0 < t < T, \end{aligned} \tag{1.5}$$

were found.

The proof of Theorem 1.1 is based on a continuity argument which can be sketched as follows.

On each vertical line $\{(\lambda u_0, \mu v_0);\ \mu > 0\}$ in the plane of initial data, with λ large, we show that v (respectively, u) blows up alone if $\mu >> 1$ (respectively, $\mu << 1$); cf. Proposition 4.1. Consequently, there is a limiting value μ^* such that $(\lambda u_0, \mu^* v_0)$ lies on the borderline of initial data yielding both types of behavior.

Now assume that T depends continuously on initial data. If we know that any blowing up component is estimated below according to (1.4), *uniformly with respect to initial data*, then we may pass to the limit along some sequences $\mu_n \to \mu^*$ to deduce that both components blow up for the initial data $(\lambda u_0, \mu^* v_0)$.

A key step is thus to establish uniform lower estimates for blowing up components (Proposition 2.1). These estimate are in fact consequences of the corresponding *upper* blow-up estimates in (1.4), also with *uniform constants* (Proposition 2.2). These uniform upper estimates, in turn, also imply the continuity of the existence time T. This is shown in Section 3, as a particular case of a result on general abstract semiflows. The more detailed version of Theorem 1.1 is stated and proved in Section 4.

2. UNIFORM BLOW-UP ESTIMATES

In this section we establish the upper and lower uniform blow-up estimates on (u, v) which are necessary to show the continuity of the existence time and the existence of a simultaneous blow-up solution. By the term "uniform", we here mean that these estimates involve constants which are independent of initial data (provided ε in (1.2) is fixed) and that they hold on the whole existence interval (or at least on an interval near T whose length is independent of initial data). This feature is essential for our purpose of proving existence of simultaneous blow-up solutions by a continuity argument (see Section 4).

Let us begin with the uniform upper blow-up estimate.

Proposition 2.1. *Let (u_0, v_0) satisfy (1.2). Then $T < \infty$ and it holds that*

$$\begin{aligned} \|u(t)\|_\infty &\le C_r(\varepsilon)(T-t)^{-\frac{1}{r-1}}, \quad 0<t<T, \\ \|v(t)\|_\infty &\le C_s(\varepsilon)(T-t)^{-\frac{1}{s-1}}, \quad 0<t<T, \end{aligned} \tag{2.1}$$

where $C_r(\varepsilon) = ((r-1)\varepsilon)^{-1/(r-1)}$.

Let us stress that in the above Proposition, we do not assume that u and v both blow up. Although such an estimate might look trivial for a bounded component, this is actually not quite so, because the constant in the right-hand side is uniform and thus provides information which is not contained in the mere boundedness of this component.

Proof. The proof uses the idea of [9] (see also [5] for a similar idea for the system (1.1) without the terms u^r, v^s). Define the auxiliary functions

$$G = u_t - \varepsilon(u^r + v^p), \quad J = v_t - \varepsilon(v^s + u^q)$$

in $Q_T = \Omega \times (0,T)$. Since $u, v > 0$ in Q_T by the strong maximum principle, parabolic regularity implies that $G, J \in C^{2,1}(Q_T)$, and we also have $G, J \in C(\overline{\Omega} \times [0,T))$, due to $(1.2)_1$. We compute

$$G_t = u_{tt} - \varepsilon r u^{r-1} u_t - \varepsilon p v^{p-1} v_t,$$

and

$$\begin{aligned} \Delta G &= \Delta u_t - \varepsilon r u^{r-1} \Delta u - \varepsilon p v^{p-1} \Delta v \\ &\quad - \varepsilon r(r-1) u^{r-2} |\nabla u|^2 - \varepsilon p(p-1) v^{p-2} |\nabla v|^2. \end{aligned}$$

Using $u_t = G + \varepsilon(u^r + v^p)$ and $v_t = J + \varepsilon(v^s + u^q)$, it follows that

$$\begin{aligned} G_t - \Delta G &\ge (u^r + v^p)_t - \varepsilon r u^{r-1}(u^r + v^p) - \varepsilon p v^{p-1}(v^s + u^q) \\ &\ge r u^{r-1}(G + \varepsilon(u^r + v^p)) - \varepsilon r u^{r-1}(u^r + v^p) \\ &\quad + p v^{p-1}(J + \varepsilon(v^s + u^q)) - \varepsilon p v^{p-1}(v^s + u^q). \end{aligned}$$

Hence,

$$G_t - \Delta G \ge r u^{r-1} G + p v^{p-1} J. \tag{2.2}$$

Similarly, we get

$$J_t - \Delta J \geq s v^{s-1} J + q u^{q-1} G. \tag{2.3}$$

Observe that, due to $u, v \geq 0$, the linear system (2.2)-(2.3) has nonnegative coefficients and therefore satisfies the maximum principle. Since, on the other hand, we have $u_t = v_t = 0$ hence $G = J = 0$ on $S_T = \partial\Omega \times (0, T)$, and $J, G \geq 0$ at $t = 0$ due to our hypothesis (1.2), we deduce that

$$J, G \geq 0 \quad \text{in } Q_T. \tag{2.4}$$

This implies that, for each $x \in \Omega$,

$$(u^{1-r})_t = -(r-1) u^{-r} u_t \leq -(r-1)\varepsilon.$$

Integrating between t and T leads to

$$u^{1-r}(t,x) \geq u^{1-r}(t,x) - u^{1-r}(T,x) \geq (r-1)\varepsilon(T-t),$$

hence $T < \infty$ and $u(t,x) \leq C_r(\varepsilon)(T-t)^{-1/(r-1)}$. The estimate for v is obtained similarly. □

With the uniform upper blow-up estimates at hand, we are now able to establish the uniform lower blow-up estimates.

Proposition 2.2. *Let (u_0, v_0) satisfy (1.2). Assume $s > p+1$ and u blows up (respectively $r > q+1$ and v blows up). Then it holds that*

$$\|u(t)\|_\infty \geq C'_r (T-t)^{-\frac{1}{r-1}}, \quad 0 < T - t < \tau_1, \tag{2.5}$$

respectively

$$\|v(t)\|_\infty \geq C'_s (T-t)^{-\frac{1}{s-1}}, \quad 0 < T - t < \tau_2, \tag{2.6}$$

with $C'_r = 2^{-(r+1)/(r-1)}$, $\tau_1 = \tau_1(p, r, s, \varepsilon) > 0$ and $\tau_2 = \tau_2(q, r, s, \varepsilon) > 0$.

Proof. Assume for instance that $s > p+1$ and u blows up and let $0 < t < T$. We first claim that

$$\|u(t)\|_\infty \leq (T-t)(2\|u(t)\|_\infty)^r + c_1 (T-t)^{1-p/(s-1)}. \tag{2.7}$$

To prove (2.7), we modify an argument of [22]. Using the variation-of-constants formula, (2.1) and $s > p+1$, we have, for all $t < \tilde{t} < T$,

$$\begin{aligned} \|u(\tilde{t})\|_\infty &\leq \|u(t)\|_\infty + \int_t^{\tilde{t}} \|u(\sigma)\|_\infty^r \, d\sigma + C_s^p(\varepsilon) \int_t^{\tilde{t}} (T-\sigma)^{-p/(s-1)} \, d\sigma \\ &\leq \|u(t)\|_\infty + \int_t^{\tilde{t}} \|u(\sigma)\|_\infty^r \, d\sigma + c_1 (T-t)^{1-p/(s-1)}, \end{aligned}$$

with

$$c_1 = \frac{C_s^p(\varepsilon)}{1 - p/(s-1)}.$$

Since u blows up, there is a first $\tilde{t} \in [t, T)$ such that $\|u(\tilde{t})\|_\infty = 2\|u(t)\|_\infty$. It follows that

$$\|u(\tilde{t})\|_\infty = 2\|u(t)\|_\infty \le \|u(t)\|_\infty + (\tilde{t} - t)(2\|u(t)\|_\infty)^r + c_1(T - t)^{1-p/(s-1)} \tag{2.8}$$

hence (2.7) follows. Next, due to (2.7), we have either

$$(T - t)(2\|u(t)\|_\infty)^r \le c_1(T - t)^{1-p/(s-1)} \tag{2.9}$$

so that

$$\|u(t)\|_\infty \le 2c_1(T - t)^{1-p/(s-1)}, \tag{2.10}$$

or

$$(T - t)(2\|u(t)\|_\infty)^r \ge c_1(T - t)^{1-p/(s-1)}, \tag{2.11}$$

so that

$$(T - t)(2\|u(t)\|_\infty)^r \ge (1/2)\|u(t)\|_\infty,$$

hence,

$$\|u(t)\|_\infty \ge 2^{-(r+1)/(r-1)}(T - t)^{-1/(r-1)}. \tag{2.12}$$

Since u blows up, (2.10) and hence (2.9) cannot hold for t close to T. If (2.11) and hence (2.12) hold on $[0, T)$, then we are done. Otherwise there is a largest $t = t_1$ such that (2.9) holds and we then get

$$(T - t_1)(2\|u(t_1)\|_\infty)^r = c_1(T - t_1)^{1-p/(s-1)},$$

hence,

$$2^{-(r+1)/(r-1)}(T - t_1)^{-1/(r-1)} \le \|u(t_1)\|_\infty \le 2c_1(T - t_1)^{1-p/(s-1)},$$

which implies

$$(T - t_1)^{r/(r-1)-p/(s-1)} \ge c_1^{-1} 2^{-2r/(r-1)}.$$

We conclude that there exists $\tau_1(p, r, s, \varepsilon) > 0$, such that for all $0 < T - t < \tau_1$, (2.11) and hence (2.12) hold. We have thus proved (2.5). □

Proof of Theorem 1.2. The upper inequalities are given by Proposition 2.1. Under hypotheses of simultaneous blow-up and (1.3), the lower inequalities follow from Proposition 2.2. In case of nonsimultaneous blow-up of u, an obvious modification of the proof of (2.7) yields

$$\|u(t)\|_\infty \le (T - t)(2\|u(t)\|_\infty)^r + C, \tag{2.13}$$

for some $C > 0$ (depending on (u, v)). For t close to T, we have $\|u(t)\|_\infty \ge 2C$, so that (2.13) implies (2.12). The lower inequality follows. □

3. Continuity of the existence time

In this section we prove the continuity of the existence time with respect to initial data for solutions of (1.1). More precisely, let $E = C_0(\Omega) \times C_0(\Omega)$, equipped with the $L^\infty \times L^\infty$ norm, and let

$$X := \big\{(u_0, v_0) \in E;\ u_0, v_0 \geq 0 \text{ and (1.2) is satisfied}\big\}.$$

We have:

Theorem 3.1. *The existence time T of solutions of problem* (1.1) *is continuous as a function of the initial data* (u_0, v_0) *from X to* $(0, \infty]$.

Note that Theorem 3.1 gives the continuity of the existence time regardless of whether the blow-up is simultaneous or not. Also, the result of Theorem 3.1 remains valid if the terms u^r, v^s are dropped in system (1.1) (and in assumption (1.2) of course).

The continuous dependence of the existence time has been studied, in [2, 12, 13, 15, 18], for scalar superlinear parabolic problems, typically of the form $u_t - \Delta u = u^p$, e.g., with Dirichlet boundary conditions. Recall that continuity [18], or even Hölder continuity [12], is true in general for subcritical p, i.e., $(N-2)p < (N+2)$, and that it may fail for some supercritical p (see [11]).

Let us remark that in the above references, the continuity of T relies on energy properties, or on some uniform estimates of the blow-up rate, like the ones obtained in the previous section for system (1.1). Although there are a number of results on blow-up rates for systems (see for instance [1, 4, 5, 7, 8, 21] for system (1.1) without the terms u^r, v^s), the question of the continuity of the existence time for systems has not been addressed before, as far as we know.

Theorem 3.1 will be proved as a particular case of a result on general abstract semiflows. Although the underlying argument is close to those in, e.g., [12, 18], we think it may be useful in view of future application to state the result in this more general and natural context.

Let E be a Banach space, with norm $\|\cdot\|$, and let X be a subset of E. By a (local) semiflow $(S(t))_{t\geq 0}$ on X, we mean the following. For each $U_0 \in X$, we are given an existence time $T(U_0) \in (0, \infty]$ and a trajectory

$$U(\cdot; U_0) \in C([0, T(U_0)); X), \qquad \text{with } U(0; U_0) = U_0.$$

We then set $S(t)U_0 := U(t;U_0)$ for each $U_0 \in X$ and each $t \in [0, T(U_0))$. Finally, we denote

$$X_t := C([0,t];X), \qquad \|U\|_{X_t} := \sup_{s\in[0,t]} \|U(s)\|.$$

On the semiflow we impose the following hypotheses (where we put $U(t) = S(t)U_0$, $V(t) = S(t)V_0$).

Continuation property:

$$(H1) \qquad \limsup_{t\to T(U_0)} \|U(t)\| = \infty, \quad \text{for each } U_0 \in X \text{ such that } T(U_0) < \infty.$$

Continuous dependence:

$$(H2) \qquad \begin{aligned} &\|U-V\|_{X_t} \le F\big(U_0, t, \|U_0 - V_0\|, \|U\|_{X_t} + \|V\|_{X_t}\big),\\ &\qquad 0 < t < \min(T(U_0), T(V_0)), \quad U_0, V_0 \in X,\\ &\text{where, for each } U_0 \in X,\ \lim_{h\to 0} F(U_0, t, h, M) = 0,\\ &\qquad \text{uniformly for } t, M \text{ bounded},\\ &\text{and } F(U_0, t, h, M) \text{ is nondecreasing in the last variable.}\end{aligned}$$

Uniform upper blow-up estimate:

$(H3)$

$$\|U(t)\| \le G\Big(\|U_0\|, \frac{1}{T(U_0) - t}\Big),$$

$$0 < t < T(U_0), \quad \text{for each } U_0 \in X \text{ such that } T(U_0) < \infty,$$

where G is nondecreasing in both variables.

Theorem 3.2. *Let $(S(t))_{t\ge 0}$ be a semiflow satisfying (H1), (H2) and (H3). Then the existence time T is upper semicontinuous from X to $(0, \infty]$.*

Proof. Fix U_0 such that $T = T(U_0)$ is finite (otherwise there is nothing to prove). Let U_n be a sequence of trajectories such that $U_n(0) \to U_0$ as $n \to \infty$, and set $e_n(t) = \|U_n(t) - U(t)\|$. Since $e_n(0) \to 0$, we may assume $e_n(0) < 1$. It is enough to consider n such that $T_n > T$.

Assume that for all $t < T$, $e_n(t) < 1$, then $T_n \le T$ due to *(H1)*; but this is impossible. Hence, there exists a minimal $t_n < T$ such that $e_n(t_n) = 1$. Our hypotheses (H2), (H3) imply that

$$1 = e_n(t_n) \le F\Big(U_0, t_n, e_n(0), 1 + 2G\Big(\|U_0\| + 1, \frac{1}{T_n - t_n}\Big)\Big).$$

Since $F(U_0, t, h, M) \to 0$, $h \to 0$, uniformly for t, M bounded, and $e_n(0) \to 0$ as $n \to \infty$, it follows that

$$G\Big(\|U_0\| + 1, \frac{1}{T_n - t_n}\Big) \to \infty, \quad n \to \infty.$$

Consequently, $T_n - t_n \to 0$ as $n \to \infty$. Since $T_n - T < T_n - t_n$ we get

$$\limsup_{n\to\infty} T_n \le T,$$

as we wanted to prove. □

As is well known, the lower semicontinuity is just a standard consequence of continuation and continuous dependence properties. We recall the following for completeness.

Proposition 3.1. *Let $(S(t))_{t\ge 0}$ be a semiflow satisfying (H1) and (H2). Then the existence time T is lower semicontinuous from X to $(0,\infty]$.*

Proof. Fix $U_0 \in X$, with $T := T(U_0) \le \infty$. With the notation of the previous proof, we may assume $e_n(0) \to 0$, $e_n(0) < 1$ and we need only consider n such that T_n is finite and $T_n < T$.

By (H1), there is a first $t_n < T_n$ such that $e_n(t_n) = 1$ and (H2) implies that

$$1 = e_n(t_n) \le F\big(U_0, t_n, e_n(0), 1 + 2\|U\|_{X_{t_n}}\big).$$

Due to (H2) and $e_n(0) \to 0$, if $T < \infty$, then $\|U\|_{X_{t_n}} \to \infty$, hence $t_n \to T$; whereas if $T = \infty$, then $t_n + \|U\|_{X_{t_n}} \to \infty$, hence $t_n \to \infty$. Since $T_n > t_n$, it follows that

$$\liminf_{n\to\infty} T_n \ge T,$$

as we wanted to prove. □

Proof of Theorem 3.1. Take E and X as defined at the beginning of this section and consider the semiflow defined by $S(t)U_0 = (u(t), v(t))$, the solution of (1.1), where $U_0 = (u_0, v_0)$. Properties *(H1)* and *(H2)* are standard for parabolic systems like (1.1). Property *(H3)* follows from Proposition 2.1. The result is thus a consequence of Theorem 3.2 and Proposition 3.1. □

4. SIMULTANEOUS BLOW-UP IN THE NONSIMULTANEOUS RANGE

Our main result Theorem 1.1 is a consequence of the following more precise statement.

Theorem 4.1. *Assume* (1.3) *and let* (u_0, v_0) *satisfy* (1.2) *and* $\|u_0\|_\infty, \|v_0\|_\infty \geq 1$. *Then there exists* $\lambda_0 \geq 1$ *such that for all* $\lambda \geq \lambda_0$ *and for some* $\mu = \mu(\lambda) \geq 1$, *the solution* (u, v) *of* (1.1) *with initial data* $(\lambda u_0, \mu v_0)$ *has simultaneous blow-up. Moreover,* (u, v) *satisfies the upper and lower blow-up estimates* (2.1), (2.5) *and* (2.6).

Remark 4.1. We may choose $\lambda_0 = \lambda_0(\|v_0\|_\infty, q, r, s, \varepsilon)$.

Remark 4.2. To ensure condition (1.2) on the initial data, it is enough for $0 \leq u_0 \in C^2(\overline{\Omega})$ to satisfy $\Delta u_0 + (1-\varepsilon)u_0^r \geq 0$ in Ω and $\Delta u_0 = 0$ on $\partial\Omega$ (and a similar condition for v_0). It is easy to check that such u_0 (moreover satisfying $\|u_0\|_\infty \geq 1$) can be found under the form $u_0 = k\varphi_1 - k^{3/4}\chi$, where $\varphi_1 > 0$ is a first eigenfunction of the Dirichlet Laplacian in Ω, χ is the solution to $-\Delta\chi = \varphi_1^{1/2}$ in Ω, with $\chi = 0$ on $\partial\Omega$, and the number $k > 0$ is large enough.

Remark 4.3. Actually, assumption (1.2) in Theorem 4.1 can be replaced by its weak formulation, namely:

$$\int_\Omega \left\{u_0\Delta\varphi + (1-\varepsilon)(u_0^r + v_0^p)\varphi\right\} dx \geq 0, \quad 0 \leq \varphi \in C^2(\overline{\Omega}), \quad \varphi = 0 \text{ on } \partial\Omega$$

(and a similar second inequality). The only change is in the proof of Proposition 2.1, where G, J no longer belong to $C(\overline{\Omega} \times [0, T))$, but where a suitable weak (instead of classical) form of the maximum principle, using duality, can be applied. For this, we only require $u_0, v_0 \in C_0(\Omega)$ to be Hölder continuous.

Before proving Theorem 4.1, we shall first establish the following result (used in the proof of Theorem 4.1), which confirms that $s > p+1$ or $r > q+1$ is indeed the range of nonsimultaneous blow-up, in the sense that if v_0 is much larger than u_0 in a suitable sense, then v blows up while u remains bounded (and vice-versa). This fact was proved in [21], for $\Omega = \mathbb{R}^N$, under restrictive assumptions on the initial data (implying $N = 1$ or u, v spatially homogeneous).

Proposition 4.1. *Assume* $s > p + 1$ *and let* (u_0, v_0) *satisfy* (1.2). *There exists a constant* $C_3 > 0$ *depending only on* p, r, s, ε *such that, if*

$$\|u_0\|_\infty \geq 1 \quad \text{and} \quad \|v_0\|_\infty > C_3\|u_0\|_\infty^{(r-1)/(s-1)}, \tag{4.1}$$

then v *blows up and* u *remains bounded.*

Remark 4.4. The symmetric statement holds when $r > q + 1$ by interchanging the roles of u, v, of p, q, and of r, s.

Proof. By the argument leading to (2.8) (applied with $t = 0$), if there exists a first $T_0 \in [0, T)$ such that $\|u(T_0)\|_\infty = 2\|u_0\|_\infty$, then it satisfies

$$\|u_0\|_\infty \le T_0(2\|u_0\|_\infty)^r + c_1 T^{1-p/(s-1)}. \tag{4.2}$$

Assume (4.1) with $C_3 > (2c_1)^{1/(s-1-p)} C_s(\varepsilon)$, where $C_s(\varepsilon)$ is given by (2.1). Using (2.1) with $t = 0$, we have

$$T \le C_s^{s-1}(\varepsilon)\|v_0\|_\infty^{1-s} \le (C_s(\varepsilon)/C_3)^{s-1} < (2c_1)^{(1-s)/(s-1-p)}.$$

Therefore

$$\|u_0\|_\infty \ge 1 \ge 2c_1 T^{1-p/(s-1)}.$$

But (4.2) then implies that $\|u_0\|_\infty \le 2T_0(2\|u_0\|_\infty)^r$. Assuming further that $C_3 > 2^{(r+1)/(s-1)} C_s(\varepsilon)$, we get that

$$T_0 \ge 2^{-r-1}\|u_0\|_\infty^{1-r} > 2^{-r-1} C_3^{s-1}\|v_0\|_\infty^{1-s} \ge T.$$

It follows that $\|u(t)\|_\infty < 2\|u_0\|_\infty$ on $[0, T)$ and the result is proved. □

Proof of Theorem 4.1. First observe that for all $\lambda, \mu \ge 1$, $(\lambda u_0, \mu v_0)$ also satisfies (1.2). By Proposition 4.1 in the case $r > q + 1$, there exists $\lambda_0 = \lambda_0(\|v_0\|_\infty, q, r, s, \varepsilon) \ge 1$ such that for all $\lambda \ge \lambda_0$ and $\mu \in [1, 2]$, the solution of (1.1) with initial data $(\lambda u_0, \mu v_0)$ has nonsimultaneous blow-up with v bounded.

Fix $\lambda \ge \lambda_0$ and denote by (u_μ, v_μ) the solution of (1.1) with initial data $(\lambda u_0, \mu v_0)$ and by T_μ its blow-up time. By Theorem 3.1, T_μ is a continuous function of μ for $\mu \ge 1$. Applying Proposition 4.1 for $s > p + 1$, if μ is large enough, then (u_μ, v_μ) has nonsimultaneous blow-up with u_μ bounded. Let then

$$E = \{\mu > 1;\ u_\mu \text{ is bounded and } v_\mu \text{ blows up}\}.$$

We thus have $\mu^* := \inf E \in [2, \infty)$.

Let $T_{\mu^*} - \tau_1 < t < T_{\mu^*}$, where τ_1 is defined in Proposition 2.2. By continuity of T_μ, we have $T_\mu - \tau_1 < t < T_\mu$ for μ close to μ^*. By (2.5), it follows that

$$\|u_\mu(t)\|_\infty \ge C_r'(T_\mu - t)^{-1/(r-1)}, \qquad \text{for } \mu < \mu^* \text{ close to } \mu^*. \tag{4.3}$$

Letting $\mu \to \mu^*-$ in (4.3) and using continuity of T_μ and of u_μ, we get

$$\|u_{\mu^*}(t)\|_\infty \ge C_r'(T_{\mu^*} - t)^{-1/(r-1)}.$$

It follows that u_{μ^*} blows up.

Let now $T_{\mu^*} - \tau_2 < t < T_{\mu^*}$, where τ_2 is defined in Proposition 2.2, and pick a sequence $\mu_j \in E$ with $\mu_j \to \mu^*$. Again by continuity of T_μ, we have $T_{\mu_j} - \tau_2 < t < T_{\mu_j}$ for j large. By (2.6), we know that

$$\|v_{\mu_j}(t)\|_\infty \geq C'_s(T_{\mu_j} - t)^{-1/(s-1)}.$$

Letting $j \to \infty$ and using the continuity of T_μ and of v_μ, we get

$$\|v_{\mu^*}(t)\|_\infty \geq C'_s(T_{\mu^*} - t)^{-1/(s-1)}.$$

It follows that v_{μ^*} blows up. The proof is complete. □

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