Differential and Integral Equations Volume 18, Number 4, Pages 379–404

LOCAL WELL POSEDNESS OF THE CAUCHY PROBLEM FOR THE LANDAU-LIFSHITZ EQUATIONS

Atsushi Fuwa and Masayoshi Tsutsumi
Department of Applied Physics, School of Science and Engineering
Waseda University, Tokyo 169-8555, Japan

(Submitted by: Reza Aftabizadeh)

1. Introduction

In 1935 Landau and Lifshitz derived an equation which describes the dynamics of the magnetization in continuum ferromagnetic bodies. The equation is of the form

$$\begin{cases} \dfrac{\partial u}{\partial t} = u \times \Delta u - \lambda u \times (u \times \Delta u), \\ u \ : \ \mathbb{R}^d \to \mathbb{S}^2, \end{cases} \tag{1.1}$$

where $\mathbb{S}^2$ denotes the two-dimensional sphere ([13]). The positive number λ is a damping parameter. (1.1) is usually called the Landau-Lifshitz equation. In case $\lambda = 0$, (1.1) is called the Heisenberg equation. This paper is devoted to studying the well posedness of the Landau-Lifshitz equation for $\lambda > 0$. Since it is a fully nonlinear system of parabolic partial differential equations and has a supercritical nonlinearity, many basic questions on the well posedness remain unsolved especially for higher spacial dimensional cases. We here assume that $\lambda = 1$.

The main previous results for the above equations are as follows. The first mathematical work on the Heisenberg equation is done by P. L. Sulem, C. Sulem, and C. Bardos [14] who establish the time local existence, uniqueness, and smoothness on $\mathbb{R}^d (d \geq 1)$. As to the Landau-Lifshitz equation Carbou and Fabrie [6] establish the time local existence, global existence with small initial data, uniqueness, and smoothness on $\mathbb{R}^3$. Zhou Yulin, Guo Boling, and Tan Shaobin prove global existence, uniqueness, and smoothness on $\mathbb{R}^1$, $\mathbb{T}^1 (\mathbb{T} = \mathbb{R}/\mathbb{Z})$ in [16]. Guo Boling and Min-Chun Hong establish global

Accepted for publication: October 2004.
AMS Subject Classifications: 35Q99, 35K45.

existence with small initial data, uniqueness, and smoothness on $\mathbb{T}^2$ and $\mathbb{R}^2$ in [11].

In this paper we investigate the Cauchy problem for the Landau-Lifshitz equation defined on $\mathbb{R}^d$ or $\mathbb{T}^d (d \geq 1)$. First, we establish time local existence, time global existence with small initial data, uniqueness, and smoothness of solutions of (1.1) on $\mathbb{R}^d$ by the finite difference method. Next, we establish time local existence, uniqueness, and smoothness of solutions of (1.1) on $\mathbb{T}^d$ by applying the theory of abstract nonlinear evolution equations on L^p spaces, which needs to establish L^p estimates for the corresponding linear elliptic equations. We here remark that the direct application to the equation faces some difficulty of supercriticality of nonlinearity. We need a manipulation to overcome it (see Section 4 below).

2. Main results

Let Ω be $\mathbb{R}^d$ or $\mathbb{T}^d (d \geq 1)$. The Landau-Lifshitz equation is a partial differential equation for a vector field $u = (u_1, u_2, u_3) = u(t,x) : \Omega \to \mathbb{S}^2$ and has the following form

$$\frac{\partial u}{\partial t} = u \times \Delta u - u \times (u \times \Delta u), \quad x \in \Omega \tag{2.1}$$

$$u(0,x) = u_0(x), \quad x \in \Omega. \tag{2.2}$$

Let us mention our main results:

Theorem 1 (Local well posedness on $\mathbb{R}^d$)**.** *Let $\Omega = \mathbb{R}^d$. Assume that*

$$|u_0| = 1, \qquad \nabla u_0 \in H^k(\mathbb{R}^d) \quad \Big(k \geq k_d = \max\big\{[\frac{d}{2}] + 1, 2\big\}\Big)$$

where $[\frac{d}{2}]$ stands for the largest integer that is not larger than $\frac{d}{2}$. Then, there exists $T > 0$ depending only on $\|\nabla u_0\|_{H^{k_d-1}(\mathbb{R}^d)}$ such that there is a unique solution u satisfying the following conditions (i)-(iv)*.*

(i) $|u| = 1$,
(ii) $\nabla u \in L^\infty(0,T; H^{k-1}(\Omega)) \cap L^2(0,T; H^k(\Omega))$,
(iii) $\dfrac{\partial u}{\partial t} \in L^\infty(0,T; H^{k-2}(\Omega)) \cap L^2(0,T; H^{k-1}(\Omega))$,
(iv) *u satisfies* (2.1).

Furthermore, the following alternatives hold:

(i) $T = +\infty$.
(ii) $T < +\infty$ *and*

$$\limsup_{t \to T} \|\nabla u(t)\|_{H^{k_d-1}(\mathbb{R}^d)} = \infty \tag{2.3}$$

Here and in the sequel we denote vector-valued function spaces by the same symbols as the corresponding scalar-valued function spaces if there is no confusion.

Corollary 1. *If $u_0 \in C^\infty(\mathbb{R}^d)$ with $\nabla u_0 \in \bigcap_{s=0}^\infty H^s(\mathbb{R}^d)$, then there exists $T > 0$ depending only on $\|\nabla u_0\|_{H^{k_d-1}(\mathbb{R}^d)}$ such that there is a unique solution $u \in C^\infty([0,T] \times \mathbb{R}^d)$ with $\nabla u \in \bigcap_{s=0}^\infty C((0,T]; H^s(\mathbb{R}^d))$.*

Theorem 2 (Global well posedness for small data on $\mathbb{R}^d$)**.** *Assume that $k \geq k_d$ and*

$$|u_0| = 1, \qquad \nabla u_0 \in H^{k-1}(\mathbb{R}^d).$$

Then, there exists $\delta > 0$ such that if $\|\nabla u_0\|_{H^{k_d-1}(\mathbb{R}^d)} < \delta$, then the solution u given by Theorem 1 exists for any $T > 0$.

Theorem 3 (Local well posedness on $\mathbb{T}^d$)**.** *Assume that $|u_0| = 1$, $u_0 \in W^{3,p}(\mathbb{T}^d)$, $p > d$ and $p \geq 2$. Then, there exists $T^* > 0$ such that the Cauchy problem* (2.1) *has a unique solution satisfying the following conditions:*

(i) $|u| = 1$,
(ii) $u \in C([0,T^*]; W^{2,p}(\mathbb{T}^d)) \cap C((0,T^*]; W^{3,p}(\mathbb{T}^d)) \cap C^1((0,T^*]; W^{1,p}(\mathbb{T}^d))$,
(iii) *u satisfies* (2.1).

3. Proofs of Theorems 1 and 2

3.1. **Space Discretization.** Fix the mesh size $h > 0$. We set $x_j^h = jh$ ($j \in \mathbb{Z}$). For $I = (i_1, i_2, \cdots, i_d) \in \mathbb{Z}^d$, we denote $x_I^h = (x_{i_1}^h, x_{i_2}^h, \cdots, x_{i_d}^h)$. Let us define the cell

$$C_I^h = \left\{(x_1, x_2, \cdots, x_d) \in \mathbb{R}^d,\ x_{i_k}^h \leq x_k < x_{i_k}^h + h, k = 1, 2, \cdots d\right\}.$$

$\mathbb{Z}_h^d = \left\{x_I^h \in \mathbb{R}^d,\ I \in \mathbb{Z}^d\right\}$ denotes the set of lattice points. $\phi = \phi(x_I^h)$ or ϕ_I^h stands for a function defined on the lattice. For $\phi : \mathbb{Z}_h^d \to \mathbb{R}^m (m \geq 1)$, we define the following operators.

$$\tau_{k,+}\phi_I^h = \phi^h(i_1, i_2, \cdots, i_k + 1, \cdots, i_d), \qquad D_{k,+}\phi^h = \frac{1}{h}(\tau_{k,+}\phi_I^h - \phi_I^h),$$

$$\tau_{k,-} = (\tau_{k,+})^{-1}, \qquad D_{k,-} = \tau_{k,-} \circ D_{k,+}, \qquad D_\pm = (D_{1,\pm}, D_{2,\pm}, \cdots, D_{d,\pm}),$$

$$\tilde{\triangle}\phi^h = \sum_{i=1}^d D_{i,+}(D_{i,-}\phi^h) = \sum_{i=1}^d D_{i,-}(D_{i,+}\phi^h).$$

Define the discrete integration:

$$\int u^h = \int_{\mathbb{Z}_h^d} u^h = \sum_{I\in\mathbb{Z}_h^d} h^d u_I^h.$$

We use the following classical notations:

$$\|u^h\|_{l^p} = \|u^h\|_p = \Big(\int |u^h|^p\Big)^{\frac{1}{p}}, \quad 1\le p<\infty,$$

$$\|u^h\|_{l^\infty} = \|u^h\|_\infty = \sup_{I\in\mathbb{Z}_h^d} |u_I^h|,$$

$$\|D_+^k u^h\|_{l^p} = \|D_+^k u^h\|_p = \sum_{|\xi|=k} \|D_+^\xi u^h\|_p,$$

$$\|u^h\|_{W_h^{m,p}} = \|u^h\|_{m,p} = \sum_{0\le k\le m} \|D_+^k u^h\|_p,$$

where ξ is a multi-index $\xi = (\xi_1, \xi_2, \cdots \xi_d)$. We use the notations

$$D_+^\xi u_I^h = D_{1,+}^{\xi_1} D_{2,+}^{\xi_2} \cdots D_{d,+}^{\xi_d} u_I, \qquad |\xi| = \xi_1+\xi_2+\cdots+\xi_d.$$

In case that $p=2$, we denote $H_h^m = W_h^{m,2}$. Clearly, we have

$$\begin{aligned} \|D_+^2 u^h\|_2 &= \|\tilde{\triangle} u^h\|_2, \\ \int_{\mathbb{Z}_h^d} (D_{k,+}\phi^h)\psi^h &= -\int_{\mathbb{Z}_h^d} \phi^h (D_{k,-}\psi^h), \\ D_{k,+}(\phi^h\psi^h) &= D_{k,+}\phi^h\ \tau_{k,+}\psi^h + \phi^h D_{k,+}\psi^h. \end{aligned}$$

We recall the discrete Sobolev inequalities.

Lemma 1. *Assume that $s<\frac{d}{p}$. There exists a constant C independent of h such that for all $\phi^h : \mathbb{Z}_h^d \to \mathbb{R}^m$, if $D_+^s\phi^h \in l^p$, then*

$$\|\phi^h\|_{\frac{pd}{d-sp}} \le C\|D_+^s\phi^h\|_p. \tag{3.1}$$

Proof. See Ladyzhenskaya [12] (also see [10]). □

3.2. **Interpolating operators.** We introduce the following interpolating operators (Ladyzhenskaya [12]). For $x = (x_1, x_2, \cdots x_d) \in C_I^h$, we put

$$q_h u^h(x) = u^h(x_I^h),$$

$$p_h u^h(x) = u^h(x_I) + \sum_{j=1}^d D_{j,+}u^h(x_I)(x_l - x_{i_j}^h) + \cdots$$

$$+\sum_{j=1}^{d} D_{1,+}\cdots D_{j-1,+}D_{j+1,+}\cdots D_{d,+}u^h(x_I)\prod_{\substack{l=1\\ l\neq j}}^{d}(x_l-x_{i_l}^h)$$

$$+D_{1,+}\cdots D_{d,+}u^h(x_I)\prod_{l=1}^{d}(x_l-x_{i_l}^h),$$

$$r_h^m u^h(x)=u^h(x_I)+\sum_{\substack{j=1\\ j\neq m}}^{d} D_{j,+}u^h(x_I)(x_j-x_{i_j}^h)+\cdots$$

$$+D_{1,+}\cdots D_{m-1,+}D_{m+1,+}\cdots D_{d,+}u^h(x_I)\prod_{\substack{l=1\\ l\neq m}}^{d}(x_l-x_{i_l}).$$

By direct computation we have

$$\frac{\partial}{\partial x_m}(p_h u^h)=r_h^m(D_{m,+}u^h). \tag{3.2}$$

Furthermore, we have:

Lemma 2. [12]

(i) *If one of the interpolates $p_h u^h, q_h u^h$, or $r_h u^h$ converges strongly in L^2 when h goes to zero, then the two others also converge to the same limit in L^2 strongly.*

(ii) *If one of the interpolates $p_h u^h, q_h u^h$, or $r_h u^h$ converges weakly in L^2 when h goes to zero, then the two others also converge to the same limit in L^2 weakly.*

Proof. See Ladyzhenskaya [12]. □

3.3. **Discretization.** For all $h>0$ we consider the discretization of u_0 which satisfies the following conditions:

(i) $|u_0^h|=1$ on $\mathbb{Z}^d$,

(ii) $q_h u_0^h \to u_0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^d)$,

(iii) $\dfrac{1}{\alpha}\|D_+u_0^h\|_{H_h^{k-1}}\le\|\nabla u_0\|_{H^{k-1}}\le\alpha\|D_+u_0^h\|_{H_h^{k-1}}$, where α is a positive constant independent of h.

Let us prepare the spatial discretization of (2.1).

$$\frac{du^h}{dt}=u^h\times\tilde{\Delta}u^h-u^h\times(u^h\times\tilde{\Delta}u^h),\qquad u^h(t=0)=u_0^h. \tag{3.3}$$

Since the map $u^h \mapsto u^h \times \tilde{\triangle} u^h - u^h \times (u^h \times \tilde{\triangle} u^h)$ is locally Lipshitz continuous in the space $E = \{u^h; \|u^h\|_\infty, \|D_+u^h\|_2 < +\infty\}$, by making use of the Cauchy-Lipshitz theorem, we can construct a unique solution u^h of (3.3) in E.

3.4. **Estimates.** We only consider the case when $d \geq 4$. See Carbou, Fabrie [6] in the case that $d = 3$. We omit the cases $d = 1$ or 2 since they are rather easy. Multiplying (3.3) by u^h, we obtain

$$\frac{d}{dt}|u^h|^2 = 0,$$

from which we deduce $|u^h| = 1$. Since $|u^h| = 1$, we note that

$$2u^h \cdot \tilde{\triangle} u^h + |D_+u^h|^2 + |D_-u^h|^2 = 0. \tag{3.4}$$

Hence, (3.3) can be rewritten as

$$\frac{du^h}{dt} = u^h \times \tilde{\triangle} u^h + \tilde{\triangle} u^h + \frac{1}{2}(|D_+u^h|^2 + |D_-u^h|^2)u^h. \tag{3.5}$$

Multiplying (3.5) by $-\tilde{\triangle} u$ and summing on $\mathbb{Z}_h^d$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|D_+u^h\|_2^2 + \|D_+^2u^h\|_2^2 \leq C\|D_+^2u^h\|_2 I_1, \tag{3.6}$$

where

$$I_1 = \|D_+u^h\|_{2p_0^1}\|D_+u^h\|_{2p_0^2}, \quad p_0^1, p_0^2 \geq 1, \ \ \frac{1}{p_0^1} + \frac{1}{p_0^2} = 1.$$

Multiplying (3.5) by $(-1)^k\tilde{\triangle}^k u^h (k \geq 2)$ and summing on $\mathbb{Z}_h^d$, we obtain

$$\frac{1}{2}\frac{d}{dt}\|D_+^k u^h\|_2^2 + \|D_+^{k+1}u^h\|_2^2 = \int \Big\{ \sum_{|\xi|=k-1} D_+^\xi \tilde{\triangle} u^h \cdot D_+^\xi (u^h \times \tilde{\triangle} u^h)$$
$$+ \frac{1}{2}(-1)^k \left(|D_+u^h|^2 + |D_+u^h|^2\right) u^h \cdot \tilde{\triangle}^k u^h \Big\}. \tag{3.7}$$

We have

$$\int \sum_{|\xi|=k-1} D_+^\xi \tilde{\triangle} u^h \cdot D_+^\xi (u^h \times \tilde{\triangle} u^h) = -\int \sum_{|\xi|=k-1} \Big\{ D_+^\xi \tilde{\triangle} u^h \cdot (u^h \times D_+^\xi \tilde{\triangle} u^h)$$
$$+ D_+^\xi \tilde{\triangle} u^h \cdot \sum_{0<\eta\leq\xi} \begin{pmatrix} \xi \\ \eta \end{pmatrix} D_+^\eta u^h \times \tau_+^\eta D_+^{\xi-\eta} \tilde{\triangle} u^h \Big\} \leq C\|D_+^{k+1}u^h\|_2 I_k, \tag{3.8}$$

where

$$I_k = \sum_{\alpha_1+\alpha_2=k-1} \|D_+^{1+\alpha_1}u^h\|_{2p_\alpha^1}\|D_+^{1+\alpha_2}u^h\|_{2p_\alpha^2}, \tag{3.9}$$

with $\alpha=(\alpha_1,\alpha_2)$, $p_\alpha^1, p_\alpha^2 \geq 1$, $\frac{1}{p_\alpha^1}+\frac{1}{p_\alpha^2}=1$. We also have

$$\begin{aligned}
&\frac{1}{2}\int(-1)^k(|D_+u^h|^2+|D_-u^h|^2)u^h\cdot\tilde{\triangle}^k u^h\\
&=-\frac{1}{2}\int\sum_{|\xi|=k-1}D_+^\xi\tilde{\triangle}u^h\cdot D_+^\xi\{(|D_+u^h|^2+|D_-u^h|^2)u^h\} \qquad (3.10)\\
&\leq C\|D_+^{k+1}u^h\|_2\Big\{I_k+\sum_{|\xi|=k-2}\sum_{0\leq\eta\leq\xi}\|D_+^\eta|D_+u^h|^2\|_{2q'_\eta}\|D_+D_+^{\xi-\eta}u^h\|_{2q_\eta}\Big\},
\end{aligned}$$

where $q_\eta, q'_\eta \geq 1$, $\frac{1}{q_\eta}+\frac{1}{q'_\eta}=1$. Furthermore, we have

$$\begin{aligned}
&\sum_{|\xi|=k-2}\sum_{0\leq\eta\leq\xi}\|D_+^\eta|D_+u^h|^2\|_{2q'_\eta}\|D_+D_+^{\xi-\eta}u^h\|_{2q_\eta}\\
&\leq\sum_{|\xi|=k-2}\sum_{0\leq\eta\leq\xi}\sum_{0\leq\zeta\leq\eta}\|D_+D_+^{\eta-\zeta}u^h\|_{2q'_\eta r'_\zeta}\|D_+D_+^\zeta u^h\|_{2q'_\eta r_\zeta}\|D_+D_+^{\xi-\eta}u^h\|_{2q_\eta}\\
&=CJ_k \qquad (3.11)
\end{aligned}$$

where

$$J_k=\sum_{\beta_1+\beta_2+\beta_3=k-2}\|D_+^{1+\beta_1}u^h\|_{2q_\beta^1}\|D_+^{1+\beta_2}u^h\|_{2q_\beta^2}\|D_+^{1+\beta_3}u^h\|_{2q_\beta^3} \qquad (3.12)$$

$$r_\zeta, r'_\zeta\geq 1,\ \frac{1}{r_\zeta}+\frac{1}{r'_\zeta}=1,\ \beta=(\beta_1,\beta_2,\beta_3),\ q_\beta^1,q_\beta^2,q_\beta^3\geq 1,\ \frac{1}{q_\beta^1}+\frac{1}{q_\beta^2}+\frac{1}{q_\beta^3}=1.$$

Hence, we obtain

$$\frac{1}{2}\frac{d}{dt}\|D_+^k u^h\|_2^2+\|D_+^{k+1}u^h\|_2^2\leq C\|D_+^{k+1}u^h\|_2(I_k+J_k). \qquad (3.13)$$

From (3.6) and (3.13), we deduce the following inequality:

$$\frac{1}{2}\frac{d}{dt}\|D_+u^h\|_{H_h^{k-1}}^2+\|D_+^2u^h\|_{H_h^{k-1}}^2\leq C_0\|D_+^2u^h\|_{H_h^{k-1}}(I+J), \qquad (3.14)$$

where

$$I=\sum_{j=1}^k I_j,\quad J=\sum_{j=2}^k J_j.$$

Set $\tau=\frac{2}{1-\frac{1}{d}}$. Using Lemma 1 , we obtain

$$\|\phi^h\|_\tau\leq C\|D_+\phi^h\|_2^{\frac{1}{2}}\|\phi^h\|_2^{\frac{1}{2}}. \qquad (3.15)$$

Estimate of I_1. Set $\frac{1}{2p_\alpha^1} = \frac{1}{2} - \frac{1}{d}$. We make use of Lemma 1 to have

$$\|D_+ u^h\|_{2p_\alpha^1} \le C\|D_+^2 u^h\|_2. \tag{3.16}$$

Note that

$$\frac{1}{2p_\alpha^2} = \frac{1}{d} = \begin{cases} \frac{1}{2} - \frac{(k_d-1)-1}{d}, & d \text{ even} \\ \frac{1}{\tau} - \frac{(k_d-1)-1}{d}, & d \text{ odd} \end{cases}$$

Then, from (3.15) we have

$$\|D_+ u^h\|_{2p_\alpha^2} \le \begin{cases} C\|D_+^{k_d-1} u^h\|_2, & d \text{ even} \\ C\|D_+^{k_d-1} u^h\|_\tau \quad \le C\|D_+^{k_d-1} u^h\|_2^{\frac{1}{2}} \|D_+^{k_d} u^h\|_2^{\frac{1}{2}}, & d \text{ odd.} \end{cases} \tag{3.17}$$

We can deduce the estimate

$$I_1 \le C\|D_+^2 u^h\|_2 \|D_+ u^h\|_{h^{k_d-1}}. \tag{3.18}$$

Estimate of I_j $(1 < j \le k)$. We now note that

$$I_j \le 2 \sum_{\substack{\alpha_1+\alpha_2=j-1 \\ \alpha_1 \ge \frac{j-1}{2}}} \|D_+^{1+\alpha_1} u^h\|_{2p_\alpha^1} \|D_+^{1+\alpha_2} u^h\|_{2p_\alpha^2}. \tag{3.19}$$

Setting $\frac{1}{2p_\alpha^1} = \frac{1}{\tau} - \frac{j-(1+\alpha_1)}{d}$, we observe that

$$\frac{1}{2p_\alpha^1} \ge \frac{1}{\tau} - \frac{1}{2} + \frac{1}{d} = \frac{1}{2d} > 0,$$
$$\frac{1}{2p_\alpha^2} = \begin{cases} \frac{1}{\tau} - \frac{k_d-1-(1+\alpha_2)}{d}, & d \text{ even} \\ \frac{1}{2} - \frac{k_d-(1+\alpha_2)}{d}, & d \text{ odd} \end{cases} > 0.$$

Hence, we have

$$\|D_+^{1+\alpha_1} u^h\|_{2p_\alpha^1} \le C\|D_+^j u^h\|_\tau \le C\|D_+^{j+1} u^h\|_2^{\frac{1}{2}} \|D_+^j u^h\|_2^{\frac{1}{2}}, \tag{3.20}$$

$$\|D_+^{1+\alpha_2} u^h\|_{2p_\alpha^2} \le \begin{cases} C\|D_+^{k_d-1} u^h\|_\tau \quad \le C\|D_+^{k_d-1} u^h\|_2^{\frac{1}{2}} \|D_+^{k_d} u^h\|_2^{\frac{1}{2}} & (d \text{ even}) \\ C\|D_+^{k_d} u^h\|_2 & (d \text{ odd}). \end{cases} \tag{3.21}$$

Combining (3.20) and (3.21), we have

$$I_j \le C\|D_+^{j+1} u^h\|_2^{\frac{1}{2}} \|D_+^j u^h\|_2^{\frac{1}{2}} \|D_+ u^h\|_{H_h^{k_d-1}}. \tag{3.22}$$

From (3.18) and (3.22), we obtain

$$I = I_1 + \sum_{j=2}^{k} I_j \tag{3.23}$$

$$\le C\Big(\|D_+^2 u^h\|_2 \|D_+ u^h\|_{H_h^{k_d-1}} + \sum_{j=2}^{k} \|D_+^{j+1} u^h\|_2^{\frac12} \|D_+^j u^h\|_2^{\frac12} \|D_+ u^h\|_{H_h^{k_d-1}}\Big)$$

$$\le C\Big(\|D_+ u^h\|_{H_h^{k-1}} \|D_+ u^h\|_{H_h^{k_d-1}} + \|D_+ 2u^h\|_{H_h^{k-1}}^{\frac12} \|D_+ u^h\|_{H_h^{k-1}}^{\frac12} \|D_+ u^h\|_{H_h^{k_d-1}}\Big).$$

Estimate of J_2. First, we estimate $\|D_+u^h\|_{2q_\beta^2}\|D_+u^h\|_{2q_\beta^3}$ $(\beta = (0,0,0))$. We set $2q_\beta^2 = 2q_\beta^3 = 2d$. Then we note that

$$\frac{1}{2q_\beta^2} = \frac{1}{2q_\beta^3} = \begin{cases} \frac{1}{\tau} - \frac{(k_d-1)-1}{d} > 0, & d \text{ even} \\ \frac12 - \frac{k_d-1}{d} > 0, & d \text{ odd.} \end{cases}$$

Hence, we obtain

$$\|D_+u^h\|_{2q_\beta^2}\|D_+u^h\|_{2q_\beta^3} = \|D_+u^h\|_{2d}^2$$
$$\le \begin{cases} C\|D_+^{k_d-1}u^h\|_\tau^2 \le C\|D_+^{k_d}u^h\|_2 \|D_+^{k_d-1}u^h\|_2 & d \text{ even} \\ C\|D_+^{k_d}u^h\|_2^2 & d \text{ odd.} \end{cases} \tag{3.24}$$

Next, we estimate $\|D_+u^h\|_{2q_\beta^1}$. Observing that

$$\frac{1}{2q_\beta^1} = \frac12 - \frac1d > 0,$$

we obtain

$$\|D_+u^h\|_{2q_\beta^1} \le C\|D_+^2 u^h\|_2. \tag{3.25}$$

Combining (3.24) and (3.25), we can deduce

$$J_2 \le C\|D_+^2u^h\|_2 \|D_+u^h\|_{H_h^{k_d-1}}^2. \tag{3.26}$$

Estimate of $J_j(2 < j \le k)$. We note that

$$J_j \le C \sum_{\substack{\beta_1+\beta_2+\beta_3=j-2 \\ \beta_1\ge1,\ 0\le\beta_3\le\frac{j-2}{3}}} \|D_+^{1+\beta_1}u^h\|_{2q_\beta^1} \|D_+^{1+\beta_2}u^h\|_{2q_\beta^2} \|D_+^{1+\beta_3}u^h\|_{2q_\beta^3}.$$

First, setting $\frac{1}{2q_\beta^1} = \frac{1}{\tau} - \frac{j-(1+\beta_1)}{d}$, we observe that

$$\frac{1}{2q_\beta^1} \geq \frac{1}{\tau} - \frac{1}{2} + \frac{\beta_1}{d} \geq \frac{1}{\tau} - \frac{1}{2} + \frac{1}{d} > 0.$$

Hence, we obtain

$$\|D_+^{1+\beta_1} u^h\|_{2q_\beta^1} \leq C\|D_+^j u^h\|_\tau \leq C\|D_+^{j+1} u^h\|_2^{\frac{1}{2}} \|D_+^j u^h\|_2^{\frac{1}{2}}. \tag{3.27}$$

Next, setting $\frac{1}{2q_\beta^2} = \frac{1}{\tau} - \frac{(k_d-1)-(1+\beta_2)}{d}$ and observing

$$\frac{1}{2q_\beta^2} \geq \frac{1}{\tau} - \frac{1}{2} + \frac{\beta_2}{d} + \frac{1}{d} \geq \frac{1}{2d} > 0,$$

we obtain

$$\|D_+^{1+\beta_2} u^h\|_{2q_\beta^2} \leq C\|D_+^{k_d-1} u^h\|_\tau \leq C\|D_+^{k_d} u^h\|_2^{\frac{1}{2}} \|D_+^{k_d-1} u^h\|_2^{\frac{1}{2}}. \tag{3.28}$$

Setting $m = d - \beta_3 - k_d$, we see that

$$0 \leq m \leq k_d - 1 - \beta_3$$

and m satisfies $\frac{1}{2q_\beta^3} = \frac{1}{2} - \frac{m}{d}$. Hence, we have

$$\|D_+^{1+\beta_3} u^h\|_{2q_\beta^3} \leq C\|D_+^{1+m+\beta_3} u^h\|_2. \tag{3.29}$$

Combining (3.27), (3.28), and (3.29), we can deduce that

$$J_j \leq C\|D_+^{j+1} u^h\|_2^{\frac{1}{2}} \|D_+^j u^h\|_2^{\frac{1}{2}} \|D_+ u^h\|_{H_h^{k_d-1}}^2. \tag{3.30}$$

From (3.26) and (3.30) we have

$$J = J_2 + \sum_{j=3}^{k} J_j \tag{3.31}$$

$$\leq C\Big(\|D_+^2 u^h\|_2 \|D_+ u^h\|_{H_h^{k_d-1}}^2 + \sum_{j=3}^{k} \|D_+^{j+1} u^h\|_2^{\frac{1}{2}} \|D_+^j u^h\|_2^{\frac{1}{2}} \|D_+ u^h\|_{H_h^{k_d-1}}^2\Big)$$

$$\leq C\Big(\|D_+ u^h\|_{H_h^{k-1}} \|D_+ u^h\|_{H_h^{k_d-1}}^2 + \|D_+^2 u^h\|_{H_h^{k-1}}^{\frac{1}{2}} \|D_+ u^h\|_{H_h^{k-1}}^{\frac{1}{2}} \|D_+ u^h\|_{H_h^{k_d-1}}^2\Big).$$

From (3.23) and (3.31), the right-hand side of (3.14) is estimated as

$$C_0 \|D_+^2 u^h\|_{H_h^{k-1}} (I + J)$$

$$\leq C\|D_+^2u^h\|_{H_h^{k-1}}(\|D_+u^h\|_{H_h^{k-1}} + \|D_+^2u^h\|^{\frac{1}{2}}_{H_h^{k-1}}\|D_+u^h\|^{\frac{1}{2}}_{H_h^{k-1}})$$
$$\times (\|D_+u^h\|^2_{H_h^{k_d-1}} + \|D_+u^h\|_{H_h^{k_d-1}}).$$

Applying Young's inequality, we can conclude the following proposition.

Proposition 1. *If u^h is a solution of* (3.3) *given in Subsection* 3.3*, then u^h satisfies the following inequalities:*

$$\frac{d}{dt}\|D_+u^h(t)\|^2_{H_h^{k-1}} + \|D_+^2u^h(t)\|^2_{H_h^{k-1}} \leq C_1(1 + \|D_+u^h(t)\|^8_{H_h^{k_d-1}})\|D_+u^h(t)\|^2_{H_h^{k-1}}, \tag{3.32}$$

$$\frac{d}{dt}\|D_+u^h(t)\|^2_{H_h^{k_d-1}} + (1 - C_2\|D_+u^h(t)\|^2_{H_h^{k_d-1}})\|D_+^2u^h(t)\|^2_{H_h^{k_d-1}} \leq 0, \tag{3.33}$$

where C_1, C_2 are positive constants independent of h.

We now deduce a priori estimates of u^h for establishing local existence.

Taking $k = k_d$ in (3.32), we see that there exist $T > 0$ and $K > 0$ which depend on $\|\nabla u_0\|_{H^{k_d-1}(\mathbb{R}^d)}$, not on h, and satisfy

$$\|D_+u^h\|_{L^\infty(0,T;H_h^{k_d-1})} \leq K, \tag{3.34}$$

$$\|\tilde{\triangle}u^h\|_{L^2(0,T;H_h^{k_d-1})} \leq K. \tag{3.35}$$

Then, for $k > k_d$ we again use (3.32) to have

$$\|D_+u^h\|_{L^\infty(0,T;H_h^{k-1})} \leq K, \tag{3.36}$$

$$\|\tilde{\triangle}u^h\|_{L^2(0,T;H_h^{k-1})} \leq K. \tag{3.37}$$

In view of the equation (3.3) we may assume that

$$\left\|\frac{du^h}{dt}\right\|_{L^\infty(0,T;H_h^{k-2})} \leq K. \tag{3.38}$$

We next consider a priori estimates for establishing global existence. In much the same manner as in [6], the inequality (3.33) yields that there exists a constant $\delta > 0$ such that if $\|D_+u_0^h\|_{H_h^{k_d-1}} < \delta$, then

$$\|D_+u^h\|_{L^\infty(0,+\infty;H_h^{k_d-1})} < \delta. \tag{3.39}$$

Then, from (3.32), (3.39), and (3.3), we see that there exists $\delta' = \delta'(\delta) > 0$ such that

$$\begin{aligned} \|D_+u^h\|_{L^\infty(0,+\infty;H_h^{k-1})} &\leq \delta', && (3.40)\\ \|\tilde{\triangle}u^h\|_{L^2(0,+\infty;H_h^{k-1})} &\leq \delta'. && (3.41)\\ \Big\|\frac{du^h}{dt}\Big\|_{L^\infty(0,+\infty;H_h^{k-2})} &< \delta'. && (3.42)\end{aligned}$$

3.5. **Limit as h goes to 0.** We set

$$T_0 = \begin{cases} T & \text{for local existence,}\\ \text{an arbitrary positive number} & \text{for global existence.}\end{cases}$$

From (3.34)-(3.38) (or (3.40)-(3.42)) and in view of $|u^h| = 1$, we obtain

$$\begin{aligned} \|q_h u^h\|_{L^2((0,T_0)\times\omega)^3} &\leq K,\\ \|q_h(D_{i_1,+}u^h)\|_{L^2((0,T_0)\times\mathbb{R}^d)^3} &\leq K,\\ \|q_h(D_{i_1,+}D_{i_2,+}u^h)\|_{L^2((0,T_0)\times\mathbb{R}^d)^3} &\leq K,\\ &\vdots && (3.43)\\ \|q_h(D_{i_1,+}D_{i_2,+}\cdots D_{i_{k+1},+}u^h)\|_{L^2((0,T_0)\times\mathbb{R}^d)^3} &\leq K,\\ \Big\|q_h\Big(\frac{du^h}{dt}\Big)\Big\|_{L^2((0,T_0)\times\mathbb{R}^d)^3} &\leq K,\end{aligned}$$

where ω is an arbitrary bounded open set and K is a positive constant independent of h.

From (3.43), up to subsequences, we deduce that when h goes to 0,

$$\begin{aligned} q_h u^h &\to u \text{ in } L^2_{\text{loc}}((0,T_0)\times\mathbb{R}^d)^3 \text{ weakly,}\\ q_h(D_{i_1,+}u^h) &\to u_{i_1} \text{ in } L^2((0,T_0)\times\mathbb{R}^d)^3 \text{ weakly,}\\ q_h(D_{i_1,+}D_{i_2,+}u^h) &\to u_{i_1,i_2} \text{ in } L^2((0,T_0)\times\mathbb{R}^d)^3 \text{ weakly,}\\ &\vdots && (3.44)\\ q_h(D_{i_1,+}D_{i_2,+}\cdots D_{i_{k_d+1},+}u^h) &\to u_{i_1,i_2,\cdots,i_{k+1}}\\ &\qquad \text{in } L^2((0,T_0)\times\mathbb{R}^d)^3 \text{ weakly,}\\ q_h\left(\frac{du^h}{dt}\right) &\to f \text{ in } L^2((0,T_0)\times\mathbb{R}^d)^3 \text{ weakly.}\end{aligned}$$

By virtue of Lemma 2, we have

$$p_h(D_{i_2,+}\cdots D_{i_{k+1},+}u^h) \to u_{i_2,\cdots,i_{k+1}} \quad \text{in} \quad L^2((0,T_0)\times\mathbb{R}^d)^3 \quad \text{weakly} \tag{3.45}$$

and

$$u_{i_1,i_2,\cdots,i_{k+1}} = \frac{\partial}{\partial x_{i_1}} u_{i_2,\cdots,i_{k+1}}. \tag{3.46}$$

Hence we obtain that

$$\begin{aligned} u_{i_1} &= \frac{\partial u}{\partial x_{i_1}}, \\ u_{i_1,i_2} &= \frac{\partial^2 u}{\partial x_{i_1}\partial x_{i_2}}, \\ &\vdots \\ u_{i_1,i_2,\cdots,i_{k+1}} &= \frac{\partial^{k+1}u}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_{k+1}}}, \end{aligned} \tag{3.47}$$

and

$$f = \frac{\partial u}{\partial t}. \tag{3.48}$$

Hence, in virtue of Lemma 2 we have

$$p_h u^h \to u \quad \text{in} \quad L^2(0,T_0;L^2_{\text{loc}}(\mathbb{R}^d)) \quad \text{weakly}, \tag{3.49}$$

$$\frac{\partial}{\partial t} p_h u^h \to \frac{\partial u}{\partial t} \quad \text{in} \quad L^2(0,T_0;L^2(\mathbb{R}^d)) \quad \text{weakly}, \tag{3.50}$$

$$\frac{\partial}{\partial x_i} p_h u^h \to \frac{\partial u}{\partial x_i} \quad \text{in} \quad L^2(0,T_0;L^2(\mathbb{R}^d)) \quad \text{weakly}. \tag{3.51}$$

Aubin's Lemma and Lemma 2 yield

$$p_h u^h \to u \quad \text{in} \quad L^2(0,T_0;L^2_{\text{loc}}(\mathbb{R}^d)) \quad \text{strongly}, \tag{3.52}$$

$$q_h u^h \to u \quad \text{in} \quad L^2(0,T_0;L^2_{\text{loc}}(\mathbb{R}^d)) \quad \text{strongly}. \tag{3.53}$$

Taking $\varphi \in \mathcal{D}((0,T_0)\times\mathbb{R}^d)^3$ and any open bounded set Ω such that $\operatorname{supp}\varphi \subset \Omega$, we have

$$\begin{aligned} \int_{(0,T_0)\times\Omega} q_h\left(\frac{du^h}{dt}\right)\cdot\varphi &= \int_{(0,T_0)\times\Omega} (q_h u^h \times q_h\tilde{\triangle}u^h)\cdot\varphi \\ &\quad - \int_{(0,T_0)\times\Omega} (q_h u^h \times (q_h u^h \times q_h\tilde{\triangle}u))\cdot\varphi. \end{aligned} \tag{3.54}$$

As h goes to 0, we can obtain

$$\int_{(0,T_0)\times\Omega} \frac{\partial u}{\partial t}\cdot\varphi = \int_{(0,T_0)\times\Omega} (u\times\Delta u)\cdot\varphi - \int_{(0,T_0)\times\Omega} (u\times(u\times\Delta u))\cdot\varphi. \tag{3.55}$$

Hence, we conclude that u satisfies (2.1) almost everywhere.

By (3.36), we have

$$\|q_h(D_{i_1,+}\cdots D_{i_j,+}u^h)\|_{L^\infty(0,T_0;L^2(\mathbb{R}^d))} \quad \leq \quad K,\ 1\leq j\leq k. \tag{3.56}$$

From (3.47) and (3.56), by lower semicontinuity, we have

$$\nabla u \in L^\infty(0,T_0;H^{k-1}(\mathbb{R}^d))\cap L^2(0,T_0;H^k(\mathbb{R}^d)). \tag{3.57}$$

Furthermore, in view of the equation (2.1), we obtain

$$\frac{\partial u}{\partial t}\in L^\infty(0,T_0;H^{k-2}(\mathbb{R}^d))\cap L^2(0,T_0;H^{k-1}(\mathbb{R}^d)). \tag{3.58}$$

From (3.53), up to subsequences, we deduce that for an arbitrary open bounded set Ω

$$q_h u^h \to u \quad \text{in} \quad (0,T_0)\times\Omega \ \text{ a.e..}$$

Hence, we can conclude that $|u|=1$ in $(0,T_0)\times\mathbb{R}^d$ almost everywhere.

3.6. **Continuous dependence on initial data and uniqueness.** Continuous dependence and uniqueness can be established for weaker solutions than those established in the previous subsections. Indeed we have

Theorem 4. *Let $u_{0i}:\mathbb{R}^d\to\mathbb{S}^2\,(i=1,2)$ be such that $u_{01}-u_{02}\in L^2(\mathbb{R}^d)$. Let u_1, and u_2 be two solutions of (2.1) with initial values u_{01} and u_{02}, respectively, satisfying $|u_i(t,x)|=1,(t,x)\in[0,T]\times\mathbb{R}^d$, and $\nabla u_i\in L^2(0,T: L^\infty(\mathbb{R}^d))\bigcap L^\infty(0,T;H^1(\mathbb{R}^d))$. Then we have $u_1-u_2\in L^2(\mathbb{R}^d)$ and*

$$\|u_1-u_2\|_{L^2(\mathbb{R}^d)}\leq K(t)\|u_{01}-u_{02}\|_{L^2(\mathbb{R}^d)} \tag{3.59}$$

with

$$K(t)=\exp\int_0^t\left(\|\nabla u_1(\tau)\|_\infty^2+\|\nabla u_2(\tau)\|_\infty^2\right)\,dt$$

which implies that the map $u_0\mapsto u$ is continuous from $L^2(\mathbb{R}^d)$ into $L^\infty(0,T; L^2(\mathbb{R}^d))$ and the solutions are uniquely determined by their initial data.

Proof. For a solution u of (2.1) satisfying the assumption, we have

$$-u\times(u\times\Delta u)=\Delta u+|\nabla u|^2u.$$

Putting $\bar u_0=u_{01}-u_{02}$ and $\bar u=u_1-u_2$, we have

$$\begin{cases}\dfrac{\partial\bar u}{\partial t}=\Delta\bar u+\bar u\times\Delta u_1+u_2\times\Delta\bar u+u_1\left(\nabla u_1+\nabla u_2\right)\cdot\nabla\bar u+\left|\nabla u_2\right|^2\bar u,\\ \bar u(x,0)=\bar u_0,\quad x\in\mathbb{R}^d.\end{cases} \tag{3.60}$$

For $0 < s < t < T' < T_0$ we define the mollifier θ_n by

$$(\theta_n * f)(t) = \int_{s-\frac{1}{n}}^{T'+\frac{1}{n}} \theta_n(t-\tau) f(\tau) d\tau,$$

where $\theta \in C^\infty(\mathbb{R})$ satisfies

$$\int_{-1}^{1} \theta(t) dt = 1, \quad \operatorname{supp} \theta \subset (-1,1), \quad \theta_n(t) = \frac{1}{n}\theta(\frac{t}{n}). \tag{3.61}$$

Using the mollifier for (3.60), we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\theta_n * (\zeta_m \bar{u})\|_2^2 + \|\theta_n * \zeta_m \nabla \bar{u}\|_2^2 \\
&\quad = -2\int_{\mathbb{R}^d} \theta_n * (\zeta_m \bar{u}) \cdot \Big(\theta_n * \sum_{j=1}^d \frac{\partial \zeta_m}{\partial x_j}\frac{\partial \bar{u}}{\partial x_j}\Big) dx \\
&\qquad + \int_{\mathbb{R}^d} \theta_n * (\zeta_m \bar{u}) \cdot \theta_n * (\zeta_m \bar{u} \times \Delta u_1)\, dx \\
&\qquad -2\int_{\mathbb{R}^d} \theta_n * (\zeta_m \bar{u}) \cdot \theta_n * \Big(u_2 \times \sum_{j=1}^d \frac{\partial \zeta_m}{\partial x_j}\frac{\partial \bar{u}}{\partial x_j}\Big) dx \\
&\qquad - \int_{\mathbb{R}^d} \zeta_m^2 \sum_{j=1}^d \Big(\theta_n * \frac{\partial \bar{u}}{\partial x_j}\Big) \cdot \Big(\theta_n * \Big(u_2 \times \frac{\partial \bar{u}}{\partial x_j}\Big)\Big) dx \\
&\qquad - \int_{\mathbb{R}^d} \theta_n * (\zeta_m \bar{u}) \cdot \theta_n * \Big(\sum_{j=1}^d \frac{\partial u_2}{\partial x_j} \times \zeta_m \frac{\partial \bar{u}}{\partial x_j}\Big) dx \\
&\qquad + \int_{\mathbb{R}^d} \theta_n * (\zeta_m \bar{u}) \cdot \zeta_m \left(\theta_n * (u_1 \nabla(u_1 + u_2) \cdot \nabla \bar{u})\right) dx \\
&\qquad + \int_{\mathbb{R}^d} \theta_n * (\zeta_m \bar{u}) \cdot \zeta_m \left(\theta_n * |\nabla u_2|^2 \bar{u}\right) dx,
\end{aligned} \tag{3.62}$$

where $\zeta_(x) = e^{-|x|/m}$.

Integrating (3.62) and taking the limit as n goes to ∞, we have

$$\begin{aligned}
&\frac{1}{2}\|\zeta_m \bar{u}(t)\|_2^2 + \int_s^t \|\zeta_m \nabla \bar{u}(\tau)\|_2^2 \, d\tau \\
&= \frac{1}{2}\|\zeta_m \bar{u}(s)\|_2^2 - 2\int_s^t \int_{\mathbb{R}^d} \zeta_m \bar{u} \cdot \Big(\sum_{j=1}^d \frac{\partial \zeta_m}{\partial x_j}\frac{\partial \bar{u}}{\partial x_j}\Big) dx d\tau
\end{aligned}$$

$$- 2\int_s^t \int_{\mathbb{R}^d} \zeta_m \bar{u} \cdot \Big(u_2 \times \sum_{j=1}^d \frac{\partial \zeta_m}{\partial x_j} \frac{\partial \bar{u}}{\partial x_j} \Big) dx d\tau$$

$$- \int_s^t \int_{\mathbb{R}^d} \zeta_m \bar{u} \cdot \Big(\sum_{j=1}^d \frac{\partial u_2}{\partial x_j} \times \zeta_m \frac{\partial \bar{u}}{\partial x_j} \Big) dx d\tau$$

$$+ \int_s^t \int_{\mathbb{R}^d} \zeta_m \bar{u} \cdot \zeta_m \left(u_1 \nabla (u_1 + u_2) \cdot \nabla \bar{u} \right) dx d\tau + \int_s^t \int_{\mathbb{R}^d} |\zeta_m \bar{u}|^2 \, |\nabla u_2|^2 dx d\tau.$$

Thus, we have

$$\begin{aligned}
& \frac{1}{2} \|\zeta_m \bar{u}(t)\|_2^2 + \frac{1}{2} \int_s^t \|\zeta_m \nabla \bar{u}(\tau)\|_2^2 \, d\tau \\
\leq \; & \frac{1}{2} \|\zeta_m \bar{u}(s)\|_2^2 + 4 \int_s^t \|\zeta \bar{u}(\tau)\|_2 \, \|\nabla \zeta_m \nabla \bar{u}(\tau)\|_2 \, d\tau \\
& + \int_s^t \|\zeta_m \bar{u}(\tau)\|_2 \, \|\nabla u_2(\tau)\|_\infty \, \|\zeta_m \nabla \bar{u}(\tau)\|_2 \, d\tau \\
& + \int_s^t \|\zeta_m \bar{u}(\tau)\|_2 \left(\|\nabla u_1(\tau)\|_\infty + \|\nabla u_2(\tau)\|_\infty \right) \|\zeta_m \nabla \bar{u}(\tau)\|_2 \, d\tau \\
& + \int_s^t \|\zeta_m \bar{u}(\tau)\|_2^2 \, \|\nabla u_2(\tau)\|_\infty^2 \, d\tau.
\end{aligned}$$

Taking the limit as $s \to 0+$ and by using Young's inequality, we deduce that

$$\begin{aligned}
& \|\zeta_m \bar{u}(t)\|_2^2 + \int_0^t \|\zeta_m \nabla \bar{u}(\tau)\|_2^2 \, d\tau - \int_0^t \|\nabla \zeta_m \nabla \bar{u}(\tau)\|_2^2 d\tau \\
& \qquad \leq \|\zeta_m \bar{u}_0\|_2^2 + C \int_0^t \|\zeta_m \bar{u}(\tau)\|_2^2 \left(1 + \|\nabla u_1(\tau)\|_\infty^2 + \|\nabla u_2(\tau)\|_\infty^2 \right) d\tau.
\end{aligned}$$

Since $1 + \|\nabla u_1(t)\|_\infty^2 + \|\nabla u_2(t)\|_\infty^2 \in L^1(0,T)$, we can use Gronwall's inequality to have

$$\|\zeta_m \bar{u}(t)\|_{L^2(\mathbb{R}^d)} \leq K(t) \|\zeta_m \bar{u}_0\|_{L^2(\mathbb{R}^d)}, \quad t \in \; [0, T_0].$$

Letting m go to ∞, we obtain (3.59).

4. PROOF OF THEOREM 3

We apply the well-known results on nonlinear evolution equations in a Banach space X.

Consider the abstract Cauchy problem in X :

$$\begin{aligned} \frac{dP(t)}{dt} + A(t,P(t))P(t) &= f(t,P(t)), && (4.1)\\ P(0) &= P_0, && (4.2) \end{aligned}$$

where (4.1) is, in general, a nonlinear equation with respect to P. We shall make the following assumptions:

(A1): The operator $A_0 = A(0,P_0)$ is a closed operator with a domain D_0 dense in X and

$$\|(\lambda I - A_0)^{-1}\|_{B(X)} \le \frac{C}{1+|\lambda|} \quad \text{for all } \lambda \text{ with } \Re\lambda \le 0\ . \tag{4.3}$$

(A2): For some $\alpha \in [0,1)$, $R > 0$, $T_0 > 0$ and for any $P \in X$ with $\|P\|_X < R$ the operator $A(t, A_0^{-\alpha}P)$ is well defined on D_0, for all $t \in [0,T_0]$. Furthermore, for any $t,\tau \in [0,T_0]$ and $P,P' \in X$ with $\|P\|_X < R, \|P'\|_X < R$,

$$\begin{aligned} &\| \left(A(t,A_0^{-\alpha}P) - A(\tau,A_0^{-\alpha}P')\right) A(\tau,A_0^{-\alpha}P')^{-1}\|_{B(X)} \\ &\qquad \le C(R)\left(|t-\tau|^{\sigma} + \|P-P'\|_X\right) \end{aligned} \tag{4.4}$$

where $0 < \sigma \le 1$.

(A3): For every $t,\tau \in [0,T_0]$, $P,P' \in X$ with $\|P\|_X < R$, $\|P'\|_X < R$,

$$\|f(t,A_0^{-\alpha}P) - f(t,A_0^{-\alpha}P')\|_X \le C(R)\left(|t-\tau|^{\sigma} + \|P-P'\|_X\right). \tag{4.5}$$

(A4): $P_0 \in D(A_0^{\beta})$ for some $\beta > \alpha$ and

$$\|A_0^{\alpha}P_0\|_X < R, \tag{4.6}$$

where $B(X)$ denotes the Banach space whose elements are the bounded linear operators in X.

The following Theorem is due to Friedman [9].

Theorem 5. *Let the assumptions* (A1) − (A4) *hold. Then, there exists* $T^* \in (0,T_0]$ *such that the Cauchy problem* (4.1), (4.2) *has a unique solution* P *satisfying*

(i) $A_0^{\gamma}P \in C([0,T^*])$ *for* $\forall\gamma \in [0,\alpha]$,
(ii) $P \in C^1((0,T^*])$.

We now consider the following Cauchy problem

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Big(\delta + (I + e^{-\delta t} u\times)\Delta\Big)\, u \;+\; e^{-2\delta t} u \displaystyle\sum_{k=1}^{d} |p_k|^2, \\ \dfrac{\partial p_i}{\partial t} = \Big(\delta + (I + e^{-\delta t} u\times)\Delta\Big)\, p_i \;+\; e^{-\delta t} p_i \times \displaystyle\sum_{k=1}^{d} \dfrac{\partial p_k}{\partial x_k} \\ \qquad\qquad + 2e^{-2\delta t} u \displaystyle\sum_{k=1}^{d} p_k \cdot \dfrac{\partial p_k}{\partial x_i} + e^{-2\delta t} \dfrac{\partial u}{\partial x_i} \displaystyle\sum_{k=1}^{d} |p_k|^2 \quad (i = 1, \cdots, d), \\ u(0) = u_0, \\ p_i(0) = \left(\dfrac{\partial u_0}{\partial x_1}, \cdots, \dfrac{\partial u_0}{\partial x_d}\right)^t. \end{cases} \tag{4.7}$$

Let E_d denote the $d \times d$ unit matrix. In (4.1) and (4.2), setting $X = L^p(\mathbb{T}^d)^{3(1+d)}, P = (u, p_1, \cdots, p_d)^t$, $A(t, P) = (\delta + (I + e^{-\delta t} u\times)\Delta)E_d$, and

$$f(t, P) = \begin{pmatrix} e^{-2\delta t} u \displaystyle\sum_{k=1}^{d} |p_k|^2 \\ e^{-\delta t} p_1 \times \displaystyle\sum_{k=1}^{d} \dfrac{\partial p_k}{\partial x_k} + 2e^{-2\delta t} \Big(\displaystyle\sum_{k=1}^{d} p_k \cdot \dfrac{\partial p_k}{\partial x_1}\Big) u + e^{-2\delta t} \displaystyle\sum_{k=1}^{d} |p_k|^2 \dfrac{\partial u}{\partial x_1} \\ \vdots \\ e^{-\delta t} p_d \times \displaystyle\sum_{k=1}^{d} \dfrac{\partial p_k}{\partial x_k} + 2e^{-2\delta t} \Big(\displaystyle\sum_{k=1}^{d} p_k \cdot \dfrac{\partial p_k}{\partial x_d}\Big) u + e^{-2\delta t} \displaystyle\sum_{k=1}^{d} |p_k|^2 \dfrac{\partial u}{\partial x_d} \end{pmatrix},$$

we can regard (4.7) as the abstract Cauchy problem (4.1), (4.2). We here take $\delta > 0$ sufficiently large.

In order to prove Theorem 3, we shall check the conditions (A1)-(A4) to solve the Cauchy problem (4.1), (4.2).

As to (A1) we need the following lemma, an analogy of the well-known result of Agmon, Douglis, Nirenberg [1].

Lemma 3. *Assume that $a_{kl}^{ij} \in C^1(\mathbb{T}^d)$ $(1 \le i, j \le m, 1 \le k, l \le d)$ satisfies*

$$\sum_{i,j=1}^{d} \sum_{k,l=1}^{m} a_{kl}^{ij} \xi_i \bar{\xi}_j \eta^k \bar{\eta}^l > 0 \tag{4.8}$$

for any $\xi_i, \xi_j, \eta^k, \eta^l \in \mathbb{C}\backslash\{0\}$. Then there exists $C_0 > 0$ such that for any $f = (f_1, \cdots, f_m) \in L^p(\mathbb{T}^d)$, any $\lambda \in \mathbb{C}$ with $|\lambda| \geq C_0$ and $\eta < \arg\lambda < 2\pi - \eta$ with $\eta \in \left(0, \frac{\pi}{2}\right)$, there exists $u = (u_1, \cdots, u_m) \in W^{2,p}(\mathbb{T}^d)$ which uniquely solves

$$-\sum_{i,j=1}^{d}\sum_{l=1}^{m} a_{kl}^{ij}\frac{\partial^2 u^l}{\partial x_i \partial x_j} - \lambda u^k = f^k, \quad k = 1, \cdots, m, \tag{4.9}$$

and satisfies for some $C > 0$

$$|\lambda|\|u\|_{L^p(\mathbb{T}^d)} + \|u\|_{W^{2,p}(\mathbb{T}^d)} \leq C\|f\|_{L^p(\mathbb{T}^d)}. \tag{4.10}$$

Setting

$$A_0 = -\delta - (I + u_0\times)\Delta, \tag{4.11}$$

by virtue of Lemma 3 we have

$$\left\|(\lambda I - A_0)^{-1}\right\|_{B(L^p)} \leq \frac{C}{1+|\lambda|}, \quad (\Re\lambda \leq 0). \tag{4.12}$$

We now consider (A2). For $\alpha \in (\frac{1}{2}, 1)$, we have

$$\|\nabla A_0^{-\alpha} u\|_p \leq C\|u\|_p \quad \forall u \in L^p(\mathbb{T}^d)$$

(see [9]). Hence, we have

$$\left\|A_0^{-\alpha}u\right\|_\infty \leq C\left\|A_0^{-\alpha}u\right\|_{1,p} \leq C\|u\|_p \tag{4.13}$$

and

$$\begin{aligned}
&\Big\|\Big\{(-\delta - (I + e^{-\delta t}A_0^{-\alpha}u\times)\Delta) - (-\delta - (I + e^{-\delta\tau}A_0^{-\alpha}u'\times)\Delta)\Big\}\\
&\qquad (-\delta - (I + e^{-\delta\tau}A_0^{-\alpha}u'\times)\Delta)^{-1}\Big\|_{B(L^p)}\\
= \;& \Big\|\Big\{A_0^{-\alpha}(e^{-\delta t}u - e^{-\delta\tau}u')\times\Big\}\frac{1}{1+|e^{-\delta\tau}A_0^{-\alpha}u'|^2}(I - (e^{-\delta\tau}A_0^{-\alpha}u'\times)\\
&\qquad + e^{-2\delta\tau}A_0^{-\alpha}u'(A_0^{-\alpha}u'\cdot))\\
&\quad \Big(-(I + e^{-\delta\tau}A_0^{-\alpha}u'\times)\Delta\Big)\Big(-\delta - (I + e^{-\delta\tau}A_0^{-\alpha}u'\times)\Delta\Big)^{-1}\Big\|_{B(L^p)}\\
\leq \;& C\left\|A_0^{-\alpha}(e^{-\delta t}u - e^{-\delta\tau}u')\right\|_{1,p}\\
&\qquad \left(1 + \delta\left\|\left(-\delta - \left(I + e^{-\delta\tau}A_0^{-\alpha}u'\times\right)\Delta\right)^{-1}\right\|_{B(L^p)}\right)\\
\leq \;& C(R)\left(\|u-u'\|_p + |t-\tau|\right).
\end{aligned} \tag{4.14}$$

We will check (A3). We write

$$\|f(t, A_0^{-\alpha}P) - f(\tau, A_0^{-\alpha}P')\|_{(L^p)^{1+d}} \leq F_1 + F_2 + F_3 + F_4, \tag{4.15}$$

where

$$\begin{aligned}
F_1 &= \left\| e^{-2\delta t}\left(A_0^{-\alpha}u\right)\sum_{k=1}^{d}|A_0^{-\alpha}p_k|^2 - e^{-2\delta\tau}\left(A_0^{-\alpha}u'\right)\sum_{k=1}^{d}|A_0^{-\alpha}p'_k|^2\right\|_{L^p},\\
F_2 &= \sum_{i=1}^{d}\left\| e^{-\delta t}\left(A_0^{-\alpha}p_i\right)\times\sum_{k=1}^{d}\frac{\partial}{\partial x_k}A_0^{-\alpha}p_k \right.\\
&\qquad\qquad \left. -e^{-\delta\tau}\left(A_0^{-\alpha}p'_i\right)\times\sum_{k=1}^{d}\frac{\partial}{\partial x_k}A_0^{-\alpha}p'_k\right\|_{L^p},\\
F_3 &= 2\sum_{i=1}^{d}\left\| e^{-2\delta t}\left(A_0^{-\alpha}u\right)\sum_{k=1}^{d}\left(A_0^{-\alpha}p_k\right)\cdot\frac{\partial}{\partial x_i}A_0^{-\alpha}p_k \right.\\
&\qquad\qquad \left. -e^{-2\delta\tau}\left(A_0^{-\alpha}u'\right)\sum_{k=1}^{d}\left(A_0^{-\alpha}p'_k\right)\cdot\frac{\partial}{\partial x_i}A_0^{-\alpha}p'_k\right\|_{L^p},\\
F_4 &= \sum_{i=1}^{d}\left\| e^{-2\delta t}\sum_{k=1}^{d}|A_0^{-\alpha}p_k|^2\frac{\partial}{\partial x_i}A_0^{-\alpha}u \right.\\
&\qquad\qquad \left. -e^{-2\delta\tau}\sum_{k=1}^{d}|A_0^{-\alpha}p'_k|^2\frac{\partial}{\partial x_i}A_0^{-\alpha}u'\right\|_{L^p}.
\end{aligned}$$

We estimate each term as

$$\begin{aligned}
F_1 &\leq \sum_{k=1}^{d}\left\| A_0^{-\alpha}p_k\|_\infty^2\|A_0^{-\alpha}(e^{-\delta t}u - e^{-\delta\tau}u')\right\|_p\\
&\quad + \sum_{k=1}^{d}\|A_0^{-\alpha}u'\|_p\left(\|A_0^{-\alpha}p_k\|_\infty + \|A_0^{-\alpha}p'_k\|_\infty\right)\|A_0^{-\alpha}(e^{-\delta t}p_k - e^{-\delta\tau}p'_k)\|_\infty\\
&\leq C(R)\|e^{-\delta t}P - e^{-\delta\tau}P'\|_p,
\end{aligned} \tag{4.16}$$

$$F_2 \leq \sum_{i,k=1}^{d}\|A_0^{-\alpha}(e^{-\delta t}p_i - e^{-\delta\tau}p'_i)\|_\infty\left\|\frac{\partial}{\partial x_k}A_0^{-\alpha}p_k\right\|_p$$

$$
\begin{aligned}
&+\sum_{i,k=1}^{d}\|A_0^{-\alpha}p_i'\|_p\left\|\frac{\partial}{\partial x_k}A_0^{-\alpha}(e^{-\delta t}p_k-e^{-\delta\tau}p_k')\right\|_\infty\\
\leq\ & C(R)\|e^{-\delta t}P-e^{-\delta\tau}P'\|_p, \qquad (4.17)
\end{aligned}
$$

$$
\begin{aligned}
F_3 \leq\ & C\sum_{i,k=1}^{d}\left\|A_0^{-\alpha}(e^{-\delta t}u-e^{-\delta\tau}u')\right\|_\infty\|A_0^{-\alpha}p_k\|_\infty\left\|\frac{\partial}{\partial x_i}A_0^{-\alpha}p_k\right\|_p\\
&+C\sum_{i,k=1}^{d}\left\|A_0^{-\alpha}u'\right\|_\infty\left\|A_0^{-\alpha}(e^{-\delta t}p_k-e^{\delta\tau}p_k')\right\|_\infty\left\|\frac{\partial}{\partial x_i}A_0^{-\alpha}p_k\right\|_p\\
&+C\sum_{i,k=1}^{d}\left\|A_0^{-\alpha}u'\right\|_\infty\left\|A_0^{-\alpha}p_k'\right\|_\infty\left\|\frac{\partial}{\partial x_i}A_0^{-\alpha}(e^{-\delta t}p_k-e^{-\delta\tau}p_k')\right\|_p\\
\leq\ & C(R)\|e^{-\delta t}P-e^{-\delta\tau}P'\|_p, \qquad (4.18)
\end{aligned}
$$

and

$$
\begin{aligned}
F_4 \leq\ & \sum_{i,k=1}^{d}\left\|A_0^{-\alpha}p_k\right\|_\infty^2\left\|\frac{\partial}{\partial x_i}A_0^{-\alpha}(e^{-2\delta t}u-e^{-2\delta\tau}u')\right\|_p\\
&+\sum_{i,k=1}^{d}\left\|A_0^{-\alpha}(e^{-\delta t}p_k-e^{-\delta\tau}p_k')\right\|_\infty\\
&\qquad\left(\|A_0^{-\alpha}p_k\|_\infty+\|A_0^{-\alpha}p_k'\|_\infty\right)\left\|\frac{\partial}{\partial x_i}A_0^{-\alpha}u'\right\|_p\\
\leq\ & C(R)\|e^{-\delta t}P-e^{-\delta\tau}P'\|_p. \qquad (4.19)
\end{aligned}
$$

Combining (4.15) and (4.16)-(4.19), we obtain

$$
\left\|f(t,A_0^{-\alpha}P)-f(\tau,A_0^{-\alpha}P')\right\|_{(L^p)^{1+d}}\leq C(R)\left(\|P-P'\|_{(L^p)^{1+d}}+|t-\tau|\right). \qquad (4.20)
$$

The final assumption (A4) is obviously satisfied. Applying Theorem 5, we easily see that there exists $T^*>0$ such that the problem (4.7) has a unique solution $P=(u,p_1,\cdots,p_d)^t$ satisfying

$$
P\in C([0,T^*];W^{1,p}(\mathbb{T}^d))\cap C((0,T^*];W^{2,p}(\mathbb{T}^d))\cap C^1((0,T^*];L^p(\mathbb{T}^d)).
$$

Replacing P by $e^{-\delta t}P$, we may assume $\delta=0$ in (4.7).

We now verify that

$$p_i = \frac{\partial u}{\partial x_i} \qquad (i = 1, 2, \cdots d). \tag{4.21}$$

Put $v_i = p_i - \frac{\partial u}{\partial x_i}$ $(1 \le i \le d)$ and set $0 < s < \zeta \le t \le T' < T^*$. Let us define the mollifier as

$$(\theta_n * f)(t) = \int_{\zeta - \frac{1}{n}}^{T' + \frac{1}{n}} \theta_n(t - \tau) f(\tau) dt,$$

where θ_n is defined in (3.61). Putting $v_i^n = \theta_n * v_i$, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|v_i^n(t)\|_2^2 = \int_{\mathbb{T}^d} v_i^n \cdot \theta_n * \Big[(I + u\times)\,\Delta p_i + \sum_{k=1}^{d} p_i \times \frac{\partial p_k}{\partial x_k} \\
+ 2u\Big(\sum_{k=1}^{d} p_k \cdot \frac{\partial p_k}{\partial x_i}\Big) + \frac{\partial u}{\partial x_i}\sum_{k=1}^{d} |p_k|^2 \Big] dx \\
+ \int_{\mathbb{T}^d} \frac{\partial v_i^n}{\partial x_i} \cdot \frac{\partial}{\partial x_i}\Big(\theta_n * \Big[(I + u\times)\Delta p_i + u\sum_{k=1}^{d} |p_k|^2\Big]\Big) dx \\
= \quad & \int_{\mathbb{T}^d} v_i^n \cdot \theta_n * \Big[(I + u\times)\,\Delta p_i + p_i \times \sum_{k=1}^{d} \frac{\partial p_k}{\partial x_k}\Big] dx \\
& + \int_{\mathbb{T}^d} \frac{\partial v_i^n}{\partial x_i} \cdot \ \theta_n * [(I + u\times)\,\Delta u]\, dx \\
= \quad & g_1(t) + g_2(t) + g_3(t),
\end{aligned} \tag{4.22}$$

where

$$\begin{aligned}
g_1(t) &= \int_{\mathbb{T}^d} v_i^n \cdot \theta_n * (I + u\times)\,\Delta p_i dx, \\
g_2(t) &= \int_{\mathbb{T}^d} \frac{\partial v_i^n}{\partial x_i} \cdot \theta_n * (I + u\times)\,\Delta u dx + \int_{\mathbb{T}^d} v_i^n \cdot \theta_n * \Big(\frac{\partial u}{\partial x_i} \times \Delta u\Big) dx, \\
g_3(t) &= \int_{\mathbb{T}^d} v_i^n \cdot \theta_n * \Big(v_i \times \sum_{k=1}^{d} \frac{\partial p_k}{\partial x_k}\Big) dx \\
&\quad + \int_{\mathbb{T}^d} v_i^n \cdot \theta_n * \Big(\frac{\partial u}{\partial x_i} \times \sum_{k=1}^{d} \frac{\partial v_k}{\partial x_k}\Big) dx.
\end{aligned}$$

Continuing computation, we get

$$\begin{aligned} g_1(t) &= -\int_{\mathbb{T}^d} \sum_{k=1}^{d} \frac{\partial v_i^n}{\partial x_k} \cdot \Big[\theta_n * (I + u\times) \frac{\partial p_i}{\partial x_k}\Big] dx \\ &\qquad - \int_{\mathbb{T}^d} v_i^n \cdot \theta_n * \Big(\sum_{k=1}^{d} \frac{\partial u}{\partial x_k} \times \frac{\partial p_i}{\partial x_k}\Big) dx, \\ g_2(t) &= \int_{\mathbb{T}^d} \sum_{k=1}^{d} \frac{\partial v_i^n}{\partial x_k} \cdot \theta_n * (I + u\times) \frac{\partial}{\partial x_k}\Big(\frac{\partial u}{\partial x_i}\Big) dx \\ &\qquad - \int_{\mathbb{T}^d} v_i^n \cdot \theta_n * \Big(\sum_{k=1}^{d} \frac{\partial u}{\partial x_k} \times \frac{\partial}{\partial x_k}\Big(\frac{\partial u}{\partial x_i}\Big)\Big) dx. \end{aligned}$$

Hence, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_i^n(t)\|_2^2 = -\int_{\mathbb{T}^d} \sum_{k=1}^{d} \frac{\partial v_i^n}{\partial x_k} \cdot \theta_n * (I + u\times) \frac{\partial v_i}{\partial x_k} dx \\ &- \int_{\mathbb{T}^d} v_i^n \cdot \theta_n * \Big(\sum_{k=1}^{d} \frac{\partial u}{\partial x_k} \times \frac{\partial v_i}{\partial x_k}\Big) dx + \int_{\mathbb{T}^d} v_i^n \cdot \theta_n * \Big(v_i \times \sum_{k=1}^{d} \frac{\partial p_k}{\partial x_k}\Big) dx \\ &+ \int_{\mathbb{T}^d} v_i^n \cdot \theta_n * \Big(\frac{\partial u}{\partial x_i} \times \sum_{k=1}^{d} \frac{\partial v_k}{\partial x_k}\Big) dx. \end{aligned} \tag{4.23}$$

Integrating (4.23) on $[\zeta, t]$ and taking the limit as $n \to \infty$, we obtain

$$\begin{aligned} &\frac{1}{2}\|v_i(t)\|_2^2 + \int_\zeta^t \|\nabla v_i(\tau)\|_2^2 \, d\tau \\ = \;& \frac{1}{2}\|v_i(\zeta)\|_2^2 - \int_\zeta^t \int_{\mathbb{T}^d} v_i \cdot \Big(\sum_{k=1}^{d} \frac{\partial u}{\partial x_k} \times \frac{\partial v_i}{\partial x_k}\Big) dx d\tau \\ &+ \int_\zeta^t \int_{\mathbb{T}^d} v_i \cdot \Big(\frac{\partial u}{\partial x_i} \times \sum_{k=1}^{d} \frac{\partial v_k}{\partial x_k}\Big) dx d\tau. \end{aligned} \tag{4.24}$$

Summing (4.24) over i and estimating, we have

$$\frac{1}{2}\sum_{i=1}^{d} \|v_i(t)\|_2^2 + \int_\zeta^t \sum_{i=1}^{d} \|\nabla v_i(\tau)\|_2^2 d\tau \tag{4.25}$$

$$\leq \quad \frac{1}{2}\sum_{i=1}^{d}\|v_i(\zeta)\|_2^2 + C\int_{\zeta}^{t}\Big[\sum_{i,k=1}^{d}\|v_i(\tau)\|_{\frac{2p}{p-2}}\|\nabla v_k(\tau)\|_2\|\nabla u(\tau)\|_p\Big]d\tau.$$

Applying the Gagliardo-Nirenberg inequality to (4.25), we obtain

$$\sum_{i=1}^{d}\|v_i(t)\|_2^2 + \int_{\zeta}^{t}\sum_{i=1}^{d}\|\nabla v_i(\tau)\|_2^2 d\tau$$
$$\leq \sum_{i=1}^{d}\|v_i(\zeta)\|_2^2 + C\int_{\zeta}^{t}\left(1+\|\nabla u\|_p^{\frac{2d}{1-\frac{d}{p}}}\right)\sum_{i=1}^{d}\|v_i\|_2^2 d\tau.$$

Since $u, p_i \in C([0,T^*];W^{1,p}(\mathbb{T}^d))$, we can take the limit as $\zeta \to 0+$ to obtain

$$\sum_{i=1}^{d}\|v_i(t)\|_2^2 \leq C\int_0^t\left(1+\|\nabla u(\tau)\|_p^{\frac{2d}{1-\frac{d}{p}}}\right)\sum_{i=1}^{d}\|v_i(\tau)\|_2^2 d\tau. \tag{4.26}$$

By using Gronwall's inequality, we have

$$v_i(t) = p_i(t) - \frac{\partial u}{\partial x_i}(t) = 0. \tag{4.27}$$

Hence, for $u_0 \in W^{3,p}(\mathbb{T}^d)$, there exists at least one u such that

$$u \in C([0,T^*];W^{2,p}(\mathbb{T}^d)) \cap C((0,T^*];W^{3,p}(\mathbb{T}^d)) \cap C^1((0,T^*];W^{1,p}(\mathbb{T}^d))$$

and

$$\begin{cases} \dfrac{\partial u}{\partial t} = (I + u\times)\,\Delta u + |\nabla u|^2 u, \quad t \in (0,T^*],\ x \in \mathbb{T}^d, \\ u(0,x) = u_0(x), \quad x \in \mathbb{T}^d. \end{cases} \tag{4.28}$$

Next, we verify that if $u_0 \in W^{3,p}(\mathbb{T}^d)$ and $|u_0(x)| = 1$, $x \in \mathbb{T}^d$, then

$$|u(t,x)| = 1, \quad t \in [0,T^*],\ x \in \mathbb{T}^d.$$

Setting $w = |u|^2 - 1$, w satisfies the following equation:

$$\frac{\partial w}{\partial t} - \Delta w = 2u\cdot\frac{\partial u}{\partial t} - 2u\cdot\Delta u - 2|\nabla u|^2 = 2|\nabla u|^2 w. \tag{4.29}$$

Multiplying (4.29) by w, we can obtain the following inequality:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w(t)\|_2^2 + \|\nabla w(t)\|_2^2 &\leq 2\|\nabla u(t)\|_\infty^2\|w(t)\|_2^2 \\ &\leq C\|u\|_{C([0,T^*];W^{2,p}(\mathbb{T}^d))}^2\|w(t)\|_2^2. \end{aligned}$$

By using Gronwall's inequality and recalling $\|w(0)\|_2 = 0$, we have $w(t) = 0$, $t \in [0, T^*]$, from which we conclude that $|u(t,x)| = 1$, $t \in [0, T^*]$, $x \in \mathbb{T}^d$. Hence, we have

$$-u \times (u \times \Delta u) = \Delta u + |\nabla u|^2 u.$$

The existence is verified.

4.1. **Uniqueness.** Let u_1, u_2 be solutions given in the previous section. Set $\bar{u} = u_1 - u_2$. Then we see that $\bar{u} \in C([0,T^*]; W^{2,p}(\mathbb{T}^d)) \cap C((0,T^*]; W^{3,p}(\mathbb{T}^d)) \cap C^1((0,T^*]; W^{1,p}(\mathbb{T}^d))$ and

$$\frac{\partial \bar{u}}{\partial t} = \Delta \bar{u} + u_1 \times \Delta \bar{u} + \bar{u} \times \Delta u_2 + |\nabla u_1|^2 \bar{u} + u_2 \left(\nabla u_1 + \nabla u_2\right) \cdot \nabla \bar{u}, \quad t \in (0, T^*],\ x \in \mathbb{T}^d, \tag{4.30}$$

$$\bar{u}(0,x) = 0, \quad x \in \mathbb{T}^d. \tag{4.31}$$

Multiplying (4.30) by $\bar{u}$, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\bar{u}(t)\|_2^2 + \|\nabla \bar{u}(t)\|_2^2 \\ &\le \|\nabla u_1(t)\|_\infty \|\bar{u}(t)\|_2 \|\nabla \bar{u}(t)\|_2 + \|\nabla u_1(t)\|_\infty^2 \|\bar{u}(t)\|_2^2 \\ &\quad + \|u_2(t)\|_\infty \|\nabla (u_1(t) + u_2(t))\|_\infty \|\bar{u}(t)\|_2 \|\nabla \bar{u}(t)\|_2, \end{aligned}$$

from which we deduce, for $t \in [0, T^*]$,

$$\begin{aligned} &\frac{d}{dt}\|\bar{u}(t)\|_2^2 + \|\nabla \bar{u}(t)\|_2^2 \\ \le\ & C\|\bar{u}(t)\|_2^2 \left(1 + \|u_1\|^4_{C([0,T^*];W^{2,p}(\mathbb{T}^d))} + \|u_2\|^4_{C([0,T^*];W^{2,p}(\mathbb{T}^d))}\right). \end{aligned}$$

Gronwall's inequality yields that

$$\bar{u}(t) = u_1(t) - u_2(t) = 0, \quad t \in [0, T^*].$$

Hence, the uniqueness assertion is proved.

REFERENCES

[1] S. Agmon, A. Douglis, and L. Nirenberg, *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I,* Comm. Pure Appl. Math., 12 (1959), 623–727.

[2] Francois Alouges and Alain Soyeur, *On global weak solutions for Landau Lifschitz equations: existence and uniqueness,* Nonlinear Analysis, Thoery, Methods & Applications, 18 (1992), 1071–1084.

[3] Herbert Amann, *Quasilinear Parabolic Systems Under Nonlinear Boundary Conditions,* Arch. Rat. Mech Anal., 92 (1986), 153–192.

[4] Hebert Amann, *Dyanmic Theory of Quasilinear Parabolic Equations II. Reaction-Diffusion Systems,* Differential and Integral Equations, 3 (1990), 13–75.
[5] Gilles Carbou and Pierre Fabrie, *Time average in micromagnetism,* J. Differential Equations, 147 (1998), 383–409.
[6] Gilles Carbou and Pierre Fabrie, *Regular solutions for Landau-Lifschitz equation in* $\mathbb{R}^3$*,* Communications in Applied Analysis, 5 (2001), 17–30.
[7] Gilles Carbou and Pierre Fabrie, *Regular solutions for Landau-Lifschitz equation in a bounded domain,* Differential and Integral Equations, 14 (2001), 213–229.
[8] Gilles Carbou, Pierre Fabrie, and Oliver Guès, *Couche limite dans un modèle de ferromagnètisme,* Communications in Applied Analysis, 27 (2002), 1467–1495.
[9] Avner Friedman, "Partial Differental Equations," 1983.
[10] A. Fuwa and M. Tsutsumi, *On discrete versions of Gagliardo-Nirenberg-Sobolev inequalities*, preprint.
[11] Boling Guo and Min-Chun Hong, *The Landau-Lifshitz equation of the ferromagnetism spin chain and harmonic maps,* Calc. Var., 1 (1993), 311–334.
[12] O.A. Ladyzhenskaya, *The Boundary Value Problems of Mathematical Physics,* Applied Mathematical Sciences, Vol., 49 Springer, Berlin, 1985.
[13] D. Landau and E.M. Lifshitz, *On the theory of the dispersion of magnetix permeability in ferromagnetic bodies,* Phys. Z. Sowj., 8 (1935), 153.
[14] P. L. Sulem, C. Sulem, and C. Bardos, *On the Continuous Limit for a System of Classical Spins,* Physics, (1986), 431–454.
[15] A. Visintin, *On Landau-Lifshitz' Equations for Ferromagnetism,* Japan J. Appl. Math., (1985), 69–84.
[16] Zhou Yulin, Guo Boling, and Tan Shaobin, *Existence and uniqueness of smooth solution for system of ferromagnetic chain,* Science in China (Ser. A (1991), 257–266.
[17] Chen Yunmei, Ding shijin, and Boling Guo, *Partial Regularity for Two Dimensional Landau-Lifshitz Equations,* Acta Mathematica Sinica, 14 (1998), 423–432.
[18] Chen Yunmei, *Dirichlet boundary value problems of Landau-Lifshitz equaiton,* Commun. in partial differential equations, 25 (2000), 101–124.