NECESSARY AND SUFFICIENT CONDITIONS FOR EXISTENCE AND UNIQUENESS OF BOUNDED OR ALMOST-PERIODIC SOLUTIONS FOR DIFFERENTIAL SYSTEMS WITH CONVEX POTENTIAL

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Abstract. We give necessary and sufficient conditions for the existence and uniqueness of bounded or almost-periodic solutions of the first-order differential system: $u' + \nabla\Phi(u) = e(t)$, when $\nabla\Phi$ denotes the gradient of a convex function on $\mathbb{R}^N$. We also study the relations of continuity between the forcing term e and the solution u. Then we give similar results for the second-order differential system: $u'' = \nabla\Phi(u) + e(t)$.

1. Introduction

The numerical space $\mathbb{R}^N$ is endowed with its standard inner product $(x, y) := \sum_{k=1}^{N} x_k y_k$ and $| \,.\, |$ denotes the associated Euclidian norm. From continuous maps $e : \mathbb{R} \longrightarrow \mathbb{R}^N$ and $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$, we formulate the following forced differential equation:

$$u' + F(u) = e(t), \tag{1.1}$$

where F is a monotone map on $\mathbb{R}^N$: $(F(x_1) - F(x_2), x_1 - x_2) \geq 0$ for all x_1 and $x_2 \in \mathbb{R}^N$. A special class, of the dissipative equation (1.1), is the case where the field F is derived from a convex potential Φ:

$$u' + \nabla\Phi(u) = e(t). \tag{1.2}$$

For the dissipative equation (1.1), Biroli, Dafermos, Haraux, Huang and Ishii have given important contributions to the question of almost-periodic solutions [4, 11-14] which are valid even for abstract evolution equations.

In this paper, we give necessary and sufficient conditions for the existence and uniqueness of the bounded (respectively almost-periodic) solution of equation (1.2) when the forcing term e is bounded (respectively almost

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periodic). We also study the relations of continuity between the bounded (respectively almost-periodic) forcing term e and the bounded (respectively almost-periodic) solution of equation (1.2). In the scalar case: $N = 1$, Slyusarchuk established similar results in [18]. The conditions which are established for equation (1.2) do not hold for equation (1.1), even in the linear case. However, we give a sufficient condition, then a necessary condition, for the existence and uniqueness of the bounded (respectively almost-periodic) solution of equation (1.1).

Then we give similar results for the following second-order differential equation:

$$u'' = \nabla\Phi(u) + e(t), \tag{1.3}$$

where Φ is a convex map on $\mathbb{R}^N$. It is natural to ask if these results hold for the following differential equation:

$$u'' = F(u) + e(t), \tag{1.4}$$

where F is a monotone map on $\mathbb{R}^N$. It is a difficult problem in which significant differences appear in comparison with the first-order case. For example, in the linear case, these results are valid for equation (1.4). In the nonlinear case, we can extend only some partial results to equation (1.4).

The question of existence and uniqueness of bounded or almost-periodic solutions is treated by Berger and Chen [3] for equation (1.3). In [7], Carminati has stated a local version of results of Berger and Chen, assuming that Φ is convex only near the minimum of Φ. Aftabizadeh [1] and Corduneanu [9] have given sufficient conditions for the existence and uniqueness of bounded or almost-periodic solutions of

$$u'' = \nabla_u\Phi(t,u), \tag{1.5}$$

where $\nabla_u\Phi(t,.)$ denotes the gradient of the convex function $\Phi(t,.)$. In [9], these sufficient conditions are used to study the solutions of some nonlinear elliptic equations. Zakharin and Parasyuk [19] have established the existence of almost-periodic (in a weak sense) solutions of (1.5) by using a variational method on a Hilbert space of Besicovitch. In all these papers mentioned above, it is assumed that the gradient $\nabla\Phi$ is strongly monotone. While in [8], for the following equation:

$$u'' = F(t,u), \tag{1.6}$$

where the partial function $F(t,.)$ is a monotone map on $\mathbb{R}^N$, the author has proved sufficient conditions for the existence of almost-periodic solutions without assuming that $F(t,.)$ is strictly monotone. Recall also for the following system with the presence of a linear damping

$$u'' + [b(t)I + B(t)]u' = F(t, u) \tag{1.7}$$

the existence and uniqueness of bounded or almost-periodic solutions are studied by Blot et al. [5] and Mawhin [16].

We denote by $BC^0(\mathbb{R}, \mathbb{R}^N)$ the Banach space of continuous bounded functions from $\mathbb{R}$ to $\mathbb{R}^N$ endowed with the norm $\| u \|_\infty := \sup_{t\in\mathbb{R}} | u(t) |$. $AP^0(\mathbb{R}^N)$ denotes the closed subspace of $BC^0(\mathbb{R}, \mathbb{R}^N)$ consisting of the Bohr almost-periodic functions from $\mathbb{R}$ to $\mathbb{R}^N$ [10, Chapter VI]. When k is a non-negative integer, $BC^k(\mathbb{R}, \mathbb{R}^N)$ (respectively $AP^k(\mathbb{R}^N)$) is the space of functions in $BC^0(\mathbb{R}^N) \cap C^k(\mathbb{R}, \mathbb{R}^N)$ (respectively $AP^0(\mathbb{R}^N) \cap C^k(\mathbb{R}, \mathbb{R}^N)$) such that all their derivatives, up to order k, are bounded (respectively almost-periodic) functions. When $u \in BC^1(\mathbb{R}, \mathbb{R}^N)$, we set $\| u \|_{C^1} := \| u \|_\infty + \| u' \|_\infty$ and when $u \in BC^2(\mathbb{R}, \mathbb{R}^N)$, we set $\| u \|_{C^2} := \| u \|_\infty + \| u' \|_\infty + \| u'' \|_\infty$. $AP^1(\mathbb{R}^N)$ (respectively $AP^2(\mathbb{R}^N)$) is a closed subspace of $BC^1(\mathbb{R}, \mathbb{R}^N)$ (respectively $BC^2(\mathbb{R}, \mathbb{R}^N)$).

Now we give necessary and sufficient conditions for the existence and uniqueness of bounded or almost-periodic solutions of equation (1.2). Let $F \in C^0(\mathbb{R}^N, \mathbb{R}^N)$. For each $u \in BC^1(\mathbb{R}, \mathbb{R}^N)$, the function $t \longrightarrow u'(t) + F(u(t))$ belongs to $BC^0(\mathbb{R}, \mathbb{R}^N)$, so we can define the following operator $\mathcal{F}_1 : BC^1(\mathbb{R}, \mathbb{R}^N) \longrightarrow BC^0(\mathbb{R}, \mathbb{R}^N)$ with $\mathcal{F}_1(u)(t) := u'(t) + F(u(t))$ for all $u \in BC^1(\mathbb{R}, \mathbb{R}^N)$ and $t \in \mathbb{R}$. For $u \in AP^1(\mathbb{R}^N)$, the function $t \longrightarrow u'(t) + F(u(t))$ belongs to $AP^0(\mathbb{R}^N)$ [10, Theorem 1.7, page 13], so we can define the restriction operator of $\mathcal{F}_1$ to $AP^1(\mathbb{R}^N)$ by $\mathcal{F}_2 : AP^1(\mathbb{R}^N) \longrightarrow AP^0(\mathbb{R}^N)$ with $\mathcal{F}_2(u) = \mathcal{F}_1(u)$ for all $u \in AP^1(\mathbb{R}^N)$.

For the first-order differential equation (1.2), we will state the following equivalences.

Theorem 1.1. *Let Φ be a convex and continuous differentiable function on $\mathbb{R}^N$. We assume that $F = \nabla\Phi$. The following assertions are equivalent:*

- (A1) $\mathcal{F}_1 : \big(BC^1(\mathbb{R}, \mathbb{R}^N), \| \,.\, \|_{C^1}\big) \longrightarrow \big(BC^0(\mathbb{R}, \mathbb{R}^N), \| \,.\, \|_\infty\big)$ *is a homeomorphism;*
- (A2) $\mathcal{F}_2 : \big(AP^1(\mathbb{R}^N), \| \,.\, \|_{C^1}\big) \longrightarrow \big(AP^0(\mathbb{R}^N), \| \,.\, \|_\infty\big)$ *is a homeomorphism;*
- (A3) $\nabla\Phi : \big(\mathbb{R}^N, | \,.\, |\big) \longrightarrow \big(\mathbb{R}^N, | \,.\, |\big)$ *is a homeomorphism;*
- (A4) Φ *is a strictly convex map on* $\mathbb{R}^N$ *such that*

$$\lim_{|x|\to+\infty} \frac{(\nabla\Phi(x), x)}{| x |} = +\infty. \tag{1.8}$$

The proof of Theorem 1.1 will be given in Section 2.

Remark on the convexity of the potential. By making the change of variables $\tau = -t$, it is easy to verify that Theorem 1.1 is valid if the potential Φ is concave instead of convex and the assertion (A4) becomes (A4b): Φ *is a strictly concave map such that* $\lim_{|x|\to+\infty} \frac{(\nabla\Phi(x),x)}{|x|} = -\infty$.

Remark on Assertion (A4). The following conditions are equivalent:

$$(1.8) \Longleftrightarrow \lim_{|x|\to+\infty} \frac{\Phi(x)}{\mid x \mid} = +\infty \Longleftrightarrow \lim_{|x|\to+\infty} \mid \nabla\Phi(x) \mid = +\infty$$

[6, Proposition 2.14, page 42].

Remark on the implication (A3) $\Longrightarrow$ (A4). On a Hilbert space H, the gradient $\nabla\Phi$ is surjective if and only if $\lim_{|x|\to+\infty} \Phi(x) - (p,x) = +\infty$, for all $p \in H$ [6, Proposition 2.13, page 41]. When the dimension of H is finite, the surjectivity of $\nabla\Phi$ is equivalent to (1.8). In the case of infinite dimension, (1.8) is not a necessary condition for the surjectivity of $\nabla\Phi$ [6, Remarque 2.3, page 43]; for this reason, we cannot extend Theorem 1.1 to the infinite-dimensional case.

Remark on the implications (A1) $\Longrightarrow$ (A2) and (A2) $\Longrightarrow$ (A3). For these implications, we will give a proof which does not require the convexity of Φ or the monotonicity of the gradient $\nabla\Phi$, therefore these implications are valid for equation (1.1) where $F \in C^0(\mathbb{R}^N, \mathbb{R}^N)$, without hypotheses of monotonicity on F (c.f. proof of Proposition 1.2).

Comments. Here we explain why Theorem 1.1 provides a generalization to the vectorial case of the main result of Slyusarchuk [18]. This result is as follows: let $F \in C^0(\mathbb{R})$. The following assertions are equivalent

$$(A1) \Longleftrightarrow (A2) \Longleftrightarrow F : \mathbb{R} \longrightarrow \mathbb{R} \text{ is a homeomorphism.}$$

Our result provides a generalization of [18, Theorem], because if $F : \mathbb{R} \longrightarrow \mathbb{R}$ is a homeomorphism, then F is strictly increasing or decreasing, therefore $\Phi(x) := \int_0^x F(\tau)\, d\tau$ is convex or concave.

For equation (1.1), Theorem 1.1 does not hold, even in the linear case (c.f. the remarks below Proposition 1.2). However, we can extend some results to equation (1.1). Recall that F is a strictly monotone map on $\mathbb{R}^N$ if $(F(x_1) - F(x_2), x_1 - x_2) > 0$ for all x_1 and $x_2 \in \mathbb{R}^N$ such that $x_1 \neq x_2$. Consider the following assertions

(A5) $F : \big(\mathbb{R}^N, |\,.\,|\big) \longrightarrow \big(\mathbb{R}^N, |\,.\,|\big)$ is a homeomorphism;

(A6) F is a strictly monotone map on $\mathbb{R}^N$ such that

$$\lim_{|x|\to+\infty} \frac{(F(x),x)}{\mid x \mid} = +\infty. \tag{1.9}$$

We will prove the following.

Proposition 1.2. *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous map. Then the following implications hold:*

$$(A6) \Longrightarrow (A1) \Longrightarrow (A2) \Longrightarrow (A5).$$

The proof of Proposition 1.2 will be given in section 2.

Remark on Assertion (A5). Even when F is a monotone map on $\mathbb{R}^N$, Assertion (A5) is not a sufficient condition for the existence or the uniqueness of a bounded or almost-periodic solution of equation (1.1). For example, consider the map $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $F(x_1, x_2) := Ax = [-x_2, x_1]$. F is a homeomorphism and a monotone map, but every solution of $u' + Au = 0$ is 2π-periodic, so $\mathcal{F}_2$ (therefore $\mathcal{F}_1$) is not injective. Let $v \neq 0$ be a solution of $v' + Av = 0$, for example $v(t) = [\sin(t), \cos(t)]$. The equation $u' + Au = v(t)$ has no bounded solution, because $u(t) = tv(t)$ is an unbounded solution, therefore $\mathcal{F}_1$ and $\mathcal{F}_2$ are not surjective.

Remark on Assertion (A6). Assertion (A6) is not a necessary condition for the existence or the uniqueness of a bounded or almost-periodic solution of equation (1.1). For example, consider the map $F : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ defined by $F(x_1, x_2) := Bx = [-x_2, x_1 + x_2]$. The map F is monotone and does not satisfy (A6). However the eigenvalues of B are conjugate and their real parts are equal to $\frac{1}{2}$, therefore the linear system $u' + Bu = 0$ has an exponential dichotomy: namely there exists $k > 0$ such that $\| \exp(-Bt) \|_{\mathcal{L}(\mathbb{R}^N)} \leq k \exp(-\frac{t}{2})$ for all $t \geq 0$. As a consequence the system $u' + Bu = 0$ has precisely one bounded solution on $\mathbb{R}$: $u(t) \equiv 0$; this implies the injectivity of $\mathcal{F}_1$. Moreover the following function $u(t) := \int_{-\infty}^{t} \exp(-B(t-s)e(s)ds$ is a solution of $u' + Bu = e(t)$ for $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$ and satisfies $| u(t) | \leq 2k \| e \|_\infty$ for all $t \in \mathbb{R}$; this implies the surjectivity of $\mathcal{F}_1$. Since $\mathcal{F}_1$ is a bounded linear map between Banach spaces, which is bijective, then $\mathcal{F}_1$ is an isomorphism between $BC^1(\mathbb{R}, \mathbb{R}^N)$ and $BC^0(\mathbb{R}, \mathbb{R}^N)$, therefore Assertion (A1) is fulfilled.

Comments. Here, we recall some results of the existence and uniqueness of the solution which are known. We start by recalling that F is a strongly monotone map on $\mathbb{R}^N$ with modulus $c > 0$ if

$$(F(x_1) - F(x_2), x_1 - x_2) \geq c \mid x_1 - x_2 \mid^2$$

for all x_1 and $x_2 \in \mathbb{R}^N$. Evidently a strongly monotone map F satisfies (A6). When F is a strongly monotone map on $\mathbb{R}^N$ with modulus c, by [6, Lemma

3.1, page 64], one has

$$| u(t) - v(t) | \leq | u(s) - v(s) | e^{c(s-t)} + \int_s^t | \mathcal{F}_1(u)(\tau) - \mathcal{F}_1(v)(\tau) | e^{c(\tau-t)} d\tau$$

for every u and $v \in BC^1(\mathbb{R}, \mathbb{R}^N)$ and for all s and $t \in \mathbb{R}$ such that $s \leq t$; then we can deduce that

$$\| u - v \|_\infty \leq \frac{\| \mathcal{F}_1(u) - \mathcal{F}_1(v) \|_\infty}{c}. \tag{1.10}$$

When F is strictly monotone, in the almost-periodic case, namely the forcing term e is almost periodic, contrary to the bounded case, the uniqueness of the bounded solution of equation (1.1) is a result known. For example, Dafermos [11, Corollary 2.10] established a result of the same type, which is valid even for contractive almost-periodic processes. Evidently, in the strongly monotone case, by using (1.10), one has the uniqueness of the bounded solution. In the strongly monotone case, for example, Biroli [4, Th. 9 and Th. 10] and Dafermos [11, Theorem 5.1] stated results of existence and uniqueness of the bounded or almost-periodic solutions for abstract evolution equations. In the strongly monotone case, by using (1.10), it is easy to verify that $\mathcal{F}_1^{-1}$ is a continuous map.

Now we give necessary and sufficient conditions for the existence and uniqueness of the bounded or almost-periodic solution of equation (1.3). Let $F \in C^0(\mathbb{R}^N, \mathbb{R}^N)$. For each $u \in BC^2(\mathbb{R}, \mathbb{R}^N)$, the function $t \longrightarrow u''(t) - F(u(t))$ belongs to $BC^0(\mathbb{R}, \mathbb{R}^N)$, so we can define the following operator $\mathcal{G}_1 : BC^2(\mathbb{R}, \mathbb{R}^N) \longrightarrow BC^0(\mathbb{R}, \mathbb{R}^N)$ with $\mathcal{G}_1(u)(t) := u''(t) - F(u(t))$ for all $u \in BC^2(\mathbb{R}, \mathbb{R}^N)$ and $t \in \mathbb{R}$. For $u \in AP^2(\mathbb{R}^N)$, the function $t \longrightarrow u''(t) - F(u(t))$ belongs to $AP^0(\mathbb{R}^N)$, so we can define the restriction operator of $\mathcal{G}_1$ to $AP^2(\mathbb{R}^N)$ by $\mathcal{G}_2 : AP^2(\mathbb{R}^N) \longrightarrow AP^0(\mathbb{R}^N)$ with $\mathcal{G}_2(u) = \mathcal{G}_1(u)$ for all $u \in AP^2(\mathbb{R}^N)$.

For the second-order differential equation (1.3), we will state the following equivalences.

Theorem 1.3. *Let Φ be a convex and continuously differentiable function on $\mathbb{R}^N$. We assume that $F = \nabla\Phi$. The following assertions are equivalent:*

- (A7) $\mathcal{G}_1 : (BC^2(\mathbb{R}, \mathbb{R}^N), \| \, . \, \|_{C^2}) \longrightarrow (BC^0(\mathbb{R}, \mathbb{R}^N), \| \, . \, \|_\infty)$ *is a homeomorphism;*
- (A8) $\mathcal{G}_2 : (AP^2(\mathbb{R}^N), \| \, . \, \|_{C^2}) \longrightarrow (AP^0(\mathbb{R}^N), \| \, . \, \|_\infty)$ *is a homeomorphism;*
- (A3) $\nabla\Phi : (\mathbb{R}^N, | \, . \, |) \longrightarrow (\mathbb{R}^N, | \, . \, |)$ *is a homeomorphism.*

The proof of Theorem 1.3 will be given in Section 3.

Remark. By using Landau's inequality:

$$\| u' \|_\infty \leq 2\sqrt{\| u'' \|_\infty}\sqrt{\| u \|_\infty}$$

and by using the fact that if an almost-periodic function admits a derivative which is uniformly continuous on $\mathbb{R}$, then the derivative is almost periodic [10, Theorem 1.8, page 13], we deduce that if $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$ (respectively $e \in AP^0(\mathbb{R}^N)$) and if u is a bounded (respectively almost-periodic) solution of equation (1.4); i.e., $u \in C^2(\mathbb{R}, \mathbb{R}^N) \cap BC^0(\mathbb{R}, \mathbb{R}^N)$ (respectively $u \in C^2(\mathbb{R}, \mathbb{R}^N) \cap AP^0(\mathbb{R}^N)$), then $u \in BC^2(\mathbb{R}, \mathbb{R}^N)$ (respectively $u \in AP^2(\mathbb{R}^N)$).

For equation (1.4), the situation is in fact more complicated. In the linear case, Theorem 1.3 holds for equation (1.4) (c.f. Proposition 1.6). But in the nonlinear case, it is difficult to determine if Theorem 1.3 is valid for equation (1.4). However, we can extend some results to equation (1.4). We will prove the following.

Proposition 1.4. *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous map. Then the following implications hold:*

$$(A6) \Longrightarrow (A7) \Longrightarrow (A8) \Longrightarrow (A5).$$

The proof of Proposition 1.4 will be given in Section 3.

For the nonlinear case, we state the following necessary and sufficient condition for the uniqueness of the bounded or almost-periodic solution of equation (1.4).

Proposition 1.5. *Let F be a continuous and monotone function on $\mathbb{R}^N$. The following assertions are equivalent:*

(A9) $\mathcal{G}_1 : BC^2(\mathbb{R}, \mathbb{R}^N) \longrightarrow BC^0(\mathbb{R}, \mathbb{R}^N)$ *is an injective map;*
(A10) $\mathcal{G}_2 : AP^2(\mathbb{R}^N) \longrightarrow AP^0(\mathbb{R}^N)$ *is an injective map;*
(A11) $F : (\mathbb{R}^N, |\,.\,|) \longrightarrow (\mathbb{R}^N, |\,.\,|)$ *is an injective map.*

The proof of Proposition 1.5 will be given in Section 3.

For the linear case, Theorem 1.3 is valid for equation (1.4); i.e., the linear operator is not necessary symmetric.

Proposition 1.6. *Let $A \in \mathcal{L}(\mathbb{R}^N, \mathbb{R}^N)$ and $(Ax, x) \geq 0$ for all $x \in \mathbb{R}^N$. We assume that $F(x) = Ax$. The following assertions are equivalent:*

(A7) $\mathcal{G}_1 : (BC^2(\mathbb{R}, \mathbb{R}^N), \|\,.\,\|_{C^2}) \longrightarrow (BC^0(\mathbb{R}, \mathbb{R}^N), \|\,.\,\|_\infty)$ *is an isomorphism;*
(A8) $\mathcal{G}_2 : (AP^2(\mathbb{R}^N), \|\,.\,\|_{C^2}) \longrightarrow (AP^0(\mathbb{R}^N), \|\,.\,\|_\infty)$ *is an isomorphism;*
(A12) *the linear operator A is nonsingular.*

The proof of Proposition 1.6 will be given in section 3.

Comments. In the strongly monotone case, results of existence and uniqueness of the bounded or almost-periodic solution of equation (1.4) are known [5, 8, 16]. Moreover the question of the continuity between the forcing term e and the solution u is treated in [5] for equation (1.7). Results of existence of bounded solutions with conditions of the type of (1.9) (without assumption of strongly monotony), are given in [8] for equation (1.6) and in [16] for equation (1.7). In the strictly monotone case, the uniqueness of the almost-periodic solution is stated in [8].

2. Proofs of results on first-order equations

The object of this section is to prove Theorem 1.1 and Proposition 1.2. When F is only strictly monotone, instead of strongly monotone, the relation (1.10) is not valid. In the strictly monotone case, we state a weaker relation than (1.10) (Lemma 2.2), which permits us to establish our results. For that, we generalize a result of Slyusarchuk to the vectorial case.

Lemma 2.1. *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous and strictly monotone map. Then for every compact subset K of $\mathbb{R}^N$ and for every $\epsilon > 0$, there exists $c > 0$ such that*

$$(F(x_1) - F(x_2), x_1 - x_2) \geq c \mid x_1 - x_2 \mid^2 \tag{2.1}$$

for all x_1 and $x_2 \in K$ such that

$$\mid x_1 - x_2 \mid \geq \epsilon. \tag{2.2}$$

Remark. Lemma 2.1 generalizes a result of Slyusarchuk to the vectorial case [18, Lemma 5]. This result is as follows: let g be a continuous and strictly increasing function on $[a, b]$. Then, for every $\epsilon > 0$, there exists $c > 0$ such that $g(x_1) - g(x_2) \geq c(x_1 - x_2)$ for all x_1 and $x_2 \in [a, b]$ such that $x_1 - x_2 \geq \epsilon$.

Proof. Assume the contrary. Then there exist two sequences $(x_n)_n$ and $(y_n)_n$ such that for all $n \in \mathbb{N}^*$, x_n and $y_n \in K$,

$$(F(x_n) - F(y_n), x_n - y_n) < \frac{1}{n} \mid x_n - y_n \mid^2 \tag{2.3}$$

and

$$\mid x_n - y_n \mid \geq \epsilon. \tag{2.4}$$

Since K is a compact set, then the sequence $(x_n)_n$ (respectively $(y_n)_n$) possesses a cluster point x_* (respectively y_*). By (2.3), continuity, and the monotonicity of F, we obtain

$$(F(x_*) - F(y_*), x_* - y_*) = 0.$$

Since F is strictly monotone, then $x_* = y_*$; this contradicts (2.4). □

Lemma 2.2. *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous and strictly monotone map. Then for every compact subset K of $\mathbb{R}^N$ and for every $\epsilon > 0$, there exists $c > 0$ such that*

$$\| u - v \|_\infty \leq \epsilon \quad \text{or} \quad \| \mathcal{F}_1(u) - \mathcal{F}_1(v) \|_\infty > \frac{c\epsilon}{2}$$

for all u and $v \in C^1(\mathbb{R}, \mathbb{R}^N)$ such that $u(t)$ and $v(t) \in K$ for all $t \in \mathbb{R}$.

Proof. Let K be a compact subset of $\mathbb{R}^N$ and $\epsilon > 0$. By Lemma 2.1, there exists $c > 0$ such that for all x_1, $x_2 \in K$ and

$$| x_1 - x_2 | \geq \frac{\epsilon}{2}, \tag{2.5}$$

one has (2.1). Let u and $v \in C^1(\mathbb{R}, \mathbb{R}^N)$ be such that $u(t)$ and $v(t) \in K$ for all $t \in \mathbb{R}$. Assume that

$$\| u - v \|_\infty > \epsilon. \tag{2.6}$$

There exists $\tau_1 \in \mathbb{R}$ such that

$$| u(\tau_1) - v(\tau_1) | > \epsilon. \tag{2.7}$$

Moreover, there exists $\tau \in \mathbb{R}$ such that $\tau < \tau_1$ and $| u(t) - v(t) | > \frac{\epsilon}{2}$ for all $t \in (\tau, \tau_1)$. Let

$$\tau_0 := \inf\{\tau \in (-\infty, \tau_1) \ ; \ | u(t) - v(t) | > \frac{\epsilon}{2}, \forall t \in (\tau, \tau_1)\}.$$

If $\tau_0 = -\infty$, then $\lim_{t \to \tau_0} | u(t) - v(t) | e^{c(t-\tau_1)} = 0$, because u and v are bounded on $\mathbb{R}$. If $\tau_0 \in \mathbb{R}$, then $| u(\tau_0) - v(\tau_0) | = \frac{\epsilon}{2}$, therefore $\lim_{t \to \tau_0} |u(t) - v(t)| e^{c(t-\tau_1)} = \frac{\epsilon}{2} e^{c(\tau_0 - \tau_1)}$. In any case, one has

$$\lim_{t \to \tau_0} | u(t) - v(t) | e^{c(t-\tau_1)} = l \in [0, \frac{\epsilon}{2}). \tag{2.8}$$

By definition of τ_0, one has $| u(t) - v(t) | > \frac{\epsilon}{2}$ for all $t \in (\tau_0, \tau_1)$. By (2.1) and (2.5), we deduce

$$(F(u(t)) - F(v(t)), u(t) - v(t)) \geq c \, | u(t) - v(t) |^2 \tag{2.9}$$

on (τ_0, τ_1). Moreover, one has

$$\frac{d}{dt}\Big(\frac{1}{2} \, | u(t) - v(t) |^2 \Big) = \big(u'(t) - v'(t), u(t) - v(t)\big)$$

$$= (\mathcal{F}_1(u)(t) - \mathcal{F}_1(v)(t), u(t) - v(t)) - (F(u(t)) - F(v(t)), u(t) - v(t))$$

on (τ_0, τ_1). By (2.9), we deduce

$$\frac{d}{dt}\Big(\frac{1}{2} \mid u(t) - v(t) \mid^2\Big)$$
$$\leq \parallel \mathcal{F}_1(u) - \mathcal{F}_1(v) \parallel_\infty \mid u(t) - v(t) \mid - c \mid u(t) - v(t) \mid^2,$$

therefore,

$$\frac{d}{dt}\Big(\frac{1}{2} \mid u(t) - v(t) \mid^2 e^{2ct}\Big) \leq \parallel \mathcal{F}_1(u) - \mathcal{F}_1(v) \parallel_\infty \mid u(t) - v(t) \mid e^{2ct}$$

on (τ_0, τ_1). By integrating on $[t, \tau_1]$, for $\tau_0 < t < \tau_1$, one has

$$\frac{1}{2} \mid u(\tau_1) - v(\tau_1) \mid^2 e^{2c\tau_1} \leq \frac{1}{2} \mid u(t) - v(t) \mid^2 e^{2ct}$$
$$+ \int_t^{\tau_1} \parallel \mathcal{F}_1(u) - \mathcal{F}_1(v) \parallel_\infty \mid u(s) - v(s) \mid e^{2cs} ds.$$

By [6, Lemma A.5, page 157], we obtain

$$\mid u(\tau_1) - v(\tau_1) \mid e^{c\tau_1} \leq \mid u(t) - v(t) \mid e^{ct} + \int_t^{\tau_1} \parallel \mathcal{F}_1(u) - \mathcal{F}_1(v) \parallel_\infty e^{cs} ds;$$

we deduce

$$\mid u(\tau_1) - v(\tau_1) \mid \leq \mid u(t) - v(t) \mid e^{c(t-\tau_1)} + \frac{\parallel \mathcal{F}_1(u) - \mathcal{F}_1(v) \parallel_\infty}{c}$$

on (τ_0, τ_1), therefore

$$\mid u(\tau_1) - v(\tau_1) \mid \leq l + \frac{\parallel \mathcal{F}_1(u) - \mathcal{F}_1(v) \parallel_\infty}{c},$$

with $l = \lim_{t \to \tau_0} \mid u(t) - v(t) \mid e^{c(t-\tau_1)}$. By (2.7) and (2.8), we obtain

$$\parallel \mathcal{F}_1(u) - \mathcal{F}_1(v) \parallel_\infty > \frac{\epsilon c}{2}. \tag{2.10}$$

In conclusion, we have proved that (2.6) implies (2.10), therefore Lemma 2.2 is proved. □

Lemma 2.3. *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous map satisfying (1.9). Then, for all $\delta > 0$, there exists $R > 0$ such that for all $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$ and $\parallel e \parallel_\infty \leq \delta$, there exists $u \in BC^1(\mathbb{R}, \mathbb{R}^N)$ such that $\mathcal{F}_1(u) = e$ and $\parallel u \parallel_\infty \leq R$.*

Proof. Let $\delta_0 > 0$. By (1.9), we deduce the existence of $R_0 > 0$ such that

$$\mid x \mid \geq R_0 \Longrightarrow \frac{(F(x), x)}{\mid x \mid} \geq \delta_0 + 1. \tag{2.11}$$

Let $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$ such that $\parallel e \parallel_\infty \leq \delta_0$. To prove the existence of a bounded solution of equation (1.1), we use the method of guiding functions as developed in the book of Krasonel'skii and Zabreiko [15]. Now we prove that $V(x) := -\frac{1}{2} \mid x \mid^2$ is a guiding function of equation (1.1): there exists $R > 0$ such that

$$(\nabla V(x), e(t) - F(x)) > 0 \tag{2.12}$$

for all $t \in \mathbb{R}$ and $x \in \mathbb{R}^N$ such that $\mid x \mid \geq R$. Note that

$$(\nabla V(x), e(t) - F(x)) \geq (F(x), x) - \delta_0 \mid x \mid .$$

By (2.11), we deduce that (2.12) is satisfied for $\mid x \mid \geq R_0$, thus V is a guiding function of equation (1.1). Moreover, the guiding function V satisfies

$$\lim_{|x| \to +\infty} V(x) = -\infty, \tag{2.13}$$

therefore, by [15, Theorem 13.6, page 56], equation (1.1) has at least one solution $u \in C^1(\mathbb{R}, \mathbb{R}^N) \cap BC^0(\mathbb{R}, \mathbb{R}^N)$. Evidently $u \in BC^1(\mathbb{R}, \mathbb{R}^N)$, therefore $\mathcal{F}_1(u) = e$, therefore $\mathcal{F}_1$ is surjective. Moreover by combining [15, Lemma 13.5, page 52 and Theorem 13.6, page 56], we obtain a bound on u: $\|u\|_\infty \leq R_1$ where R_1 is a constant depending only on R_0 and $\| \; e \; \|_\infty$, but not depending on the specific forcing term e. This constant R_1 is defined by $R_1 > R_0$

$$\mid x \mid \geq R_1 \Longrightarrow V(x) < \min\{V(x); \mid x \mid \leq R_0\}. \tag{2.14}$$

By the choice of the guiding function $V(x) := -\frac{1}{2} \mid x \mid^2$, every constant R_1 in the interval $(R_0, +\infty)$ satisfies (2.14), therefore, one has $\| \; u \; \|_\infty \leq R_1$ for all $R_1 > R_0$, thus

$$\| \; u \; \|_\infty \leq R_0. \tag{2.15}$$

Note that for the particular case of the guiding function satisfying (2.13), Alonso and Ortega give a direct proof for the existence of a bounded solution and the existence of the bound R_1 defined by (2.14) in [2, Proposition 3.1]. This proof is elementary and does not require the use of degree theory. □

Proof of Proposition 1.2. (A6) $\Longrightarrow$ (A1). a) $\mathcal{F}_1$ is continuous. The Nemitski operator $\mathcal{N}_F$ built on F is defined by $\mathcal{N}_F(u) := F \circ u$. The operator $\mathcal{N}_F$ is a continuous map from $\left(BC^0(\mathbb{R}, \mathbb{R}^N), \| \, . \, \|_\infty\right)$ to itself, because F is continuous and $u(\mathbb{R})$ is relatively compact. We note that $\mathcal{F}_1 = D + \mathcal{N}_F \circ I$, where D is the derivative operator and I is the canonical injection from $\left(BC^1(\mathbb{R}, \mathbb{R}^N), \| \, . \, \|_{C^1}\right)$ to $\left(BC^0(\mathbb{R}, \mathbb{R}^N), \| \, . \, \|_\infty\right)$. Since the linear maps D and I are bounded and since $\mathcal{N}_F$ is continuous, we see that $\mathcal{F}_1$ is continuous.

b) $\mathcal{F}_1$ is injective. Let u and $v \in BC^1(\mathbb{R}, \mathbb{R}^N)$ such that $\mathcal{F}_1(u) = \mathcal{F}_1(v)$. There exists a compact subset K of $\mathbb{R}^N$ such that $u(t)$ and $v(t) \in K$ for all $t \in \mathbb{R}$. By Lemma 2.2, we deduce that $\| \; u - v \; \|_\infty \leq \epsilon$ for all $\epsilon > 0$, thus $u = v$.

c) $\mathcal{F}_1$ is surjective. It is a consequence of Lemma 2.3.

d) $(\mathcal{F}_1)^{-1}$ is continuous. Let $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$ and $(e_n)_n$ a sequence such that $e_n \in BC^0(\mathbb{R}, \mathbb{R}^N)$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to+\infty} \| e_n - e \|_\infty = 0. \tag{2.16}$$

Let u_n and $u \in BC^1(\mathbb{R}, \mathbb{R}^N)$ be such that $\mathcal{F}_1(u_n) = e_n$ and $\mathcal{F}_1(u) = e$. We want to prove that

$$\lim_{n\to+\infty} \| u_n - u \|_{C^1} = 0. \tag{2.17}$$

By (2.16), there exists $\delta_0 > 0$ such that $\| e \|_\infty \leq \delta_0$ and $\| e_n \|_\infty \leq \delta_0$ for all $n \in \mathbb{N}$. By Lemma 2.3 and the injectivity of $\mathcal{F}_1$, there exists $R_0 > 0$ such that $\| u_n \|_\infty$ and $\| u \|_\infty \leq R_0$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. By Lemma 2.2, there exists $c > 0$ such that

$$\| u_n - u \|_\infty \leq \epsilon \tag{2.18}$$

or

$$\| e_n - e \|_\infty > \frac{c\epsilon}{2}, \tag{2.19}$$

for all $n \in \mathbb{N}$ (the compact subset considered is the closed ball $\bar{B}(0, R_0)$). By (2.16), there exists $n_1 \in \mathbb{N}$ such that$\| e_n - e \|_\infty \leq \frac{c\epsilon}{2}$ for all $n \geq n_1$, thus by (2.18) and (2.19), we deduce that (2.18) is satisfied for all $n \geq n_1$, therefore

$$\lim_{n\to+\infty} \| u_n - u \|_\infty = 0. \tag{2.20}$$

Since the Nemitski operator $\mathcal{N}_F$ built on F is a continuous map from $(BC^0(\mathbb{R}, \mathbb{R}^N), \|.\|_\infty)$ to itself, then

$$\lim_{n\to+\infty} \| \mathcal{N}_F(u_n) - \mathcal{N}_F(u) \|_\infty = 0. \tag{2.21}$$

By (2.16) and (2.21), we deduce that

$$\lim_{n\to+\infty} \| u_n' - u' \|_\infty = 0 \tag{2.22}$$

and, by (2.20) and (2.22), we obtain (2.17), thus $(\mathcal{F}_1)^{-1}$ is continuous.

(A1) $\Longrightarrow$ (A2). It suffices to prove that $\mathcal{F}_2$ is surjective.

Let $e \in AP^0(\mathbb{R}^N)$. By hypothesis, there exists $u \in BC^1(\mathbb{R}, \mathbb{R}^N)$ such that $\mathcal{F}_1(u) = e$. Now we prove that u is almost periodic. Let $K := \overline{u(\mathbb{R})}$. By hypothesis K is a compact subset of $\mathbb{R}^N$. Recall that the *hull* of e is denoted by $H(e)$ and is defined by: $e_* \in H(e)$ if and only if there exists a real sequence $(\tau_n)_n$ such that we have

$$\lim_{n\to+\infty} \sup_{t\in\mathbb{R}} | e(t+\tau_n) - e_*(t) | = 0.$$

By [10, Lemma 4.3, page 104], for each $e_* \in H(e)$, there exists a solution u_* of $\mathcal{F}_1(u_*) = e_*$ such that $u_*(\mathbb{R}) \subset K$. Since $\mathcal{F}_1$ is injective, this bounded solution u_* is unique. By [10, Theorem 4.6, page 106], we obtain that the solution u is almost periodic. Finally $u \in AP^0(\mathbb{R}^N)$. Since $u'(t) = e(t) - F(u(t))$, then $u' \in AP^0(\mathbb{R}^N)$, therefore $u \in AP^1(\mathbb{R}^N)$.

(A2) $\Longrightarrow$ (A5). If we denote by $\mathcal{C}$ the set of constant mappings from $\mathbb{R}$ to $\mathbb{R}^N$, one has $\mathcal{C} \subset AP^1(\mathbb{R}^N)$ and, for $u \in \mathcal{C}$, the function $\mathcal{F}_2(u) \in \mathcal{C} \subset AP^0(\mathbb{R}^N)$ ($\mathcal{F}_2(u)(t) = F(u(0))$ for all $t \in \mathbb{R}$), so we can define the restriction operator of $\mathcal{F}_2$ to $\mathcal{C}$ by $\mathcal{F}_3 : \mathcal{C} \longrightarrow \mathcal{C}$ with $\mathcal{F}_3(u)(t) = F(u(0))$ for all $u \in \mathcal{C}$ and all $t \in \mathbb{R}$. For $u \in \mathcal{C}$, one has $\| u \|_{C^1} = | u(0) |$ and $\| \mathcal{F}_3(u) \|_\infty = | F(u(0)) |$; then it is equivalent to prove that $\mathcal{F}_3$ or F is a homeomorphism. It remains to prove that $\mathcal{F}_3$ is surjective. Let $e \in \mathcal{C}$. By hypothesis, there exists $u \in AP^1(\mathbb{R}^N)$ such that $\mathcal{F}_2(u) = e$. We want to prove that $u \in \mathcal{C}$. For that we denote by $u_\tau(t) = u(t+\tau)$ for all t and $\tau \in \mathbb{R}$. Note that $\mathcal{F}_2(u_\tau) = e$ for all $\tau \in \mathbb{R}$. By injectivity of $\mathcal{F}_2$, we deduce that $u_\tau = u$ for all $\tau \in \mathbb{R}$, therefore $u \in \mathcal{C}$. □

Proof of Theorem 1.1. (A4) $\Longrightarrow$ (A1) $\Longrightarrow$ (A2) $\Longrightarrow$ (A3). It is a consequence of Proposition 1.2 with $F = \nabla\Phi$.

(A3) $\Longleftrightarrow$ (A4). By [17, Corollary 23.5.1, page 219 and 25.5.1, page 246, Theorem 26.6, page 259 and Lemma 26.7, page 260], one has (A3) equivalent to Φ being strictly convex and Φ satisfying

$$\lim_{|x|\to+\infty} | \nabla\Phi(x) | = +\infty. \tag{2.23}$$

By [6, Proposition 2.14, page 42], the relation (1.8) and (2.23) are equivalent, thus (A3) and (A4) are equivalent. □

3. Proofs of results on second-order equations

The object of this section is to prove Theorem 1.3 and Proposition 1.4, 1.5 and 1.6.

Lemma 3.1. *Let $\alpha \in \mathbb{R}$, $\beta > 0$ and $\gamma \in \mathbb{R}$. If $r \in BC^0(\mathbb{R},\mathbb{R}) \cap C^2(\mathbb{R},\mathbb{R})$ satisfies the following assertion*

$$r(t) \geq \alpha \qquad \Longrightarrow \qquad r''(t) \geq \beta r(t) - \gamma, \tag{3.1}$$

then

$$\sup_{t\in\mathbb{R}} r(t) \leq \max(\frac{\gamma}{\beta}, \alpha). \tag{3.2}$$

Proof. Assume the contrary. Then there exists $t_0 \in \mathbb{R}$ such that

$$r(t_0) > R_0 := \max(\frac{\gamma}{\beta}, \alpha). \tag{3.3}$$

Let $O := \{t \in \mathbb{R}; r(t) > R_0\}$ and let I_{t_0} be the component of O containing t_0. Consequently, I_{t_0} is an open interval such that

$$r(t) > R_0, \qquad \forall t \in I_{t_0}. \tag{3.4}$$

By (3.1) and (3.4), we deduce that

$$r''(t) > 0, \qquad \forall t \in I_{t_0}, \tag{3.5}$$

so r is strictly convex. Assume that the interval I_{t_0} is bounded: $I_{t_0} = (a, b)$ with $-\infty < a < b < +\infty$. In this case, the convex function r satisfies (3.4) and $r(a) = r(b) = R_0$, which is impossible. Therefore the interval I_{t_0} is unbounded. Assume that the interval $I_{t_0} = \mathbb{R}$. In this case, r is a convex and bounded function on $\mathbb{R}$, therefore r is a constant function, which is a contradiction with (3.5). Assume that $I_{t_0} = (a, +\infty)$ with $a \in \mathbb{R}$. In this case, the convex function r satisfies (3.4) and $r(a) = R_0$, then $\sup_{t\in\mathbb{R}} r(t) = +\infty$, which is a contradiction with $r \in BC^0(\mathbb{R}, \mathbb{R})$. By a similar way, we prove that the case $I_{t_0} = (-\infty, b)$, with $b \in \mathbb{R}$, is not possible. In conclusion, (3.2) is satisfied. □

Lemma 3.2. *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous and strictly monotone map. Then for every compact subset K of $\mathbb{R}^N$ and for every $\epsilon > 0$, there exists $c > 0$ such that*

$$\| u - v \|_\infty \leq \max\left(\frac{\| \mathcal{G}_1(u) - \mathcal{G}_1(v) \|_\infty}{c}, \epsilon\right) \tag{3.6}$$

for all u and $v \in C^2(\mathbb{R}, \mathbb{R}^N)$ such that $u(t)$ and $v(t) \in K$ for all $t \in \mathbb{R}$.

Proof. Let K be a compact subset of $\mathbb{R}^N$ and $\epsilon > 0$. By Lemma 2.1, there exists $c > 0$ such that (2.1) and (2.2) are fulfilled. Let u and $v \in C^2(\mathbb{R}, \mathbb{R}^N)$ be such that $u(t)$ and $v(t) \in K$ for all $t \in \mathbb{R}$. We denote by

$$h := v - u \quad \text{and} \quad r(t) := \frac{1}{2} \mid h(t) \mid^2 . \tag{3.7}$$

The function r is of class C^2 with

$$r'(t) = \big(h(t), h'(t)\big) ,$$
$$r''(t) = \big(h(t), h''(t)\big) + \mid h'(t) \mid^2 . \tag{3.8}$$

By definition of $\mathcal{G}_1$, one has

$$r''(t) = (F(v(t)) - F(u(t)), h(t)) + (\mathcal{G}_1(v)(t) - \mathcal{G}_1(u)(t), h(t)) + \mid h'(t) \mid^2,$$

therefore,

$$r''(t) \geq (F(v(t)) - F(u(t)), h(t)) - \| \mathcal{G}_1(v) - \mathcal{G}_1(u) \|_\infty \| h \|_\infty . \tag{3.9}$$

By (2.1), (2.2), and (3.9), one has

$$r(t) \geq \frac{\epsilon^2}{2} \qquad \Longrightarrow \qquad r''(t) \geq 2cr(t) - \| \mathcal{G}_1(v) - \mathcal{G}_1(u) \|_\infty \| h \|_\infty .$$

By Lemma 3.1, we deduce

$$\frac{\| h \|_\infty^2}{2} \leq \max\Big(\frac{\| \mathcal{G}_1(u) - \mathcal{G}_1(v) \|_\infty \| h \|_\infty}{2c}, \frac{\epsilon^2}{2} \Big),$$

so we obtain (3.6). □

Lemma 3.3. *Let $F : \mathbb{R}^N \longrightarrow \mathbb{R}^N$ be a continuous map satisfying* (1.9). *Then, for all $\delta > 0$, there exists $R > 0$ such that for all $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$ with $\| e \|_\infty \leq \delta$, there exists $u \in BC^2(\mathbb{R}, \mathbb{R}^N)$ such that $\mathcal{G}_1(u) = e$ and $\| u \|_\infty \leq R$.*

Proof. Let $\delta_0 > 0$. By (1.9), we deduce the existence of $R_0 > 0$ such that (2.11) is satisfied. Let $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$ such that $\| e \|_\infty \leq \delta_0$. Note that $(F(x) + e(t), x) \geq (F(x), x) - \delta_0 \mid x \mid$. By (2.11), we deduce that $(F(x) + e(t), x) \geq 0$ for all $t \in \mathbb{R}$ and $\mid x \mid = R_0$. By [16, Theorem 3.1] or [8, Lemma 3.2 and Remark 1.1], equation (1.4) has at least one solution $u \in C^2(\mathbb{R}, \mathbb{R}^N) \cap BC^0(\mathbb{R}, \mathbb{R}^N)$ such that $\| u \|_\infty \leq R_0$. One has $u \in BC^2(\mathbb{R}, \mathbb{R}^N)$ (c.f. the remark below Theorem 1.3). □

Proof of Proposition 1.4. (A6) $\Longrightarrow$ (A7). a) $\mathcal{G}_1$ is continuous. The proof is similar to (A6) $\Longrightarrow$ (A1) of Proposition 1.2.

b) $\mathcal{G}_1$ is injective. Let u and $v \in BC^2(\mathbb{R}, \mathbb{R}^N)$ be such that $\mathcal{G}_1(u) = \mathcal{G}_1(v)$. There exists a compact subset K of $\mathbb{R}^N$ such that $u(t)$ and $v(t) \in K$ for all $t \in \mathbb{R}$. By Lemma 3.2, we deduce that $\| u - v \|_\infty \leq \epsilon$ for all $\epsilon > 0$, thus $u = v$.

c) $\mathcal{G}_1$ is surjective. It is a consequence of Lemma 3.3.

d) $(\mathcal{G}_1)^{-1}$ is continuous. Let $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$ and $(e_n)_n$ a sequence such that $e_n \in BC^0(\mathbb{R}, \mathbb{R}^N)$ for all $n \in \mathbb{N}$ and

$$\lim_{n \to +\infty} \| e_n - e \|_\infty = 0. \tag{3.10}$$

Let u_n and $u \in BC^2(\mathbb{R}, \mathbb{R}^N)$ be such that $\mathcal{G}_1(u_n) = e_n$ and $\mathcal{G}_1(u) = e$.We want to prove that

$$\lim_{n \to +\infty} \| u_n - u \|_{C^2} = 0. \tag{3.11}$$

By (3.10), there exists $\delta_0 > 0$ such that $\| e \|_\infty \leq \delta_0$ and $\| e_n \|_\infty \leq \delta_0$ for all $n \in \mathbb{N}$. By Lemma 3.3 and the injectivity of $\mathcal{G}_1$, there exists $R_0 > 0$ such that $\| u_n \|_\infty$ and $\| u \|_\infty \leq R_0$ for all $n \in \mathbb{N}$. Let $\epsilon > 0$. By Lemma 3.2, there exists $c > 0$ such that

$$\| u_n - u \|_\infty \leq \max\left(\frac{\| e_n - e \|_\infty}{c}, \epsilon \right) \tag{3.12}$$

for all $n \in \mathbb{N}$ (the compact subset considered is the closed ball $\bar{B}(0, R_0)$). By (3.10) and (3.12), we deduce that

$$\lim_{n \to +\infty} \| u_n - u \|_\infty = 0. \tag{3.13}$$

Since the Nemitski operator $\mathcal{N}_F$ built on F is a continuous map from $\big(BC^0(\mathbb{R}, \mathbb{R}^N), \| \, . \, \|_\infty\big)$ to itself, then

$$\lim_{n \to +\infty} \| \mathcal{N}_F(u_n) - \mathcal{N}_F(u) \|_\infty = 0. \tag{3.14}$$

By (3.10) and (3.14), we deduce that

$$\lim_{n \to +\infty} \| u_n'' - u'' \|_\infty = 0. \tag{3.15}$$

By (3.13), (3.15), and Landau's inequality:

$$\| u_n' - u' \|_\infty \leq 2 \sqrt{\| u_n'' - u'' \|_\infty} \sqrt{\| u_n - u \|_\infty},$$

we deduce that

$$\lim_{n \to +\infty} \| u_n' - u' \|_\infty = 0 \tag{3.16}$$

By (3.13), (3.15), and (3.16), we obtain (3.11), thus $(\mathcal{F}_1)^{-1}$ is continuous.

(A7) $\Longrightarrow$ (A8). We consider a system equivalent to equation (1.4)

$$\begin{cases} x' = y \\ y' = F(x) + e(t). \end{cases} \tag{3.17}$$

To state that the bounded solution of the first-order system (3.17) is almost periodic, the proof is similar to (A1) $\Longrightarrow$ (A2) of Proposition 1.2.

(A8) $\Longrightarrow$ (A5). The proof is similar to (A2) $\Longrightarrow$ (A5) of Proposition 1.2. □

Proof of Theorem 1.3. (A4) $\Longrightarrow$ (A7) $\Longrightarrow$ (A8) $\Longrightarrow$ (A3). It is a consequence of Proposition 1.4 with $F = \nabla \Phi$. By Theorem 1.1, one has **(A3)** $\Longleftrightarrow$ **(A4)**, therefore **(A7)** $\Longleftrightarrow$ **(A8)** $\Longleftrightarrow$ **(A3)**. □

Proof of Proposition 1.5. (A9) $\Longrightarrow$ (A10) $\Longrightarrow$ (A11). These implications are trivial.

(A11) $\Longrightarrow$ (A9). Let u and $v \in BC^2(\mathbb{R}, \mathbb{R}^N)$ be such that $\mathcal{G}_1(u) = \mathcal{G}_1(v)$. The function h and the numerical function r defined by (3.7) satisfy (3.8), therefore

$$r''(t) = (F(v(t)) - F(u(t)), h(t)) + | \, h'(t) \, |^2 \, . \tag{3.18}$$

By the monotonicity of F and (3.18), one has $r''(t) \geq 0$, then r is a convex and bounded function on $\mathbb{R}$, therefore r is a constant function; i.e., $r''(t) = 0$ for all $t \in \mathbb{R}$. By the monotonicity of F and (3.18), we obtain that $h'(t) = 0$, therefore $h''(t) = 0$ for all $t \in \mathbb{R}$. Since u and v are two solutions of equation (1.4), we deduce that $F(u(t)) = F(v(t))$ for all $t \in \mathbb{R}$. By the injectivity of F, one has $u(t) = v(t)$ for all $t \in \mathbb{R}$. □

Proof of Proposition 1.6. (A7) $\Longrightarrow$ (A8) $\Longrightarrow$ (A12). It is a consequence of Proposition 1.4 with $F(x) = Ax$.

(A12) $\Longrightarrow$ (A7). Obviously, $\mathcal{G}_1$ is linear and continuous. It suffices to prove that $\mathcal{G}_1$ is bijective; i.e., for each $e \in BC^0(\mathbb{R}, \mathbb{R}^N)$, the linear differential system

$$u'' = Au + e(t) \tag{3.19}$$

admits a unique bounded solution. The differential system (3.19) can be put in the form of a first-order differential system by letting $v(t) = u'(t)$ and $U(t) = [u(t), v(t)]$. The linear system which is equivalent to equation (3.19) is

$$U' = MU + E(t) \tag{3.20}$$

with

$$E(t) = [0, e(t)] \qquad \text{and} \qquad M = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}.$$

Let us prove now that the matrix M has no eigenvalue with real part zero. Assume the contrary: there exists at least one eigenvalue λ for which $\Re(\lambda) = 0$: $\lambda = i\omega$ with $\omega \in \mathbb{R}$. We deduce that the homogeneous equation associated to equation (3.20): $U' = MU$ admits a non trivial $\frac{2\pi}{\omega}$-periodic solution. Thus, there exists $u \neq 0$ a $\frac{2\pi}{\omega}$-periodic solution of $u'' = Au$, therefore $\mathcal{G}_1$ is not injective. By Proposition 1.5, $\mathcal{G}_1$ is injective, which is a contradiction with the last sentence. In conclusion the matrix M has no eigenvalue with real part zero and by [10, Theorem 4.3, page 95], $\mathcal{G}_1$ is bijective. □

REFERENCES

[1] A.R. Aftabizadeh, *Bounded solutions for some gradient type systems*, Libertas Math., 2 (1982), 121–130.

[2] J.M. Alonso and R. Ortega, *Global asymptotic stability of forced Newtonian system with dissipation*, J. Math. Anal. Appl., 196 (1995), 965–986.

[3] M.S. Berger and Y.Y. Chen, *Forced quasiperiodic and almost periodic solution for nonlinear systems*, Nonlinear Anal., 21 (1993), 949–965.

[4] M. Biroli, *Sur les solutions bornées ou presque-périodiques des équations d'évolution multivoques sur un espace de Hilbert*, Ricerche Mat., 21 (1972), 17–47 (French).

[5] J. Blot, P. Cieutat, and J. Mawhin, *Almost-periodic oscillations of monotone second-order systems*, Adv. Differential Equations, 2 (1997), 693–714.

[6] H. Brézis, *Opérateurs maximaux momonotones et semi-groupes de contractions dans les espaces de Hilbert*, North-Holland, Amsterdam, 1973 (French).
[7] C. Carminati, *Forced systems with almost periodic and quasiperiodic forcing term*, Nonlinear Anal., 32 (1998), 727–739.
[8] P. Cieutat, *Almost periodic solutions of second-order systems with monotone fields on a compact subset*, Nonlinear Anal., 53 (2003), 751–763.
[9] C. Corduneanu, *Bounded and almost periodic solutions of certain nonlinear elliptic equations*, Tôhoku Math. J., 32 (1980), 265–278.
[10] C. Corduneanu, *Almost periodic functions*, Wiley, New York, 1968. Reprinted, Chelsea, New York, 1989.
[11] C. M. Dafermos, *Almost periodic processes and almost periodic solutions of evolution equations*, Dynamical systems (Proc. Internat. Sympos., Univ. Florida, Gainesville, Fla., 1976), 43-57. Academic Press, New York, 1977.
[12] A. Haraux, *Nonlinear evolution equations - global behavior of solutions*, Lecture Notes in Mathematics, 841, Springer-Verlag, Berlin-New York, 1981.
[13] F. L. Huang, *Existence of almost periodic solutions for dissipative differential equations*, Ann. Differential Equations, 6 (1990), 271–279.
[14] H. Ishii, *On the existence of almost periodic complete trajectories for contractive almost periodic processes*, J. Differential Equations, 43 (1982), 66–72.
[15] M.A. Krasnosel'skii, P.P. Zabreiko, *Geometrical methods of nonlinear analysis*, Springer-Verlag, Berlin, 1984.
[16] J. Mawhin, *Bounded and almost periodic solutions of nonlinear differential equations: variational vs nonvariational approach*, Calculus of variations and differential equations (Haifa, 1998), 167-184, Chapman & Hall/CRC Res. Notes Math., 410, Chapman & Hall/CRC, Boca Raton, FL, 2000.
[17] RT. Rockafellar, *Convex analysis*, Princeton University Press, New-York, 1970.
[18] V. E. Slyusarchuk, *Necessary and sufficient conditions for existence and uniqueness of bounded and almost-periodic solutions of nonlinear differential equations*, Acta Appl. Math., 65 (2001), 333–341.
[19] S. F. Zakharin and I.O. Parasyuk, *Generalized and classical almost periodic solutions of Lagrangian systems,* Funkcial. Ekvac., 42 (1999), 325–338.