Differential and Integral Equations Volume 18, Number 5 (2005), 567–582

STEADY STATES FOR A SYSTEM DESCRIBING SELF-GRAVITATING FERMI-DIRAC PARTICLES

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Abstract. In this paper we obtain existence, nonexistence, and multiplicity results for the Dirichlet boundary-value problem $-\Delta u = f_\alpha(u+c)$ in a bounded domain $\Omega \subset \mathbb{R}^d$, with a nonlocal condition $\int_\Omega f_\alpha(u+c) = M$. The solutions of this BVP are steady states for some evolution system describing self-gravitating Fermi-Dirac particles.

1. Introduction

We study the following nonlocal boundary-value problem

$$\begin{aligned} -\Delta u &= f_\alpha(u+c), \quad \text{in } \Omega \\ u|_{\partial\Omega} &= 0, \\ M &= \int_\Omega f_\alpha(u+c), \end{aligned} \tag{1.1}$$

with given $M > 0$, where $f_{\alpha,c}(u) = f_\alpha(u+c) = I_\alpha(e^{-u-c})$, $c \in \mathbb{R}$ (we use the notation $f_\alpha = f_{\alpha,0}$) and

$$I_\alpha(\lambda) = \int_0^\infty \frac{z^\alpha}{1+\lambda e^z} dz$$

is the Fermi function and $\alpha = \frac{d}{2} - 1$, $d \geq 2$ is the dimension of the ambient space. We look for its positive solutions. The domain Ω is assumed to be a bounded subset of $\mathbb{R}^d$, where $d \geq 3$, with sufficiently smooth boundary $\partial\Omega$.

We show that for sufficiently small mass $M > 0$ there exists a solution in any dimension $d \geq 3$, while there is no solution for strictly star-shaped

Accepted for publication: October 2004.
AMS Subject Classifications: 35J60, 35J65, 47J05, 47J30.

domains for $d \geq 5$ and sufficiently large M. In the case of an annular domain Ω we prove the existence of two solutions for sufficiently small mass $M > 0$ and any dimension $d \geq 3$. We also present known results for the existence in dimension $d = 3$ with any mass $M > 0$, or $d = 4$ with sufficiently small masss $M > 0$, obtained by a variational approach, cf. [2], [22].

The solutions of (1.1) are steady states of the respective parabolic system:

$$\begin{aligned} &n_t = \nabla \cdot (D(\nabla p + n\nabla\varphi)) \text{ in } U, \\ &\Delta\varphi = n, \end{aligned} \tag{1.2}$$

with no-flux boundary condition

$$\frac{\partial p}{\partial \nu} + n\frac{\partial \varphi}{\partial \nu} = 0, \quad \varphi_{|\partial U} = 0, \tag{1.3}$$

and initial condition

$$n(x,0) = n_0(x),$$

where

$$\begin{aligned} &n(x,t) = \frac{\mu}{2}\vartheta^{d/2} I_{d/2-1}(\lambda(x,t)), \\ &p(x,t) = \frac{\mu}{d}\vartheta^{d/2+1} I_{d/2}(\lambda(x,t)), \\ &\mu = \eta_0 G\sigma_d 2^{\frac{d}{2}}, \quad D = -\frac{I_{d/2-1}(\lambda(x,t))}{\lambda(x,t)I'_{d/2-1}(\lambda(x,t))}, \end{aligned} \tag{1.4}$$

where G is the gravitational constant, ϑ is the temperature, σ_d is the area of the unit sphere in $\mathbb{R}^d$ and η_0 is the bound on the density and U is a bounded domain, with sufficiently smooth boundary. The function $\lambda(x,t)$ is introduced to express implicit relations between n and p and as such could be eliminated. The above nonlinear parabolic system with nonlinear diffusion D describes the evolution of a cloud of self-gravitating particles that obey the Fermi-Dirac statistics. This model has appeared for the first time in [10] as the consequence of research on kinetic equations. The static problem has been studied numerically in [6] and [9]. These equations belong to a generalized class of drift-diffusion equations proposed in [7].

For the discussion and motivation of this evolution problem we refer to [4] and [10]. In particular, in [2] a priori bounds and global existence has been proved in the dimension $d = 3$ for any mass $M > 0$ and in the dimension $d = 4$ for sufficiently small mass $M > 0$. Accumulation points of the evolution problem correspond to stationary solutions; i.e., solutions for problem (1.1). Therefore the existence of stationary solutions often indicates the global existence of the evolution system and thus no existence of blow-up. For discussion on the possibility of gravitational collapse in microcanonical setting (with the fixed energy) we refer to [3], [4] and [21], and in a radial a radial case to [21].

Remark 1. We adapted notation from [2], where n, p, E, and the left-hand side of the second equation from (1.2) are divided by the factor $\sigma_d G$ present in [10].

In particular, for the system (1.2), the mass $M_0 = \int_\Omega n(x,t)dx$ and the energy

$$E = \frac{d}{2}\int_\Omega p(x,t)dx + \frac{1}{2}\int_\Omega n(x,t)\varphi(x,t)dx \tag{1.5}$$

are conserved.

Looking for stationary solutions of (1.2), $\varphi(x,t) = \varphi(x)$, we are led, after multiplication of (1.2) by $\vartheta\ln(\lambda) - \varphi$ and integration, to

$$\vartheta\ln(\lambda) - \varphi = -c_1,$$

for some constant c_1, whence

$$\Delta\varphi = \frac{\mu}{2}\vartheta^{\frac{d}{2}} I_{d/2-1}(e^{\frac{\varphi - c_1}{\vartheta}}), \tag{1.6}$$

with $M_0 = \int_\Omega \frac{\mu}{2}\vartheta^{d/2} I_{d/2-1}(\lambda(x,t))dx$. We rescale this equation by introducing new variables: $c = c_1/\vartheta$, $u(x) = -\frac{\varphi(x/a)}{\vartheta}$, where $a \stackrel{\mathrm{df}}{=} \left(\frac{\mu}{2}\vartheta^{d/2-1}\right)^{1/2}$. Then (1.6) can be expressed as (1.1) on $\Omega \stackrel{\mathrm{df}}{=} aU$ where $M = a^d M_0 \frac{2}{\mu}\vartheta^{-d/2}$.

We begin with some properties of the composition of the Fermi function with exponential nonlinearity. Let us first recall the properties of the Fermi function. Namely, it satisfies the following inequalities and has the following limits:

$$\frac{\Gamma(\alpha+1)}{\lambda+1} \le I_\alpha(\lambda) \le \frac{\Gamma(\alpha+1)}{\lambda}, \tag{1.7}$$

and

$$\lim_{\lambda\to 0^+} \frac{(-\ln(\lambda))^{\alpha+1}}{(\alpha+1)I_\alpha(\lambda)} = 1, \qquad \lim_{\lambda\to\infty} \frac{\Gamma(\alpha+1)}{\lambda I_\alpha(\lambda)} = 1. \tag{1.8}$$

The first limit follows immediately after the substitution $z = -\ln(\lambda) + w$ in the integral $I_\alpha(\lambda) = \int_0^\infty \frac{z^\alpha}{1+\lambda e^z}dz$, while the second one follows from inequalities (1.7); cf., e.g. [2].

Now, we can state the following

Lemma 1.1. *The functions f_α, defined above for any $\alpha > 0$, are increasing, convex, and possess the following asymptotics*

$$\lim_{u\to\infty} \frac{u^{\alpha+1}}{(\alpha+1)f_\alpha(u)} = 1, \qquad \lim_{u\to-\infty} \frac{\Gamma(\alpha+1)e^u}{f_\alpha(u)} = 1. \tag{1.9}$$

Proof. First notice that, from the very definition of f_α, its monotonicity follows. By substitution $p = e^{z-u}$ one obtains

$$f_\alpha(u) = \int_0^\infty \frac{z^\alpha}{1+e^{z-u}}dz = \int_{e^{-u}}^\infty \frac{(u+\ln(p))^\alpha}{p(1+p)}dp.$$

Therefore, $f'_\alpha(u) = \alpha f_{\alpha-1}(u)$ guarantees that f'_α is increasing and thus f_α is convex, since $\alpha - 1 = \frac{d}{2} - 2 > -1$ if $d \geq 3$. Also note that

$$\frac{\Gamma(\alpha+1)}{e^{-u}+1} \leq f_\alpha(u) \leq \Gamma(\alpha+1)e^u, \tag{1.10}$$

To prove the asymptotic property of f_α it suffices to use the asymptotics (1.8) of I_α. □

From monotonicity and asymptotic properties of f it follows that for given $M > 0$ and arbitrary $u \in C(\bar{\Omega})$ there exists only one c such that $\int_\Omega f_\alpha(u + c) = M$. We shall denote it by $c(u)$. Moreover $c : C(\bar{\Omega}) \to \mathbb{R}$ is a continuous (Lipschitz) functional. Indeed, by the mean-value theorem we have

$$\begin{aligned} 0 &= \int_\Omega f_\alpha\big(u_1(x) + c(u_1)\big) - f_\alpha\big(u_2(x) + c(u_2)\big)dx \\ &= \int_\Omega f'_\alpha\big(\theta(x)\big)\big(u_1(x) - u_2(x) + c(u_1) - c(u_2)\big)dx, \end{aligned}$$

whence

$$|c(u_1) - c(u_2)| \leq \|u_1 - u_2\|_\infty. \tag{1.11}$$

Therefore the BVP with constraint (1.1) is equivalent to

$$\begin{aligned} -\Delta u &= f_\alpha(u + c(u)), \\ u|_{\partial\Omega} &= 0. \end{aligned} \tag{1.12}$$

Moreover, looking for positive solutions (which are the only ones by positivity of f_α and $(-\Delta)^{-1}$) leads to the following useful inequalities

$$f_\alpha^{-1}\left(\frac{M}{|\Omega|}\right) - \|u\|_\infty \leq c(u) \leq f_\alpha^{-1}\left(\frac{M}{|\Omega|}\right). \tag{1.13}$$

Indeed, since we have $u \geq 0$ one obtains

$$M = \int_\Omega f_\alpha\big(u + c(u)\big) \geq \int_\Omega f\big(c(u)\big),$$

whence $c(u) \leq f_\alpha^{-1}\big(\frac{M}{|\Omega|}\big)$. Similarly,

$$M = \int_\Omega f_\alpha\big(u + c(u)\big) \leq \int_\Omega f\big(c(u) + \|u\|_\infty\big)$$

gives the second inequality.

2. Nonexistence results for strictly star-shaped domains

We recall the well-known Pohozaev identity [17]:

Lemma 2.1. *Let $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$ satisfy, for a continuous function f,*

$$-\Delta u = f(u),$$

with

$$u = 0 \text{ on } \partial\Omega,$$

where $\Omega \subset \mathbb{R}^d$ is a bounded domain with sufficiently smooth boundary. Then the identity

$$d \int_\Omega F(u) + \frac{2-d}{2} \int_\Omega f(u)u = \frac{1}{2} \int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 (x \cdot \nu)$$

holds.

Applying this to (1.1) one gets the following

Theorem 2.1. *Assume that Ω is a strictly star-shaped domain; i.e., for some positive $\beta > 0$ and any $x \in \partial\Omega$ we have $x \cdot \nu \geq \beta > 0$. Then for sufficiently large mass $M > 0$ and $d \geq 5$ there exists no solution for the BVP (1.1).*

Proof. Applying Lemma 2.1 to our case, namely with $f(u) = f_\alpha(u + c(u))$, if $x \cdot \nu \geq \beta$ we obtain

$$\int_{\partial\Omega} \left(\frac{\partial u}{\partial \nu}\right)^2 (x \cdot \nu) \geq \frac{\beta}{|\partial\Omega|} \left(\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\right)^2 \geq \frac{M^2 \beta}{|\partial\Omega|},$$

whence

$$\begin{aligned} &\frac{d}{\alpha+1} \int_\Omega \Big(f_{\alpha+1}\big(u(x) + c(u)\big) - f_{\alpha+1}\big(c(u)\big)\Big) dx \\ &+ \frac{2-d}{2} \int_\Omega f_\alpha\big(u(x) + c(u)\big)\big(u(x) + c(u) - c(u)\big) dx \geq \frac{\beta}{2|\Omega|} M^2. \end{aligned} \tag{2.1}$$

Furthermore, using asymptotics (1.9) we obtain for $\varepsilon \overset{\text{df}}{=} \frac{d^2-4d-4}{d^2+4d-4}$ (which is positive provided $d > 2(1+\sqrt{2})$) existence of $t_\varepsilon > 0$ such that, for any $t \geq t_\varepsilon$

$$f_{\alpha+1}(t) \leq \frac{1+\varepsilon}{\alpha+2} t^{\alpha+2}, \qquad f_\alpha(t) \geq \frac{1-\varepsilon}{\alpha+1} t^{\alpha+1}. \tag{2.2}$$

Let us define $\Omega_\varepsilon = \{x \in \Omega : u(x)+c(u) < t_\varepsilon\}$. Next, we make use of estimates (2.2). The coefficient before $\int_{\Omega\setminus\Omega_\varepsilon}(u(x)+c(u))^{\alpha+2}dx$; i.e.,

$$\frac{d(1+\varepsilon)}{(\alpha+1)(\alpha+2)} + \frac{(2-d)(1-\varepsilon)}{2(\alpha+1)},$$

vanishes by the choice of ε and thus (2.1) yields

$$\int_{\Omega_\varepsilon}\Big(\frac{d}{\alpha+1}f_{\alpha+1}\big(u(x)+c(u)\big) + \frac{2-d}{2}f_\alpha\big(u+c(u)\big)\big(u(x)+c(u)\big)\Big)$$
$$+\frac{d-2}{2}c(u)M \geq \frac{\beta}{2|\Omega|}M^2.$$

Hence, by $c(u) \leq f_\alpha^{-1}(\frac{M}{|\Omega|})$ ((1.13)), we get

$$D_\varepsilon|\Omega| + \frac{d-2}{2}Mf_\alpha^{-1}\left(\frac{M}{|\Omega|}\right) \geq \frac{\beta}{2|\Omega|}M^2, \tag{2.3}$$

where $D_\varepsilon = \sup_{t\leq t_\varepsilon}|\frac{d}{\alpha+1}f_{\alpha+1}(t) + \frac{2-d}{2}f_\alpha(t)t|$. Therefore, by sublinearity of f_α^{-1}, i.e. the second inequality from (2.2), the condition (2.3) gives an upper bound on M, which proves to be a necessary condition for the existence of solutions for (1.12). □

Remark 2. We also have the following a priori estimate from below for the sup norm of the solution of (1.12)

$$\|u\|_\infty \geq \frac{\beta}{(d+2)|\Omega|}M. \tag{2.4}$$

Proof. From (2.1), by the mean-value theorem applied to $f_{\alpha+1}$, we obtain, for some $\theta \in (0,1)$

$$\begin{aligned}\|u\|_\infty\frac{d+2}{2}M &\geq \int_\Omega \frac{d+2}{2}f_\alpha\big(u(x)\big)u(x)dx\\ &\geq d\int_\Omega f_\alpha\big(\theta u(x)+c(u)\big)u(x)dx + \frac{2-d}{2}\int_\Omega f_\alpha\big(u(x)+c(u)\big)u(x)dx \\ &\geq \frac{\beta}{2|\Omega|}M^2.\end{aligned} \tag{2.5}$$

3. Existence results

We shall formulate a corresponding fixed-point problem and look for its solution in the cone of nonnegative function in the space of continuous functions $C(\Omega)$ with the sup norm.

Thus, we are left to examine the existence of solutions for the following integral equation

$$u(x) = \int_\Omega G(x,y) f_\alpha(u(y) + c(u))dy, \tag{3.1}$$

in the space $X = C(\bar{\Omega})$, equipped with the sup norm, since Lipschitz continuity of f_α guarantees the existence of a classical solution to the BVP (1.1). Here the function G is the Green function for BVP (1.1). From [13] it follows that it is positive, symmetric, and enjoys the following estimates

$$|G(x,y)| \le C|x-y|^{2-n}, \qquad |\nabla_x G(x,y)| \le C|x-y|^{1-n}, \tag{3.2}$$

for any $x \ne y \in \Omega$. Let P be the cone of nonnegative functions in $C(\bar{\Omega})$, and let T be defined on P by the formula

$$T(u)(x) = \int_\Omega G(x,y) f_\alpha(u(y) + c(u))dy. \tag{3.3}$$

Due to the first estimate (1.1) and Lipschitz continuity of f_α and c ((1.11)), this is a well defined operator $T : C(\bar{\Omega}) \to C(\bar{\Omega})$. Notice that, since the Green's function for (1.1) is positive, the operator T maps P into itself. Moreover, by the second inequality (1.1), it maps $C(\bar{\Omega})$ into $C^1(\bar{\Omega})$, whence from the compact inclusion $C^1(\bar{\Omega}) \hookrightarrow C(\bar{\Omega})$ the compactness of T follows.

We look for fixed points of T in P, since from [13] and by elliptic regularity, cf. [12, pages 241-244], they are positive solutions of BVP (1.1).

Now, we are ready to prove the main existence result:

Theorem 3.1. *There exists a value $M_0 > 0$ such that for $M < M_0$ the BVP* (1.1) *has at least one positive solution.*

Proof. We shall prove that some convex set of the form $P \cap B(0,N)$, where $B(0,N)$ is a ball in $C(\bar{\Omega})$ of radius N, is mapped by T into itself. Then an application of the Schauder theorem shall yield the existence of a fixed point of T. Let

$$\mathcal{G} \overset{\mathrm{df}}{=} \sup_{x \in \bar{\Omega}} \int_\Omega G(x,y)dy, \tag{3.4}$$

and

$$M_0 = M_0(\mathcal{G}) \overset{\mathrm{df}}{=} |\Omega| f_\alpha\Big(f_{\alpha-1}^{-1}\big(\frac{1}{\mathcal{G}\alpha}\big) - \mathcal{G} f_\alpha\big(f_{\alpha-1}^{-1}\big(\frac{1}{\mathcal{G}\alpha}\big)\big)\Big). \tag{3.5}$$

Now we shall prove that for $M \le M_0$ the integral operator T maps some ball into itself. Fix $N_0 = \mathcal{G} f_\alpha(f_{\alpha-1}^{-1}(\frac{1}{\mathcal{G}\alpha}))$. Take any $u \ge 0$ such that $\|u\|_\infty \le N$, then using (1.13) one gets

$$\|Tu\|_\infty = \sup_{x\in\bar\Omega}\int_\Omega G(x,y)f_\alpha(c(u)+u(y))dy \le \mathcal{G} f_\alpha(f_\alpha^{-1}(\frac{M}{|\Omega|})+N).$$

The inequality $\|Tu\|_\infty \le N$ holds provided the following inequality is satisfied:

$$f_\alpha^{-1}(M) \le f_\alpha^{-1}(\frac{N}{\mathcal{G}}) - N. \tag{3.6}$$

The supremum of the left-hand side of the above inequality is attained at $N = N_0$ and is equal to $f_\alpha^{-1}(M_0)$. Therefore, for $M \le M_0$, the condition $\|u\|_\infty \le N$ implies $\|Tu\|_\infty \le N$ where $N = N_0$. □

The above result can be improved in the dimension $d = 3$ as the following theorem states:

Theorem 3.2. *For any $M < M_0$ and $d \ge 3$ the BVP (1.1) has at least one positive solution. In dimension $d = 3$ the result holds for any mass $M > 0$.*

Proof. Fix a positive number $\frac{d-2}{d} < \beta < \frac{2}{d}$. Fix any $N > 0$. Then, for any $\|u\|_\infty \le N$, we have

$$\begin{aligned}
\|Tu\|_\infty &= \sup_{\bar\Omega}\int_\Omega G(x,y)f_\alpha\big(u(y)+c(u)\big)dy\\
&\le \sup_{\bar\Omega}\int_\Omega \Big(G(x,y)f_\alpha\big(c(u)+u(y)\big)^\beta\Big)\Big(f_\alpha\big(c(u)+u\big)^{1-\beta}\Big)dy\\
&\le \sup_{\bar\Omega}\left(\int_\Omega G(x,y)^{\frac{1}{\beta}}f_\alpha\big(c(u)+u(y)\big)dy\right)^\beta\left(\int_\Omega f_\alpha\big(c(u)+u(y)\big)dy\right)^{1-\beta}\\
&\le \sup_{\bar\Omega}\left(\int_\Omega G(x,y)^{\frac{1}{\beta}}f_\alpha\big(c(u)+u(y)\big)dy\right)^\beta \cdot M^{1-\beta}\\
&\le \sup_{\bar\Omega}\left(\int_\Omega G(x,y)^{\frac{1}{\beta}}\right)^\beta\left(f_\alpha\Big(f_\alpha^{-1}(\frac{M}{|\Omega|})+N\Big)\right)^\beta M^{1-\beta}.
\end{aligned}$$

The last term is smaller than N, provided that N is sufficiently large. Indeed, it suffices to recall asymptotic the behaviour of f_α in (1.9) together with the condition $\beta < \frac{2}{d}$. □

Remark 3. Note that the above proof could also be applied to the case $d = 4$ for small values of $M > 0$.

Remark 4. In fact, in Theorem 3.1 it is important that the value of $\frac{M}{|\Omega|}$ should be sufficiently small. However in theorem (3.2) the smallness of

$(\frac{M}{|\Omega|})^{1-\beta}|\Omega|^{\frac{d}{2}}$ seems to be crucial for the existence of the solution of the BVP (1.1) at least for homothetic domains.

Remark 5. Instead of the Schauder theorem we could also apply the cone-compression Theorem 4.1, since condition (1.13) implies

$$\|Tu\|_\infty \geq T\Big(f_\alpha^{-1}\big(\frac{M}{|\Omega|}\big)\Big)$$

for any $u \in P$.

Remark 6. Notice that translation of the function f_a by $f_\alpha^{-1}(\frac{M}{|\Omega|})$, for $M > 0$ small enough, is helpful in forcing the nonlinearity to be sublinear on some interval and thus making a fixed-point approach feasible.

Proof. We shall now make the above comment more precise. Indeed, let us define the point at which f_α has derivative equal to 1; i.e., $c_1 = f_\alpha^{-1}(1) = f_{\alpha-1}^{-1}\big(\frac{1}{\alpha}\big)$, and the critical value of translation $c_0 = c_1 - f_\alpha(c_1)$.

For $M < |\Omega| f(c_0)$, we have the following sup- and sublinearity conditions:

$$\begin{aligned} &f_\alpha(t) \leq t \ \text{ for any } \ a(c) \leq t \leq b(c),\\ &f_\alpha(t) \leq f_\alpha(a(c)) = a(c) \ \text{ for any } \ t \leq a(c), \end{aligned} \tag{3.7}$$

where $a(c) = f_\alpha(a(c)) < b(c) = f_\alpha(b(c))$ for $c < c_0$. The functions $a, b : (-\infty, c_0] \to \mathbb{R}$ are defined implicitly

$$c(t) = f_{\alpha,c_0}(u_0 + t) - (u_0 + t) \ \text{ and } \ a(c(t)) = f_{\alpha,c_0}(u_0 + t)$$

$$c(t) = f_{\alpha,c_0}(u_0 - t) - (u_0 - t) \ \text{ and } \ b(c(t)) = f_{\alpha,c_0}(u_0 - t)$$

for any $t \geq 0$. The point u_0 is the unique fixed point of f_{α,c_0}. □

Remark 7. The existence of solutions in the dimension $d = 4$ with large mass $M > 0$ seems to be an open question.

4. Multiplicity results for an annulus

In this section we shall prove the existence of at least two radial solutions for the BVP (1.12) in the case of an annular domain Ω.

We shall apply the following cone-compression and cone-expansion theorem due to M. Krasnosielski [15], in the form taken from [14]:

Theorem 4.1. *Let P be a cone in a Banach space, U_1 and U_2 two bounded open neighbourhoods of zero such that $\overline{U_1} \subset U_2$, let $T : P \cap (\overline{U_2} \setminus U_1) \to P$ be a compact (nonlinear) operator. If one of the following conditions*

$$\begin{aligned} &||T(u)|| \leq ||u|| \qquad \textit{for} \quad u \in \partial U_1,\\ &||T(u)|| \geq ||u|| \qquad \textit{for} \quad u \in \partial U_2, \end{aligned} \qquad (\textit{expansion})$$

or reversely

$$\begin{aligned} &||T(u)|| \geq ||u|| \qquad \textit{for} \quad u \in \partial U_1, \\ &||T(u)|| \leq ||u|| \qquad \textit{for} \quad u \in \partial U_2, \end{aligned} \qquad (\textit{compression})$$

is satisfied then the operator T has a fixed point.

Consider, for fixed values of $R > \rho > 0$, an annulus $\Omega \stackrel{\mathrm{df}}{=} \{x \in \mathbb{R}^d : \rho \leq |x| \leq R\}$. Then looking for radial solutions $v(r) = u(|x|)$ of (1.12) leads to

$$\begin{aligned} -v'' - \frac{d-1}{r}v' = f_\alpha(v + c(v)) \ \text{ in } \ (\rho, R), \\ v(\rho) = v(R) = 0. \end{aligned} \tag{4.1}$$

Integrating twice the above equation and using boundary conditions one gets the integral equation:

$$v(r) = Tv(r) \stackrel{\mathrm{df}}{=} \int_\rho^R G(r,s) f_{\alpha,c(v)}(v(s))ds. \tag{4.2}$$

Then $T : C([\rho, R]) \to C([\rho, R])$ is a well-defined, completely continuous operator on the space $C([\rho, R])$ equipped with the sup norm $\| \cdot \|_\infty$. The function G, called the Green's function for (4.1), has the following form

$$G(r,s) = \frac{s(R^{d-2} - \max(r,s)^{d-2})(\min(r,s)^{d-2} - \rho^{d-2})}{(d-2)r^{d-2}(R^{d-2} - \rho^{d-2})}. \tag{4.3}$$

Similar integral equations with the same Green's function but with different nonlinearity were considered in [11]. Let us fix some $R > R_1 > \rho_1 > \rho > 0$ and consider a cone $P = \{v \geq 0 : \inf_{r \in [\rho_1, R_1]} \geq D\|v\|_\infty\}$, where

$$D = \frac{\inf_{s,r \in [\rho_1, R_1]} G(r,s)}{\sup_{s,r \in [\rho, R]} G(r,s)}.$$

Then $T : C([\rho, R]) \to P$.

To prove the first of the expansion conditions we proceed as in a nonradial case to show that, for any $\|v\|_\infty \leq N_0$ (N_0 to be defined in a similar way as in the nonradial case), we have $\|Tv\|_\infty \leq N_0$. Indeed, for any $\|v\|_\infty \leq N_0$ we obtain

$$\sup_{r \in [\rho, R]} \int_\rho^R G(r,s) f_\alpha(c(v) + v(s))ds \leq \mathcal{G} f_\alpha\Big(f_\alpha^{-1}\Big(\frac{M}{|\Omega|}\Big) + N_0\Big) \leq N_0.$$

To prove the second of the expansion conditions one has to use superlinearity of the nonlinear term (1.9). Let

$$\mathcal{G}_1 \stackrel{\mathrm{df}}{=} \sup_{r\in[\rho_1,R_1]} \int_\rho^R G(r,s)ds$$

and let N_1 be, by (1.9), the maximal positive number such that

$$f_\alpha^{-1}\left(\frac{M}{|\Omega|}\right) \geq f_\alpha^{-1}\left(\frac{N_1}{\mathcal{G}_1}\right) + (1-D)N_1.$$

Now we take any $v \in P$ such that $\|v\|_\infty = N_1$:

$$\begin{aligned}
&\sup_{r\in[\rho,R]} \int_\rho^R G(r,s) f_\alpha\big(c(v)+v(s)\big)ds \\
&\geq \sup_{r\in[\rho_1,R_1]} \int_\rho^R G(r,s) f_\alpha\Big(f_\alpha^{-1}\big(\frac{M}{|\Omega|}\big) - \|v\|_\infty + D\|v\|_\infty\Big)ds \qquad (4.4)\\
&\geq \mathcal{G}_1 f_\alpha\Big(f_\alpha^{-1}\big(\frac{M}{|\Omega|}\big) + (D-1)N_1\Big) \geq N_1.
\end{aligned}$$

The latter inequality holds by the choice of N_1. Obviously we could also take any $N \leq N_1$ instead of N_1, provided it is greater than N_0. This is true if $N \leq \min(N_0, N_1)$.

Thus we have proven the following

Theorem 4.2. *For sufficiently small mass $M > 0$, the BVP* (4.1) *has at least two positive solutions.*

Proof. Indeed, we have only to define sets $U_1 = B(0, N_0)$ and $U_2 = B(0, N_1)$, and by direct application of the second part of Theorem 4.1 we get a solution v_1 such that $N_0 \leq \|v_1\|_\infty \leq N_1$. The second solution v_2 could be obtained either by the first part of Theorem 4.1 with an additional set U_0 (defined as indicated in Remark 5) or simply enough by the Schauder theorem in U_1. As a matter of fact, to ascertain that v_2 is different from v_1 (it might happen that $\|v_1\|_\infty = \|v_2\|_\infty$), one should take U_1 with a bit smaller radius to get v_2. □

5. Existence results by variational methods

In this section we present known results [2], following the approach in [22], but using notation which is consistent with the one from previous sections. We also present the main idea of the proofs.

Consider, on $P = \{\rho \in L^{1+\alpha}(\Omega) : \int_\Omega \rho = 1, \rho \geq 0\}$, the following entropy functional:

$$W_M(\rho) = \int_\Omega w(\rho) = E(\rho) + S_M(\rho),$$

where

$$E(\rho) = \int_\Omega e(\rho) = \int_\Omega \frac{1}{2}\rho(x)(-\Delta)^{-1}(\rho)(x)dx,$$

and

$$\begin{aligned} S_M(\rho) &= \int_\Omega s_M(\rho) \\ &= \int_\Omega \frac{1}{M}(\frac{1}{M(\alpha+1)} f_{\alpha+1}(f_\alpha^{-1}(M\rho(x))) - \rho(x) f_\alpha^{-1}(M\rho(x)))dx. \end{aligned} \tag{5.1}$$

For the above entropy functional the Euler-Lagrange equation is exactly (1.1), where the constant $\mu = -c$ is a Lagrange multiplier. Indeed, if

$$H_M(\rho) = \frac{1}{M}\Big(\int_\Omega \rho - 1\Big),$$

then from the equation

$$W'_M(\rho)\psi - \mu H'_M(\rho)\psi = 0,$$

for any $\psi \in L^{1+\alpha}(\Omega)$, we get

$$\int_\Omega \psi\Big((-\Delta)^{-1}\rho - \frac{1}{M} f_\alpha^{-1}(M\rho) - \frac{\mu}{M}\Big) = 0$$

and after the substitution $u = M(-\Delta)^{-1}\rho$ one obtains

$$f_\alpha(u - \mu) = M\rho = -\Delta u.$$

Then, by $\int_\Omega \rho = 1$, one has $\int_\Omega f_\alpha(u-\mu) = M$ but, by monotonicity, asymptotics, and continuity of f_α, there exists exactly one such μ. One has to note however that the functional S_M has the singular derivative at 0. Therefore we are left to prove that a maximizer of the functional S_M over P is in fact positive (we shall prove it later cf. (iii)).

Notice the following properties of the function s_M

$$\begin{aligned} s'_M(\rho) &= -\tfrac{1}{M} f_\alpha^{-1}(M\rho), \\ s''_M(\rho) &= -\tfrac{1}{f'_\alpha(f_\alpha^{-1}(M\rho))} = -\tfrac{1}{\alpha f_{\alpha-1}(f_\alpha^{-1}(M\rho))} < 0. \end{aligned} \tag{5.2}$$

Furthermore, s_M increases on $(0, \frac{f_\alpha(0)}{M})$ and s_M decreases on $(\frac{f_\alpha(0)}{M}, \infty)$. We also have the following asymptotics :

$$\lim_{\rho\to\infty} \frac{f_\alpha^{-1}(\rho)}{(\frac{d}{2}\rho)^{2/d}} = 1. \tag{5.3}$$

We shall prove the existence of a maximizer of W_M over P; that is, the existence of $\rho_0 \in P$ such that

$$\sup_{\rho\in P} W_M(\rho) = W_M(\rho_0),$$

which will be a solution of the BVP (1.1). It can be achieved by standard steps:

(i) Coerciveness of $-W_M$ (thus boundedness of $-W_M$ from below),
(ii) Weak upper-semicontinuity of W_M,
(iii) Positivity of ρ_0.

Note that (i) and (ii) give the existence of a maximizer on $L^{1+\frac{2}{d}}(\Omega)$, which by (iii) and Lagrange multiplier method turns out to be a solution of the BVP (1.1).

(i) Using asymptotic properties (1.9) of f_α and (5.3) one obtains, for $|\rho|_{L^{1+\frac{2}{d}}}$ sufficiently large

$$W_M(\rho) \le c_1 M^{\frac{5}{6}-\frac{d}{12}} |\rho|^{\frac{7}{6}+\frac{d}{12}}_{L^{1+\frac{2}{d}}} - c_2 M^{\frac{2}{d}-1} |\rho|^{1+\frac{2}{d}}_{L^{1+\frac{2}{d}}},$$

where c_1 is a positive constant from the Sobolev inclusion $W^{2,1+\frac{2}{d}}(\Omega) \subset L^6(\Omega)$ and c_2 is any positive constant smaller than $(\frac{d}{2})^{\frac{2}{d}+1}\left(\frac{d}{2}+1\right)^{-1}$. Therefore, $-W_M$ is coercive; i.e., $|\rho|_{L^{1+\frac{2}{d}}} \to \infty$ implies $W_M(\rho) \to -\infty$. This forces maximizing sequences to be bounded.

(ii) Since, by (5.2), S_M is concave and, by asymptotics (1.9) and (5.3), it is a well-defined, continuous functional on $L^{1+\frac{2}{d}}(\Omega)$, it must be also, by the Mazur theorem, weakly upper-semicontinuous. As for E, one applies

$$\int_\Omega (\rho-\rho_0)(-\Delta^{-1})(\rho-\rho_0) \ge 0$$

to get its weak upper-semicontinuity. Reflexivity of $L^{1+\frac{2}{d}}(\Omega)$ therefore is sufficient to ensure the existence of a maximizer of W_M.

(iii) The idea of the following reasoning comes from the one applied for the micro-canonical setting (with fixed energy E, cf. [5]). In our case, that of canonical ensemble, the reasoning can be considerably simplified.

Suppose that, on the contrary, $\sup_{\rho\in P} W(\rho) = W(\rho_0)$ and $\rho_0 = 0$ on the set Ω_0 of positive measure. We shall show that there exists ρ such that $W(\rho) > W(\rho_0)$. We shall achieve this by moving part of the mass to the set Ω_0 thus increasing S_M much more than decreasing E. Consider a one-parameter family

$$\rho_\lambda = \begin{cases} (1-\lambda)\rho_0 \text{ on } \Omega\setminus\Omega_0, \\ \frac{\lambda}{|\Omega_0|} \text{ on } \Omega_0. \end{cases}$$

We will prove that for sufficiently small $0 < \lambda < 1$ we have

$$W_M(\rho_\lambda) > W_M(\rho_0). \tag{5.4}$$

Note that the functional E is differentiable at $\rho = \rho_0$, as is the functional $S_M^1(\rho) = \int_{\Omega\setminus\Omega_0} s_M(\rho)$. On Ω_0 we define $S_M^2(\rho) = \int_{\Omega_0} s_M(\rho)$ and apply the mean-value theorem for the function s_M, thus obtaining for some $\theta_\lambda \in (0,1)$

$$S_M^2(\rho_\lambda) - S_M^2(\rho_0) = \int_{\Omega_0} s_M(\rho_\lambda) - s_M(\rho_0) = \frac{\lambda}{|\Omega_0|} s_M'\Big(\theta_\lambda \frac{\lambda}{|\Omega_0|}\Big).$$

But this, due to (5.2) and (1.9), implies that

$$\lim_{\lambda\to 0^+} \left(S_M^2(\rho_\lambda) - S_M^2(\rho_0)\right)/\lambda = \infty,$$

which combined with differentiability of E and S_M^1 proves the claim (5.4), that is contradictory to our supposition of ρ_0 being the maximum of W_M.

The results of this section can be summarized as

Theorem 5.1. *For any $M > 0$ and $d = 3$ the BVP (1.1) has at least one positive solution. In dimension $d = 4$ the result holds for sufficiently small mass $M > 0$.*

6. Final comments

Remark 8. In an essentially unchanged way, instead of the Dirichlet condition in the BVP (1.1), the free boundary-value problem can be treated to get the existence of solutions as in Theorem 3.2. In that case, we consider the integral operator with kernel $E(x) = \frac{1}{(d-2)\omega_d}|x|^{2-d}$, which enjoys the same estimates as the Green's function (3.2). In the dimensions $d = 1, 2$ the existence holds for small $M > 0$.

Also a similar version of Theorem 2.1 for free boundary condition can be stated.

Remark 9. The existence of solutions for (1.1), in the case $d = 4$ and large $M > 0$, seems to be an open question.

Remark 10. The existence of solutions for the BVP (1.1) satisfying the energy constraint (1.5) is being examined in [3] and [21], while corresponding evolution equation is discussed in [4].

Acknowledgements. I would like to thank Piotr Biler and T. Nadzieja for suggesting this problem to me and many fruitful discussions and suggestions, which greatly improved this paper. I would like also to express my gratitude to Bogdan Przeradzki for introducing me to fixed-point methods in nonlinear analysis.

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