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ON 2D ZAKHAROV SYSTEM IN A BOUNDED DOMAIN

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Abstract. The paper deals with initial boundary-value problems for the Zakharov system arising in plasma physics in two-dimensional domains under various boundary conditions. We prove the global well-posedness of these problems in some Sobolev-type classes and study properties of the solutions. In the dissipative case the existence of a global attractor is established.

Introduction

The following system of coupled equations

$$\begin{cases} n_{tt} - \Delta\left(n + |E|^2\right) = 0, \quad x \in \Omega,\ t > 0, \\ iE_t + \Delta E - nE = 0, \quad x \in \Omega,\ t > 0, \end{cases} \tag{0.1}$$

was proposed by Zakharov [25] for the description of wave phenomena in plasma. Here $E(x,t)$ is a complex function and $n(x,t)$ is a real one.

In the case $\Omega = \mathbb{R}^d$ $(d \geq 1)$ the Cauchy problem for (0.1) was studied by many authors (see, e.g., [4, 6, 11] and the references therein). The local well posedness was shown in appropriate Sobolev classes (see [11] for the most advanced results) and there are solutions which blow up in finite time for $d = 2$ [12]. In the case $d \leq 3$ for some special classes of initial data the global well posedness is also known [6]. At present there is no criteria known which identifies those initial data leading to blow up and those leading to global-in-time solutions.

The key idea which leads to the best known well posedness results in the case $\Omega = \mathbb{R}^d$ (see [11] and the literature quoted therein) involves the method developed by Bourgain to study the Cauchy problem for nonlinear dispersive evolution equations (see [4] and the references therein). In fact, Bourgain's method [5] relies on the analysis of the problem after a Fourier transformation. Formally this method also covers spatially periodic problems, however

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its application to problem (0.1) in $\mathbb{R}^d$ gives a better result in comparison with the periodic case (see the discussion in [11]).

In this paper we consider the problem

$$\begin{cases} n_{tt} - \Delta\left(n + |E|^2\right) + \alpha n_t + \beta n = f(x), \quad x \in \Omega,\ t > 0, \\ iE_t + \Delta E - nE + i\gamma E = g(x), \quad x \in \Omega,\ t > 0, \end{cases} \tag{0.2}$$

in a bounded domain $\Omega \subset \mathbb{R}^2$, where α, β, and γ are nonnegative parameters and $f(x)$ and $g(x)$ are given (real and complex) functions. We equip problem (0.2) with boundary conditions either of Dirichlet type

$$n(x,t) = 0, \quad E(x,t) = 0 \quad \text{for} \quad x \in \partial\Omega,\ t > 0, \tag{0.3}$$

or Neumann

$$\frac{\partial n(x,t)}{\partial \nu} = 0, \quad \frac{\partial E(x,t)}{\partial \nu} = 0 \quad \text{for} \quad x \in \partial\Omega,\ t > 0, \tag{0.4}$$

where ν is the outward normal vector, or else, if $\Omega = (0, l_1) \times (0, l_2)$ is a rectangle, (l_1, l_2)-periodic type boundary conditions:

$$n(x,t) \text{ and } E(x,t) \text{ are } (l_1, l_2)\text{-periodic on } \mathbb{R}^2,\ t > 0. \tag{0.5}$$

In this paper we prove that the Cauchy problem for (0.2) equipped with one of the boundary conditions mentioned has a unique solution such that

$$(n_t; n; E) \in C\left([0,T];\ L_2(\Omega) \times H^1(\Omega) \times H^2(\Omega)\right)$$

for initial data $(n_1; n_0; E_0)$ with a restriction on the L_2 norms of g and E_0 only and with the existence time $T \sim \|g\|_{L_2}^{-1}$ in the conservative case ($\gamma = 0$) and $T = \infty$ in the case when either $g \equiv 0$ or $\gamma > 0$. To prove the existence of solutions we use Galerkin approximations and the standard compactness method. The proof of their uniqueness relies on the method which was developed in the theory of shallow shells for the case of critical nonlinearities (see [20] and also [7]). We also prove the continuous dependence of solutions on initial data and show that these solutions generate a continuous semiflow S_t on a closed subset of the space $L_2(\Omega) \times H^1(\Omega) \times H^2(\Omega)$. In particular this result means that problem (0.1) is well posed in some Sobolev class for a set of initial data with the only restriction on the size of initial datum E_0 for $E(x,t)$.

In the dissipative case ($\alpha > 0$, $\gamma > 0$) we study long-time behaviour of the semiflow S_t and prove the existence of a compact global attractor under some restrictions on external forces f and g. These restrictions can be essentially relaxed when the domain Ω is thin.

We note that the existence and properties of a global attractor for the one-dimensional version of problem (0.2) were studied earlier in [9, 13] and also in [21].

1. Statements of the main results

Let $\Omega \subset \mathbb{R}^2$ be either a smooth bounded domain, or a rectangle $(0, l_1) \times (0, l_2)$. Let A be the self-adjoint operator generated in $\mathcal{H} = L_2(\Omega)$ by the Dirichlet form

$$a(u, v) = \int_\Omega \nabla u \nabla v \, dx, \quad u, v \in V,$$

where either $V = H_0^1(\Omega)$ (the Dirichlet case), or $V = H^1(\Omega)$ (the Neumann case) or else $V = H_{per}^1(\Omega)$ with $\Omega = (0, l_1) \times (0, l_2)$ (the periodic case). In all these cases the operator A is nonnegative and has a compact resolvent such that there exists an orthonormal basis $\{e_k\}$ consisting of eigenfunctions of A:

$$Ae_k = \lambda_k e_k, \quad 0 \le \lambda_1 \le \lambda_2 \le \dots, \quad \lim_{k \to \infty} \lambda_k = \infty.$$

We also clearly have $Au = -\Delta u$ for any $u \in D(A)$.

For $s \in [-2, 2]$ we define the Sobolev-type spaces $\mathcal{H}_s$ by the formula $\mathcal{H}_s = D((I + A)^{s/2})$ with the graph norm $\| \cdot \|_s \equiv \|(I + A)^{s/2} \cdot \|$ for $0 \le s \le 2$ and $\mathcal{H}_s = [\mathcal{H}_{-s}]'$ is the completion of $\mathcal{H}_0 \equiv \mathcal{H} = L_2(\Omega)$ with respect to $\| \cdot \|_s \equiv \|(I + A)^{s/2} \cdot \|$ when $-2 \le s < 0$. By interpolation (see [17]) we have that

$$c_s^{-1} \|v\|_s \le \|v\|_{H^s(\Omega)} \le c_s \|v\|_s, \quad v \in \mathcal{H}_s, \quad s \in [-2, 2], \ s \ne \pm\tfrac{1}{2}, \tfrac{3}{2}, \tag{1.1}$$

for some constant $c_s > 0$. Below we denote by $(\cdot, \cdot)_s$ the inner product in $\mathcal{H}_s$. In the case $s = 0$ we also use the notations $(\cdot, \cdot) = (\cdot, \cdot)_0$ and $\| \cdot \| = \| \cdot \|_0$. We also mention that we deal with both real and complex versions of the spaces $\mathcal{H}_s$. We keep the notation $\mathcal{H}_s$ for the real case and denote by $\overline{\mathcal{H}}_s$ its complexification. For norms and inner products we use the same notation.

For the boundary conditions described above (either (0.3), or (0.4), or (0.5)) problem (0.2) can be written in the unified form:

$$n_{tt} + A\left(n + |E|^2\right) + \alpha n_t + \beta n = f(x), \tag{1.2}$$

and

$$iE_t - AE - nE + i\gamma E = g(x). \tag{1.3}$$

We equip equations (1.2) and (1.3) with initial data

$$n_t|_{t=0} = n_0, \ n|_{t=0} = n_1, \ E|_{t=0} = E_0, \tag{1.4}$$

and assume that α, β, and γ are nonnegative constants, and the real $f(x)$ and the complex $g(x)$ functions are given and belong to $L_2(\Omega)$.

We understand solutions to problem (1.2)–(1.4) in the sense of the following definition.

Definition 1. A pair $(n; E)$ is said to be a *semi-strong* solution to problem (1.2)–(1.4) on an interval $[0, T]$ if and only if

$$(n_t; n; E) \in L^\infty([0, T]; \mathcal{E}), \quad \mathcal{E} \equiv \mathcal{H} \times \mathcal{H}_1 \times \overline{\mathcal{H}}_2, \tag{1.5}$$

and (i) relations (1.2) and (1.3) are satisfied in the sense of distributions, (ii) initial data (1.4) hold.

We call this solution "semi-strong" because it is weak with respect to the variable n and strong with respect to E.

We also note that by (1.2) and (1.3) and (1.5) for any semi-strong solution $(n; E)$ we have that

$$n_{tt} \in L^\infty([0, T]; \mathcal{H}_{-1}) \quad \text{and} \quad E_t \in L^\infty\left([0, T]; \overline{\mathcal{H}}\right). \tag{1.6}$$

Therefore (after changing on a set of zero measure in $[0, T]$) the triple $(n_t; n; E)$ is a weakly continuous function with values in $\mathcal{H} \times \mathcal{H}_1 \times \overline{\mathcal{H}}_2$ (see [17, Lemma 8.1]). In particular, this means that initial data (1.4) have a sense. Moreover, (1.5) and (1.6), Aubin's embedding theorem (see [22, Corollary 4]) and also (1.3) imply that

$$(n_t; n; E_t; E) \in C\left([0, T]; \mathcal{H}_{-\sigma} \times \mathcal{H}_{1-\sigma} \times \overline{\mathcal{H}}_{-\sigma} \times \overline{\mathcal{H}}_{2-\sigma}\right) \tag{1.7}$$

for every $\sigma > 0$.

Our first result is the following uniqueness theorem for semi-strong solutions to problem (1.2)–(1.4). It also states that relation (1.7) holds for $\sigma = 0$ and gives an estimate for the L_2 norm of $E(x, t)$.

Theorem 1. *Problem* (1.2)–(1.4) *on every interval* $[0, T]$ *may have at most one semi-strong solution.*

Assume that $(n_1; n_0; E_0) \in \mathcal{E}$, $f \in \mathcal{H}$, *and* $g \in \overline{\mathcal{H}}$. *Then any semi-strong solution* $(n; E)$ *on an interval* $[0, T]$ *to problem* (1.2)–(1.4) (*if it exists*) *possesses the property*

$$(n_t; n; E_t; E) \in C\left([0, T]; \mathcal{H} \times \mathcal{H}_1 \times \overline{\mathcal{H}} \times \overline{\mathcal{H}}_2\right) \tag{1.8}$$

and satisfies for any $t \in [0, T]$ *the relations*

$$\|E(t)\| \le \|E_0\| e^{-\gamma t} + \gamma^{-1} \|g\| \left(1 - e^{-\gamma t}\right) \tag{1.9}$$

in the case $\gamma > 0$ *and*

$$\|E(t)\| \le \|E_0\| + t\|g\|, \tag{1.10}$$

if $\gamma \geq 0$.

To prove this theorem in Section 2 we use the method developed by Sedenko [20] in the theory of elastic shells (see also [7]).

For further properties of semi-strong solutions we also refer to Proposition 1 and Proposition 2 in Section 2.

To formulate the existence result we need the following well-known Sobolev-type estimate.

Lemma 1. *There exist positive constants* $c_0(\Omega)$ *and* $c_1(\Omega)$ *such that*

$$\|u\|^4_{L_4(\Omega)} \leq c_0(\Omega)\|u\|^2 \cdot \|\nabla u\|^2 + c_1(\Omega)\|u\|^4 \tag{1.11}$$

for any $u \in \mathcal{H}_1$. *If* $\mathcal{H}_1 = H^1_0(\Omega)$ (*the Dirichlet case*), *then* $c_0(\Omega) = 2$ *and* $c_1(\Omega) = 0$.

For the sake of completeness we give the proof of this lemma in the Appendix.

Remark 1. As we will see in Theorem 2 the constant $c_0(\Omega)$ determines the size of a set of initial data for which we can prove well posedness. If $\Omega = (0, l_1) \times (0, l_2)$ is a rectangle, then $c_0(\Omega)$ is any number greater than 2. In general $c_0(\Omega)$ may depend on Ω. We refer to the Appendix for details.

Our next result is the following assertion on well posedness of problem (1.2)–(1.4).

Theorem 2. *Let* $(n_1, n_0, E_0) \in \mathcal{E} \equiv \mathcal{H} \times \mathcal{H}_1 \times \overline{\mathcal{H}}_2$ *and* $(f,\, g) \in \mathcal{H} \times L_q(\Omega)$ *for some* $q > 2$. *Let* $r_\Omega = \sqrt{2/c_0(\Omega)}$, *where* $c_0(\Omega)$ *is the constant from* (1.11). *Then problem* (1.2)–(1.4) *has a unique semi-strong solution on an interval* $[0, T]$ *provided that*

- *either* $\gamma \geq 0$ *and* $\|E_0\| + T \cdot \|g\| < r_\Omega$,
- *or else* $\gamma > 0$, *and* $\max\{\|E_0\|, \gamma^{-1}\|g\|\} < r_\Omega$ *and* $T > 0$ *is arbitrary.*

This solution possesses property (1.8) *and depends continuously on initial data; i.e.,*

$$\lim_{k\to\infty} \max_{[0,T]} \left(\|n^k_t(t) - n_t(t)\|^2 + \|n^k(t) - n(t)\|^2_1 + \|E^k(t) - E(t)\|^2_2 \right) = 0, \tag{1.12}$$

where $(n; E)$ *is the solution with initial data* $U \equiv (n_1; n_0; E_0)$ *and* $(n^k; E^k)$ *is the solution with initial data* $U^k \equiv (n^k_1; n^k_0; E^k_0)$ *such that* $U^k \to U$ *in the space* $\mathcal{E} = \mathcal{H}_0 \times \mathcal{H}_1 \times \overline{\mathcal{H}}_2$ *as* $k \to \infty$.

The following assertion is an immediate consequence of Theorem 1 and Theorem 2.

Corollary 1. *Assume that either* $g \equiv 0$ *or* $\gamma > 0$ *and* $\gamma^{-1}\|g\| < r_\Omega$. *Let*

$$\mathcal{B}_r = \{U \equiv (n_1; n_0; E_0) \in \mathcal{E} \;:\; \|E_0\| < r\} \subset \mathcal{E}, \tag{1.13}$$

where $\gamma^{-1}\|g\| \le r < r_\Omega$. *Then problem* (1.2)–(1.4) *is globally well posed for any initial data* $U = (n_1; n_0; E_0)$ *from* $\mathcal{B}_r$. *The corresponding solution* $(n; E)$ *possesses the property*

$$(n_t(t); n(t); E(t)) \in \mathcal{B}_r \quad \text{for any} \quad t \ge 0. \tag{1.14}$$

Moreover, the mapping $S_t : \mathcal{B}_r \mapsto \mathcal{B}_r$ *given by the formula*

$$S_t U = (n_t(t); n(t); E(t)), \quad t \ge 0,\; U = (n_1; n_0; E_0) \in \mathcal{B}_r,$$

is a strongly continuous semigroup of continuous mappings in the topology induced by $\mathcal{E}$ *on* $\mathcal{B}_r$.

Proof. The global existence of semi-strong solutions for initial data from $\mathcal{B}_r$ follows from Theorem 2. Invariance property (1.14) follows from (1.10) in the case $g \equiv 0$ and from (1.9) in the case $\gamma > 0$. Thus the mapping S_t is well defined on $\mathcal{B}_r$. The semigroup property follows from the uniqueness statement of Theorem 1. □

Our third result deals with a global attractor of the dynamical system $(\mathcal{B}_r, S_t)$ generated by problem (0.2) with the Dirichlet boundary condition (0.3) on the set $\mathcal{B}_r$ given by (1.13). We recall that $\mathcal{A}$ is a global attractor for a dynamical system (X, S_t) on a complete metric space X if $\mathcal{A}$ is strictly invariant and uniformly attracts every bounded set from X. We refer to monographs [2, 8, 23] for basic facts concerning attractors.

Theorem 3. *Assume that* $\alpha > 0$, $\beta = 0$ *and* $\gamma \ge \gamma_0 > 0$. *Let* $(f, g) \in \mathcal{H} \times L_q(\Omega)$, $q > 2$, *and* $\lambda_1 > 0$ *be the smallest eigenvalue of the operator* $-\Delta$ *with the Dirichlet boundary condition. Let* $\delta = \frac{1}{4}\min\{\lambda_1, 8\gamma_0, 2\}$. *Then there exist positive constants* b_1 *and* b_2 *depending on* r *only and* d_Ω *depending on* Ω *only such that under the conditions*

$$\alpha^2 \ge \max\{1\,,\,\delta\}\,, \quad \|g\|^2 \le b_1 \delta \alpha^{-1} \gamma \tag{1.15}$$

and

$$d_\Omega^2 \cdot \left[\left(1 + \frac{\gamma\alpha}{\delta}\right)\|g\| + \frac{\|f\|^2}{2\lambda_1\delta}\left(1 + \frac{1}{2\lambda_1}\right)\right] \le b_2 \alpha\gamma, \tag{1.16}$$

the dynamical system $(\mathcal{B}_r, S_t)$ *generated by problem* (0.2) *and* (0.3) *possesses a compact global attractor. If* $\Omega = (0, l_1) \times (0, l_2)$ *is a rectangle, then* $d_\Omega < 3$.

To prove this theorem we use the idea due to Ball [3] in the form presented in [18]. We also note that a result similar to Theorem 3 remains true in the case $\beta > 0$ for *all* boundary conditions mentioned above (either (0.3), (0.4), or (0.5)). However, we choose for presentation the Dirichlet case with $\beta = 0$ only for some (technical) simplifications of the proof and the corresponding analogs of conditions (1.15) and (1.16).

The following assertion is an application of Theorem 3 to thin domains.

Corollary 2. *Assume that*

$$\Omega = \{(x_1, x_2) \,:\, 0 < x_2 < \varepsilon,\ 0 < x_1 < 1\}.$$

Let $f(x)$ and $g(x)$ be bounded functions. Then there exists $\varepsilon_0 > 0$ such that under the condition $\alpha \geq 1$ the dynamical system $(\mathcal{B}_r, S_t)$ generated by problem (0.2) *and* (0.3) *possesses a compact global attractor for every $0 < \varepsilon \leq \varepsilon_0$.*

Proof. In the case considered we have that $\lambda_1 = \pi^2(1 + \varepsilon^{-2})$, $d_\Omega < 3$ and

$$\|g\|^2 \leq C_1 \mathrm{Vol}\,(\Omega) = C_1\varepsilon, \quad \|f\|^2 \leq C_2 \mathrm{Vol}\,(\Omega) = C_2\varepsilon.$$

Thus we can choose ε_0 such that (1.15) and (1.16) hold for every $0 < \varepsilon \leq \varepsilon_0$. □

We note that the study of the dynamics of nonlinear evolutionary PDEs on thin domains was started in [14, 15] and continued by many authors (see, e.g., [1, 19, 24] and the references therein). The dynamics of the Zakharov system on a thin domain was not considered before.

2. Proof of Theorem 1

We split the proof into several steps.

2.1. **Uniqueness.** We use the method suggested by Sedenko [20] in the theory of elastic shells (see also [7]). Bellow we need the following lemma.

Lemma 2. (*a*) *Let $u(x) \in \mathcal{H}_1$ and P_N be the orthoprojector on Span$\{e_k : k = 1, 2, \dots, N\}$. Then there exists $N_0 > 0$ such that*

$$\max_{x\in\Omega} |(P_N u)(x)| \leq C \left[\log(1 + \lambda_N)\right]^{1/2} \|u\|_{H^1} \tag{2.1}$$

for all $N > N_0$, where the constant C does not depend on N.

(*b*) *Let $u \in H^s(\Omega)$ and $v \in H^1(\Omega) \bigcap L_\infty(\Omega)$, where $0 < s < 1$. Then*

$$\|uv\|_{H^s} \leq C\|u\|_{H^s} \left(\|v\|_{L_\infty} + \|v\|_{H^1}\right). \tag{2.2}$$

Proof. To obtain (2.1) one can use the same argument as in [7, Lemma 2.2], where this inequality was proved for the spectral projector P_N related to the biharmonic operator.

As for relation (2.2), it follows from the standard description of fractional Sobolev spaces (see, e.g., [17]) and from the embedding $H^1(\Omega) \subset L_p(\Omega)$ for all $1 \le p < \infty$. □

Let $(n^{(i)}; E^{(i)})$, $i = 1, 2$, be two solutions to problem (1.2)–(1.4). We set $n = n^{(1)} - n^{(2)}$ and $E = E^{(1)} - E^{(2)}$ and denote $A_1 = I + A$. Then the pair $(n; E)$ solves the equations

$$n_{tt} + A_1 n = F(t) \quad \text{with} \quad F(t) \equiv \Delta \left(|E^{(1)}|^2 - |E^{(2)}|^2 \right) - \alpha n_t + (1 - \beta) n, \tag{2.3}$$

and

$$iE_t - A_1 E = G(t) \quad \text{with} \quad G(t) \equiv nE^{(1)} + n^{(2)} E - (1 + i\gamma) E \tag{2.4}$$

with zero initial data. It follows from (1.7) that the right-hand sides F and G in (2.3) and (2.4) possess the properties

$$F(t) \in C\left([0, T]; \mathcal{H}_{-\sigma}\right), \quad G(t) \in C\left([0, T]; \overline{\mathcal{H}}_{1-\sigma}\right) \cap C^1\left([0, T]; \overline{\mathcal{H}}_{-\sigma}\right) \tag{2.5}$$

for every $\sigma > 0$.

Let P_N be the same as in Lemma 2. We consider the function $n^N(t) = P_N n(t)$ as a solution to the linear finite-dimensional problem

$$n_{tt} + A_1 n = P_N F(t). \tag{2.6}$$

Multiplying (2.6) by $2A_1^{-2s} n_t^N$ in $L_2(\Omega)$ one can easily see that

$$\|n_t^N(t)\|_{-s}^2 + \|n^N(t)\|_{1-s}^2 \le 2 \int_0^t \|P_N F(\tau)\|_{-s} \cdot \|n_t^N(\tau)\|_{-s} d\tau.$$

Since $P_N \to I$ strongly in every space $\mathcal{H}_\sigma$, this implies that

$$\|n_t(t)\|_{-s}^2 + \|n(t)\|_{1-s}^2 \le 2 \int_0^t \|F(\tau)\|_{-s} \cdot \|n_t(\tau)\|_{-s} d\tau. \tag{2.7}$$

for any $0 < s < 1$. Similarly, the function $E^N(t) = P_N E(t)$ solves the linear finite-dimensional problem

$$iE_t - A_1 E = P_N G(t).$$

Therefore, we can find that

$$\frac{d}{dt} \|E^N\|_{1-s}^2 = 2\Im(iE_t^N - A_1 E^N, E^N)_{1-s} = 2\Im(P_N G(t), E^N)_{1-s}$$

and

$$\frac{d}{dt}\|E_t^N\|_{-s}^2 = 2\Im(iE_{tt}^N - A_1E_t^N, E_t^N)_{-s} = 2\Im(P_N G_t(t), E_t^N)_{-s}$$

for any $0 < s < 1$. After integration with respect to t and a limit transition $N \to \infty$ we obtain that

$$\begin{aligned}&\|E_t(t)\|_{-s}^2 + \|E(t)\|_{1-s}^2 \\ &\le 2\int_0^t (\|G_t(\tau)\|_{-s}\|E_t(\tau)\|_{-s} + \|G(\tau)\|_{1-s}\|E(\tau)\|_{1-s})\, d\tau.\end{aligned}$$

Therefore, using (2.7) we obtain

$$\Psi(t) \le 2\int_0^t \Phi(\tau)\cdot\sqrt{\Psi(\tau)}d\tau. \tag{2.8}$$

Here we denote

$$\Psi(t) = \|n_t(t)\|_{-s}^2 + \|n(t)\|_{1-s}^2 + \|E_t(t)\|_{-s}^2 + \|E(t)\|_{1-s}^2 \tag{2.9}$$

and

$$\Phi(t) = \|F(t)\|_{-s} + \|G(t)\|_{1-s} + \|G_t(t)\|_{-s}, \tag{2.10}$$

where $F(t)$ and $G(t)$ are defined in (2.3) and (2.4).

Now using the relation

$$\operatorname*{ess\,sup}_{t\in[0,T]}\left\{\|n_t^{(i)}(t)\|_0^2 + \|n^{(i)}(t)\|_1^2 + \|E_t^{(i)}(t)\|_0^2 + \|E^{(i)}(t)\|_2^2\right\} < C, \quad i = 1,2, \tag{2.11}$$

which follows from (1.5), we estimate the function $\Phi(t)$.

Estimate for $\|F(t)\|_{-s}$. Taking into account that $H^{2-s}(\Omega)$ is an algebra for the two-dimensional case and using (2.11) we obtain that

$$\begin{aligned}\|F(t)\|_{-s} &\le C\left(\left\||E^{(1)}|^2 - |E^{(2)}|^2\right\|_{2-s} + \|n_t\|_{-s} + \|n\|_{-s}\right) \\ &\le C\left(\|E\|_{2-s} + \|n_t\|_{-s} + \|n\|_{-s}\right).\end{aligned}$$

From (2.4) we also have that

$$\|E\|_{2-s} \le \|E_t\|_{-s} + \|nE^{(1)}\|_{-s} + \|n^{(2)}E\|_{-s} + C\|E\|_{-s}.$$

Therefore,

$$\|F(t)\|_{-s} \le C\left(\|nE^{(1)}\|_{-s} + \|n^{(2)}E\|_{-s} + \sqrt{\Psi(t)}\right). \tag{2.12}$$

Using relations (1.1) and (2.11), embeddings

$$L_{2/(1+s)}(\Omega) \subset H^{-s}(\Omega) \quad \text{and} \quad H^{1-s}(\Omega) \subset L_{2/s}(\Omega), \quad 0 < s < 1, \tag{2.13}$$

and also Hölder's inequality for $p = \frac{s+1}{s}$ and $q = s+1$, we get

$$\begin{aligned} \|nE^{(1)}\|_{-s} + \|n^{(2)}E\|_{-s} &\le C\left\{\|nE^{(1)}\|_{L_{2/(1+s)}} + \|n^{(2)}E\|_{L_{2/(1+s)}}\right\} \\ &\le C\left\{\|n\|_{L_{2p/(1+s)}}\|E^{(1)}\|_{L_{2q/(1+s)}} + \|n^{(2)}\|_{L_{2q/(1+s)}}\|E\|_{L_{2p/(1+s)}}\right\} \\ &\le C\left\{\|n\|_{1-s} + \|E\|_{1-s}\right\}. \end{aligned} \tag{2.14}$$

Therefore, it follows from (2.12) that

$$\|F(t)\|_{-s} \le C \cdot \sqrt{\Psi(t)}, \quad t \in [0, T]. \tag{2.15}$$

Estimate for $\|G(t)\|_{1-s}$. We obviously have that

$$\|G(t)\|_{1-s} \le \|nE^{(1)}\|_{1-s} + \|n^{(2)}E\|_{1-s} + C\|E\|_{1-s}.$$

It follows from (2.2) and (2.11) that

$$\|nE^{(1)}\|_{1-s} \le C\|n\|_{1-s}\left(\|E^{(1)}\|_{L_\infty} + \|E^{(1)}\|_{H^1}\right) \le C\|n\|_{1-s}. \tag{2.16}$$

For the second term we have that

$$\|n^{(2)}E\|_{1-s} \le \|P_N(n^{(2)})E\|_{1-s} + \|Q_N(n^{(2)})E\|_{1-s}.$$

By Lemma 2 we obtain that

$$\begin{aligned} \|P_N(n^{(2)})E\|_{1-s} &\le C\|E\|_{1-s}\left(\max_{x\in\Omega}|P_N n^{(2)}| + \|P_N n^{(2)}\|_1\right) \\ &\le C\left[\log(1+\lambda_N)\right]^{1/2}\|E\|_{1-s} \end{aligned} \tag{2.17}$$

for $N \ge N_0$, where N_0 is large enough. Taking into account that

$$\|Q_N u\|_{s_1} \le \lambda_{N+1}^{-(s_2-s_1)/2}\|Q_N u\|_{s_2} \quad \text{for all} \quad u \in \mathcal{H}_{s_2},\ 0 < s_1 < s_2, \tag{2.18}$$

and using (2.2) and (2.11), we obtain that

$$\begin{aligned} \|Q_N(n^{(2)})E\|_{1-s} &\le C\|Q_N n^{(2)}\|_{1-s}\left(\|E\|_{L_\infty} + \|E\|_1\right) \\ &\le C\lambda_{N+1}^{-s/2}\|n^{(2)}\|_1\|E\|_2 \le C\lambda_{N+1}^{-s/2}. \end{aligned} \tag{2.19}$$

Thus, (2.16), (2.17), and (2.19) imply that

$$\|G(t)\|_{1-s} \le C_1\left[\log(1+\lambda_N)\right]^{1/2}\sqrt{\Psi(t)} + C_2\lambda_{N+1}^{-s/2}, \quad N \ge N_0. \tag{2.20}$$

Estimate for $\|G_t(t)\|_{-s}$. It is clear that

$$\|G_t(t)\|_{-s} \le \|n_t E^{(1)}\|_{-s} + \|n_t^{(2)}E\|_{-s} + \|nE_t^{(1)}\|_{-s} + \|n^{(2)}E_t\|_{-s} + C\|E_t\|_{-s}.$$

By the same argument as in (2.14) one can easily see that

$$\|n_t^{(2)}E\|_{-s} + \|nE_t^{(1)}\|_{-s} \le C\left\{\|E\|_{1-s} + \|n\|_{1-s}\right\}. \tag{2.21}$$

Taking into account (2.2) and (2.11) we obtain that

$$\begin{aligned}\|n_t E^{(1)}\|_{-s} &= \sup_{\phi\in\mathcal{H}_s,\|\phi\|_s=1}\left|(n_t E^{(1)},\phi)_0\right| \le \|n_t\|_{-s}\sup_{\|\phi\|_s=1}\left\|E^{(1)}\phi\right\|_s \\ &\le C\|n_t\|_{-s}\left(\|E^{(1)}\|_{L_\infty}+\|E^{(1)}\|_{H^1}\right)\le C\|n_t\|_{-s}. \qquad (2.22)\end{aligned}$$

We also have that

$$\|n^{(2)}E_t\|_{-s}\le\|P_N(n^{(2)})E_t\|_{-s}+\|Q_N(n^{(2)})E_t\|_{-s} \qquad (2.23)$$

and from Lemma 2 we obtain

$$\begin{aligned}\|P_N(n^{(2)})E_t\|_{-s} &\le \|E_t\|_{-s}\sup_{\|\phi\|_s=1}\|P_N(n^{(2)})\phi\|_s \\ &\le C\|E_t\|_{-s}\left(\max_{x\in\Omega}|P_N n^{(2)}|+\|P_N n^{(2)}\|_{H^1}\right) \\ &\le C\left[\log(1+\lambda_N)\right]^{1/2}\|E_t\|_{-s},\quad N\ge N_0. \qquad (2.24)\end{aligned}$$

Taking into account (2.18), (2.11), and (2.13), from Lemma 2 and Hölder's inequality we obtain

$$\|Q_N(n^{(2)})E_t\|_{-s}\le C\|E_t\|_0\|Q_N(n^{(2)})\|_{L_{2/s}}\le C\lambda_{N+1}^{-s/2}. \qquad (2.25)$$

Thus from estimates (2.21)–(2.25) we conclude that

$$\|G_t(t)\|_{-s}\le C_1\left[\log(1+\lambda_N)\right]^{1/2}\sqrt{\Psi(t)}+C_2\lambda_{N+1}^{-s/2},\quad N\ge N_0. \qquad (2.26)$$

Concluding step. By (2.10), (2.15), (2.20) and (2.26) we have that

$$\Phi(t)\le C_1\left[\log(1+\lambda_N)\right]^{1/2}\sqrt{\Psi(t)}+C_2\lambda_{N+1}^{-s/2},\quad N\ge N_0.$$

Substituting this relation into (2.8) and using Gronwall's lemma we therefore infer that

$$\Psi(t)\le C_2\lambda_{N+1}^{-s}\exp\left\{C_3\left[\log(1+\lambda_N)\right]^{1/2}t\right\},\quad t\in[0,T].$$

For every $\delta>0$ there exists $C_\delta>0$ such that

$$\left[\log(1+\lambda_N)\right]^{1/2}\le\delta\log(1+\lambda_N)+C_\delta.$$

Hence,

$$\Psi(t)\le C_2 e^{C_\delta T}\lambda_{N+1}^{-s}(1+\lambda_N)^{C_3T\delta},\quad t\in[0,T].$$

Now if we choose δ such that $C_3T\delta<s$ and send $N\to\infty$, we obtain that $\Psi(t)\equiv0$ for all $t\in[0,T]$. This implies the uniqueness of semi-strong solutions.

2.2. **Smoothness.** Now we prove (1.8). Let $(n(t); E(t))$ be a semi-strong solution to problem (1.2)–(1.4) on an interval $[0, T]$. In this case the function $n(t)$ is a weak (variational) solution to the linear problem

$$n_{tt} + An = F(t) \equiv \Delta|E(t)|^2 - \alpha n_t(t) - \beta n(t) + f(x). \tag{2.27}$$

It is clear that $F(t) \in L_\infty(0, T; \mathcal{H})$. Therefore, it follows from [17, Theorem 8.2] that

$$(n_t; n) \in C\left([0, T]; \mathcal{H} \times \mathcal{H}_1\right). \tag{2.28}$$

It follows from (2.28) and (1.7) that the nonlinear term $n(t)E(t)$ from (1.3) possesses the property

$$n(t)E(t) \in C^1(0, T; \overline{\mathcal{H}}_{-\sigma}), \quad \sigma > 0.$$

Therefore, the function $\widetilde{E} \equiv E_t$ is a solution (in the sense of distributions) to the problem

$$i\widetilde{E}_t + \Delta\widetilde{E} - n\widetilde{E} = G(t) \equiv -i\gamma E_t + n_t E, \quad \widetilde{E}|_{t=0} = E_1, \tag{2.29}$$

where $E_1 = -i\left(AE_0 + n_0E_0 - i\gamma E_0 + g(x)\right)$. We consider the following sesquilinear form

$$a(t, u, v) = \int_\Omega \nabla u(x)\nabla\overline{v(x)}\,dx + \int_\Omega n(x, t)u(x)\overline{v(x)}\,dx$$

on $\overline{\mathcal{H}}_1$. Since

$$\begin{aligned}\Big|\int_\Omega nu\overline{v}\,dx\Big| &\le \|n\| \cdot \|u\|_{L_4(\Omega)} \cdot \|v\|_{L_4(\Omega)} \\ &\le C\|n\| \cdot [\|u\| \cdot \|u\|_1 \cdot \|v\| \cdot \|v\|_1]^{1/2} \\ &\le \delta\|u\|_1 \cdot \|v\|_1 + C\delta^{-1}\|n\|^2 \cdot \|u\| \cdot \|v\|\end{aligned}$$

for every $\delta > 0$, the form $a(t, u, v)$ is bounded and coercive in the following sense: there exists positive ω_1 and ω_2 such that

$$a(t, u, u) + \omega_1\|u\|^2 \ge \omega_2\|u\|_1^2, \quad u \in \overline{\mathcal{H}}_1.$$

It is also continuously differentiable with respect to t. Therefore, since $E_1 \in \overline{\mathcal{H}}$ and $G(t) \in L_\infty(0, T; \overline{\mathcal{H}})$, by [17, Theorem 11.1] problem (2.29) is uniquely solvable in the sense of distributions.

Let $\{\widetilde{E}_1^N\} \subset \overline{\mathcal{H}}_1$ and $\{G^N(t)\} \subset C^1(0, T; \overline{\mathcal{H}})$ be sequences such that

$$\lim_{N\to\infty}\left\{\|\widetilde{E}_1^N - E_1\| + \int_0^T \|G^N(t) - G(t)\|^2 dt\right\} = 0.$$

Consider the problem

$$i\widetilde{E}^N_t + \Delta\widetilde{E}^N - n\widetilde{E}^N = G^N(t), \quad \widetilde{E}|_{t=0} = \widetilde{E}^N_1.$$

By [17, Theorem 10.1] this problem has a unique solution

$$\widetilde{E}^N(t) \in C(0,T;\overline{\mathcal{H}}_1) \cap C^1(0,T;\overline{\mathcal{H}}_{-1}).$$

It follows from the results given in [17, Chapter 3] that

$$\|\widetilde{E}^N(t)\|^2 = \|\widetilde{E}^N_1\|^2 + 2\Im\int_0^t (G^N(\tau), \widetilde{E}^N(\tau))d\tau, \tag{2.30}$$

and also

$$\|\widetilde{E}^{N_1}(t) - \widetilde{E}^{N_2}(t)\|^2 = \|\widetilde{E}^{N_1}_1 - \widetilde{E}^{N_2}_1\|^2 + 2\Im\int_0^t (G^{N_1} - G^{N_2}, \widetilde{E}^{N_1} - \widetilde{E}^{N_2})d\tau, \tag{2.31}$$

for any N, N_1 and N_2. By Gronwall's lemma, from (2.31) we have that

$$\max_{[0,T]} \|\widetilde{E}^{N_1}(t) - \widetilde{E}^{N_2}(t)\|^2 \to 0 \quad \text{as} \quad N_1, N_2 \to \infty.$$

Thus there exists a function $E^*(t) \in C(0,T;\overline{\mathcal{H}})$ such that

$$\lim_{N\to\infty} \max_{[0,T]} \|\widetilde{E}^N(t) - E^*(t)\|^2 = 0.$$

By the uniqueness of solutions to problem (2.29) with a fixed $G(t)$ we can conclude that $E^*(t) \equiv \widetilde{E}(t)$. Thus $E_t(t) \equiv \widetilde{E}(t) \in C(0,T;\overline{\mathcal{H}})$. Consequently, using equation (1.3) and also relation (2.28) we obtain (1.8).

For further use we note that after the limit transition in (2.30) we also obtain that

$$\|E_t(t)\|^2 = \|E_1\|^2 + 2\Im\int_0^t (-i\gamma E_t(\tau) + n_t(\tau)E(\tau), E_t(\tau))d\tau. \tag{2.32}$$

2.3. **Energy-type relations.** In this subsection we establish relations (1.9) and (1.10) and also prove some kind of energy relations for semi-strong solutions which we will use in the proofs of Theorem 2 and Theorem 3.

Proposition 1. *Let* $(n_1; n_0; E_0) \in \mathcal{E}$, $f \in \mathcal{H}$ *and* $g \in \overline{\mathcal{H}}$. *Then any semi-strong solution* $(n; E)$ *on an interval* $[0,T]$ *to problem* (1.2)–(1.4) *possesses the properties:*

- *Relations* (1.9) *and* (1.10) *hold.*

- *For any $t \in [0,T]$ we have that*

$$V_0(n_t(t),n(t)) + \|E_t(t)\|^2 + \int_0^t \left(\alpha\|n_t(\tau)\|^2 + 2\gamma\|E_t(\tau)\|^2\right) d\tau$$
$$= V_0(n_1,n_0) + \|E_1\|^2 + 2\int_0^t R(n_t(\tau),n(\tau),E(\tau))d\tau, \quad (2.33)$$

where $V_0(n_t,n)$ is given by

$$V_0(n_t,n) = \frac{1}{2}\left[\|n_t\|^2 + \|\nabla n\|^2 + \beta\|n\|^2 - 2(f,n)\right], \quad (2.34)$$

the element $E_1 \in \overline{\mathcal{H}}$ is defined by n_0 and E_0 from (1.3); *i.e.,*

$$E_1 = -i\left(AE_0 + n_0E_0 - i\gamma E_0 + g(x)\right), \quad (2.35)$$

and

$$R(n_t,n,E) = (n_t, n|E|^2 + |\nabla E|^2) + \Re(n_t, g\overline{E}). \quad (2.36)$$

Proof. Multiplying equation (1.3) by $E(t)$ in $\overline{\mathcal{H}}$ and using the smoothness properties of a semi-strong solution $(n;E)$ one can see that

$$\frac{d}{dt}\|E(t)\|^2 + 2\gamma\|E(t)\|^2 = 2\Im(g,E(t)) \le 2\|E(t)\|\|g\|.$$

This relation implies (1.9) and (1.10).

It follows from [17, Theorem 8.2] that $(n_t;n) \in C([0,T];\mathcal{H}\times\mathcal{H}_1)$ satisfies the energy relation:

$$\frac{1}{2}\left[\|n_t(t)\|^2 + \|\nabla n(t)\|^2\right] = \frac{1}{2}\left[\|n_1\|^2 + \|\nabla n_0\|^2\right] + \int_0^t (F(\tau),n_t(\tau))d\tau \quad (2.37)$$

for any $t \in [0,T]$, where $F(t)$ is the same as in (2.27). Therefore from the structure of $F(t)$ and relations (2.28) and (2.37) one can easily derive the equality

$$V_0(n_t(t),n(t)) + \alpha\int_0^t \|n_t(\tau)\|^2 d\tau = V_0(n_1,n_0) + \int_0^t (\Delta|E(\tau)|^2, n_t(\tau))d\tau, \quad (2.38)$$

where $V_0(n_t,n)$ is given by (2.34). Relation (2.32) implies

$$\|E_t(t)\|^2 + 2\gamma\int_0^t \|E_t(\tau)\|^2 d\tau = \|E_1\|^2 + 2\Im\int_0^t (n_t(\tau)E(\tau),E_t(\tau))d\tau, \quad (2.39)$$

where $E_1 \in \mathcal{H}$ is given by (2.35). Since

$$(\Delta|E|^2, n_t) = 2(n_t, |\nabla E|^2) + 2\Re(n_tE, \Delta E),$$

from (1.3) we can find that

$$\begin{aligned}2\Im(n_tE, E_t) + (\Delta|E|^2, n_t) &= 2(n_t, |\nabla E|^2) + 2\Re(n_tE, iE_t + \Delta E)\\ &= 2(n_t, |\nabla E|^2) + 2\Re(n_tE, nE + g) = 2R(n_t, n, E).\end{aligned}$$

Therefore (2.33) follows from (2.38) and (2.39). □

Below we also need the following assertion.

Proposition 2. *Let* $(n_1; n_0; E_0) \in \mathcal{E}$, $f \in \mathcal{H}$, *and* $g \in \overline{\mathcal{H}}$. *Assume that there exists a semi-strong solution* $(n; E)$ *on an interval* $[0, T]$ *to problem* (1.2)–(1.4). *Then for any* $t \in [0, T]$ *we have the relation*

$$\begin{aligned}&V_0^\delta(n_t(t), n(t)) + V_1(n(t), E(t)) + \int_0^t \left(\alpha\|A_\delta^{-1/2}n_t\|^2 + 2\gamma V_1(n, E)\right) d\tau\\ &= V_0^\delta(n_1, n_0) + V_1(n_0, E_0) + \int_0^t \left(-2\gamma\Re(g, E) + (f + \delta|E|^2, A_\delta^{-1}n_t)\right) d\tau,\end{aligned} \tag{2.40}$$

where $A_\delta = A + \delta I$ *with* $\delta > 0$ (*in the Dirichlet case we can also take* $\delta = 0$), $V_0^\delta(n_t, n)$ *is given by*

$$V_0^\delta(n_t, n) = \frac{1}{2}\left[\|A_\delta^{-1/2}n_t\|^2 + \|n\|^2 + (\beta - \delta)\|A_\delta^{-1/2}n\|^2\right], \tag{2.41}$$

and

$$V_1(n, E) = \|\nabla E\|^2 + (n, |E|^2) + 2\Re(g, E). \tag{2.42}$$

Proof. Since $n(t)$ is a weak (variational) solution to the linear problem

$$n_{tt} + \alpha n_t + A_\delta n + (\beta - \delta)n = F(t) \equiv -A_\delta(|E(t)|^2) + \delta|E(t)|^2 + f(x)$$

with $F(t) \in C(0, T; \mathcal{H})$, as in [17, Chapter 3] it is easy to find that

$$\begin{aligned}&V_0^\delta(n_t(t), n(t)) + \alpha\int_0^t \|A_\delta^{-1/2}n_t\|^2 d\tau\\ &\qquad = V_0^\delta(n_1, n_0) + \int_0^t \left[-(n_t, |E|^2) + (f + \delta|E|^2, A_\delta^{-1}n_t)\right] d\tau.\end{aligned} \tag{2.43}$$

One can easily show that

$$V_1(n(t), E(t)) = V_1(n_0, E_0) + \int_0^t \left((n_t, |E|^2) - 2\gamma\Im(E, E_t)\right) d\tau. \tag{2.44}$$

Formally, this relation can be obtained by multiplication of (1.3) by E_t in $\overline{\mathcal{H}}$ and integration of the real part over the interval $[0, t]$. Using the smoothness property (1.8) and the method described in [17, Chapter 3] it is easy to justify this procedure. In a similar way, multiplication of (1.3) by E in $\overline{\mathcal{H}}$ leads to the relation

$$\int_0^t V_1(n,E)d\tau = \int_0^t \left(-\Im(E_t,E) + \Re(g,E)\right) d\tau. \tag{2.45}$$

Therefore, (2.44) and (2.45) imply that

$$\begin{aligned} &V_1(n(t),E(t)) + 2\gamma\int_0^t V_1(n,E)d\tau \\ &= V_1(n_0,E_0) + \int_0^t \big((n_t,|E|^2) - 2\gamma\Re(g,E)\big)\, d\tau. \end{aligned} \tag{2.46}$$

Taking the sum of (2.43) and (2.46), we get (2.40). □

3. Existence theorem

In this section we prove Theorem 2 by the compactness method.

We write equations (1.2) and (1.3) in the form

$$n_{tt} + A_1 n + A_1\left(|E|^2\right) = -\alpha n_t + (1-\beta)n + |E|^2 + f(x), \tag{3.1}$$

and

$$iE_t - AE - nE + i\gamma E = g(x), \tag{3.2}$$

where, as above, $A = -\Delta$ with the boundary conditions considered and $A_1 = I + A$. We equip equations (3.1) and (3.2) with initial data (1.4).

We start with the Galerkin approximations of the Zakharov problem:

$$\begin{cases} n^N_{tt} + A_1\left(n^N + P_N|E^N|^2\right) = -\alpha n^N_t + (1-\beta)n^N + P_N|E|^2 + P_N f, \\ iE^N_t - AE^N - P_N(n^N E^N) + i\gamma E^N = P_N g. \end{cases} \tag{3.3}$$

Here P_N is the orthoprojector on $\operatorname{span}\{e_k : k=1,2,\dots,N\}$, where $\{e_k\}$ is the sequence of eigenfunctions of A. To simplify notations we omit below the index N.

Let us multiply the second equation from (3.3) by $2\overline{E}$ and integrate the imaginary part of the result over Ω. Then we obtain that

$$\frac{d}{dt}\|E\|^2 + 2\gamma\|E\|^2 = 2\Im(g,E) \le 2\|E\|\|g\|.$$

From this relation we conclude that relations (1.9) and (1.10) hold for this approximate solution. Thus we have that

$$\|E(t)\| \le \kappa(E_0,g,T) \equiv \kappa, \quad t\in[0,T], \tag{3.4}$$

where either $\kappa = \|E_0\| + T\|g\|$ or else

$$\kappa = \begin{cases} \|E_0\| + T\|g\|, & \text{if } \gamma = 0; \\ \max\{\|E_0\|, \gamma^{-1}\|g\|\}, & \text{if } \gamma > 0. \end{cases} \tag{3.5}$$

Now we take the first equation from (3.3) and multiply it by $2A_1^{-1}n_t$ in $\mathcal{H}$. We obtain that

$$\frac{d}{dt}\left\{\|n_t\|_{-1}^2 + \|n\|^2 - 2(f,n)_{-1}\right\} + 2(n_t, |E|^2) = G_1(t), \tag{3.6}$$

where

$$G_1(t) = -2\alpha\|n_t\|_{-1}^2 + 2(1-\beta)(n_t, n)_{-1} + 2(n_t, |E|^2)_{-1}. \tag{3.7}$$

Then we multiply the second equation from (3.3) by $4\overline{E}_t + 4\gamma\overline{E}$ and integrate the real part of the result over Ω; as in the proof of Proposition 2 one can see that

$$\frac{d}{dt}\left\{2\|\nabla E\|^2 + 4\Re(g,E) + 2(n,|E|^2)\right\} + 4\gamma\|\nabla E\|^2 - 2(n_t, |E|^2) = G_2(t), \tag{3.8}$$

where

$$G_2(t) = -4\gamma\left[(n,|E|^2) + \Re(g,E)\right]. \tag{3.9}$$

Taking the sum of (3.6) and (3.8), we get

$$\frac{d}{dt}V(n(t), n_t(t), E(t)) + 4\gamma\|\nabla E\|^2 = G_1(t) + G_2(t), \tag{3.10}$$

where

$$V(n, n_t, E) = \|n_t\|_{-1}^2 + \|n\|^2 - 2(f,n)_{-1} + 2\|\nabla E\|^2 + 4\Re(g,E) + 2(n,|E|^2). \tag{3.11}$$

It follows from (3.10) that the functional $V(t) \equiv V(n(t), n_t(t), E(t))$ satisfies the inequality

$$V(t) \le V_0 + \int_0^t (G_1(\tau) + G_2(\tau))\, d\tau, \tag{3.12}$$

where V_0 does not depend on N and G_1 and G_2 are given by (3.7) and (3.9). Below we use this inequality to obtain the second a priori estimate.

Let us estimate the term $2(n,|E|^2)$ in (3.11). Using (1.11) and (3.4) we find that

$$\begin{aligned} 2(n,|E|^2) &\le 2\|n\| \cdot \|E\|_{L_4(\Omega)}^2 \\ &\le 2\kappa\sqrt{c_0(\Omega)}\|n\| \cdot \|\nabla E\| + \eta\|n\|^2 + C_\eta \end{aligned} \tag{3.13}$$

for every $\eta > 0$. We also obviously have that

$$(f, n)_{-1} \le \eta\|n\|^2 + C_\eta(f) \quad \text{and} \quad (g, E) \le \eta\|\nabla E\|^2 + C_{\eta,\kappa}(g)$$

for every $\eta > 0$. Consequently, using (3.11) and (3.13) we have that

$$\begin{aligned} V(n, n_t, E) \quad \ge \quad & \|n_t\|_{-1}^2 + (1 - 2\eta)\|n\|^2 + (2 - \eta)\|\nabla E\|^2 \\ & -2\kappa\sqrt{c_0(\Omega)}\|n\| \cdot \|\nabla E\| - C_{\eta,\kappa} \end{aligned}$$

for every $\eta > 0$. Thus under the condition $\kappa^2 c_0(\Omega) < 2$ we can choose $\eta > 0$ and $\delta > 0$ such that

$$V(n, n_t, E) \ge \delta\left(\|n_t\|_{-1}^2 + \|n\|^2 + \|\nabla E\|^2\right) - C_\kappa(f, g).$$

In a similar way it is easy to find that

$$G_i(t) \le C_1\left(\|n_t\|_{-1}^2 + \|n\|^2 + \|\nabla E\|^2\right) + C_{2,\kappa}(f, g), \quad i = 1, 2.$$

Therefore, using Gronwall's lemma we obtain from (3.12) the second a priori estimate:

$$\|n_t(t)\|_{-1}^2 + \|n(t)\|^2 + \|\nabla E(t)\|^2 \le C_\kappa(T, f, g), \quad t \in [0, T], \tag{3.14}$$

provided $\kappa < r_\Omega = \sqrt{2/c_0(\Omega)}$. From (1.11) we also have that

$$\|E(t)\|_{L_4(\Omega)}^2 \le C_\kappa(T, f, g), \quad t \in [0, T]. \tag{3.15}$$

Now we are in position to obtain the main a priori estimate.

Let us take now the second equation from (3.3), multiply it by $4\Delta(\overline{E}_t + \gamma\overline{E})$, and integrate the real part of the result over Ω. Then we obtain that

$$\frac{d}{dt}\left(2\|\Delta E\|^2 - 4\Re(g, \Delta E)\right) + 4\gamma\|\Delta E\|^2 + 4\Re(nE, -\Delta E_t) = R_1(t), \tag{3.16}$$

where

$$R_1(t) = 4\gamma\Re(g + nE, \Delta E).$$

Then we multiply the first equation of (3.3) by $2n_t$ and integrate the result over Ω

$$\frac{d}{dt}\left(\|n_t\|^2 + \|\nabla n\|^2 - 2(f, n)\right) + 2(n_t, -\Delta|E|^2) = 2R_2(t), \tag{3.17}$$

where

$$R_2(t) = -\alpha\|n_t\|^2 - \beta(n, n_t).$$

After straightforward computations we can see that

$$\begin{aligned} & 2\Re(nE, -\Delta E_t) + (n_t, -\Delta|E|^2) \\ & \qquad = \frac{d}{dt}\left\{2\Re(nE, -\Delta E) + \|P_N(nE)\|^2\right\} - R_3(t), \end{aligned} \tag{3.18}$$

where

$R_3(t) = 2\Re(P_N(nE), n_tE) + 2(n_t, |\nabla E|^2) + 2\Im(n[g - i\gamma E], -\Delta E + P_N(nE))$.

Let

$$H_0(t) = \|n_t\|^2 + \|\nabla n\|^2 + 2\|\Delta E\|^2 + 2\|P_N(nE)\|^2$$

and

$$H_1(t) = -2(f, n) - 4\Re(g, \Delta E) + 4\Re(nE, -\Delta E).$$

Taking the sum of (3.16) and (3.17) and substituting (3.18) we get

$$\frac{d}{dt}(H_0(t) + H_1(t)) + 4\gamma\|\Delta E\|^2 = R_1(t) + 2R_2(t) + 2R_3(t).$$

By (1.11) and (3.14)

$$\|n\|_{L_4} \le C_1\|\nabla n\|^{1/2} + C_2,$$

where C_i depends on κ and T. Therefore by (3.15) we have that

$$|(nE, -\Delta E)| \le \|n\|_{L_4}\|E\|_{L_4}\|\Delta E\| \le \eta\left(\|\Delta E\|^2 + \|\nabla n\|^2\right) + C_\eta$$

for every $\eta > 0$. Taking into account that

$$|(f, n)| \le \eta\|n\|^2 + C_\eta\|f\|^2 \quad \text{and} \quad |(g, \Delta E)| \le \eta\|\Delta E\|^2 + C_\eta\|g\|^2,$$

after an appropriate choice of η we obtain that

$$|H_1(t)| \le \frac{1}{2}H_0(t) + C, \quad t \in [0, T]. \tag{3.19}$$

Similarly, we have that

$$|R_i(t)| \le C_1H_0(t) + C_2, \quad t \in [0, T],\ i = 1, 2. \tag{3.20}$$

Now we estimate $R_3(t)$. Since $H^s(\Omega) \subset L_p(\Omega)$ for $s = 1 - 2/p$, $2 \le p < \infty$, by interpolation we have that

$$\|u\|_{L_p} \le C\|u\|_{H^{1-2/p}} \le C\|u\|^{2/p}\|u\|_1^{1-2/p}.$$

One can also easily see that

$$\|vw\|_{H^s} \le C\|v\|_1\|w\|_1 \quad \text{for} \quad 0 < s < 1.$$

Therefore, from (3.14) and (3.15) we obtain that, for some $s \in (1/2, 1)$,

$$\begin{aligned} |(P_N(nE), n_tE)| &\le \|n_t\|\|E\|_{L_4}\|P_N(nE)\|_{L_4} \le C\|n_t\|\|P_N(nE)\|_{H^{1/2}} \\ &\le C\|n_t\|\|nE\|_{H^s} \le C\|n_t\|\|n\|_1\|E\|_1 \\ &\le C_1(\|n_t\|^2 + \|\nabla n\|^2) + C_2 \le C_1H_0(t) + C_2. \end{aligned} \tag{3.21}$$

Similarly, since $g \in L_q(\Omega)$ for some $q > 2$, we obtain that

$$|(n[g - i\gamma E], -\Delta E + P_N(nE))| \le C_1(\|n_t\|^2 + \|\nabla n\|^2 + \|\Delta E\|^2) + C_2$$

$$\leq \quad C_1 H_0(t) + C_2. \tag{3.22}$$

We also have that

$$\begin{aligned} |(n_t, |\nabla E|^2)| &\leq \|n_t\| \|\nabla E\|_{L_4}^2 \leq C\|n_t\| \|\nabla E\| \|E\|_2 \leq C\|n_t\| \|E\|_2 \\ &\leq C_1(\|n_t\|^2 + \|\Delta E\|^2) + C_2 \leq C_1 H_0(t) + C_2. \end{aligned} \tag{3.23}$$

Therefore, (3.21), (3.22), and (3.23) imply that

$$|R_3(t)| \leq C_1 H_0(t) + C_2, \quad t \in [0, T]. \tag{3.24}$$

Consequently, using (3.19), (3.20), and (3.24) we obtain that

$$H_0(t) \leq C_1 \int_0^t H_0(\tau) d\tau + C_2, \quad t \in [0, T].$$

Therefore, Gronwall's lemma yields that

$$\|n_t\|^2 + \|\nabla n\|^2 + 2\|\Delta E\|^2 \leq C, \quad t \in [0, T].$$

Taking into account this estimate and also (3.4) and (3.14) we can pass to the limit as $N \to \infty$ and obtain the existence of a semi-strong solution on the interval $[0, T]$.

Now we prove (1.12), i.e. continuous dependence of semi-strong solutions on initial data. We will use the standard approach (see, e.g., [17, Chapter 3] for linear equations) based on an energy relation (this is (2.33) in our case).

Let $(n; E)$ be a semi-strong solution with initial data $U \equiv (n_1; n_0; E_0)$ and $(n^k; E^k)$ be a solution with initial data $U^k \equiv (n_1^k; n_0^k; E_0^k)$ such that $U^k \to U$ in the space $\mathcal{E} = \mathcal{H}_0 \times \mathcal{H}_1 \times \overline{\mathcal{H}}_2$ as $k \to \infty$.

Relying on Propositions 1 and 2 and using the same arguments as in the proof of the existence theorem we can conclude that

$$\sup_{[0,T]} \left(\|n_t^k(t)\|^2 + \|n^k(t)\|_1^2 + \|E_t^k(t)\| + \|E^k(t)\|_2^2 \right) \leq C, \quad k = 1, 2, \ldots. \tag{3.25}$$

Therefore, using the uniqueness theorem one can find that

$$(n_t^k; n^k; E^k) \to (n_t; n; E) \quad \text{weakly}^* \text{ in } \quad L_\infty(0, T, \mathcal{E}).$$

By Aubin's embedding theorem (see [22, Corollary 4]) we also have that

$$\lim_{k\to\infty} \max_{[0,T]} \left(\|n_t^k(t) - n_t(t)\|_{-\sigma}^2 + \|n^k(t) - n(t)\|_{1-\sigma}^2 \right) = 0, \tag{3.26}$$

and

$$\lim_{k\to\infty} \max_{[0,T]} \left(\|E_t^k(t) - E_t(t)\|_{-\sigma}^2 + \|E^k(t) - E(t)\|_{2-\sigma}^2 \right) = 0, \tag{3.27}$$

for every $\sigma > 0$. In particular, this implies that

$$(n_t^k(t); n^k(t); E_t^k(t); E^k(t)) \to (n_t(t); n(t); E_t(t); E(t)) \quad \text{for each} \quad t \in [0, T] \tag{3.28}$$

weakly in $\mathcal{H} \times \mathcal{H}_1 \times \overline{\mathcal{H}} \times \overline{\mathcal{H}}_2$.

Lemma 3. *Let*

$$F(t) = F(n_t(t), n(t), E_t(t)), \quad F^k(t) = F(n_t^k(t), n^k(t), E_t^k(t)),$$

where

$$F(n_t, n, E_t) = \frac{1}{2}\left[\|n_t\|^2 + \|\nabla n\|^2 + \beta\|n\|^2\right] + \|E_t\|^2.$$

Then

$$\lim_{k\to\infty} \max_{[0,T]} \left| F^k(t) - F(t) \right| = 0. \tag{3.29}$$

Proof. It follows from Proposition 1 that

$$\begin{aligned} F^k(t) &+ \int_0^t g^k(\tau)d\tau - (f, n^k(t)) \\ &= F^k(0) - (f, n_0^k) + 2\int_0^t R(n_t^k(\tau), n^k(\tau), E^k(\tau))d\tau, \end{aligned}$$

where $R(n_t, n, E)$ is given by (2.36). Here and below we use the notation

$$g^k(t) = g(n_t^k(t), E_t^k(t)), \quad g(t) = g(n_t(t), E_t(t)),$$

where $g(n_t, E_t) = \alpha\|n_t\|^2 + 2\gamma\|E_t\|^2$. Using relations (3.26) and (3.27) one can see that

$$\lim_{k\to\infty} \max_{[0,T]} \left| F^k(t) + \int_0^t g^k(\tau)d\tau - F(t) - \int_0^t g(\tau)d\tau \right| = 0. \tag{3.30}$$

Since, by the property of weak convergence,

$$\int_0^t g(\tau)d\tau \le \liminf_{k\to\infty} \int_0^t g^k(\tau)d\tau,$$

from (3.30) we have that

$$\begin{aligned} \liminf_{k\to\infty} F^k(t) &+ \int_0^t g(\tau)d\tau \\ &\le \lim_{k\to\infty} \left[F^k(t) + \int_0^t g^k(\tau)d\tau \right] = F(t) + \int_0^t g(\tau)d\tau \end{aligned}$$

for each $t \in [0,T]$. Thus $\liminf_{k\to\infty} F^k(t) \le F(t)$ for every $t \in [0,T]$. However the weak convergence (3.28) implies that $F(t) \le \liminf_{k\to\infty} F^k(t)$. Therefore

$$F(t) = \lim_{k\to\infty} F^k(t), \quad t \in [0,T].$$

Consequently, by (3.30) we have that

$$\lim_{k\to\infty} \left| \int_0^t g^k(\tau)d\tau - \int_0^t g(\tau)d\tau \right| = 0 \text{ for each } t \in [0,T]. \tag{3.31}$$

Using (3.25) and the Ascoli theorem we conclude that the convergence in (3.31) is uniform with respect to $t \in [0,T]$. Therefore (3.30) implies (3.29). □

Now we are in position to prove (1.12).

We consider $W(t) \equiv (n_t(t), n(t), E_t(t))$ and $W^k(t) \equiv (n_t^k(t), n^k(t), E_t^k(t))$ as elements in the space $C(0,T;\widetilde{\mathcal{E}})$ where $\widetilde{\mathcal{E}} = \mathcal{H} \times \mathcal{H}_1 \times \overline{\mathcal{H}}$. It is easy to find that

$$\max_{[0,T]} |W(t) - W^k(t)|^2_{\widetilde{\mathcal{E}}} \le \max_{[0,T]} \left\{ |W^k(t)|^2_{\widetilde{\mathcal{E}}} - |W(t)|^2_{\widetilde{\mathcal{E}}} \right\}$$
$$+C \max_{[0,T]} \left| \widetilde{P}_N(W(t) - W^k(t)) \right|_{\widetilde{\mathcal{E}}} + C \max_{[0,T]} \left| (I - \widetilde{P}_N)W(t) \right|_{\widetilde{\mathcal{E}}}$$

where $\widetilde{P}_N = P_N \times P_N \times P_N$ and P_N is the spectral orthoprojector on $\text{span}\{e_k : k = 1,2,\dots,N\}$. From (3.26)–(3.29) we conclude that

$$\limsup_{k\to\infty} \max_{[0,T]} |W(t) - W^k(t)|^2_{\widetilde{\mathcal{E}}} \le C \max_{[0,T]} \left| (I - \widetilde{P}_N)W(t) \right|_{\widetilde{\mathcal{E}}}. \tag{3.32}$$

Let $h_N(t) = \left| (I - \widetilde{P}_N)W(t) \right|_{\widetilde{\mathcal{E}}}$. The sequence $\{h_N(t)\}$ is a nonincreasing sequence of continuous functions such that $h_N(t) \to 0$ for each $t \in [0,T]$ as $N \to \infty$. Thus, by the Dini theorem $h_N(t) \to 0$ uniformly in t and therefore from (3.32) we obtain that

$$\lim_{k\to\infty} \max_{[0,T]} |W(t) - W^k(t)|^2_{\widetilde{\mathcal{E}}} = 0$$

which implies (1.12). This completes the proof of Theorem 2.

4. PROOF OF THEOREM 3

According to the general theory of dissipative systems (see, e.g., [2, 8, 16, 23]) to prove the existence of a compact global attractor we need to establish dissipativity and asymptotic compactness of the dynamical system $(\mathcal{B}_r, S_t)$ generated by problem (0.2) and (0.3) in the set $\mathcal{B}_r$ given by (1.13).

4.1. **Dissipativity.** In this subsection we prove that the dynamical system $(\mathcal{B}_r, S_t)$ generated by problem (0.2) and (0.3) is dissipative in the topology of the space $\mathcal{E}$.

By Theorem 1 we have the following dissipativity property in the variable E:

$$\limsup_{t\to\infty} [\sup\{\|E(t)\| \ : \ (n_1; n_0; E) \in B\}] \le \gamma^{-1}\|g\| \tag{4.1}$$

for any bounded set B in $\mathcal{B}_r$. We use it to obtain dissipativity in $\mathcal{E}$. We start with the following assertion.

Lemma 4. *Let $\gamma \ge \gamma_0 > 0$ for some positive number γ_0. Then there exists an absolute constant $b_1 > 0$ such that under condition* (1.15) *we have that*

$$\limsup_{t\to\infty} \left[\sup\left\{\|n_t(t)\|_{-1}^2 + \|n(t)\|^2 + \|\nabla E(t)\|^2 \ : \ (n_1; n_0; E) \in B\right\}\right] \le R_* \tag{4.2}$$

for any bounded set B in $\mathcal{B}_r$, where

$$R_* = c_r\left[\left(1 + \frac{\gamma\alpha}{\delta}\right)\|g\| + \frac{\|f\|^2}{2\lambda_1\delta}\left(1 + \frac{1}{2\lambda_1}\right)\right] \tag{4.3}$$

with the constant c_r depending on r only and with $\delta = \frac{1}{4}\min\{\lambda_1, 8\gamma_0, 2\}$.

Proof. To prove (4.2) we consider the following Lyapunov-type function

$$\mathcal{W}(n_t, n, E) = V_0^*(n_t, n) + V_1(n, E) + \varepsilon\left[(n, A^{-1}n_t) + \frac{\alpha}{2}\|A^{-1/2}n\|^2\right],$$

where $V_0^*(n_t, n) \equiv V_0^0(n_t, n)$ is given by (2.41) with $\beta = \delta = 0$ and $V_1(n, E)$ by (2.42). The parameter $0 < \varepsilon \le \min\{\alpha, \gamma\}$ will be chosen later. Since we consider the Dirichlet case only, by Lemma 1 we have that

$$\|u\|_{L_4(\Omega)}^2 \le \sqrt{2}\|u\| \cdot \|\nabla u\|, \quad u \in \mathcal{H}_1 = H_0^1(\Omega).$$

Therefore, since $(n_t; n; E) \in \mathcal{B}_r$, we have that

$$(n, |E|^2) \le \sqrt{2}\|n\|\|E\|\|\nabla E\| \le \sqrt{2}r\|n\|\|\nabla E\|.$$

Then for all $t \ge 0$ we obviously have that

$$\mathcal{W}(n_t, n, E) \ge \frac{1}{2}\left(1 - \frac{\varepsilon}{\alpha}\right)\|A^{-1/2}n_t\|^2 + a_r\left(\|n\|^2 + \|\nabla E\|^2\right) - 2\|g\|, \tag{4.4}$$

where $a_r > 0$ depends on r only, and also

$$\frac{d}{dt}\left[(n, A^{-1}n_t) + \frac{\alpha}{2}\|A^{-1/2}n(t)\|^2\right]$$

$$= \|A^{-1/2}n_t\|^2 - \|n\|^2 - (n, |E|^2) + (f, A^{-1}n)$$

for almost all $t \geq 0$. Therefore, from (2.40) in Proposition 2 we have that

$$\begin{aligned} &\frac{d}{dt}\mathcal{W}(n_t, n, E) + \varepsilon\mathcal{W}(n_t, n, E) \\ &\quad = \quad -(2\gamma - \varepsilon)\|\nabla E\|^2 - 2\gamma(n, |E|^2) - 2(\gamma - \varepsilon)\Re(g, E) \\ &\qquad\quad -(\alpha - \frac{3\varepsilon}{2})\|A^{-1/2}n_t\|^2 + (f, A^{-1}n_t) - \frac{\varepsilon}{2}\|n\|^2 + \varepsilon(f, A^{-1}n) \\ &\qquad\quad +\varepsilon^2\left[(n, A^{-1}n_t) + \frac{\alpha}{2}\|A^{-1/2}n\|^2\right]. \end{aligned}$$

Let ρ be any number possessing the property $\gamma^{-1}\|g\| < \rho < r_\Omega = 1$. By (4.1) we have that $\|E(t)\| \leq \rho$ for $t \geq t_B$. Therefore we obtain that

$$2\gamma(n, |E|^2) \leq 2\sqrt{2}\rho\gamma\|n\|\|\nabla E\| \leq \gamma\|\nabla E\|^2 + 2\rho^2\gamma\|n\|^2$$

for $t \geq t_B$. Similarly

$$-2(\gamma - \varepsilon)\Re(g, E) \leq 2\gamma\rho\|g\| \leq 2\gamma\|g\|, \quad t \geq t_B.$$

Therefore, taking into account that

$$\begin{aligned} \big(n, A^{-1}n_t\big) + \frac{\alpha}{2}\|A^{-1/2}n\|^2 &\leq \frac{\alpha}{\lambda_1}\|n\|^2 + \frac{1}{2\alpha}\|A^{-1/2}n_t\|^2, \\ \big(f, A^{-1}n\big) &\leq \frac{1}{4}\|n\|^2 + \frac{1}{\lambda_1^2}\|f\|^2, \\ \big(f, A^{-1}n_t\big) &\leq \frac{1}{2\alpha\lambda_1}\|f\|^2 + \frac{\alpha}{2}\|A^{-1/2}n_t\|^2, \end{aligned}$$

we get

$$\begin{aligned} &\frac{d}{dt}\mathcal{W}(n_t, n, E) + \varepsilon\mathcal{W}(n_t, n, E) \\ &\quad \leq \quad 2\gamma\|g\| - \left(\frac{\alpha}{2} - \frac{3\varepsilon}{2} - \frac{\varepsilon^2}{2\alpha}\right)\|A^{-1/2}n_t\|^2 + \frac{\|f\|^2}{2\alpha\lambda_1} \\ &\qquad\quad -\varepsilon\left(\frac{1}{4} - \frac{\alpha\varepsilon}{\lambda_1} - 2\rho^2\frac{\gamma}{\varepsilon}\right)\|n\|^2 + \varepsilon\frac{\|f\|^2}{\lambda_1^2} - (\gamma - \varepsilon)\|\nabla E\|^2 \end{aligned}$$

for $t \geq t_B$. If we take $\alpha \geq 1$ and $\varepsilon\alpha = \delta_0 \equiv \frac{1}{8}\min\{\lambda_1, 8\gamma_0, 2\}$, then

$$\frac{\alpha}{2} - \frac{3\varepsilon}{2} - \frac{\varepsilon^2}{2\alpha} \geq \frac{1}{2}\left(\frac{\alpha}{4} - \frac{\delta_0^2}{\alpha^3}\right)$$

and

$$\frac{1}{4} - \frac{\varepsilon\alpha}{\lambda_1} - 2\rho^2\frac{\gamma}{\varepsilon} \geq \frac{1}{8} - 2\rho^2\frac{\gamma\alpha}{\delta_0}.$$

Therefore, under the conditions

$$\alpha^2 \geq \max\{1\,,\,2\delta_0\}\,, \quad 16\gamma\alpha\rho^2 \leq \delta_0,$$

we obtain that

$$\frac{d}{dt}\mathcal{W}(n_t, n, E) + \frac{\delta_0}{\alpha}\mathcal{W}(n_t, n, E) \leq 2\gamma\|g\| + \frac{\|f\|^2}{2\alpha\lambda_1}\cdot\left(1+\frac{1}{2\lambda_1}\right)$$

for $t \geq t_B$. Therefore, for $\mathcal{W}(t) \equiv \mathcal{W}(n_t(t), n(t), E(t))$ we obtain the estimate

$$\mathcal{W}(t) \leq \mathcal{W}(t_B)\exp\left\{-\frac{\delta_0(t-t_B)}{\alpha}\right\} + \frac{1}{\delta_0}\left[2\gamma\alpha\|g\| + \frac{\|f\|^2}{2\lambda_1}\left(1+\frac{1}{2\lambda_1}\right)\right]$$

for all $t \geq t_B$. Consequently relation (4.2) follows from (4.4). □

Now we prove the following assertion.

Lemma 5. *Assume that relation* (4.2) *holds with the parameter* R_* *given by* (4.3). *Then there exists a constant* d_Ω (*possibly depending on the domain* Ω *only*) *such that under the condition*

$$d_\Omega^2 \cdot R_* < 2\gamma \cdot \alpha \tag{4.5}$$

the dynamical system $(\mathcal{B}_r, S_t)$ *generated by problem* (0.2) *and* (0.3) *is dissipative in the topology of the space* $\mathcal{E}$. *In the case when* $\Omega = (0, l_1) \times (0, l_2)$ *is a rectangle we have that* $d_\Omega \leq 3$.

Proof. On the trajectories $(n_t; n; E)$ we consider the following Lyapunov function

$$W(n_t; n; E_t) = W_0(n_t, n, E_t) - (f, n) + \varepsilon\left[(n, n_t) + \frac{\alpha}{2}\|n\|^2\right], \tag{4.6}$$

where

$$W_0(n_t, n, E_t) = \frac{1}{2}\left[\|n_t\|^2 + \|\nabla n\|^2\right] + \|E_t\|^2.$$

The parameter $0 < \varepsilon \leq \alpha/2$ will be chosen later. It is clear that

$$c_0 W_0(n_t, n, E_t) - c_1\|f\|^2 \leq W(n_t; n; E_t) \leq c_2 W_0(n_t, n, E_t) + c_3\|f\|^2 \tag{4.7}$$

for some positive constants c_i. Below we use the notation

$$W(t) = W(n_t(t); n(t); E_t(t)) \quad \text{and} \quad W_0(t) = W_0(n_t(t), n(t), E_t(t)).$$

Since

$$\frac{d}{dt}\left[(n, n_t) + \frac{\alpha}{2}\|n\|^2\right] = \|n_t\|^2 - \|\nabla n\|^2 + (n, \Delta|E|^2 + f),$$

from Proposition 1 we obtain that

$$\frac{d}{dt}W(t) = -(\alpha-\varepsilon)\|n_t\|^2 - \varepsilon\|\nabla n\|^2 - 2\gamma\|E_t\|^2 + Q_\varepsilon(n_t;n;E) \tag{4.8}$$

for almost all $t \geq 0$, where

$$Q_\varepsilon(n_t;n;E) = 2(n_t, |\nabla E|^2) + \widetilde{Q}_\varepsilon(n_t;n;E)$$

with

$$\widetilde{Q}_\varepsilon(n_t;n;E) = 2(n_t, n|E|^2 + \Re\{g\overline{E}\}) + \varepsilon(n, \Delta|E|^2 + f).$$

Fix $\delta > 0$. Then by Lemma 4 there exists $t_{B,\delta} \geq 0$ such that

$$\|n(t)\|^2 + \|\nabla E(t)\|^2 \leq (1+\delta)^2 R_* \quad \text{for} \quad t \geq t_{B,\delta}.$$

Therefore, after simple calculations it is easy to see that

$$|\widetilde{Q}_\varepsilon(n_t;n;E)| \leq C_1 \left(\|n_t\|^2 + \|\nabla n\|^2 + \|\Delta E\|^2\right)^\nu + C_2, \quad t \geq t_{B,\delta},$$

where $0 < \nu < 1$ and C_1 and C_2 are positive constants. Directly from (1.3) we have that

$$\|iE_t + \Delta E\| \leq \|g\| + \|nE\| + \gamma\|E\| \leq C_1(R_*) \cdot \|\nabla n\|^{1/2} + C_2(R_*), \quad t \geq t_{B,\delta}.$$

Therefore,

$$\|\Delta E\| \leq \|E_t\| + C_1(R_*) \cdot \|\nabla n\|^{1/2} + C_2(R_*), \quad t \geq t_{B,\delta}. \tag{4.9}$$

Consequently

$$|\widetilde{Q}_\varepsilon(n_t;n;E)| \leq C_1 W_0(t)^\nu + C_2, \quad t \geq t_{B,\delta},$$

with $0 < \nu < 1$. It is clear from Lemma 1 that there exist constants d_Ω and C_Ω such that

$$\|E_{x_1}\|^2_{L_4} + \|E_{x_2}\|^2_{L_4} \leq d_\Omega \|\nabla E\| \cdot \|\Delta E\| + C_\Omega \|\nabla E\|^2. \tag{4.10}$$

Therefore, since

$$2(n_t, |\nabla E|^2) = 2(n_t, |E_{x_1}|^2 + |E_{x_2}|^2) \leq 2\|n_t\| \left(\|E_{x_1}\|^2_{L_4} + \|E_{x_2}\|^2_{L_4}\right),$$

we conclude that

$$Q_\varepsilon(n_t;n;E) \leq 2d_\Omega(1+\delta)\sqrt{R_*}\|n_t\|\|\Delta E\| + C_1 W_0(t)^\nu + C_2, \quad t \geq t_{B,\delta},$$

for some $0 < \nu < 1$. Consequently, from (4.8) and (4.9) we have that

$$\begin{aligned} \frac{d}{dt}W(t) \quad &\leq \quad -(\alpha-\varepsilon)\|n_t\|^2 - 2\gamma\|E_t\|^2 + 2d_\Omega(1+\delta)\sqrt{R_*}\|n_t\|\|E_t\| \\ &\quad -\varepsilon\|\nabla n\|^2 + C_1 W_0(t)^\nu + C_2, \quad t \geq t_{B,\delta}. \end{aligned} \tag{4.11}$$

Under condition (4.5) we can choose δ and ε such that

$$d_\Omega^2(1+\delta)^2 R_* < 2\gamma(\alpha-\varepsilon).$$

Therefore (4.7) and (4.11) imply that

$$\frac{d}{dt}W(t) \le -\omega W(t) + C, \quad t \ge t_{B,\delta},$$

for some $\omega > 0$. Using (4.7) and (4.9), as in the proof of Lemma 4, we can conclude that $(\mathcal{B}_r, S_t)$ is dissipative in the topology of $\mathcal{E}$ under condition (4.5).

For the estimate of d_Ω in the case $\Omega = (0,l_1)\times(0,l_2)$ we refer to the Appendix. □

4.2. **Asymptotic compactness.** Now we prove that the system $(\mathcal{B}_r, S_t)$ is asymptotically compact; i.e. we prove that for any bounded sequence $\{U_j\}$ from $\mathcal{B}_r$ and for any sequence $\{t_j\} \subset \mathbb{R}_+$ such that $t_j \to \infty$ the set $\{S_{t_j}U_j\}$ is precompact in $\mathcal{E}$ (see, e.g., [16, 23] or [8]).

We use the idea due to Ball [3] (see also [18]) and rely on the following assertion.

Lemma 6. *Let $\{U_k\} \subset \mathcal{B}_r$ and $U_k \to U$ weakly in $\mathcal{E}$. Then the following properties holds.*

(i) *For any $T > 0$ we have that*

$$U_k(t) \to U(t) \quad \text{weakly}^* \text{ in} \quad L_\infty(0,T;\mathcal{E}), \quad k\to\infty, \tag{4.12}$$

where $U_k(t) = S_tU_k$ and $U(t) = S_tU$. Moreover

$$U_k(t) \to U(t) \quad \text{strongly in} \quad C(0,T;\mathcal{H}_{-\sigma}\times\mathcal{H}_{1-\sigma}\times\overline{\mathcal{H}}_{2-\sigma}) \tag{4.13}$$

for every $\sigma > 0$ when $k\to\infty$.

(ii) *If for some $t > 0$ we have that $S_tU_k \to V$ weakly in $\mathcal{E}$, then $V = S_tU$; i.e. the semiflow S_t is weakly closed.*

Proof. As in the proof of Theorem 2, using Propositions 1 and 2 one can see that

$$\max_{[0,T]} |U_k(t)|_{\mathcal{E}} \le C_T, \quad k = 1,2\ldots \tag{4.14}$$

Thus there exists a subsequence $\{k_j\}$ such that

$$U_{k_j}(t) \to V(t) \quad \text{weakly}^* \text{ in} \quad L_\infty(0,T;\mathcal{E}), \quad j\to\infty,$$

where $V(t) = (n_t; n; E)$ is a semi-strong solution with initial data U. By the uniqueness of Theorem 1 $V(t)$ does not depend on the subsequence $\{k_j\}$ and, moreover, $V(t) \equiv U(t)$. Thus (4.12) holds. Property (4.13) follows from Aubin's embedding theorem (see [22, Corollary 4]) and from relation (4.14).

The second part of the statement easily follows from (4.13). □

We choose $\varepsilon > 0$ such that $\alpha - 2\varepsilon \geq 0$ and $\gamma - \varepsilon \geq 0$ and for $U = (n_t; n; E)$ denote

$$L(U) \equiv L(n_t, E_t) = (\alpha - 2\varepsilon)\|n_t\|^2 + 2(\gamma - \varepsilon)\|E_t\|^2$$

and

$$\begin{aligned} K(U) \equiv K(n_t; n; E) \quad &= \quad 2(n_t, |\nabla E|^2 + n|E|^2 + \Re\{gE\}) \\ &\quad +\varepsilon(n, \Delta|E|^2 + f) + 2\varepsilon^2(n, n_t) + \alpha\varepsilon^2\|n\|^2. \end{aligned}$$

Let $W(U) = W(n_t; n; E_t)$, where $W(n_t; n; E_t)$ is given by (4.6). Then it follows from (4.8) that

$$\begin{aligned} &W(S_t U) + \int_\sigma^t e^{-2\varepsilon(t-\tau)} L(S_\tau U) d\tau \\ &= e^{-2\varepsilon(t-\sigma)} W(S_\sigma U) + \int_\sigma^t e^{-2\varepsilon(t-\tau)} K(S_\tau U) d\tau \end{aligned} \tag{4.15}$$

for any $t \geq \sigma \geq 0$. Now we apply the idea of Theorem 3.1 from [18].

Let $\{U_j\}$ be a bounded sequence in $\mathcal{B}_r$ and let $\{t_j\}$ be a sequence of positive real numbers which converges to infinity as $j \to \infty$. Since $(\mathcal{B}_r, S_t)$ is dissipative, we can choose subsequences $\{U_{j_k}\}$ and $\{t_{j_k}\}$, a number $T > 0$, and elements $V_m \in \mathcal{B}_r$ such that

$$S_{t_{j_k} - mT} U_{j_k} \to V_m \quad \text{weakly in} \quad \mathcal{E}, \quad m = 0, 1, \dots \tag{4.16}$$

By Lemma 6(ii) we have that $S_{mT} V_m = V_0$. From (4.15) we have that

$$\begin{aligned} &W(S_{t_{j_k}} U_{j_k}) + \int_0^{mT} e^{-2\varepsilon(mT-\tau)} L(S_\tau S_{t_{j_k} - mT} U_{j_k}) d\tau \\ &= e^{-2\varepsilon mT} W(S_{t_{j_k} - mT} U_{j_k}) + \int_0^{mT} e^{-2\varepsilon(mT-\tau)} K(S_\tau S_{t_{j_k} - mT} U_{j_k}) \mathrm{d}\tau \end{aligned} \tag{4.17}$$

for every m and $t_{j_k} \geq mT$. It follows from relation (4.16) and Lemma 6(i) that

$$\liminf_{k \to \infty} \int_0^{mT} e^{-2\varepsilon(mT-\tau)} L(S_\tau S_{t_{j_k} - mT} U_{j_k}) d\tau \geq \int_0^{mT} e^{-2\varepsilon(mT-\tau)} L(S_\tau V_m) d\tau$$

and

$$\lim_{k \to \infty} \int_0^{mT} e^{-2\varepsilon(mT-\tau)} K(S_\tau S_{t_{j_k} - mT} U_{j_k}) d\tau = \int_0^{mT} e^{-2\varepsilon(mT-\tau)} K(S_\tau V_m) d\tau.$$

Therefore, from (4.17) we have that

$$\limsup_{k\to\infty} W(S_{t_{j_k}}U_{j_k}) + \int_0^{mT} e^{-2\varepsilon(mT-\tau)}L(S_\tau V_m)d\tau \\ \leq C\cdot e^{-2\varepsilon mT} + \int_0^{mT} e^{-2\varepsilon(mT-\tau)}K(S_\tau V_m)\mathrm{d}\tau. \qquad (4.18)$$

On the other hand, since $V_0 = S_{mT}V_m$, from (4.15) we also obtain that

$$W(V_0) + \int_0^{mT} e^{-2\varepsilon(mT-\tau)}L(S_\tau V_m)d\tau \\ = e^{-2\varepsilon mT}W(V_m) + \int_0^{mT} e^{-2\varepsilon(mT-\tau)}K(S_\tau V_m)\mathrm{d}\tau. \qquad (4.19)$$

Comparing (4.18) and (4.19) it is easy to see that

$$\limsup_{k\to\infty} W(S_{t_{j_k}}U_{j_k}) \leq W(V_0) + C\cdot e^{-2\varepsilon mT}, \quad m = 1,2,\dots$$

Therefore, since $S_{t_{j_k}}U_{j_k} \to V_0$ weakly in $\mathcal{E}$, we can conclude that

$$\lim_{k\to\infty} W(S_{t_{j_k}}U_{j_k}) = W(V_0).$$

This, and also the structure of W, imply that $S_{t_{j_k}}U_{j_k} \to V_0$ strongly in $\mathcal{E}$. Thus $(\mathcal{B}_r, S_t)$ is asymptotically compact. This completes the proof of Theorem 3.

5. Appendix

Proof of Lemma 1. We start with the case when $\Omega = (0,l_1)\times(0,l_2)$ is a rectangle and rely on the idea presented in [10].

We first note that for any function $\phi \in H^1(0,l)$ of a single variable the following estimate holds

$$\max_{[0,l]} |\phi(x)|^2 \leq 2\|\phi\|_{L_2(0,l)}\cdot\|\phi'\|_{L_2(0,l)} + \frac{\sigma}{l}\|\phi\|^2_{L_2(0,l)}, \qquad (5.1)$$

where $\sigma = 0$ when $\phi \in H_0^1(0,l)$ and $\sigma = 1$ otherwise. Indeed, we can write the representation

$$|\phi(x)|^2 = |\phi(y)|^2 + 2\Re\int_y^x \phi'(z)\overline{\phi(z)}dz, \quad x,y\in[0,l].$$

This formula implies that

$$|\phi(x)|^2 \leq |\phi(y)|^2 + 2\|\phi\|_{L_2(0,l)}\cdot\|\phi'\|_{L_2(0,l)}, \quad x,y\in(0,l). \qquad (5.2)$$

After integrating with respect to y we obtain (5.1) with $\sigma = 1$. In the case $\phi \in H_0^1(0,l)$ we can take $y = 0$ in (5.2) and, since $\phi(0) = 0$, obtain (5.1) with $\sigma = 0$.

Now, for $u \in H^1(\Omega)$ with $\Omega = (0,l_1) \times (0,l_2)$ we have that

$$\begin{aligned}\int_\Omega |u(x,y)|^4 dxdy &\leq \int_0^{l_1} dx \int_0^{l_2} dy \max_x |u(x,y)|^2 \cdot \max_y |u(x,y)|^2 \\ &= \int_0^{l_2} \max_x |u(x,y)|^2 dy \cdot \int_0^{l_1} \max_y |u(x,y)|^2 dx.\end{aligned}$$

Therefore, application of (5.1) gives the inequality

$$\int_\Omega |u(x,y)|^4 dxdy \leq \|u\|^2 \Big(2\|u_x\| + \frac{\sigma}{l_1}\|u\|\Big)\Big(2\|u_y\| + \frac{\sigma}{l_2}\|u\|\Big), \tag{5.3}$$

which implies the statements of Lemma 1 and Remark 1 in the case when Ω is a rectangle. To prove (1.11) in the general case we first extend the function u from $H^1(\Omega)$ to a function $\widetilde{u} \in H^1(\Pi)$ on some rectangle $\Pi \supset \Omega$ (if $u \in H_0^1(\Omega)$ we suppose $\widetilde{u} = 0$ on $\Pi \setminus \Omega$). Applying (5.3) to $\widetilde{u}$ and using continuity of the extension operator we obtain the statement of Lemma 1.

Estimate for d_Ω from (4.10) in the case when $\Omega = (0,l_1)\times(0,l_2)$. Since $E_{x_1} \in H^1(\Omega)$ we have from (5.3) that

$$\|E_{x_1}\|_{L_4}^4 \leq 4\|E_{x_1}\|^2 \|E_{x_1x_1}\| \|E_{x_1x_2}\| + C\|E_{x_1}\|^3 \|\Delta E\|$$

and a similar estimate for $\|E_{x_2}\|_{L_4}^4$. Therefore,

$$\begin{aligned}&\|E_{x_1}\|_{L_4}^4 + \|E_{x_2}\|_{L_4}^4 \\ &\leq 2\|\nabla E\|^2 \left(\|E_{x_1x_1}\|^2 + 2\|E_{x_1x_2}\|^2 + \|E_{x_2x_2}\|^2\right) + C\|\nabla E\|^3 \|\Delta E\|.\end{aligned}$$

Since in the case of a rectangle $\Omega = (0,l_1) \times (0,l_2)$ we have

$$E_{x_1x_1}\big|_{x_2=0,l_2} = E_{x_1}\big|_{x_2=0,l_2} = 0, \quad E_{x_2x_2}\big|_{x_1=0,l_1} = E_{x_2}\big|_{x_1=0,l_1} = 0,$$

for smooth functions E from $H_0^1(\Omega)$, integration by parts gives us that

$$2\|E_{x_1x_2}\|^2 = 2\Re \int_\Omega E_{x_1x_1}(x)\overline{E_{x_2x_2}(x)}dx.$$

Therefore, we obtain that

$$\|E_{x_1}\|_{L_4}^4 + \|E_{x_2}\|_{L_4}^4 \leq 2\|\nabla E\|^2 \|\Delta E\|^2 + C\|\nabla E\|^3 \|\Delta E\|.$$

This implies that for $\Omega = (0,l_1) \times (0,l_2)$ (4.10) holds with $d_\Omega = 2 + \delta$, where $\delta > 0$ can be chosen arbitrarily small.

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