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POSITIVE SOLUTION BRANCH FOR ELLIPTIC PROBLEMS WITH CRITICAL INDEFINITE NONLINEARITY

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Abstract. In this paper, we study the semilinear elliptic problem with critical nonlinearity and an indefinite weight function, namely $-\Delta u = \lambda u + h(x)u^{\frac{n+2}{n-2}}$ in a smooth domain bounded (respectively, unbounded) $\Omega \subseteq \mathbb{R}^n$, $n > 4$, for $\lambda \geq 0$. Under suitable assumptions on the weight function, we obtain the positive solution branch, bifurcating from the first eigenvalue $\lambda_1(\Omega)$ (respectively, the bottom of the essential spectrum).

1. Introduction

In this paper, we study the following (critical exponent) semilinear elliptic problem in a smooth domain $\Omega \subseteq \mathbb{R}^n$:

$$\left.\begin{array}{rcl} -\Delta u & = & \lambda u + h(x)u^{\frac{n+2}{n-2}} \quad \text{in } \Omega \\ u & > & 0 \quad \text{in } \Omega; \quad u = 0 \quad \text{in } \partial\Omega \end{array}\right\} \tag{1.1}$$

for dimensions $n > 4$, λ a nonnegative parameter and h a C^2 function which changes sign. If Ω is unbounded, the boundary condition translates to looking for classical solutions in the space C_0 of continuous functions in $\mathbb{R}^n$ vanishing at infinity.

Considering (1.1), we prove that there exists a continuum of solutions (λ, u) in $\mathbb{R} \times C_0(\Omega)$ bifurcating from the first eigenvalue $\lambda_1(\Omega)$(of $-\Delta$ with

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Dirichlet condition) if Ω is bounded, (respectively, from the bottom of the essential spectrum if Ω is unbounded) and reaching $\{\lambda = 0\} \times C_0(\Omega) \setminus \{0\}$. We stress that this result is new for the semilinear problem involving the critical exponent and indefinite nonlinearities, both for bounded as well as unbounded domains.

Indeed, T.Ouyang in [14] has proved the existence of a local branch of positive solutions of (1.1) in a bounded domain, bifurcating from $\lambda_1(\Omega)$ using Rabinowitz local bifurcation theory. He uses the assumption

$$\int_\Omega h(x)(\phi_1(x))^{p+1}dx < 0$$

to show that the branch turns back at some λ_0. While we also use the same theory, because of our apriori bounds, our proof gives the existence of a connected branch beginning from $(\lambda_1(\Omega), 0)$ going back all the way to $(0, u_0)$ where $u_0 > 0$ and for all dimensions $n > 4$.

For $h = 1$, using variational methods, Brezis and Nirenberg have proved the existence of a branch bifurcating from $\lambda_1(\Omega)$ and blowing up at $\lambda^* > 0$ for $n = 3$, while for $n \geq 4$ the branch blows up at $\lambda^* = 0$. Note that, in [13], Y. Li and M. Zhu have considered the same problem as in (1.1), for a compact Riemannian manifold in a different context. Also, Cerqueti and Grossi in [6] have studied a similar problem when the linear term goes to 0 and $h \equiv$ constant.

In the case of Ω unbounded, the unique result about the existence of a continuum of solutions is due to J. Toland. In [16], for $h > 0$ and radial, he proves the existence of an unbounded continuum of radial solutions in $\{0\} \times L^p(\mathbb{R}^n)$ bifurcating from $(0, 0)$, for a suitable p.

In [12], the author considers the case of $h > 0$ and $\lambda = 0$, in S^n, and carries out a sharp blow up analysis. Whereas, in [7], they consider the indefinite case. Combining the blow up analysis of Y. Li in Ω^+ with estimates in Ω^- and in a neighborhood of Γ, they get a priori estimates for the solutions. Here we extend these results for $\lambda > 0$ independently of λ and for Ω bounded or unbounded. This extension also involves new techniques and is nontrivial. In our case, the a priori estimates are more delicate in Ω^+ because of the critical exponent and here we need to restrict the dimension to $n > 4$. It is possible that finer estimates would remove this restriction.

Using the a priori estimates, the existence of the branch follows from the global bifurcation theorem of Rabinowitz, for a bounded domain. For Ω unbounded, the above a priori estimates and topological arguments help us to obtain the branch as the limit of the branches obtained for bounded domains approximating Ω. Such an approach has been successfully worked out in

[4] for subcritical nonlinearities in $\mathbb{R}^n$. Here we get in fact two nontrivial solutions for $\lambda > 0$ and small, unlike the case of Toland, since $\Omega^+ := \{x \in \Omega : h(x) > 0\}$ is bounded in our case.

As in [4], we assume that

(H1) $h \in C^2(\mathbb{R}^n, \mathbb{R})$, the set $\Omega^+ := \{x \in \mathbb{R}^n : h(x) > 0\}$ is bounded, and $\overline{\Omega^+} \subset \Omega$; and

(H2) for all $x \in \Gamma := \{x \in \mathbb{R}^n : h(x) = 0\}, \nabla h(x) \neq 0$.

From these, it follows that $\Gamma = \overline{\Omega^+} \cap \overline{\Omega^-}$ is bounded, where $\Omega^- = \{x \in \mathbb{R}^n : h(x) < 0\}$. Also, note that (H1) implies that a neighbourhood of $\partial\Omega$ is contained in Ω^-. We further assume that

(H3) If $\mathcal{S} = \{x \in \mathbb{R}^n : h(x) > 0, \nabla h(x) = 0\}$, then for $x_0 \in \mathcal{S}$, and for $n - 2 < \theta < n$, there exists $\sigma > 0$ such that in $B_\sigma(x_0)$, the following holds:

$$h(x) = h(x_0) + \sum_{j=1}^{n} a_j |x^j - x_0^j|^{\theta-1}(x^j - x_0^j) + R(x),$$

where $|\nabla R(x)||x|^{-\theta}$ tends to 0 as x tends to x_0.

A condition similar to (H3) is used in [12] and also in [8]. If Ω is unbounded, we will need

(H4) $h(x) \to -\infty$ when $|x|$ goes to ∞.

Our main results are the following two theorems regarding the branch of positive solutions of (1.1) when Ω is bounded and also when it is unbounded.

Theorem 1.1. *Consider the equation* (1.1) *in a bounded smooth domain* $\Omega \subset \mathbb{R}^n$, $n > 4$. *Assume* ($H1$), ($H2$), *and* ($H3$). *Then there exists* $\mathcal{C}^+$, *a branch of nontrivial solutions, connected in* $\mathbb{R} \times C_0(\Omega)$, *bifurcating from* $(\lambda_1(\Omega), 0)$. *Moreover, in this case, the projection of* $\mathcal{C}^+$ *on* $\mathbb{R}$,

$$\prod_{\mathbb{R}} \mathcal{C}^+ = [0, \lambda_0],$$

where $\lambda_1(\Omega) \leq \lambda_0 < \lambda_1(\Omega^+)$.

For unbounded domains with infinite measure, we prove

Theorem 1.2. *Consider the equation* (1.1) *in* $\Omega \subset \mathbb{R}^n$, $n > 4$, *unbounded and of infinite measure, with smooth boundary. Assume* (H1), (H2), (H3), and (H4). *Then there exists* $\mathcal{C}^+$, *a branch of nontrivial solutions, connected in* $\mathbb{R} \times C_0(\Omega)$, *bifurcating from the bottom of the essential spectrum* 0. *Moreover, in this case, the projection of* $\mathcal{C}^+$ *on* $\mathbb{R}$,

$$\prod_{\mathbb{R}} \mathcal{C}^+ = [0, \lambda_0],$$

where $0 < \lambda_0 < \lambda_1(\Omega^+)$ *. More precisely,*

(i) *there exists* $u_0 \in C_0(\Omega)$ *such that* $(0, u_0) \in \mathcal{C}^+$ *and* $u_0 > 0$;
(ii) *if* $(0, u) \in \mathcal{C}^+ \setminus (\Lambda, 0)$, *then* $\|u\| > c > 0$.

We want to stress that this result is true if $\Omega = \mathbb{R}^n$, $n > 4$, and in this case we get the branch bifurcating from $(0, 0)$. Note that, from Theorem 1.2, we get a multiplicity result in the case of unbounded domains : for $\lambda > 0$ small, there exist at least two solutions to (1.1). There are unbounded domains with finite measure for which the imbedding $H_0^1 \subset L^2$ is compact. (See [1], Chapter 6, for example). In such cases, Theorem 1.1 will go through.

2. An outline of the proof

The main ingredient of the proof of Theorem 1.1 and Theorem 1.2 is an a priori estimate for the solutions of (1.1) in bounded domains. To get a priori estimates in a bounded domain Ω, we subdivide it into three regions, for a fixed small $\delta > 0$, as in [7]:

(1) $\Omega_\delta^- := \Omega \cap \Omega^- \cap \{x : \text{dist}(x, \Gamma) > \delta > 0\}$,
(2) $\Gamma_\delta := \{x : \text{dist}(x, \Gamma) \leq \delta\}$,
(3) $\Omega_\delta^+ := \Omega^+ \cap \{x : \text{dist}(x, \Gamma) > \delta > 0\}$.

We show that, in each of the above regions, the solution is uniformly bounded by a constant depending only on n, h, Ω^+, Γ. These proofs are contained in Sections 3, 4, and 5 respectively.

For a bounded domain Ω, the existence of a bifurcation branch in the cone of positive solutions in C_0 follows from direct application of the Rabinowitz global bifurcation theorem and the above a priori estimates. This is the idea of the proof of Theorem 1.1 in Section 6.

For unbounded domains, we consider the problem in smooth bounded domains Ω_i with $\Omega_i \subset \Omega_{i+1}$ and $\cup_i \Omega_i = \Omega$. Theorem 1.1 ensures the existence of a bifurcation branch $\mathcal{C}_i$ in each of Ω_i. We show that these solutions are uniformly bounded and then use Whyburn's theorem (see [17]) to pass to the limit as i goes to infinity and get the branch of solutions for (1.1). We recall the following results from [17].

Definition. [Whyburn] Let G be any infinite collection of point sets. The set of all points x such that every neighborhood of x contains points of infinitely many sets of G is called *the superior limit of* G *(*$\limsup G$*)*. The set of all points y such that every neighborhood of y contains points of all but a finite number of sets of G is called *the inferior limit of* G *(*$\liminf G$*)*.

Theorem 2.1 (Whyburn)**.** *Let* $\{A_i\}_{i\in\mathbb{N}}$ *be a sequence of connected closed sets such that* $\liminf\{A_i\} \not\equiv \emptyset$. *Then, if the set* $\cup_{i\in\mathbb{N}} A_i$ *is relatively compact,* $\limsup\{A_i\}$ *is a closed, connected set.*

We apply Theorem 2.1 as follows: Let A_i be the connected component containing $(0,0)$ in $\{0 \leq \lambda\} \times C_0(\mathbb{R}^n) \cap \mathcal{C}_i$. From (i) of Theorem 1.1 we have $\Pi_{\mathbb{R}} A_i = [0\,, \lambda_0^i]$. First, we will prove that λ_0^i converges to $\lambda_0 > 0$. Then, passing to the limit $i \to \infty$, using Theorem 2.1, we get that $\mathcal{C} := \limsup_{i\to+\infty} A_i$ is connected, closed, and bifurcating from $(\Lambda, 0)$. Furthermore, if $(\lambda, u) \in \mathcal{C}$, then (λ, u) is a solution to (1.1)

Also, we will show in Section 6 that for any solution (λ, u) of (1.1), we must have $\lambda \leq \lambda_1(\Omega^+)$. Hence, in the following estimates, we will always consider $\lambda \in [0, \lambda_1(\Omega^+)]$

3. Estimates in Ω_δ^-

Here we will obtain a priori bounds for the solution u of (1.1) in the region Ω_δ^-. We begin with the following estimate which in fact is true in both the larger sets Ω^- and Ω^+:

Proposition 3.1. *Given* $x_0 \in \Omega^\pm$, $\epsilon > 0$, *and* $B_\epsilon(x_0) \subset\subset \Omega^\pm$, *there exists* $C = C(\epsilon, \lambda)$ *such that*

$$\int_{B_{\frac{\epsilon}{2}}(x_0)} u^{\frac{n+2}{n-2}} dx \leq \Big(\frac{C(C+|\lambda|)}{\inf_{B_\epsilon}|h|}\Big)^{\frac{n+2}{4}}. \tag{3.1}$$

Proof. We consider on the ball $B_\epsilon = B_\epsilon(x_0)$ an eigenfunction ϕ associated to the first eigenvalue $\lambda_1(\epsilon)$ which satisfies :

$$\begin{cases} -\Delta\phi &= \lambda_1(\epsilon)\phi \text{ in } B_\epsilon, \quad \phi &= 0 \text{ on } \partial B_\epsilon, \\ \quad\phi &> 0 \text{ in } B_\epsilon, \quad \|\phi\|_{C^1} &\leq 1. \end{cases}$$

For convenience of notation, denote $p = \frac{n+2}{n-2}$. Multiply the equation in (1.1) by ϕ^α and choose $\alpha \geq \frac{2p}{p-1}$. We obtain

$$\int_{B_\epsilon} (-\Delta u)\phi^\alpha = \int_{B_\epsilon} \{\lambda u + h(x)u^p\}\phi^\alpha. \tag{3.2}$$

Since $\phi|_{\partial B_\epsilon} = \frac{\partial\phi^\alpha}{\partial n}|_{\partial B_\epsilon} = 0$ (note that $\alpha > 1$), the left-hand side of (3.2) gives

$$\begin{aligned}\int_{B_\epsilon} (-\Delta u)\phi^\alpha &= -\int_{B_\epsilon} u\Delta(\phi^\alpha) = -\alpha\int_{B_\epsilon} u(\Delta\phi)\phi^{\alpha-1} - \alpha(\alpha-1)\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} \\ &= \alpha\lambda_1(\epsilon)\int_{B_\epsilon} u\phi^\alpha - \alpha(\alpha-1)\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2}. \end{aligned} \tag{3.3}$$

From (3.2) and (3.3), we have

$$\int_{B_\epsilon} hu^p\phi^\alpha = \alpha\lambda_1(\epsilon)\int_{B_\epsilon} u\phi^\alpha - \alpha(\alpha-1)\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} - \lambda\int_{B_\epsilon} u\phi^\alpha. \quad (3.4)$$

If $B_\epsilon \subset \Omega^+$, we have

$$\int_{B_\epsilon} hu^p\phi^\alpha \le \alpha\lambda_1(\epsilon)\int_{B_\epsilon} u\phi^\alpha. \quad (3.5)$$

If $B_\epsilon \subset \Omega^-$, we have

$$\begin{aligned}
&\int_{B_\epsilon} |h|u^p\phi^\alpha = \int_{B_\epsilon} (-h)u^p\phi^\alpha \\
&= -\alpha\lambda_1(\epsilon)\int_{B_\epsilon} u\phi^\alpha + \alpha(\alpha-1)\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} + \int_{B_\epsilon} \lambda u\phi^\alpha \\
&\le \alpha(\alpha-1)\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} + \int_{B_\epsilon} \lambda u\phi^\alpha. \quad (3.6)
\end{aligned}$$

Now, the right-hand side of (3.5) and (3.6) can be estimated using Hölder's inequality ($\frac{1}{p} + \frac{1}{q} = 1$) as follows

$$\int_{B_\epsilon} u\phi^\alpha \le \Big(\int_{B_\epsilon} u^p\phi^\alpha\Big)^{\frac{1}{p}}\Big(\int_{B_\epsilon} \phi^\alpha\Big)^{\frac{1}{q}};$$

$$\int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} \le \Big(\int_{B_\epsilon} u^p\phi^\alpha\Big)^{\frac{1}{p}}\Big(\int_{B_\epsilon} \phi^{\alpha-2q}|\nabla\phi|^{2q}\Big)^{\frac{1}{q}}.$$

Therefore, by choosing $\alpha \ge 2q = \frac{2p}{p-1}$ and $\|\phi\|_{C^1} \le 1$, we deduce the existence of a constant $C_0 := C(\epsilon)$ such that

$$\int_{B_\epsilon} u\phi^\alpha, \int_{B_\epsilon} u|\nabla\phi|^2\phi^{\alpha-2} \le C_0\Big(\int_{B_\epsilon} u^p\phi^\alpha\Big)^{\frac{1}{p}}. \quad (3.7)$$

Hence, from (3.5), (3.6), and (3.7), and since λ is bounded, we get the existence of a constant $C_1 := C(\epsilon,\lambda)$ such that

$$\int_{B_\epsilon} |h|u^p\phi^\alpha \le C_1\Big\{\int_{B_\epsilon} u^p\phi^\alpha\Big\}^{1/p},$$

which implies

$$\{\inf_{B_\epsilon} |h|\}\Big\{\int_{B_\epsilon} u^p\phi^\alpha\Big\}^{1-1/p} \le C_1.$$

Thus, we finally obtain

$$\inf_{B_{\frac{\epsilon}{2}}(x_0)} \phi^\alpha \int_{B_{\epsilon/2}(x_0)} u^p \le \Big\{ \frac{C_1}{\inf_{B_\epsilon(x_0)} |h|} \Big\}^{\frac{p}{p-1}},$$

from which we immediately get (3.1). □

Bound for u in Ω_δ^- when Ω is bounded. Let us define a δ neighbourhood of the boundary $\partial\Omega$, $G := \{x \in \Omega_\delta^- \ : \ dist(x, \partial\Omega) \le \delta\}$. Let $A := \{x \in \Omega_\delta^- : -\Delta u(x) < 0\}$. We split the domain into three sets $\Omega_\delta^- = (\Omega_\delta^- \setminus G) \cup (G \setminus A) \cup (G \cap A)$. We will get the a priori estimate using, in the first set , the earlier integral estimate, in the second one, a pointwise estimate, and then, in the third set, a maximum principle and the previous estimates.

For any $x \in \Omega_\delta^- \setminus G$, there exists a ball $B_{\delta/2}(x) \subset \Omega^-$ and the integral estimate (3.1) holds for u in $B_{\delta/4}(x)$. Then we use the following (Lemma 9.20 from [10]) :

Lemma 3.1. *Let $u \in W^{2,n}(\Omega)$ with $Lu \ge f$ where L is a strictly elliptic second order-operator with ellipticity constant λ_L and $f \in L^n(\Omega)$. For all $B = B_\epsilon(y) \subset \Omega$ and $p > 0$, we have*

$$\sup_{B_{\frac{\epsilon}{2}}(y)} u \le C(n, p, \epsilon)\Big(\Big(\frac{1}{|B|}\int_B (u^+)^p\Big)^{\frac{1}{p}} + \frac{\epsilon}{\lambda_L}\|f\|_{L^n(B)}\Big). \tag{3.8}$$

Note that, for this lemma, we only require the coefficients of the operator to be bounded. We combine this estimate for $f = 0$, $p = \frac{n+2}{n-2}$ and $L = \Delta + \lambda$ in the ball $B_{\delta/4}(x)$, together with the estimate (3.1), to conclude that

$$\sup_{B_{\delta/4}(x)} u \le C(n, \lambda, \delta)\Big\{\frac{1}{\inf_{B_{\delta/2}(x)} |h|}\Big\}^{\frac{n-2}{4}}. \tag{3.9}$$

Thus we have, if $x \in \Omega_\delta^- \setminus G$,

$$u(x) \le C(n, \lambda, \delta)\Big\{\frac{1}{\inf_{\Omega_{\delta/2}^-} |h|}\Big\}^{\frac{n-2}{4}}. \tag{3.10}$$

In case $x \in G \setminus A$,

$$0 \le -\Delta u(x) = \lambda u(x) + h(x) u^{\frac{n+2}{n-2}}(x),$$

and hence

$$-h(x) u^{\frac{n+2}{n-2}}(x) \le \lambda u(x).$$

Since $u(x) > 0$, we have the following *pointwise estimate*

$$u(x) \leq \Big(\frac{\lambda}{\inf_{\Omega_\delta^-} |h|}\Big)^{\frac{n-2}{4}} \text{ for all } x \in G \setminus A. \tag{3.11}$$

Using the above estimates and recalling that $u = 0$ on $\partial\Omega$, we have for points on $\partial(G \cap A)$,

$$u(x) \leq M, \tag{3.12}$$

where $M = \max\{C(n,\lambda,\delta)\{\frac{1}{\inf_{\Omega_{\delta/2}^-} |h|}\}^{\frac{n-2}{4}}, (\frac{\lambda}{\inf_{\Omega_\delta^-} |h|})^{\frac{n-2}{4}}\}$. Now we show that u has the same bound inside $(G \cap A)$ also. For that define $c(x) := \lambda + h(x)u(x)^{4/n-2}$ and consider the equation

$$\Delta v + c(x)v \geq 0 \text{ in } (G \cap A). \tag{3.13}$$

Note that, for $x \in A$, $c(x) < 0$ and that $u - M$ is a solution of (3.13). Hence, by the generalized weak maximum principle (Theorem 9.1 in [10] with $f \equiv 0$), we have

$$u(x) - M \leq 0 \text{ in } (G \cap A).$$

Combining all the cases, we have proved:

Proposition 3.2. *Let u be a solution of* (1.1) *for a bounded domain Ω and $0 \leq \lambda \leq \lambda_1(\Omega^+)$. Assuming* (H1)*, we have*

$$\sup_{\Omega_\delta^-} u(x) \leq \max\Big\{C(n,\lambda,\delta)\Big\{\frac{1}{\inf_{\Omega_{\delta/2}^-} |h|}\Big\}^{\frac{n-2}{4}}, \Big(\frac{\lambda}{\inf_{\Omega_\delta^-} |h|}\Big)^{\frac{n-2}{4}}\Big\}. \tag{3.14}$$

Remark. If Ω is unbounded, we choose Ω_i, increasing, smooth, bounded domains such that $\cup\Omega_i = \Omega$. Then, it follows from (3.14) that the solutions u_i of the approximate problem in Ω_i (see Section 6) satisfy

$$\sup_{(\Omega_i)_\delta^-} u(x) \leq \max\Big\{C(n,\lambda,\delta)\Big\{\frac{1}{\inf_{(\Omega_i)_{\delta/2}^-} |h|}\Big\}^{\frac{n-2}{4}}, \Big(\frac{\lambda}{\inf_{(\Omega_i)_\delta^-} |h|}\Big)^{\frac{n-2}{4}}\Big\}.$$

Now if $\limsup_{|x|\to+\infty} h(x) < 0$, it follows that the u_i's are uniformly bounded in Ω^-. Hence, if (H4) holds, then we get a uniform bound for $\{u_i\}$. Although (H4) is stronger, we need it to get the uniform decay of u_i (see Section 6).

4. Estimates in Γ_δ

We will get a bound for u in Γ_δ, using the bound obtained in Ω_δ^- and the moving-plane method. Let us fix $x_0 \in \Gamma$. Since Γ is compact, it is sufficient to give an a priori bound in a neighborhood of x_0. The sketch of the proof is the following:

1. By making first a transformation preserving some properties of the coefficients of the equation, we construct a convex neighborhood of x_0.

2. Applying in this domain the moving-plane method to an auxiliary function (similar to [7]), we show a "Harnack inequality" satisfied by u in a cone with x_0 as vertex. Combining this inequality with the integral estimate (3.1), we get the a priori bound.

1. A strict convex neighborhood of x_0. Up to some rotation or translation, we can suppose that $x_0 = 0$ and that Γ is tangent to the hyperplane $x_1 = 0$. Doing a Kelvin transform (take the center of the inversion on the x_1 axis such that the sphere is tangent to $x_1 = 0$), we can suppose Ω^+ is at the left of Γ and also strictly convex in the x_1 direction in a neighborhood of x_0. But, contrary to the case of [7], the equation is not preserved by the Kelvin transform. Indeed, let K be the Kelvin transform with y_0 as the center of the inversion; that is: $K : \mathbb{R}^n \setminus \{y_0\} \to \mathbb{R}^n,\ x \mapsto y_0 + \frac{x-y_0}{|x-y_0|^2}|y_0|^2$, and let $\bar{u}$ be the Kelvin transform of u; that is,

$$\bar{u}(x) = \Big(\frac{|y_0|}{|x-y_0|}\Big)^{n-2} u(K(x)).$$

Then, $\bar{u}$ satisfies the following equation :

$$-\Delta \bar{u} = \lambda a(x)\bar{u} + \tilde{h}(x)\bar{u}^{\frac{n+2}{n-2}}, \tag{4.1}$$

$$a(x) = \Big(\frac{|y_0|}{|x-y_0|}\Big)^4,\ \tilde{h}(x) = h(K(x)).$$

Given $\eta > 0$, consider the convex domain D containing x_0 enclosed by the surfaces

$$\partial^1 D := \big\{x \in \Omega^- : \ dist(x,\Gamma) = \eta\big\} \quad \text{and} \quad \partial^2 D := \{x : \ x_1 = -5\eta\}.$$

Since $y_0 \neq 0$, by choosing η such that $5\eta < |y_0|$, we have $a, \tilde{h} \in C_0(D)$. Moreover, the assumptions made on h in (H2) are inherited by $\tilde{h}$ in a neighborhood of $K(x_0) = 0$.

In the sequel, for notational convenience, we will denote $\tilde{h}$ by h and $\bar{u}$ by u.

With the aim of applying a moving-plane method to some auxiliary function in the domain D, we are led to choose η small enough in such a way that

$$\lambda_1(-\Delta - \lambda a(x), D) > 0, \tag{4.2}$$

$$\frac{\partial a}{\partial x_1}(x) \le 0 \quad x \in D, \tag{4.3}$$

$$\sup_D\{\frac{\partial h}{\partial x_1}\} < 0. \tag{4.4}$$

The condition (4.2) holds if η is small enough to ensure that $\lambda_1(-\Delta, D) > \lambda\|a\|_\infty$. The condition (4.4) is made possible by (H2).

2. Moving-plane method and Harnack inequality. Let $\tilde{u}$ be a continuous extension of u on all of ∂D such that $0 \le \tilde{u} \le \sup_{\partial^1 D} u$. Since $\partial^1 D \subset \Omega_\eta^-$, Proposition 3.2, shows that $\tilde{u} \le m$, where m is defined by (3.14). Let $C_0 > 0$ be a constant to be fixed later and $g \in C^1(\overline{D})$ a function satisfying

$$g(x) < 0 \quad \text{ and } \quad \frac{\partial g}{\partial x_1}(x) > 0 \quad \forall x \in \overline{D}, \tag{4.5}$$

(for example, $g(x) = -A + x_1$ with $A > 0$ chosen to ensure $g < 0$ in $\overline{D}$).

We consider the function w, a solution of the following problem (which is well defined thanks to (4.2)):

$$\begin{cases} -\Delta w - \lambda a(x) w &= C_0 g \quad \text{in } D \\ w &= \tilde{u} \quad \text{on } \partial D. \end{cases}$$

We introduce the auxiliary function $v = u - w$. One can see that v satisfies :

$$\begin{cases} -\Delta v &= f(x,v) \quad \text{in } D \\ v &= 0 \quad \text{on } \partial^1 D, \end{cases}$$

where $f(x,v) = \lambda a(x) v + \lambda h(x)(v+w)^{\frac{n+2}{n-2}} - C_0 g$. We claim that by choosing C_0 large enough and $\eta_1 \in (0, \eta)$ small enough, the following conditions can be realized:

$$v \ge 0 \quad \text{on } D \cap \{-\eta < x_1 < \eta\} \tag{4.6}$$

$$\frac{\partial f}{\partial x_1}(x,v) \le 0 \quad \forall x \in D \cap \{-2\eta_1 < x_1 < \eta\}, \forall v > 0. \tag{4.7}$$

To prove (4.6), we are going to estimate w and $\frac{\partial w}{\partial x_1}$ in D. To this end, let us consider (H, G) solutions of

$$\Delta H + \lambda a(x) H = 0 \quad \text{in } D, \qquad \Delta G + \lambda a(x) G = g \text{ in } D,$$
$$H = \tilde{u} \quad \text{on } \partial D, \qquad G = 0 \quad \text{on } \partial D,$$

allowing us to split w as

$$w = H - C_0 G. \tag{4.8}$$

Since $\lambda_1(-\Delta - \lambda a(x), D) > 0$ (see (4.2)), the maximum principle holds for the operator $-\Delta - \lambda a(x)$. Therefore, on the one hand, by applying Theorem I.3 in [3] which extends the Alexandrov-Bakelman-Pucci estimate for narrow domains, we obtain

$$\|H\|_{C_0(D)} \leq C \sup_{\partial D} H \leq Cm. \tag{4.9}$$

On the other hand, since $g \leq 0$ (see (4.5)), we get

$$G > 0 \quad \text{ on } D, \tag{4.10}$$

and from Hopf's Lemma,

$$\frac{\partial G}{\partial x_1} < 0 \quad \text{ on } \partial D \cap \{-\eta \leq x_1 \leq \eta\}. \tag{4.11}$$

Let $D_\eta \subset\subset D \cap \Omega^-$ be a tubular neighborhood of $\partial^1 D \cap \{-\eta \leq x_1 \leq \eta\}$ such that

$$\sup_{D_\eta} \frac{\partial G}{\partial x_1} < 0. \tag{4.12}$$

Let us first show that (4.6) holds on D_η. Since $v = 0$ on $\partial^1 D$, it is sufficient to prove that $\frac{\partial v}{\partial x_1} \leq 0$ in D_η. Clearly, by the definition of v,

$$\frac{\partial v}{\partial x_1} = \frac{\partial u}{\partial x_1} - \frac{\partial w}{\partial x_1} = \frac{\partial u}{\partial x_1} - \frac{\partial H}{\partial x_1} + C_0 \frac{\partial G}{\partial x_1}. \tag{4.13}$$

Since $D_\eta \subset\subset \Omega^-$, by the estimates obtained in the previous step and by standard elliptic estimates, we have :

$$\sup_{x \in D_\eta} \left| \frac{\partial u}{\partial x_1} \right| \leq Cm. \tag{4.14}$$

From (4.9) and Theorem 8.33 in [10], we have :

$$\left\| \frac{\partial H}{\partial x_1} \right\|_{C_0(D_\eta)} \leq C \Big(\sup_D H + \sup_{\partial D \cap \{-\eta \leq x_1 \leq \eta\}} \left| \frac{\partial \tilde{u}}{\partial x_1} \right| \Big) \leq Cm. \tag{4.15}$$

From (4.13), (4.14), and (4.15), it follows that

$$\frac{\partial v}{\partial x_1} \leq Cm + C_0 \sup_{D_\eta} \frac{\partial G}{\partial x_1}. \tag{4.16}$$

Now, using (4.12), the right-hand side of (4.16) can be made negative on D_η by choosing C_0 large enough. Combining (4.16) with $v = 0$ for x in $\partial^1 D$, we obtain $v \geq 0$ in D_η.

On the compact set $K := D \cap \{-\eta \le x_1 \le \eta\} \setminus D_\eta$, by using $u \ge 0$ and (4.9) we get

$$v \ge -w = -H + C_0 G \ge -Cm + C_0 \inf_K G. \tag{4.17}$$

Using now the property (4.10), we can choose C_0 large enough and make the right-hand side of (4.17) positive. This concludes the proof that $v \ge 0$ in $D \cap \{-\eta < x_1 < \eta\}$.

Let us now prove (4.7). A simple computation yields :

$$\begin{aligned}\frac{\partial f}{\partial x_1}(x,v) &= \lambda \frac{\partial a}{\partial x_1}(x) v + \lambda \frac{\partial h}{\partial x_1}(x)(v+w(x))^{\frac{n+2}{n-2}} \\ &+ h(x)\frac{\partial w}{\partial x_1}(x)\frac{n+2}{n-2}(v+w(x))^{\frac{4}{n-2}} - C_0 \frac{\partial g}{\partial x_1}(x).\end{aligned}$$

Using (4.3) and the assumption $\lambda \ge 0$, we get

$$\frac{\partial f}{\partial x_1}(x,v) \le \lambda \frac{\partial h}{\partial x_1}(x)(v+w)^{\frac{n+2}{n-2}} + h(x)\frac{\partial w}{\partial x_1}(x)\frac{n+2}{n-2}(v+w)^{\frac{4}{n-2}} - C_0 \frac{\partial g}{\partial x_1}. \tag{4.18}$$

We consider now two cases:

First, $h(x) \le 0$. In this case, since $\frac{\partial h}{\partial x_1} \le 0$ in D, it suffices to prove that $\frac{\partial w}{\partial x_1} \ge 0$ (for C_0 large). From (4.8) and taking into account (4.15) we obtain

$$\frac{\partial w}{\partial x_1} = -\frac{\partial H}{\partial x_1} + C_0 \frac{\partial G}{\partial x_1} \le Cm + C_0 \frac{\partial G}{\partial x_1}. \tag{4.19}$$

Now, since $\frac{\partial a}{\partial x_1} \le 0$ on D, we can apply the moving plane to the equation satisfied by G, and derive $\frac{\partial G}{\partial x_1} < 0$ on $D \cap \{-\eta < x_1 < \eta\}$ (see [11]). Hence, by choosing C_0 large enough, the right-hand side can be made negative. Now, let us consider the case where $h(x) > 0$. Since

$$h(x) \le C\eta_1 \quad \text{for } -\eta_1 < x < 0, \tag{4.20}$$

we get from (4.18) that

$$\frac{\partial f}{\partial x_1}(x,v) \le -F_1(v+w(x))^{\frac{n+2}{n-2}} + F_2(v+w(x))^{\frac{4}{n-2}} - F_3, \tag{4.21}$$

where F_i are strictly positive reals defined as

$$F_1 = \lambda |\sup_D \{\frac{\partial h}{\partial x_1}\}|, \quad F_2 := C\eta_1, \quad F_3 = C_0 \inf_D \{\frac{\partial g}{\partial x_1}\}.$$

Now, the function $F : [0,\infty) \to \mathbb{R}$, $\xi \mapsto -F_1 \xi^p + F_2 \xi^{p-1} - F_3$ satisfies

$$F(0) < 0, \quad F' > 0 \text{ near } \xi = 0, \quad \lim_{\xi \to \infty} F(\xi) = -\infty.$$

Therefore, the function F has a maximum which is negative as soon as F_2 is small enough, a condition which can be realized by choosing η_1 small enough. Hence, going back to (4.21) with this choice of η_1, we conclude that

$$\frac{\partial f}{\partial x_1}(x,v) \leq 0 \quad \forall x \in D \cap \{-2\eta_1 < x_1 < \eta\}, \quad \forall v > 0. \tag{4.22}$$

Since $v \geq 0$, $v = 0$ in $\partial^1 D$, and (4.22) is satisfied, we can apply the moving-plane method to the equation (4.6) to prove that v is monotone decreasing in the x_1 direction on the domain $D \cap \{-\eta_1 < x_1 < \eta\}$ (see for instance [11]). At this point, we conclude as in [7] (Section 3, step 4 : deriving the a priori bound). Let us just sketch the proof.

Since the function v is monotone decreasing in the x_1 direction, this is still true if we rotate the x_1-axis by a small angle. Therefore, for any $x_0 \in \Gamma$, there exists Δ_{x_0}, a cone of vertex x_0 and staying to the left of x_0, such that

$$v(x) \geq v(x_0) \quad \text{for } \; x \in \Delta_{x_0}. \tag{4.23}$$

From (4.23), we obtain

$$u(x) + C \geq u(x_0) \quad \text{for } \; x \in \Delta_{x_0}. \tag{4.24}$$

By a similar argument, one can prove that (4.24) is true for any point x in a small neighborhood of Γ. Remarking that the intersection of Δ_{x_0} with the set $\{x : h(x) \geq \delta > 0\}$ has a positive measure, and combining this with the integral estimate (3.1) we get the a priori bound in the neighborhood of Γ

$$||u||_{\infty,\Gamma_\delta} \leq C(n, \delta, \lambda_1(\Omega^+), \frac{1}{\inf_{\Omega_\delta^+} |h|}, \Gamma). \tag{4.25}$$

5. Estimates in Ω_δ^+

We will look for conditions on h which will ensure that the sequence (λ_i, v_i), solutions of (1.1), does not blow up in Ω_δ^+. Supposing that they are not bounded, we have a sequence of local maxima $x_i \in \Omega_\delta^+$ of v_i, such that $v_i(x_i) \to \infty$ as $i \to \infty$. By the earlier section, since the elements of $\{v_i\}$ are uniformly bounded near Γ it follows that the x_i's are away from the boundary $\partial\Omega_\delta^+$ of Ω_δ^+ for large i and hence $x_i \to x_0 \in \Omega_\delta^+$. Since here we are dealing with only *interior blow up points*, we may as well consider more generally a sequence (λ_i, v_i), satisfying, in Ω_δ^+,

$$-\Delta v_i(x) = \lambda_i v_i(x) + h(x) v_i^{p_i} \tag{5.1}$$

corresponding to subcritical power $1 < p_i = \frac{n+2}{n-2} - \tau_i$, $\tau_i \to 0$ and $\lambda_i \to \tilde{\lambda}$ for some $\tilde{\lambda} \in (0, \lambda_1(\Omega^+)]$, assuming that

$$\{v_i\} \text{ remains uniformly bounded on the boundary of } \Omega_\delta^+.$$

If $\tilde{\lambda} = 0$, for the final step, we *fix* $p_i = \frac{n+2}{n-2}$ for all i in the equation (5.1). Then the arguments are similar to [12], after using the estimate for the linear term as in [6]. We arrive at the L^∞ bounds for a sequence (λ_i, v_i) with $\lambda_i \to 0$ assuming condition (H3). However, if $\tilde{\lambda} \neq 0$, we remark that (H3) can be weakened to :

$(H3)'$ If $\mathcal{S} = \{x \in \mathbb{R}^n : h(x) > 0, \nabla h(x) = 0\}$, then

$$c_1[dist(x, \mathcal{S})]^{\theta-1} \leq |\nabla h(x)| \leq c_2[dist(x, \mathcal{S})]^{\theta-1}$$

for all x in $\mathcal{S}_d = \{x \in \mathbb{R}^n : h(x) > 0, |\nabla h(x)| < d\}$, for some $d > 0$ and $n - 2 < \theta \leq n$. The proofs have been given below for this case using this weaker assumption.

Observe that the condition (H3) imposes a flatness of order θ on h. The fact that $\theta > n-2$ is the right threshhold for the blowing up solutions to behave like "standard solutions" was first identified in [12], namely, the $(*)_\theta$ condition there.

In the first subsection we will give the standard blow up argument in Proposition 5.1 (see [15]) to analyze v_i, in a small neighbourhood of x_i and also derive various local estimates required later on. In the second subsection, we use these estimates to prove that a blow up point of v_i is necessarily a critical point of h. This motivates the assumption (H3). Using this assumption, we analyze the nature of the blow up points and show that in fact v_i does not blow up; i.e., the sequence $\{v_i\}$ is uniformly bounded.

5.1. Blow up points of $\{v_i\}$.

Proposition 5.1. *Suppose that $h \in C^1(\Omega_\delta^+)$ and there exist A_1 and A_2 such that, in Ω_δ^+,*

$$h(x) \geq \frac{1}{A_1}, \quad \|\nabla h(x)\| \leq A_2.$$

Then for every $0 < \varepsilon < 1$, $R > 1$, there exist positive constants C_0 and C_1 depending on $A_1, A_2, \varepsilon, R, \lambda$, and n such that if v is a positive solution of ,

$$-\Delta v(x) = \lambda v(x) + h(x) v^p, \quad v > 0 \tag{5.2}$$

with $\max_B v > C_0$, then there exists a finite number $k = k(v)$ and a set of local maxima in Ω_δ^+ of v, and a set $S(v, C_0) = \{x_1, \ldots, x_k\} \subset \Omega_\delta^+$ such that

(i) x_j *are the local maxima of* v *and* $\{B_{R\lambda_j}(x_j)\}_{1\le j\le k}$ *are disjoint balls for* $\lambda_j = v(x_j)^{-\frac{(p_i-1)}{2}}$, *and*

$$\|v(x_j)^{-1}v(x_j+\lambda_j x) - \delta_j(x)\|_{C^2(B_{2R}(0))} < \varepsilon,$$

where

$$\delta_j(x) = (1 + h_j|x|^2)^{\frac{2-n}{2}} \text{ with } h_j = (n(n-2))^{-1}h(x_j)$$

is the unique solution of

$$\begin{aligned} \Delta\delta_j + h_j\delta_j^{\frac{n+2}{n-2}} &= 0 \quad \text{in} \quad \mathbb{R}^n, \\ \delta_j &> 0 \quad \text{in} \quad \mathbb{R}^n, \quad \delta_j(0) = 1; \end{aligned}$$

(ii) $v(x) \le C_1(dist(x,S))^{-(\frac{2}{(p-1)})}, x \in \Omega_\delta^+$; *and*

(iii) $|p - \frac{n+2}{n-2}| < \varepsilon$.

Proof of Proposition 5.1. Let $\{v_i\}$ be a sequence of positive solutions of 5.1 with $\max_{\Omega_\delta^+} v_i \to \infty$. Let $v_i(x_i) = \max_{\overline{\Omega_\delta^+}} v_i$. Since $\{v_i\}$ is bounded near Γ and Ω_δ^- uniformly, we have that the elements of $\{x_i\}$ are away from Γ and hence $x_i \to x_0$ as $i \to \infty$ with $x_0 \in \Omega_\delta^+$. In fact, there exists $\overline{r} > 0$ such that $B(x_0, \overline{r}) \subset \Omega_\delta^+$. Then, for r small, $0 < r < \overline{r}$, $B_r(x_i) \subset \Omega_\delta^+$, for all i, after discarding a finite number of x_i's if necessary.

Consider the rescaled function

$$\zeta_i(x) = \frac{1}{v_i(x_i)} v_i\Big(x_i + \frac{x}{(v_i(x_i))^{\frac{p_i-1}{2}}}\Big)$$

in $B_r(x_i)$. Then ζ_i satisfies

$$-\Delta\zeta_i(x) = \frac{\lambda}{(v_i(x_i))^{p_i-1}}\zeta_i(x) + \hat{h}(x)\zeta_i^{p_i}$$

in $B_{R_i}(0), R_i = r(v_i(x_i))^{(\frac{p_i-1}{2})}$ and $\hat{h}(x) = h(x_i + \frac{x}{(v_i(x_i))^{\frac{p_i-1}{2}}})$. For any fixed compact set K, one can find R large such that $K \subset B_R(0) \subset B_{R_i}(0)$, for all i large. Since h is bounded on $\overline{\Omega}^+$ and $\zeta_i(x) \le 1$ on $B_R(0)$, one finds that the right-hand side of the equation for ζ_i is in $L^\infty(B_R(0))$ and hence in all the L^p's for $1 < p \le \infty$. Thus, by elliptic regularity theory, $\zeta_i \in W^{2,p}(B_R(0))$ for all p and hence $\{\zeta_i\}$ is uniformly bounded in $C^{2,\alpha}_{loc}$, using the Sobolev inclusion $W^{2,p}(B_R(0)) \hookrightarrow C^{2,\alpha}(B_R(0))$, for some p large and $\alpha > 0$ satisfying $\alpha < 2 - \frac{n}{p}$. Then by using Ascoli-Arzela's theorem, one finds that $\zeta_i \to \zeta$ in

C^2_{loc} where ζ is the unique solution of the equation

$$\begin{cases} -\Delta\zeta &= h^*(\zeta)^{\frac{n+2}{n-2}} \quad \text{in } \mathbb{R}^n, \\ \zeta &> 0; \quad \zeta(0) = 1, \end{cases}$$

where $h^* = h(\lim_{i\to\infty} x_i) = h(x_0)$. Then it follows that, for a given ε ,

$$\|\zeta_i(x) - \zeta(x)\|_{C^2(B_R(0))} < \varepsilon$$

holds for all large i. Combining this estimate with the fact

$$\zeta(x) = \frac{1}{(1 + h^*|x - x_0|^2)^{\frac{n-2}{2}}},$$

we get $\zeta_i(x) < \varepsilon + \zeta(x) < 2\zeta(x)$ if $\varepsilon < \min_{B_R} \zeta(x) = \zeta(R)$. It then follows that

$$v_i(x) < \frac{C}{|x - x_0|^{(\frac{2}{p_i-1})}} \quad \text{for} \quad x \in B(x_i, Rv_i(x_i)^{-\frac{p_i-1}{2}}).$$

Now consider the functions $\left\{v_i(x)|x - x_i|^{2/(p_i-1)}\right\}$. If this sequence of functions is not bounded on Ω^+_δ, then let $\{x_i^{(2)}\}$ be the maxima of these functions. Then $\{v_i(x_i^{(2)})\}$ has to go to infinity and $x_i^{(2)} \neq x_i$ and $\{x_i^{(2)}\}$ converges to some point $x_0^{(2)} \in \Omega^+_\delta$. Now, rescaling v_i in a small neighbourhood of $x_i^{(2)}$ and repeating the argument as before, we get the local estimate near $x_i^{(2)}$. The above process stops after a finite stage, after we get $\{x_i^{(1)}\}\dots\{x_i^{(k_i)}\}$, because for each v_i the energy, $\int(|\nabla(v_i)|)^2 - \lambda\int(v_i)^2$, is fixed and near each local maximum it is larger than a fixed positive number, η:

$$\begin{aligned}\int_{B_r(x_i)} h(x)v_i^{p_i+1} &= \int_{B_R(0)} \tilde{h}(x)\zeta^{\frac{2n}{n-2}} \\ &= \int_{B_R(0)} (\tilde{h}(x) - h^*)\zeta^{\frac{2n}{n-2}} + \int_{B_R(0)} h^*\zeta^{\frac{2n}{n-2}} > o(1) + \eta.\end{aligned}$$

□

The above proposition, in particular (ii), motivates the definition of an isolated blow up point.

Definition 1. A point $x_0 \in \Omega'$ is called an isolated blow up point of $\{v_i\}$, solutions of (5.1), if there exists $0 < \bar{r} < dist(x_0, \partial\Omega')$ and $C > 0$ and a sequence $\{x_i\}$ tending to x_0, such that x_i is a local maximum of $\{v_i\}$, $v_i(x_i) \to \infty$ and

$$v_i(x) \leq C|x - x_i|^{-(\frac{2}{p_i-1})} \quad \forall\, x \in B_{\bar{r}}(x_0).$$

Since we will be interested in the blow up points staying away from each other, we also need to introduce the definition of a simple isolated blow up point.

Definition 2. x_0 is an isolated simple blow up point of $\{v_i\}$, solution of (5.1), if it is an isolated blow up point such that for some $\rho > 0$ (independent of i), $\tilde{v}_i$ has precisely one critical point in $(0, \rho)$ for all large i, where

$$\tilde{v}_i(r) = r^{\frac{2}{p_i - 1}} \overline{v}_i(r), \quad \overline{v}_i(r) = \frac{1}{|\partial B_r|} \int_{\partial B_r(x_i)} v_i, \quad r > 0.$$

As a corollary of Proposition 5.1 we have

Corollary 5.1. *Let x_0 be an isolated blow up point of $\{v_i\}$. Then one can choose $R_i \to \infty$ first and then $(\varepsilon)_i \to 0^+$ depending on R_i and a subsequence $\{v_i\}$ so that*

(i) $r_i = \frac{R_i}{(v_i(x_i))^{\frac{p_i - 1}{2}}} \to 0$ and x_i is the only critical point of $v_i(x)$ in $|x - x_i| < r_i$; and

(ii) $\tilde{v}_i(r)$ has a unique critical point in $0 < r < r_i$. In particular, for simple isolated blow up points, it then follows that $\tilde{v}_i(r)$ is strictly decreasing in (r_i, ρ).

The proof will follow using the arguments of the proof of Proposition 5.1. See also [12] (Proposition 2.1 there).

We now state the two versions of Pohozaev identities which will be frequently used in the later proofs:

Lemma 5.1. (Pohozaev Identity) *Let v be a C^2 solution of* (5.1) *and for $\sigma > 0$ consider the ball $B_\sigma \subset \Omega_\delta^+$. Let ν denote the unit outer-normal vector field on the boundary ∂B_σ. We have*

$$(I) \int_{\partial B_\sigma} B(\sigma, x, v, \nabla v) = \lambda \int_{B_\sigma} v^2 - \frac{\lambda}{2} \int_{\partial B_\sigma} \sigma v^2 + \frac{1}{p+1} \int_{B_\sigma} (x \cdot \nabla h) v^{p+1}$$
$$+ \Big(\frac{n}{p+1} - \frac{n-2}{2}\Big) \int_{B_\sigma} h(x) v^{p+1} - \frac{\sigma}{p+1} \int_{\partial B_\sigma} h(x) v^{p+1},$$

where

$$B(\sigma, x, v, \nabla v) := -\frac{(n-2)}{2} v \frac{\partial v}{\partial \nu} - \frac{\sigma}{2} |\nabla v|^2 + \sigma \Big|\frac{\partial v}{\partial \nu}\Big|^2;$$

$$(II) \int_{B_\sigma} \nabla h v^{p+1}\, dx = \int_{\partial B_\sigma} ((p+1)(\frac{\lambda}{2} v^2 \nu + \nabla v \frac{\partial v}{\partial \nu} - \frac{1}{2} |\nabla v|^2 \nu) + h v^{p+1} \nu) dS_\sigma.$$

Proof. Multiplying (5.1) by $\sum_i x_i v_i$, we get

$$\frac{(2-n)}{2}\int_{B_\sigma}|\nabla v|^2+\frac{1}{2}\int_{\partial B_\sigma}x\cdot\nu|\nabla v|^2-\int_{\partial B_\sigma}(x\cdot\nabla v)\frac{\partial v}{\partial\nu}$$

$$=-n\int_{B_\sigma}(\lambda\frac{v^2}{2}+h(x)\frac{v^{p+1}}{p+1})-\int_{B_\sigma}(x\cdot\nabla h)\frac{v^{p+1}}{p+1}+\int_{\partial B_\sigma}x\cdot\nu(\lambda\frac{v^2}{2}+h(x)\frac{v^{p+1}}{p+1}).$$

Multiplying the equation by v and integrating by parts,

$$\int_{B_\sigma}|\nabla v|^2=\int_{B_\sigma}\lambda v^2+h(x)v^{p+1}\int_{\partial B_\sigma}v\frac{\partial v}{\partial\nu}.$$

Using this, we get after simplification

$$\int_{B_\sigma}(x\cdot\nabla h)\frac{v^{p+1}}{p+1}+\Big(\frac{n}{p+1}-\frac{n-2}{2}\Big)\int_{B_\sigma}h(x)v^{p+1}-\frac{\sigma}{p+1}\int_{\partial B_\sigma}h(x)v^{p+1}$$

$$+\int_{B_\sigma}\lambda v^2-\frac{\lambda}{2}\int_{\partial B_\sigma}\sigma v^2=\frac{n-2}{2}\int_{\partial B_\sigma}v\frac{\partial\nu}{\partial\nu}-\frac{\sigma}{2}\int_{\partial B_\sigma}|\nabla v|^2+\sigma\int_{\partial B_\sigma}\Big(\frac{\partial v}{\partial\nu}\Big)^2.$$

This completes the proof of the Pohozaev identity (I).

The Pohozaev identity (II) can be easily obtained by multiplying the equation (5.1) by $\frac{\partial v}{\partial x_k}$ $(1\le k\le n)$ and integrating by parts over B_σ. □

Corollary 5.2. *For $u(x)=\frac{a}{|x|^{n-2}}+b(x)$ where $a>0$ and $b(x)$ is a nonnegative differentiable function, with $b(0)>0$, we have $B(\sigma,x,u,\nabla u)<0$ on ∂B_σ, for all σ small.*

The proof follows by direct computation.

Lemma 5.2. (A Harnack inequality) *Let h satisfy*

$$\frac{1}{A_1}\le h(x)\le A_1\quad\forall\quad x\in\Omega_\delta^+ \tag{5.3}$$

and $\{v_i\}$ satisfy (5.1), having 0 as an isolated blow up point. Then for any $0<r<\frac{\bar r}{3}$, with $\bar r$ as in Definition 1, we have the Harnack inequality

$$\max_{B_{2r}\setminus B_{r/2}}v_i(y)\le C\min_{B_{2r}\setminus B_{r/2}}v_i(y) \tag{5.4}$$

with a uniform $C=C(n,\lambda,\|h\|_{L^\infty(\Omega_\delta^+)})$.

The proof of this lemma follows along the same lines as in [12] and [6].

Now we look for lower and upper bounds for v_i, in a fixed neighbourhood of the blow up point. The arguments are as in [12] (Section 2 there). The main difference is that, for the upper bound for v_i, we need to exploit specifically the extra linear term in our case, in Lemma 5.3 and in Proposition 5.3.

Proposition 5.2. *Suppose* $h \in C^1(B_2)$ *and*

$$A_1 \geq h(x) \geq \frac{1}{A_1}, \quad ||\nabla h(x)|| \leq A_2 \text{ for all } x \in B_2 \tag{5.5}$$

for some positive constants A_1, A_2. *Let* v_i *be solutions of* 5.1 *and* $x_i \to 0$ *be an isolated blow up point with*

$$v_i(x) \leq \frac{A_3}{|x - x_i|^{\frac{2}{p_i - 1}}} \text{ for all } x \in B_2. \tag{5.6}$$

Then there exists a positive constant $C = C(n, \lambda_0, A_1, A_2, A_3)$, *such that up to a subsequence,*

$$v_i(x) \geq C v_i(x_i)(1 + h_i v_i(x_i)^{p_i - 1}|x - x_i|^2)^{2-n/2} \text{ for all } |x - x_i| \leq 1, \tag{5.7}$$

where h_i *is as defined in Proposition* 5.1. *In particular, for any* $e \in \mathbb{R}^n$, $|e| = 1$, *we have*

$$v_i(x_i + e) \geq C^{-1} v_i(x_i)^{-1}. \tag{5.8}$$

The proof is similar to that of Proposition 2.2 of [12]. Under the additional assumption that the isolated blow up point is also simple, we obtain below an upper bound for v_i, in $B(0,1)$.

Proposition 5.3. *Let* h *and* $\{v_i\}$ *satisfy the conditions as in Proposition* 5.2. *Moreover, assume that* $x_i \to 0$ *is an isolated simple blow up point as defined in Definition* 2 *and* $(H3)'$ *holds. Then there exists a positive constant* $C = C(n, \lambda_0, A_1, A_2, A_3, \rho)$ *such that*

$$v_i(x) \leq v_i(x_i)^{-1}|x - x_i|^{2-n} \quad \text{for all } 0 < |x - x_i| \leq 1. \tag{5.9}$$

For the proof of Proposition 5.3 we need

Lemma 5.3. *Let* h *satisfy* (5.5) *and* 0 *be an isolated simple blow up point of* $\{v_i\}$. *Then there exists* $\delta_i > 0$, $\delta_i = O(R_i^{-2} + o(1))$ *such that*

$$v_i(y) \leq C v_i(0)^{-\alpha_i}|y|^{2-n+\delta_i} \quad \forall \quad R_i v_i(0)^{\frac{-(p_i-1)}{2}} \leq |y| \leq 1, \tag{5.10}$$

where $\alpha_i = ((n - 2 - \delta_i)(p_i - 1)/2) - 1$ *and* C *is some positive constant depending only on* $n, A_1, A_3, \rho, \lambda_0$.

Proof. Our aim here is to construct a suitable test function to compare with $\{v_i\}$, in order to get (5.10). Consider the operator

$$\mathcal{L}_i \varphi = \Delta\varphi + h v_i^{p_i - 1}\varphi + \lambda\varphi.$$

Then v_i satisfies $\mathcal{L}_i v_i = 0$. From direct calculations, for $0 \leq \mu \leq n - 2$, we have,

$$\Delta(|x - x_i|^{-\mu}) = -\mu(n - 2 - \mu)|x - x_i|^{-2-\mu} \text{ for } |x - x_i| > 0.$$

As in [12], it can be seen that

$$v_i(x)^{(p_i-1)} \le CR_i^{-2}|x-x_i|^{-2} \text{ for all } r_i \le |x-x_i| \le \rho. \tag{5.11}$$

Therefore, using (5.11), we get

$$\mathcal{L}_i(|x-x_i|^{-\mu}) \le -\frac{\mu(n-2-\mu)}{|x-x_i|^{\mu+2}} + \frac{C}{R_i^2|x-x_i|^{\mu+2}} + \frac{\lambda}{|x-x_i|^{\mu}}. \tag{5.12}$$

For $\mu < 1/2$ very small, consider the function

$$f_{\mu,\eta}(x) = |x-x_i|^{-\mu} + |x-x_i|^{-\eta}$$

for some η fixed, $0 < \eta < \frac{2}{n-2}$. Using (5.12), we have

$$\begin{aligned} \mathcal{L}_i f_{\mu,\eta}(x) \quad \le \quad & -\frac{\mu(n-2-\mu)}{|x-x_i|^{\mu+2}} + \frac{C}{R_i^2|x-x_i|^{\mu+2}} + \frac{\lambda}{|x-x_i|^{\mu}} \\ & -\frac{\eta(n-2-\eta)}{|x-x_i|^{\eta+2}} + \frac{C}{R_i^2|x-x_i|^{\eta+2}} + \frac{\lambda}{|x-x_i|^{\eta}}. \end{aligned}$$

Regrouping the terms on the right-hand side above, we get

$$\mathcal{L}_i f_{\mu,\eta}(x) \le (-\mu(n-2-\mu) + \tfrac{C}{R_i^2})\tfrac{1}{|x-x_i|^{\mu+2}} \tag{5.13}$$

$$+(-\eta(n-2-\eta) + \tfrac{C}{R_i^2} + \lambda|x-x_i|^{\eta+2-\mu} + \lambda|x-x_i|^2)\tfrac{1}{|x-x_i|^{\eta+2}}. \tag{5.14}$$

We choose

$$\mu = \delta_i = \frac{C}{R_i^2}, \tag{5.15}$$

so that the expression in (5.13) is negative. Observe that, for large i , $\delta_i < \eta$ and hence $\eta + 2 - \delta_i > 1$. Thus we have the expression in (5.14) negative, if, for example,

$$|\frac{C}{R_i^2}| < \eta(n-2-\eta)/2, \quad |x-x_i| \le \Big(\frac{\eta(n-2-\eta)}{2\lambda}\Big).$$

In particular, if we define

$$\rho_\lambda := \min\{\rho, \Big(\frac{\eta(n-2-\eta)}{2\lambda}\Big)\}. \tag{5.16}$$

then from the above discussion it follows that for $|x-x_i| \le \rho_\lambda$

$$\mathcal{L}_i f_{\delta_i,\eta} \le 0; \quad \mathcal{L}_i f_{n-2-\delta_i,n-2-\eta} \le 0.$$

For $r_i \le |x-x_i| \le \rho_\lambda$, define

$$\varphi_i = M_i \rho_\lambda^{\delta_i} f_{\delta_i,\eta} + A v_i(x_i)^{-\alpha_i} f_{n-2-\delta_i,n-2-\eta},$$

where $M_i = \max_{\partial B_{\rho_\lambda}} v_i$ and A is a large constant chosen as follows: From (5.11), we have

$$v_i(x)^{(p_i-1)/2} \le \frac{C}{R_i r_i} \text{ for } |x-x_i| = r_i.$$

In particular, we have

$$v_i(x)^{1+\alpha_i} \le \frac{C_1}{{R_i}^{n-2-\delta_i}{r_i}^{n-2-\delta_i}} \text{ for } |x-x_i| = r_i. \tag{5.17}$$

If we choose $A > \frac{C_1}{{R_i}^{n-2-\delta_i}}$, then it follows from (5.17) that for $|x-x_i| = r_i$

$$v_i(x) \le \frac{C_1}{{R_i}^{n-2-\delta_i} v_i(x)^{\alpha_i}{r_i}^{n-2-\delta_i}} \le \frac{A}{v_i(x)^{\alpha_i}{r_i}^{n-2-\delta_i}} \le \varphi_i(x).$$

Observe that on $|x-x_i| = \rho_\lambda$

$$v_i(x) \le M_i = M_i \frac{\rho_\lambda^{\delta_i}}{|x-x_i|^{\delta_i}} \le \varphi_i(x).$$

Hence we have

$$\mathcal{L}_i\varphi_i \le 0 = \mathcal{L}v_i \quad in \quad r_i < |x-x_i| < \rho_\lambda,$$

with $\varphi_i(x) \ge v_i(x)$ for $|x-x_i| = \rho_\lambda$ and $|x-x_i| = r_i$. From the maximum principle, it then follows that

$$\varphi_i(x) \ge v_i(x) \text{ in } \quad r_i < |x-x_i| < \rho_\lambda. \tag{5.18}$$

In order to prove (5.10), it is enough to show that $M_i \le C_2 v(x_i)^{-\alpha_i}$ for some constant C_2. In fact, using the Harnack inequality (5.2) and the fact that $r_i^{2/(p_i-1)}\overline{v_i}$ is strictly decreasing for any $r_i < \theta < \rho_\lambda$, we have

$$\begin{aligned} \rho_\lambda^{(p_i-1)/2} M_i &\le C\rho_\lambda^{(p_i-1)/2} \min_{\partial B_{\rho_\lambda}} v_i \le C\rho_\lambda^{(p_i-1)/2}\overline{v_i}(\rho_\lambda) \\ &\le C\theta^{(p_i-1)/2}\overline{v_i}(\theta) \le C\theta^{(p_i-1)/2}\varphi_i(\theta) \\ &= C\theta^{(p_i-1)/2}(M_i\rho_\lambda^{\delta_i}(\theta^{-\delta_i}+\theta^{-\eta}) + Av(x_i)^{-\alpha_i}(\theta^{-(n-2-\delta_i)}+\theta^{-\eta})) \\ &\le C\theta^{(p_i-1)/2}(M_i\rho_\lambda^{\delta_i}(\theta^{-\delta_i}+\theta^{-\eta}) + 2Av(x_i)^{-\alpha_i}\theta^{-(n-2-\delta_i)}). \end{aligned} \tag{5.19}$$

We need to choose $\theta = \theta(\rho_\lambda, \eta, n, A_1, A_3) > 0$ small such that

$$C\theta^{(p_i-1)/2}\rho_\lambda^{\delta_i}(\theta^{-\delta_i}+\theta^{-\eta}) < \frac{1}{2}\rho_\lambda^{(p_i-1)/2}.$$

This can be achieved if

$$C\theta^{(p_i-1)/2}\rho_\lambda^{\delta_i} 2\theta^{-\eta} < \frac{1}{2\rho_\lambda^{(p_i-1)/2}},$$

which is the same as $C(\frac{\theta}{\rho_\lambda})^{(p_i-1)/2-\eta} < \frac{\rho_\lambda^{\eta-\delta_i}}{2}$. This happens if we have $C(\frac{\theta}{\rho_\lambda})^{(n-2)/2-\eta} < \frac{\rho_\lambda^{\eta-\delta_i}}{4}$. This is possible for small enough θ, since $\eta < (n-2)/2$. Then from (5.19) we get $M_i \leq Cv(x_i)^{-\alpha_i}$. The proof can now be completed. □

Lemma 5.4. *Under the same hypotheses as in Lemma* 5.3, *we have*

$$\zeta_i(0)^{\tau_i} = 1 + o(1).$$

For the proof of this lemma, see that of Lemma 2.3 in [12].

Since our equation (5.1) has a linear term, we need some more estimates. For that we need the following limits for certain integrals near the blow up point.

Lemma 5.5. *Let* $\{v_i\}$ *be a sequence of solutions of* (5.2) *and let* $x_i \to 0$ *be a sequence of isolated blow up points converging to* 0*. Under the assumptions of Lemma* 5.3, *for* $0 < s < n$ *we have, with* r_i *as in Corollary* 5.1,

(i) $\int_{B_{r_i}} |x|^s v_i(x+x_i)^{p_i+1} = \frac{1}{v_i(x_i)^{2s/n-2}} \left\{ \int_{\mathbb{R}^n} \frac{|x|^s}{(1+k_i|x|^2)^n} dx + o(1) \right\}$,
(ii) $\int_{B_{r_i}} (v_i)^2 = O(\frac{1}{v_i(x_i)^{2^*-2}})$,
(iii) $\int_{B_{r_i}} (v_i)^{p_i} = O(\frac{1}{v_i(x_i)})$

The proof easily follows from Proposition 5.1, (i), using the change-of-variables formula and Lemma 5.4.

In the next lemma, we prove that an *isolated simple blow up point* of $\{v_i\}$ **has** to be a *critical point* of the function h. Later on, we will show that this conclusion holds even if we *do not* assume that the blow up point is simple.

Lemma 5.6. *Under the assumptions of Lemma* 5.3, *if* $\{v_i\}$, *solutions of* (5.1), *have* 0 *as an isolated simple blow up point, then we have*

$$\nabla h(0) = 0.$$

Proof. Let η be a cut off function which is 1 on $B_{1/4}$ and 0 outside $B_{1/2}$. Multiplying the equation for (5.1) by $\eta \cdot \frac{\partial v_i}{\partial x_j}$, and integrating by parts on B_1, we get

$$\frac{1}{p_i+1}\int_{B_1} v_i^{p_i+1}\frac{\partial h}{\partial x_j}\eta = -\frac{\lambda}{2}\int_{B_1} v_i^2\Big(\frac{\partial \eta}{\partial x_j}\Big) - \frac{1}{p_i+1}\int_{B_1} v_i^{p_i+1}\Big(\frac{\partial \eta}{\partial x_j}\Big)h + \frac{1}{2}\int_{B_1}|\nabla v_i|^2\frac{\partial \eta}{\partial x_j} - \int_{B_1}\frac{\partial v_i}{\partial x_j}(\nabla v_i\cdot\nabla\eta).$$

After simplification,

$$\Big|\int_{B_1}\frac{\partial h}{\partial x_j}v_i^{p_i+1}\Big|\le\frac{c\lambda}{2}\int_A v_i^2+c\int_A v_i^{p_i+1}+\frac{c}{2}\int_A|\nabla v_i|^2,\tag{5.20}$$

where A is the annulus $B_{1/2}\setminus B_{1/4}$. Using the estimate (5.10),

$$\int_A v_i^2+c\int_A v_i^{p_i+1}\le\frac{c}{v_i(0)^{2\alpha_i}}+\frac{c}{v_i(0)^{(p_i+1)\alpha_i}}.\tag{5.21}$$

To evaluate $\int_A|\nabla v_i|^2$, we use the Schauder's estimate ([10], Theorem 3.9) in the bigger annulus $A_1=\{x:\sigma_1<|x|<1\}$ with $0<\sigma_1<1/4$, to get,

$$\sup_A|\nabla v_i|\le\sup_{A_1}v_i+C\sup_{A_1}(\lambda v_i+hv_i^{p_i}).$$

Using the Harnack inequality of Lemma 5.2,

$$\sup_{A_1}v_i\le C\inf_{A_1}v_i\le Cv_i(x_i+e).$$

Hence, for some constant $C>0$, we have

$$\sup_A|\nabla v_i|^2\le Cv_i(x_i+e)^2$$

for a fixed unit vector e. From Lemma 5.3, it follows that

$$\int_A|\nabla v_i|^2\le\frac{C}{v_i(0)^{2\alpha_i}}.$$

Note that α_i tends to 1. Combining the above inequality with (5.20) and (5.21), we get

$$\Big|\int_{B_1}\frac{\partial h}{\partial x_j}v_i^{p_i+1}\Big|\le c\frac{C}{v_i(0)^{2\alpha_i}}.$$

By Proposition 5.1,(i), $(\int_{B_1}v_i^{p_i+1})$ is greater than a bounded positive constant and hence

$$\begin{aligned}\Big|\frac{\partial h}{\partial x_j}(0)\int_{B_1}v_i^{p_i+1}\Big| &\le \Big|\int_{B_1}\Big(\frac{\partial h(0)}{\partial x_j}-\frac{\partial h(x)}{\partial x_j}\Big)v_i^{p_i+1}\Big|+\Big|\int_{B_1}\frac{\partial h(x)}{\partial x_j}v_i^{p_i+1}\Big|\\ &\le \sup_{B_1}|D^2h(x)|\int_{B_1}|x|v_i^{p_i+1}+\frac{c}{(v_i(0))^{2\alpha_i}} \qquad (5.22)\\ &\le \frac{\sup|D^2h(x)|}{(v_i(x_i))^{2/n-2}}+\frac{c}{v_i(0)^{2\alpha_i}}\ \to 0\end{aligned}$$

for each j, $1\le j\le n$ and hence $\nabla h(0)=0$. □

The following estimate is necessary to manage the linear term in our equation. This lemma is similar to Proposition 3.5 of [6] but we have an extra term involving the gradient of h.

Lemma 5.7. *Let h and $\{v_i\}$ satisfy the assumptions of Proposition 5.3 and suppose that $x_i \to 0$ is an isolated simple blow up point. Assume that $n > 4$ and $(H3)'$ holds. Then there exists a positive constant $C = C(n, h, \rho)$ such that, for any $e \in \mathbb{R}^n$ with $|e| = 1$,*

$$\lambda v_i(x_i)^{\frac{2(n-4)}{n-2}} \leq C v_i(x_i)^2 v_i(x_i + e)^2 + o(1). \tag{5.23}$$

Remark. It is important to note that the above estimate is in fact true when we replace λ above by λ_i where $\lambda_i \to 0$, as in [6].

Proof: We write the Pohozaev identity (I) (Proposition 5.1) in the unit ball $B_1 = B(0,1)$ and estimate both the sides. We have

$$\int_{\partial B_1} B(x, v, \nabla v) = (\lambda \int_{B_1} v^2 - \frac{\lambda}{2} \int_{\partial B_1} \sigma v^2) + \frac{1}{p+1} \int_{B_1} (x \cdot \nabla h) v^{p+1}$$

$$(\frac{n}{p+1} - \frac{n-2}{2}) \int_{B_1} h(x) v^{p+1} - \frac{1}{p+1} \int_{\partial B_1} h(x) v^{p+1}, \tag{5.24}$$

where

$$B(x, v, \nabla v) := -\frac{(n-2)}{2} v \frac{\partial v}{\partial \nu} - \frac{1}{2} |\nabla v|^2 + \Big|\frac{\partial v}{\partial \nu}\Big|^2.$$

Again, to estimate the gradient term in the left-hand side of (5.24), using Schauder's estimate ([10], Theorem 3.9) in the annulus $A = \{x : \sigma_1 < |x| < \sigma_2\}$, $0 < \sigma_1 < 1 < \sigma_2$, we have

$$\sup_{|x - x_i| = 1} |\nabla v_i| \leq \sup_A v_i + C \sup_A (\lambda v_i + h v_i^{p_i}).$$

Using the Harnack inequality of Lemma 5.2,

$$\sup_A v_i \leq C \inf_A v_i \leq C v_i(x_i + e).$$

Hence, for some constant $C > 0$, we have

$$\sup_{|x - x_i| = 1} |\nabla v_i|^2 \leq C v_i(x_i + e)^2$$

and it follows that the left-hand side

$$\int_{\partial B_1} B(x, v, \nabla v) \leq \int_{\partial B_1} \Big\{ \frac{n-2}{2} |v_i| |\nabla v_i| + \frac{1}{2} |\nabla v_i|^2 \Big\} \, dS \leq C v_i(x_i + e)^2. \tag{5.25}$$

The right-hand side of (5.24) is greater than

$$\lambda \int_{B_1} v^2 - \frac{\lambda}{2} \int_{\partial B_1} v^2 - \frac{1}{p+1} \int_{B_1} x \cdot \nabla h v^{p_i + 1} - \frac{1}{p+1} \int_{\partial B_1} h(x) v^{p+1},$$

which we denote as $I_1+I_2+I_3+I_4$ and estimate each part as follows: Using Lemma 5.5,(ii),

$$I_1 = \lambda \int_{B_1} v_i^2\,dx \geq \lambda \int_{B_{r_i}} v_i^2\,dx = O(\frac{1}{v_i(x_i)^{2^\star-2}}) = \lambda C \frac{v_i(x_i)^{\frac{2(n-4)}{n-2}}}{v_i(x_i)^2}. \quad (5.26)$$

The integrals I_2 and I_4 can be estimated as in [6] using Lemma 5.3 and we have

$$\begin{aligned} I_2 &= \frac{\lambda}{2}\int_{\partial B_1} v_i^2(x)\,dx \leq \lambda \frac{C}{v_i(x_i)^{2\alpha_i}} \\ &\leq \lambda C \frac{v_i(x_i)^{\frac{2(n-4)}{n-2}}}{v_i(x_i)^2} \frac{1}{v_i(x_i)^{\frac{2(n-4)}{n-2}+2\alpha_i-2}} = \lambda C \frac{v_i(x_i)^{\frac{2(n-4)}{n-2}}}{v_i(x_i)^2} o(1). \end{aligned} \quad (5.27)$$

Similarly, we get

$$I_4 = \frac{1}{p_i+1}\int_{\partial B_1} h(x) v^{p_i+1}\,dx \leq \frac{C}{v_i(x_i)^{\alpha_i(p_i+1)}} \leq \frac{C}{v_i(x_i)^{\frac{2n}{n-2}-c\tau_i}} = \frac{o(1)}{v_i(x_i)^2}. \quad (5.28)$$

Using Lemma 5.6, we conclude that 0 is a critical point of h and hence, in a small neighbourhood of h, the condition (H3)' holds. Thus we have

$$I_3 = \frac{1}{p_i+1}\Big|\int_{B_1} x\cdot\nabla h v^{p_i+1}\Big|\,dx \leq C\int_{B_{r_i}} |x|^\theta v^{p_i+1}\,dx + \int_{r_i\leq|x-x_i|\leq 1} v^{p_i+1}\,dx.$$

The first integral can be estimated using Lemma 5.5, (i):

$$\int_{B_{r_i}} |x|^\theta v^{p_i+1}\,dx = o(\frac{1}{v_i(x_i)^2})$$

if $\theta > n-2$. Whereas the second integral, after using Lemma 5.3 as in (5.28), satisfies

$$\int_{r_i\leq|x-x_i|\leq 1} v^{p_i+1}\,dx = \frac{o(1)}{v_i(x_i)^2}.$$

Now the Lemma follows by combining (5.25) with the estimates for the four integrals. □

Proof of Proposition 5.3. For $|x-x_i| \leq r_i$, the inequality follows from Proposition 5.1 and Lemma 5.4. We need to prove the inequality for $r_i \leq |x-x_i| \leq 1$. We first prove it for $|x-x_i| = 1$. Fix $e \in \mathbb{R}^n$ such that $|e| = 1$ and define

$$w_i(x) = \frac{v_i(x)}{v(x_i+e)}.$$

Then w_i satisfies

$$-\Delta w_i = \lambda w_i + h v_i(x_i+e)^{p_i-1} w_i^{p_i}.$$

For $r > 0$ and i_0 fixed, consider the compact set $K_r = \{x \in B(0,2) : r \leq |x - x_{i_0}| \leq 1\}$. By Lemma 5.2, for $i \geq i_0$, we have

$$\max_{K_r} w_i \leq C \min_{K_r} w_i \leq C \min_{\partial B_1} w_i \leq C.$$

Since $x_i \to 0$, it follows that w_i is bounded on every compact subset $\mathbb{R}^n \setminus \{0\}$. By elliptic theory, there exists a nonnegative function w such that $w_i \to w$ in $C^2_{loc}(B_2 \setminus \{0\})$. Moreover, w satisfies

$$-\Delta w = \lambda w \text{ in } B_1 \setminus \{0\}. \tag{5.29}$$

Using Proposition 9.1 in the Appendix of [13], we can further classify w in a small neighbourhood $B(0,\sigma_1)$ of the origin. That is, there exists $\alpha \geq 0$ and $\sigma_1 > 0$, such that

$$w(x) = \alpha G(x) + \varphi_{\sigma_1}, \tag{5.30}$$

where $G(x) = \alpha C_n |x|^{2-n} + E(x)$ is the unique solution in the sense of distributions for the equation

$$\begin{aligned} -\Delta G &= \lambda G + \alpha \delta_0 \text{ in } B(0,\sigma_1) \\ G &= 0 \text{ on } \partial B(0,\sigma_1) \end{aligned}$$

and φ_{σ_1} is the unique C^2 solution of the boundary-value problem

$$\begin{aligned} -\Delta \varphi &= \lambda \varphi \text{ in } B(0,\sigma_1) \\ \varphi &= w \text{ on } \partial B(0,\sigma_1). \end{aligned}$$

Here σ_1 is sufficiently small such that $\lambda < \lambda_1(B(0,\sigma_1))$.

From Lemma 9.2 of [13], we further know that E satisfies the following: For all $0 < \varepsilon < 1$, there exists some constant $C(\varepsilon)$ depending only on ε, n and λ such that

$$|x|^{n-4+\varepsilon} |E(x)| + |x|^{n-3+\varepsilon} |\nabla E(x)| \leq C(\varepsilon) \text{ for all } x \in B(0,\sigma_1),\ n \geq 4$$

and

$$|x|^{\varepsilon-1} |E(x) - E(0)| + |x|^{\varepsilon} |\nabla E(x)| \leq C(\varepsilon) \text{ for all } x \in B(0,\sigma_1),\ n = 3.$$

Also, note that φ_{σ_1} is C^2.

Claim (i) $\alpha > 0$; i.e., w is singular at the origin.

For $0 < r < 2$ fixed, since $\{w_i\}$ is uniformly bounded on ∂B_r, by the dominated convergence theorem we have

$$\lim_{i\to\infty} v_i(x_i+e)^{-1} r^{2/(p_i-1)} \overline{v_i}(r) = \lim_{i\to\infty} r^{2/(p_i-1)} \overline{w_i}(r) = r^{(n-2)/2} \overline{w}(r).$$

Since 0 is an isolated simple blow up point, by (iii) of Corollary 5.1, we have that $v_i(x_i+e)^{-1}r^{2/(p_i-1)}\overline{v_i}(r)$ is strictly decreasing for $r_i < r < \rho$. Hence $r^{(n-2)/2}\overline{w}(r)$ is nonincreasing for $0 < r < \rho$.

If w is regular at the origin, then $\overline{w}$ is bounded near 0 and

$$\lim_{r\to 0^+} r^{(n-2)/2}\overline{w}(r) = 0.$$

It follows that $\overline{w}(r) = 0$ in $(0,\rho)$. But by the Harnack inequality, for $x \in K_r$,

$$v_i(x) \le \max_{K_r} v_i \le C \min_{K_r} v_i$$

and hence

$$\frac{1}{C} \le \frac{v_i(x_i + re)}{v_i(x_i+e)} \le C.$$

Again using the Harnack inequality, we have $\max_{\partial B_r} v_i \le C \min_{\partial B_r} v_i$. Therefore, for any $0 < r < 1$, $\overline{v_i} \ge \frac{1}{C} v_i(x_i + re)$. Hence, $\overline{w_i} \ge \frac{1}{C^2} > 0$ for any $0 < r < 1$, a contradiction. Therefore, w must be singular at the origin and hence $\alpha > 0$.

Claim (ii): $\{v_i(x_i)v_i(x_i+e)\}_i$ is bounded.

Proof of the claim: Suppose that $\lim_{i\to\infty} v_i(x_i)v_i(x_i+e)$ tends to ∞. Multiply equation (5.1) by $v_i(x_i+e)^{-1}$ and integrate by parts on $B(0,\sigma)$, $\sigma < \sigma_1$. We get

$$-\int_{\partial B_\sigma} \frac{\partial w_i}{\partial \nu} = -v_i(x_i+e)^{-1}\int_{B_\sigma} \Delta v_i\, dx = -v_i(x_i+e)^{-1}\int_{B_\sigma} (hv_i^{p_i} + \lambda v_i)\, dx. \tag{5.31}$$

By elliptic theory and (5.30),

$$\lim_{i\to\infty} \int_{\partial B_\sigma} \frac{\partial w_i}{\partial \nu} = \int_{\partial B_\sigma} \frac{\partial w}{\partial \nu} = \int_{\partial B_\sigma} \frac{\partial}{\partial \nu}(\alpha C_n |x|^{2-n} + E(x) + \varphi_{\sigma_1}).$$

Now

$$\begin{aligned} \frac{\partial E}{\partial \nu}(x) \quad &\le \quad |\nabla E(x)| \le \frac{C(\varepsilon)}{|x|^{n-3+\varepsilon}} \text{ for } n \ge 4 \\ \mathit{or} \quad &\le \quad \frac{C(\varepsilon)}{|x|^{\varepsilon}} \text{ for } n = 3. \end{aligned}$$

Therefore, in either case, i.e., for $n \ge 3$, we have

$$\int_{B_\sigma} \frac{\partial w}{\partial \nu} \le -\alpha C_n (n-2)\omega_n + C(\varepsilon)\omega_n \sigma^{2-\varepsilon} + ||\nabla \varphi_{\sigma_1}||_\infty \omega_n \sigma^{n-1} < 0 \tag{5.32}$$

for σ sufficiently small. Whereas, by (iii) in Lemma 5.5, we have

$$\int_{|x-x_i|\le r_i} hv_i^{p_i} \le \frac{C}{v_i(x_i)}$$

and from Lemma 5.3

$$\int_{r_i\le|x-x_i|\le\sigma} hv_i^{p_i} \le \frac{o(1)}{v_i(x_i)}.$$

Moreover,

$$\int_{B_\sigma} \lambda v_i\,dx \le |\lambda| \int_{|x-x_i|\le r_i} v_i\,dx + |\lambda| \int_{r_i\le|x-x_i|\le\sigma} v_i\,dx.$$

For the linear term changing the variable, using Proposition 5.1,

$$\begin{aligned} |\lambda| \int_{|x-x_i|\le r_i} v_i\,dx &= |\lambda| \int_{|x|\le R_i} \frac{1}{v_i(x_i)^{(p_i-1)n/2}} v_i(\frac{x}{v_i(x_i)^{(p_i-1)/2}} + x_i)\,dx \\ &\le \frac{|\lambda|}{v_i(x_i)^{(p_i-1)n/2)-1}} \int_{|x|\le R_i} \frac{dx}{(1+|x|^2)^{(n-2)/2}} + o(1) = \frac{o(1)}{v_i(x_i)} \end{aligned} \tag{5.33}$$

since $\int_{|x|\le R_i} \frac{dx}{(1+|x|^2)^{(n-2)/2}} = O(R_i^2)$ and $\frac{R_i^2}{v_i(x_i)^{p_i-1}}$ is o(1) by Corollary 5.1.

To estimate the integral in the annulus $A_i = r_i \le |x-x_i| \le \sigma$, we use Lemma 5.7. Since we have assumed that $v_i(x_i)v_i(x_i+e) \to \infty$, we have from (5.23)

$$\lambda v_i(x_i)^{\frac{2(n-4)}{n-2}} \le C v_i(x_i)^2 v_i(x_i+e)^2. \tag{5.34}$$

From Lemma 5.3, we have

$$\lambda \int_{r_i\le|x-x_i|\le\sigma} v_i\,dx \le \frac{\lambda C}{v_i(x_i)^{\alpha_i}}$$

where $\alpha_i = 1 - \frac{2\delta_i}{n-2}$. Hence from (5.34) it follows that

$$\begin{aligned} \lambda v_i(x_i+e)^{-1} \int_{r_i\le|x-x_i|\le\sigma} v_i\,dx &\le (\frac{\lambda C}{v_i(x_i+e)v_i(x_i)}) \frac{1}{v_i(x_i)^{\frac{-2\delta_i}{n-2}}} \\ &\le \frac{C\sqrt{\lambda}}{v_i(x_i)^{\frac{n-4-2\delta_i}{n-2}}} \to 0. \end{aligned} \tag{5.35}$$

We get a contradiction from equations (5.31), (5.32), (5.33), and (5.35). Hence $v_i(x_i+e)v_i(x_i) \le C$. The proof can now be completed as in [12], [6]. □

Using the above estimates, by direct calculations we have

Lemma 5.8. *Under the hypotheses of Proposition* (5.3), *we have, for* $0 < s < n$,

$$\int_{B_{r_i}} |y|^s \zeta_i(y)^{p_i+1} = \frac{1}{\zeta_i(0)^{2s/n-2}} \Big\{ \int_{\mathbb{R}} \frac{|x|^s}{(1+k_i|x|^2)^n} dx + o(1) \Big\},$$

$$\int_{r_i \le |y| \le 1} |y|^s \zeta_i(y)^{p_i+1} = o\Big(\frac{1}{\zeta_i(0)^{2s/n-2}} \Big),$$

where $r_i = \frac{R_i}{(\zeta_i(0))^{\frac{p_i-1}{2}}}$.

5.2. **Nature of blow up points of v_i.** The earlier estimates and the Pohozaev identity will now be used to derive various conclusions about the possible blow up points of $\{v_i\}$. In the following proposition, we first prove $\{v_i\}$ can blow up only at a critical point of the function h. It is important to note that here we do not assume that the blow up point is simple. Recall that in Lemma 5.6, we had used this assumption for the same conclusion.

Proposition 5.4. *Under the assumption* (5.5) *of Proposition* 5.2, *if* $\{v_i\}$, *solutions of* (5.1), *have* 0 *as an isolated blow up point, and if* $(H3)'$ *holds, then we have*

$$\nabla h(0) = 0.$$

Proof. We shall consider two cases:
Case (i) 0 *is an isolated simple blow up point:* In this case, the proof follows from that of Lemma 5.6.
Case (ii) *0 is not an isolated simple blow up point:* Without loss of generality, suppose that $v_i(0) \to \infty$. From (iii) of Corollary 5.1, we know that $\tilde{v}_i(r)$ has a unique critical point for $0 < r < r_i$. Since 0 is not isolated, there exists another critical point, say $\mu_i \ge r_i$ of $\tilde{v}_i(r)$ such that $\mu_i \to 0$. Now consider the rescaled function

$$\zeta_i(x) = \mu_i^{2/(p_i-1)} v_i(\mu_i x) \text{ in } |x| < 1/\mu_i$$

which satisfies the equation

$$-\Delta \zeta_i(x) = \lambda \mu_i^2 \zeta_i(x) + h(\mu_i x)\zeta_i^{p_i}(x) \text{ in } |x| < 1/\mu_i. \tag{5.36}$$

Note that

$$\lim_{i\to\infty} \zeta_i(0) = \infty \tag{5.37}$$

and from (5.6) we have

$$|x|^{2/(p_i-1)}\zeta_i(x) \le A_3. \tag{5.38}$$

Moreover, it can be verified that $r^{2/(p_i-1)}\bar{\zeta}_i(r)$ has precisely one critical point $(0,1)$, and that

$$\frac{d}{dr}\,|_{r=1}\; r^{2/(p_i-1)}\bar{\zeta}_i(r) = 0. \tag{5.39}$$

Hence 0 is an isolated simple blow up point of ζ_i. Define $w_i = \zeta_i(0)\zeta_i$, which satisfies the equation

$$-\Delta w_i = \lambda\mu_i^2 w_i(x) + h(\mu_i x) w_i(0)^{\frac{1-p_i}{2}} w_i^{p_i}(x) \text{ in } |x| < 1/\mu_i. \tag{5.40}$$

Step 1: $\lim_{i\to\infty} w_i(x) = \frac{a}{|x|^{n-2}} + b(x)$ where $a > 0$ is a constant.

From (5.9) and the Harnack inequality of Lemma 5.2 for w_i, we conclude that the right-hand side of (5.40) is uniformly bounded in all of $L^p_{loc}(B_{1/\mu_i} \setminus \{0\})$, for $1 < p < \infty$. By standard elliptic theory, $w_i \in W^{2,p}$ for all p and hence, by bootstrap arguments, $\{w_i\}$ is bounded in C^3_{loc} and hence converges in $C^2_{loc}(\mathbb{R}^n \setminus \{0\})$ to some w which satisfies

$$-\Delta w(y) = 0 \quad \forall\, y \in \mathbb{R}^n \setminus \{0\}.$$

Moreover, w has to have a singularity at $x = 0$. In fact, we have, by arguments as above,

$$\lim_{i\to\infty} r^{2/p_i-1}\overline{\zeta}_i(r)\zeta_i(0) = r^{\left(\frac{n-2}{2}\right)}\overline{w}(r)$$

for any $0 < r < 2$, where

$$\overline{w}(r) = \frac{1}{|\partial B_r|}\int_{\partial B_r} w(x)dS_x.$$

Since 0 is a isolated simple blow up point for $\{\zeta_i\}$, $\left(r^{2/p_i-1}\overline{\zeta_i}(r)\right)$ has only one critical point at r_i in $(0,1)$ and so does $\tilde{w}_i$. Thus $r^{\frac{n-2}{2}}\bar{w}(r)$ is nonincreasing for $0 < r < 1$. This would be possible only if w is unbounded at the origin. Thus,

$$w(y) = \frac{a}{|x|^{n-2}} + b(x) \tag{5.41}$$

where $a > 0$ for some positive constant and $b(x)$ is some regular harmonic function in $\mathbb{R}^n$.

Since w is positive, we have $\lim_{|x|\to\infty} b(x) \geq 0$. Hence, by the maximum principle, $b(x) \equiv b$ a constant. Moreover, multiplying equation (5.39) by $\zeta_i(0)$ and sending the limit to ∞ we get $\frac{d}{dr}\,|_{r=1}\; r^{\frac{n-2}{2}}\bar{w}(r) = 0$. Hence we get

$$b = a > 0. \tag{5.42}$$

Step 2: By the Pohozaev identity applied to (5.36) on B_σ, for $\sigma > 0$ small, we have

$$\int_{\partial B_\sigma} B(\sigma, x, \zeta_i, \nabla\zeta_i) = \lambda\mu_i^2 \int_{B_\sigma} \zeta_i^2 - \frac{\lambda\mu_i^2}{2}\int_{\partial B_\sigma} \sigma\zeta_i^2 + \frac{1}{p_i+1}\int_{B_\sigma}(x\cdot\nabla\tilde{h})\zeta_i^{p_i+1}$$
$$+\Big(\frac{n}{p_i+1} - \frac{n-2}{2}\Big)\int_{B_\sigma}\tilde{h}(x)\zeta_i^{p_i+1} - \frac{\sigma}{p_i+1}\int_{\partial B_\sigma}\tilde{h}(x)\zeta_i^{p_i+1}$$

where $\tilde{h}(x) = h(\mu_i x)$. Since $p_i + 1 \le \frac{2n}{n-2}$, it follows that

$$\Big(\frac{n}{p_i+1} - \frac{n-2}{2}\Big)\int_{B_\sigma}\tilde{h}(x)\zeta_i^{p_i+1} \ge 0. \tag{5.43}$$

Thus we have

$$\begin{aligned}\int_{\partial B_\sigma} B \quad (\sigma, x, \zeta_i, \nabla\zeta_i) &\ge \frac{1}{p_i+1}\int_{B_\sigma}(x\cdot\nabla\tilde{h})\zeta_i^{p_i+1}\\ &- \quad \frac{\sigma}{p_i+1}\int_{\partial B_\sigma}\tilde{h}(x)\zeta_i^{p_i+1} - \lambda\mu_i^2\frac{\sigma}{2}\int_{\partial B_\sigma}\zeta_i^2.\end{aligned} \tag{5.44}$$

Since we have, by Proposition 5.3,

$$\zeta_i(y) \le \frac{c}{\zeta_i(0)|y|^{n-2}} \quad \forall \quad |y| \le 1,$$

it follows that

$$\lambda\mu_i^2\frac{\sigma}{2}\int_{\partial B_\sigma}\zeta_i^2 \le \lambda\mu_i^2\frac{\sigma^2}{2}\frac{1}{\zeta_i(0)^2}, \tag{5.45}$$

$$\frac{\sigma}{p_i+1}\int_{\partial B_\sigma}\tilde{h}(x)\zeta_i^{p_i+1} \le c\sigma\int_{\partial B_\sigma}\frac{1}{\zeta_i(0)^{p_i+1}}\frac{dS_x}{|x|^{(n-2)(p_i+1)}} \le \frac{c\sigma}{\sigma^{2n}}\frac{1}{\zeta_i(0)^{p_i+1}}. \tag{5.46}$$

We now multiply the inequality (5.44) by $(\zeta_i(0))^2$ and pass to the limit, using the convergence of w_i to w and μ_i to 0, to get

$$\begin{aligned}\int_{\partial B_\sigma} B(\sigma, x, w, \nabla w) \quad &\ge \quad \lim_{i\to\infty}\zeta_i^2(0)\Big(\int_{B_\sigma}\frac{x\cdot\nabla\tilde{h}}{p_i+1}\zeta_i^{p_i+1} + O(\frac{1}{\zeta_i(0)^{p_i+1}})\Big)\\ &\ge \quad \lim_{i\to\infty}\Big(\zeta_i^2(0)\int_{B_\sigma}\frac{(x\cdot\nabla\tilde{h})}{p_i+1}\zeta_i^{p_i+1}\Big).\end{aligned} \tag{5.47}$$

For $\sigma > 0$ small, using Corollary 5.2,

$$\int_{\partial B_\sigma} B(\sigma, x, w, \nabla w) < 0. \tag{5.48}$$

On the other hand, we will now show that

$$\zeta_i(0)^2 \int_{B_\sigma} (x \cdot \nabla \tilde{h}) \zeta_i^{p_i+1} = o(1) \tag{5.49}$$

as $i \to \infty$, using our assumption that $|\nabla h(0)| = d \neq 0$, and the resulting contradiction will prove the proposition. Notice that $\nabla \tilde{h}(x) = \mu_i \nabla h(\mu_i x)$. The required integral is

$$\begin{aligned}
\int_{B_\sigma} (x \cdot \nabla \tilde{h}(x)) \zeta_i^{p_i+1} &= \int_{B_\sigma} x \cdot (\nabla \tilde{h}(x) - \nabla \tilde{h}(0)) \zeta_i^{p_i+1} + \int_{B_\sigma} (x \cdot \nabla \tilde{h}(0)) \zeta_i^{p_i+1} \\
&\leq \mu_i \int_{B_\sigma} x \cdot (\nabla h(\mu_i x) - \nabla h(0)) \zeta)^{p_i+1} + \mu_i d \int_{B\sigma} |x| \zeta_i^{p_i+1} \\
&\leq \mu_i (\sup_{B_\sigma} |D^2 h(x)|) \int_{B_\sigma} |x|^2 \zeta_i^{p_i+1} + \mu_i d \int_{B_\sigma} |x| \zeta_i^{p_i+1} \\
&\leq \mu_i^2 \frac{C}{\zeta_i(0)^{4/n-2}} + \mu_i d \Big(\frac{C}{\zeta_i(0)^{2/n-2}} \Big),
\end{aligned} \tag{5.50}$$

using the estimates from Lemma 5.8. If we have

$$\mu_i \leq \frac{C}{\zeta_i(0))^2} \tag{5.51}$$

then (5.49) will follow from (5.50).

To prove (5.51), we repeat the argument used in (i) above, using the cut off function, now for the equation (5.36), and arrive at the estimate (5.22) for ζ_i, namely,

$$\begin{aligned}
\Big| \frac{\partial \tilde{h}}{\partial x_j}(0) \int_{B_1} \zeta_i^{p_i+1} \Big| &\leq \Big| \int_{B_1} \Big(\frac{\partial \tilde{h}(0)}{\partial x_j} - \frac{\partial \tilde{h}(x)}{\partial x_j} \Big) \zeta_i^{p_i+1} \Big| + \Big| \int_{B_1} \frac{\partial \tilde{h}(x)}{\partial x_j} \zeta_i^{p_i+1} \Big| \\
&\leq \sup_{B_1} |D^2 \tilde{h}(x)| \int_{B_1} |x| \zeta_i^{p_i+1} + \frac{c}{(\zeta_i(0))^2} \leq \frac{(\mu_i)^2 \sup |D^2 h(x)|}{(\zeta_i(0))^{2/n-2}} + \frac{c}{\zeta_i(0)^2}.
\end{aligned}$$

Summing over j and using the Cauchy-Schwarz inequality,

$$\mu_i d = |\nabla \tilde{h}(0)| \leq c \frac{\mu_i^2}{(\zeta_i(0))^{2/n-2}} + \frac{c}{(\zeta_i(0))^2},$$

and hence $\mu_i = 0(\frac{1}{\zeta_i(0)^2})$. Hence (5.49) holds and the proposition follows. □

Proposition 5.4 indicates that we need to put conditons on critical points of h to ensure that there is no blow up. We first show that, under suitable assumptions on ∇h near the critical points of h, the blow up points are isolated and simple.

Note that, in Proposition 5.1, the number of critical points $k(v)$ depends on the function v and as we take the limit as $i \to \infty$, $k(v_i)$ may increase and two sequences of blow up points may come very close. In the following proposition, we prove that this does not occur and the blow up points are isolated and simple.

We end this section by summarizing the results for both the cases $\tilde{\lambda} = 0$ as well as $\tilde{\lambda} > 0$:

Proposition 5.5. *Under the same assumptions as in Proposition* 5.2, *suppose, in addition, there exists a positive constant* $d > 0$ *such that for* $x \in \Omega_d = \{x \in \mathbb{R}^n : h(x) > 0, |\nabla h(x)| < d\}$,

$$c_1[dist(x, \mathcal{S})]^{\theta-1} \leq |\nabla h(x)| \leq c_2[dist(x, \mathcal{S})]^{\theta-1}, \tag{5.52}$$

for $\mathcal{S} = \{x \in \mathbb{R}^n : h(x) > 0, \nabla h(x) = 0\}$, *and* $n-2 < \theta \leq n$. *Let* $\{(\lambda_i, v_i)\}_i$ *be a sequence of solutions of* (5.1) *with* $\lambda_i \to \tilde{\lambda}$. *Then we have the following.*

(*i*) *Isolated blow up points of* $\{v_i\}_i$ *are isolated simple: If* $x_i \to 0$ *is an isolated blow up point of* $\{v_i\}_i$, *then it is simple.*

(*ii*) *The blow up points of* $\{v_i\}_i$ *are isolated: More precisely, for* $\varepsilon > 0$ *and* $R > 1$, *there exists some positive constant* $r^* = r^*(n, \varepsilon, R, A_1, c_1, c_2, d,$ *modulus of continuity of* $\nabla h)$ *such that, for any solution* v_i *with* $\max_{\Omega_\delta^+} v_i > C^*$, *we have*

$$|q_l - q_j| \geq r^* \quad \forall \quad 1 \leq l \neq j \leq k,$$

where $q_l = q_l(v_i)$ *and* $k = k(v_i)$ *are as in Proposition* 5.1.

(*iii*) *If* $\tilde{\lambda} > 0$, *then in fact* $\{v_i\}_i$ *is uniformly bounded in* L^∞.

If $\tilde{\lambda} = 0$, *then further assuming* $(H3)$ *and fixing* $p_i = \frac{n+2}{n-2}$, *we arrive at the same conclusion.*

Remark. In the following, we give the proofs only for the case $\tilde{\lambda} > 0$. The main difference between the two cases occurs in the proof of (iii), where we work directly with the solutions without rescaling them. As mentioned in the beginning of this section, conclusion (iii) for $\tilde{\lambda} = 0$ follows from the estimate in Lemma 5.7 and arguments similar to those in [12] (Theorem 4.4 and Corollary 4.1 there), using also (H3). For the case $\tilde{\lambda} > 0$, (5.52) alone (with in fact $n - 2 < \theta \leq n$) is enough to obtain a contradiction in Pohozaev's identity. This is because of the presence of the linear term and the behaviour of $w(x) = \lim v_i(x)v_i(0)$, which is given by (5.66), while, in the case $\tilde{\lambda} = 0$, w is a harmonic function in a deleted neighbourhood of 0.

Proof of (i). Suppose 0 is not an isolated simple blow up point. Then there exists $\mu_i \to 0$, $\mu_i > r_i$, such that μ_i is a critical point for $\tilde{v}_i(r) = r^{2/(p_i-1)}\overline{v_i}(r)$

where $\overline{v_i}(r) := \frac{1}{|\partial B_r|}\int_{\partial B_r} v_i\, dS(x)$ is the spherical average of v_i and r_i is as in Corollary 5.1.

Define $\xi_i(x) = \mu_i^{2/p_i - 1} v_i(\mu_i x)$; then

$$\begin{aligned} -\Delta \xi_i(x) &= \mu_i^2 \lambda \xi_i(x) + h_i(x)\xi_i^{p_i}(x) \text{ in } |x| < \frac{1}{\mu_i} \\ \xi_i &> 0, \end{aligned}$$

where $h_i(x) = h(\mu_i x)$. Moreover, it can be verified that

$$|x|^{2/(p_i-1)}\xi_i(x) \leq C, \text{ in } |x| < \frac{1}{\mu_i}$$

and $\xi_i(0)$ goes to ∞ as i goes to ∞. Furthermore, $r^{2/(p_i-1)}\bar{\xi}_{i(r)}$ has precisely one critical point in $0 < r < 1$, where $\bar{\xi}_i(r) = \frac{1}{|\partial B_r|}\int_{\partial B_r} \xi_i$, and hence 0 is an isolated simple blow up point of ξ_i. We have

$$\frac{d}{dr}|_{r=1}(r^{2/(p_i-1)}\bar{\xi}_i(r)) = 0.$$

Repeating the same argument as in case (ii), Step 1 , in the proof of Proposition 5.4, it follows that

$$\xi_i(0)\xi(x) \to \frac{a}{|x|^{n-2}} + b,$$

where $0 < b = a =$ constant. Hence for $\sigma > 0$ small, using Corollary 5.2,

$$\lim_{i\to\infty} \int_{\partial B_\sigma} B(\sigma, x, \xi_i(0)\xi_i, \xi_i(0)\nabla\xi_i) < 0. \tag{5.53}$$

On the other hand, as in case (ii), Step 1, in the proof of Proposition 5.4, applying the Pohozaev identity to the equation for ξ_i and using arguments as in (5.43) and (5.45) we get

$$\begin{aligned} &\int_{\partial B_\sigma} B(\sigma, x, \xi_i(0)\xi_i, \xi_i(0)\nabla\xi_i) \qquad (5.54)\\ &\geq \frac{1}{p_i+1}\int_{B_\sigma} (x\cdot\nabla h_i)\xi_i^{p_i+1}\xi_i(0)^2 - \frac{\sigma}{p_i+1}\int_{\partial B_\sigma} h_i(x)\xi_i^{p_i+1}\xi_i(0)^2. \end{aligned}$$

Applying Proposition 5.3 to ξ_i, we conclude that

$$\xi_i(0)\xi_i(x) \leq \frac{C}{|x|^{n-2}} \;\forall\; |x| \leq 1.$$

Using this estimate, we get, as in (5.46),

$$\frac{\sigma}{p_i+1}\int_{\partial B_\sigma} h_i(x)\xi_i(0)^2\xi_i^{p_i+1}(x)dx \leq C\frac{\sigma^{-2n+1}}{\xi_i(0)^{p_i-1}} \to 0 \text{ as } i \to \infty. \tag{5.55}$$

We now prove that for the remaining term

$$\xi_i(0)^2 \int_{B_\sigma} (x \cdot \nabla h_i(x)) \xi_i(x)^{p_i+1} = o(1). \tag{5.56}$$

Since $\nabla h_i(x) = \mu_i \nabla h(\mu_i x)$ we need to consider the following two cases :

Case (i): $|\nabla h(0)| \geq \frac{d}{2}$: In this case the estimate (5.56) is similar to (5.49) in the proof of Proposition 5.4.

Case (ii): $|\nabla h(0)| < d/2$: In this case, we choose σ small such that $B(0,\sigma) \subset \Omega_d$. Then using the condition (5.52) we have for all $x \in B(0,\sigma)$

$$C|x|^{\theta-1} \leq |\nabla h(x)| \leq C_2 |x|^{\theta-1}$$

for $n-2 \leq \theta < n$. Hence

$$\begin{aligned}
&\int_{B_\sigma} (x \cdot \nabla h_i(x)) \xi_i(x)^{p_i+1} dx = \int_{B_\sigma} \mu_i (x \cdot \nabla h(\mu_i x)) \xi_i(x)^{p_i+1} dx \\
\leq & \int_{B_\sigma} \mu_i |x| |\nabla h(\mu_i x)| \xi_i(x)^{p_i+1} dx \leq C_2 \int_{B_\sigma} \mu_i^\theta |x|^\theta \xi_i(x)^{p_i+1} dx \\
= & C_2 \int_{B_{r_i}} \mu_i^\theta |x|^\theta \xi_i(x)^{p_i+1} dx + C_2 \int_{B_\sigma \setminus B_{r_i}} \mu_i^\theta |x|^\theta \xi_i(x)^{p_i+1} dx \\
= & I_1 + I_2.
\end{aligned}$$

From Lemma 5.8, we have

$$I_1 = \mu_i^\theta \frac{1}{\xi_i(0)^{2\theta/n-2}} \Big[\int_{\mathbb{R}^n} \frac{|x|^\theta}{(1+h_i|x|^2)^n} dx + o(1) \Big] = \frac{o(1)}{\xi_i(0)^2} \tag{5.57}$$

and

$$I_2 = \mu_i^\theta O(\frac{1}{\xi_i(o)^{2\theta/n-2}}) = \frac{o(1)}{\xi_i(0)^2} \tag{5.58}$$

where (5.57) and (5.58) follow from the fact that $n-2 \leq \theta < n$ and $\mu_i \to 0$. Therefore (5.56) holds in either case. Now (5.54), (5.55), and (5.56) give a contradiction to the inequality (5.53). Hence the isolated blow up point 0 of v_i, must be simple. □

Proof of (ii). Suppose there exist sequences $q_\ell(v_i), q_j(v_i)$ of distinct local maxima of v_i such that

$$|q_\ell(v_i) - q_j(v_i)| = \min_{1 \leq r \neq s \leq k(v_i)} \{|q_s(v_i) - q_r(v_i)|\} \to 0.$$

Without loss of generality, assume that $q_\ell(v_i) = 0$ for all i and that $q_j(v_i) = q_1(v_i) = q_i$. Also let $\sigma_i = |q_1(v_i)| \to 0$. Since the balls $B(0, R_i(v_i(0))^{-\frac{p_i-1}{2}})$

and $B(q_i, R_i(v_i(q_i))^{-\frac{(p_i-1)}{2}})$ are disjoint,

$$\sigma_i > \max\{R_i(v_i(0))^{-\frac{p_i-1}{2}}, R_i(v_i(q_i))^{-\frac{p_i-1}{2}}\}. \tag{5.59}$$

Define $\zeta_i(x) = \sigma_i^{2/p_i-1} v_i(\sigma_i x)$ for $|x| < 1/\sigma_i$: This function satisfies the equation

$$-\Delta\zeta_i(x) = \sigma_i^2 \lambda \zeta_i(x) + \tilde{h}(x)\zeta_i^{p_i}(x) \text{ in } |x| \leq 1/\sigma_i \tag{5.60}$$
$$\zeta_i > 0,$$

where $\tilde{h}(x) = h(\sigma_i x)$. Moreover,

$$|x|^{2/p_i-1}\zeta_i(x) = (\sigma_i|x|)^{2/p_i-1} v_i(\sigma_i x) \leq C_1 \text{ for } |x| \leq \frac{\sigma_i}{2}$$

since, by (ii) of Proposition 5.1,

$$\sigma_i^{2/p_i-1} v_i(x) \leq C_1 \text{ for all } x \in \Omega_\delta^+.$$

Similarly, it can be seen that, for $\tilde{q}_i = \frac{q_i}{|q_i|}$,

$$|x - \tilde{q}_i|^{2/p_i-1}\zeta_i(x) \leq C_1 \text{ for } |x - \tilde{q}_i| \leq \frac{\sigma_i}{2}.$$

Also, from the definition of ζ_i, it follows that $\zeta_i(0)$ and $\zeta_i(\tilde{q}_i) \to \infty$ as $i \to \infty$. Therefore, 0 and $\overline{q} := \lim \tilde{q}_i$ are isolated blow up points of $\{\zeta_i\}$. Moreover, observe that $\tilde{h}$ satisfies the condition (5.52) in a neighbourhood of the origin as well as in a neighbourhood of $\overline{q}$. Hence, from (i) above it follows that both 0 and $\overline{q}$ must be isolated simple blow up points for ζ_i.

From Proposition 5.1, there exists at most a countable set $\mathcal{S}_1 \subset \mathbb{R}$ such that $\min\{|x-y| : x, y \in \mathcal{S}_1\} \geq 1$ and

$$\begin{aligned} \lim_{i\to\infty} \zeta_i(0)\zeta_i(x) &= g(x) \text{ in } C^0_{loc}(\mathbb{R}^n \backslash \mathcal{S}_1) \\ g(y) &> 0 \text{ in } \mathbb{R}^n \backslash \mathcal{S}_1. \end{aligned}$$

Let $\mathcal{C} \subset \mathcal{S}_1$ contain those points where g is singular. Arguing as in Claim (i) of the proof of Proposition 5.3, g must be singular at the origin. In case $\frac{\zeta_i(0)}{\zeta_i(\tilde{q}_i)}$ is bounded, we write

$$\zeta_i(0)\zeta_i(x) = \frac{\zeta_i(0)}{\zeta_i(\tilde{q}_i)}\zeta_i(\tilde{q}_i)\zeta_i(x).$$

Now using the fact that $\frac{\zeta_i(0)}{\zeta_i(\tilde{q}_i)}$ is bounded and that $\tilde{q}_i$ is an isolated simple blow up point for ζ_i (arguing as in Claim (i) of the proof of Proposition 5.3), we conclude that g must be singular at $\overline{q}$.

If $\frac{\zeta_i(0)}{\zeta_i(\tilde{q}_i)} \to \infty$, then, from Proposition 5.2, we have

$$\zeta_i(0)\zeta_i(x) \geq \frac{\zeta_i(0)}{\zeta_i(\tilde{q}_i)} C^{-1} \frac{\zeta_i(\tilde{q}_i)^2}{(1 + k_i|x - \tilde{q}_i|^2)^{(n-2)/2}} \to \infty.$$

Hence we have proved that $\{0, \overline{q}\} \subset \mathcal{C}$. Now using the maximum principle, we can write

$$g(x) = \frac{a_1}{|x|^{n-2}} + \frac{a_2}{|x - \bar{q}|^{n-2}} + b(x), \tag{5.61}$$

where $a_1, a_2 > 0$ are positive constants and $b(x)$ is a nonnegative function such that

$$\begin{aligned} b(x) &\geq 0 \text{ in } \mathbb{R}^n \backslash \{\mathcal{C} \backslash \{0, \bar{q}\}\}, \\ \Delta b(x) &= 0 \text{ in } \mathbb{R}^n \backslash \{\mathcal{C} \backslash \{0, \bar{q}\}\}. \end{aligned}$$

Note that, from (5.61), we can infer that in a small neighbourhood $B(0, \sigma)$, $\sigma > 0$, of the origin,

$$g(x) = \frac{a_1}{|x|^{n-2}} + A + f(x),$$

where $A = \frac{a_2}{|\bar{q}|^{n-2}} > 0$ and f is a differentiable function with $f(0) = 0$. Hence, from Corollary 5.2, it follows that

$$\int_{\partial B_\sigma} B(\sigma, x, g, \nabla g) < 0. \tag{5.62}$$

Whereas, using the Pohozaev identity and estimates like (5.43) and (5.45) for ζ_i, we get

$$\begin{aligned} \int_{\partial B_\sigma} B(\sigma, x, g, \nabla g) &= \lim_{i \to \infty} \int_{\partial B_\sigma} B(x, \sigma, \zeta_i(0)\zeta_i, \zeta_i(0)\nabla\zeta_i) \qquad (5.63) \\ \geq \lim_{i\to\infty} \Big(\frac{1}{p_i + 1} \int_{B_\sigma} (x \cdot \nabla \tilde{h}) \zeta_i^{p_1+1} \zeta_i(0)^2 &- \frac{\sigma}{p_i + 1} \int_{\partial B_\sigma} \tilde{h}(x) \zeta_i^{p_i+1} \zeta_i(0)^2 \Big) = 0. \end{aligned}$$

Note that here $\nabla\tilde{h}(x) = \sigma_i \nabla h(\sigma_i x)$ and hence we have to consider the two cases as discussed in the proof of (i) above. Equations (5.62) and (5.63) give a contradiction. □

Proof of (iii). Suppose (without loss of generality), the sequence $\{v_i\}$ blows up at the origin. Then, by (i) and (ii) above, we know that 0 is an isolated simple blow up point. The sequence $w_i(x) = v_i(0)v_i(x)$ satisfies

$$\begin{aligned} -\Delta w_i &= \lambda w_i + h(x) v_i(0)^{(1-p_i)} w_i^{p_i}, &\qquad (5.64) \\ w_i(0) &\to \infty &\qquad (5.65) \end{aligned}$$

in, say, $B(0,1)$, and $\tilde{w}_i(r) = v_i(0)\tilde{v}_i(r)$ has precisely one critical point in $(0,\rho)$. Hence, 0 is also an isolated simple blow up point of w_i.

Using equation (5.9) and arguing as in the proof of Proposition 5.3, one sees that $w_i(x) \to w$ in $C^2_{loc}(B_1\setminus\{0\})$ where w satisfies the following equation

$$-\Delta w = \lambda w \text{ in } B_1 \setminus \{0\}. \tag{5.66}$$

We now use the Pohozaev identity in this neighbourhood $B(0,\sigma)$ to arrive at a contradiction, but in a way different from the earlier ones. In particular, we exploit the fact that the linear term tends to infinity in the limit, which was not the case for the rescaled sequences. Applying Lemma 5.1 to (5.1), we have

$$\begin{aligned}\int_{\partial B_\sigma} B(\sigma,x,v_i,\nabla v_i) &= \lambda\int_{B_\sigma} v_i^2 - \frac{\lambda}{2}\int_{\partial B_\sigma}\sigma v_i^2 + \frac{1}{p_i+1}\int_{B_\sigma}(x\cdot\nabla h)v_i^{p_i+1}\\ &+ \left(\frac{n}{p_i+1} - \frac{n-2}{2}\right)\int_{B_\sigma} h(x)v_i^{p_i+1} - \frac{\sigma}{p_i+1}\int_{\partial B_\sigma} h(x)v_i^{p_i+1}.\end{aligned} \tag{5.67}$$

Multiplying the left-hand side by $v_i(0)^2$, and taking the limit as $i \to \infty$, we get

$$\lim_{i\to\infty}\int_{\partial B_\sigma} B(\sigma,x,w_i,\nabla w_i) = \int_{\partial B_\sigma} B(\sigma,x,w,\nabla w) < \infty. \tag{5.68}$$

As in the proof of Proposition 5.4, using estimates (5.43) and (5.46), we get

$$\int_{\partial B_\sigma} B(\sigma,x,v_i,\nabla v_i) \ge \lambda v_i(0)^2\int_{B_{r_i}} v_i^2\,dx + \frac{v_i(0)^2}{p_i+1}\int_{B_\sigma}(x\cdot\nabla h)v_i^{p_i+1} + O(1). \tag{5.69}$$

Moreover,

$$\begin{aligned} v_i(0)^2\int_{B_{r_i}} v_i^2 &= v_i^4(0)\int_{B_{R_i}}\frac{dy}{(1+k^2|y|^2)^{n-2}}\frac{1}{(v_i(0))^{\frac{n(p_i-1)}{2}}} + o(1)\\ &= \frac{1}{v_i(0)^{\frac{(p_i-1)n}{2}-4}}\int_0^{R_i}\frac{r^{n-1}dr}{(1+k_i^2r^2)^{n-2}} + o(1)\\ &= \frac{1}{v_i(0)^{\frac{2(4-n)}{n-2}}}(O(1) + \frac{(R_i)^{4-n}}{4-n}) + o(1) \qquad \text{if } n>4\\ &= O(1) + \log(R) \qquad \text{if } n=4\\ &\to \infty, \end{aligned} \tag{5.70}$$

for $n \ge 4$. We know that $\nabla h(0) = 0$ and by (5.52), for σ small,

$$|\nabla h(x)| \le C|x|^{\theta-1} \text{ in } B_\sigma.$$

We split the integral over B_σ as the sum of two integrals over B_{r_i} and $B_\sigma \backslash B_{r_i}$, and evaluate the first using Lemma 5.8 (i):

$$\begin{aligned} v_i^2(0) \int_{B_{r_i}} x \cdot \nabla h(x) v_i^{p_i+1}(x) &\leq v_i^2(0) \int_{B_\sigma} |x|^{\theta-1} v_i(x)^{p_i+1} \\ &= \frac{1}{(v_i(0))^{\frac{2\theta}{n-2}-2}} \Big(\int_0^{R_i} \frac{r^\theta r^{n-1} dr}{(1+k_i r^2)^n} \Big) + o(1) \\ &= \frac{1}{(v_i(0))^{\frac{2\theta}{n-2}-2}} \Big[O(1) + \frac{1}{R_i^{n-\theta}} \Big] + o(1), \end{aligned} \tag{5.71}$$

which is bounded as $i \to \infty$ if $n-2 \leq \theta \leq n$. The other integral we evaluate as follows:

$$v_i^2(0) \int_{B_\sigma \backslash B_{r_i}} (v_i(x))^{p_i+1} (x \cdot \nabla h(x)) \leq \frac{c}{(v_i(0))^{p_i-1}} \int_{r_i}^{\sigma} \frac{|r|^\theta r^{n-1} dr}{r^{(n-2)p_i}} \tag{5.72}$$

$$\leq \frac{C}{v_i(0)^{p_i-1}} \Big(\int_{r_i}^{\sigma} \frac{dr}{r^{3-\theta}} + o(1) \Big) = \frac{c}{(v_i(0))^{p_i-1}} \Big(\sigma^{\theta-2} - r_i^{\theta-2} + o(1) \Big) = o(1).$$

Now (5.68), (5.70), (5.71), and (5.72) give a contradiction to (5.67).

6. Solution branch

Now, we can prove Theorems 1.1 and 1.2.

Proof of Theorem 1.1. By the Crandall-Rabinowitz theorem (see [9]), there exists a local branch $\mathcal{C}^+$ in $\mathbb{R}^+ \times C_0(\Omega)$ bifurcating from $\lambda_1(\Omega)$. By the Hopf maximum principle, this branch remains in the cone of positive solutions in $C_0(\Omega)$.

To prove that $\lambda_0 := \sup\{\lambda \geq 0 : (\lambda, u) \in \mathcal{C}^+\} < \lambda_1(\Omega^+)$, we use a standard argument for superlinear elliptic problems : Multiply (1.1) by ϕ_{Ω^+}, the first eigenfunction of $-\Delta$ in (Ω^+), and integrate by parts in Ω^+, to obtain :

$$\lambda_1(\Omega^+) \int_{\Omega^+} u \phi_{\Omega^+} + \int_{\partial\Omega^+} \frac{\partial \phi_{\Omega^+}}{\partial n} u = \int_{\Omega^+} h(x) u^p \phi_{\Omega^+} + \lambda \int_{\Omega^+} u \phi_{\Omega^+}. \tag{6.1}$$

From (6.1), and the Hopf lemma

$$(\lambda_1(\Omega^+) - \lambda) \int_{\Omega^+} u \phi_{\Omega^+} \geq \int_{\Omega^+} h(x) u^p \phi_{\Omega^+} > 0,$$

which implies that $\lambda < \lambda_1(\Omega^+)$.

Observe that the a priori estimates of Proposition 3.2 and equation (4.25) implies that the solutions of (1.1) are uniformly bounded on $\partial\Omega_\delta^+$. Hence, further using the blow up analysis of Section 5 for $p = \frac{n+2}{n-2}$, we have that every solution to (1.1) with $\lambda \geq 0$ is bounded in $L^\infty(\Omega)$. This bound depends only on n, h, Γ, and Ω^+. Hence the branch has to reach $\{\lambda = 0\} \times C_0(\Omega)$. Note that the branch meets the axis $\lambda = 0$ at $u_0 \neq 0$ since $(0,0)$ is not a bifurcation point. This completes the proof of Theorem 1.1. $\square$

Remark. Suppose in addition that

$$\int_\Omega h(x)\phi_1^{p+1} < 0,$$

where ϕ_1 is a positive eigenfunction associated to the first eigenvalue. A well-known argument (see [2] or [14]), based upon the Crandall-Rabinowitz theorem, shows that the branch bifurcates to the right from $(0, \lambda_1(\Omega))$. Then $\lambda_0 > \lambda_1(\Omega)$.

Proof of Theorem 1.2. Let $\{\Omega_i\}$ be such that $\Omega_i \subset \Omega_{i+1}$, $\cup\Omega_i = \Omega$ and let the approximate problem on Ω_i be:

$$\begin{cases} -\Delta u &= \lambda u + h(x)u^{\frac{n+2}{n-2}} \quad \text{in } \Omega_i \\ u &> 0 \quad \text{in } \Omega_i \\ u &= 0 \quad \text{in } \partial\Omega_i. \end{cases} \tag{6.2}$$

By Theorem 1.1, there exists $\mathcal{C}_i^+$, a branch of positive solutions to (6.2), bifurcating from $(\lambda_1(\Omega_i), 0)$ such that

$$\Pi_{\mathbb{R}}\mathcal{C}_i^+ \supset [0, \lambda_1(\Omega_i)]. \tag{6.3}$$

From a priori estimates in Proposition 3.2, equation 4.25 and Section 5, we have that $\cup_i\mathcal{C}_i^+$ is a bounded set in $\mathbb{R} \times C_0(\Omega)$. Let us prove that $\cup_i\mathcal{C}_i^+$ is relatively compact in $\mathbb{R} \times C_0(\Omega)$. For this, let $(\lambda_i, u_i) \in \mathcal{C}_i^+$. Then, for any compact set K of Ω, it is easy to see that there exists $(\lambda, u) \in \mathbb{R} \times C_0(\Omega)$ such that (up to a subsequence), $\lambda_i \to \lambda$ and $u_i \to u$ in $C_0(K)$. Now, by (H4) and (3.9), $u_i(x) \to 0$ uniformly when $|x| \to +\infty$. Then, $\|u_i - u\|_{C_0(\Omega)} \to 0$ when $i \to +\infty$. Thus, the existence of $\mathcal{C}^+$ follows from Whyburn's results and the fact that $(0,0) \in \liminf_{i\to+\infty} \mathcal{C}_i$. Now, let us prove (i). For this, note that, from Theorem 1.1, there exists $(0, u_i) \in \mathcal{C}_i$ such that $u_i \neq 0$. Then,

$$\int_{\Omega^+} |\nabla u_i|^2 \leq \int_{\Omega_i} |\nabla u_i|^2 \leq \|h\|_{L^\infty(\Omega^+)} \int_{\Omega^+} u_i^{\frac{2N}{N-2}} \leq C \tag{6.4}$$

which implies by Sobolev imbeddings

$$\|u_i\|_{C_0(\Omega_i)} \geq \frac{1}{|\Omega^+|}(\|u_i\|_{L^{\frac{2N}{N-2}}(\Omega^+)}) \geq C(\Omega^+, \|h\|_{L^\infty(\Omega^+)}) > 0. \tag{6.5}$$

By the arguments above, $\{u_i\}$ admits a convergent subsequence, converging to $\bar{u}$ in $C_0(\Omega)$ with

$$\|\bar{u}\|_{C_0(\Omega)} \geq \limsup_{i\to+\infty} \frac{1}{|\Omega^+|}(\lim \|u_i\|_{L^{\frac{2N}{N-2}}(\Omega^+)}) \geq C(\Omega^+, \|h\|_{L^\infty(\Omega^+)}) > 0 \quad (6.6)$$

which completes the proof of (i). Observe that, if (ii) does not hold, there exists a sequence $\{(0, u_{\tau_i})\}$ in $\mathcal{C}^+ \setminus (0,0)$ such that $\|u_{\tau_i}\|_{C_0(\Omega)} \to 0$ when $i \to +\infty$. Now using similar arguments as (6.4) and (6.5) for u_{τ_i} in Ω, we get

$$\|u_{\tau_i}\|_{C_0(\Omega)} \geq C(\Omega^+, \|h\|_{L^\infty(\Omega^+)}) > 0$$

which is a contradiction. This completes the proof of Theorem 1.2. □

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Appendix

Bounds when $\lambda_i \to 0$: Here we assume that $\lambda_i \to \tilde{\lambda} = 0$. We will follow the arguments as in Proposition 4.4 of [12] that we adapt here for our equation in the bounded domain Ω_δ^+. Let $\{y_i\} \in \Omega_\delta^+$ be a sequence of local maxima of solutions (λ_i, u_i) of (1.1). Without loss of generality, we can assume that $y_i \to 0$. For a $\sigma > 0$ to be chosen later, by the Pohozaev identity, we have :

$$\int_{B_\sigma} \nabla h(x)(u_i(x))^{\frac{2n}{n-2}} = I_1 + I_2 + I_3$$

where

$$\begin{aligned} I_1 &= \frac{2n}{n-2}\int_{\partial B_\sigma}(\frac{\partial u}{\partial \nu}\nabla u - \frac{1}{2}|\nabla u|^2\nu), \\ I_2 &= \frac{\lambda_i}{2}\int_{\partial B_\sigma} u_i^2\nu, \qquad I_3 = \int_{\partial B_\sigma} h(x)(u_i(x))^{\frac{2n}{n-2}}\nu, \end{aligned}$$

where ν is the unit outward normal. Our aim is to show that for each $i \in \{1,2,3\}$,

$$|I_i| \leq \frac{C}{u(y_i)^{\frac{2n}{n-2}}} \text{ for } n \geq 6 \text{ and } |I_i| \leq \frac{C}{u(y_i)^{\frac{8}{3}}} \text{ for } n = 5.$$

Using Proposition 5.3 and Lemma 5.7, we have

$$|I_2| \leq \frac{C(\sigma)}{(u_i(y_i))^{2+\frac{2(n-4)}{n-2}}} \leq \frac{C(\sigma)}{(u_i(y_i))^{\frac{2n}{n-2}}} \text{ for } n \geq 6 \quad (6.7)$$

and

$$|I_3| \leq \frac{C(\sigma)}{(u_i(y_i))^{\frac{2n}{n-2}}}. \tag{6.8}$$

To estimate the integral I_1, we proceed as follows: Fix $0 < \sigma_1 < 1/2$ small and let $A = \{x : \sigma_1 \leq |x| \leq 1/2\}$ denote the annulus. Let $\varepsilon_i = \frac{1}{u_i(y_i)}$. We construct $\eta_i(r)$, $r = |x|$, a cut off function with support in A and such that

$$\begin{aligned} \eta_i(x) &= 1 \text{ for all } A_i = \{x : \sigma_1 + \varepsilon_i \leq |x| \leq 1/2 - \varepsilon_i\} && (6.9)\\ &= 0 \text{ for } \mathbb{R}^n \setminus A && (6.10)\\ |\nabla \eta_i(x)| &\leq \frac{c}{\varepsilon_i} \text{ for } x \in A \setminus A_i. && (6.11) \end{aligned}$$

Multiplying the equation (5.1) by $\eta_i u_i$ and integrating by parts we get

$$\int_A |\nabla u_i|^2 \eta_i + \int_A (\nabla u_i \cdot \nabla \eta_i) u_i = \lambda_i \int_A u_i^2 \eta_i + \int_A h u_i^{2n/n-2} \eta_i.$$

Hence

$$\int_{A_i} |\nabla u_i|^2 \leq 1/2 \int_{A \setminus A_i} |\nabla (u_i^2) \cdot \nabla \eta_i| + \lambda_i \int_A u_i^2 \eta_i + \int_A h u_i^{2n/n-2} \eta_i. \tag{6.12}$$

Note that the last two terms above can be estimated using Proposition 5.3 and Lemma 5.7. Again, using Proposition 5.3 and the definition of η_i, we have

$$\frac{1}{2} \int_{A \setminus A_i} |\nabla (u_i^2) \cdot \nabla \eta_i| \leq C \Big(\int_{A \setminus A_i} |\nabla (u_i^2)|^2 \Big)^{\frac{1}{2}} \Big(\int_{A \setminus A_i} |\nabla \eta_i|^2 \Big)^{\frac{1}{2}} = \frac{C}{u_i(y_i)^{n+1}}. \tag{6.13}$$

(Recall that here we use the Schauder estimate and the Harnack inequality as in the proof of Lemma 5.6 to get $\sup_A |\nabla u_i| \leq C \inf_A u_i(x)$ and then use Proposition 5.3.)

Substituting (6.13) in (6.12), we get

$$\begin{aligned} \int_{A_i} |\nabla u_i|^2 &\leq \frac{C}{u_i(y_i)^{n+1}} + \frac{C(\sigma)}{(u_i(y_i))^{2+\frac{2(n-4)}{n-2}}} + \frac{C(\sigma)}{(u_i(y_i))^{\frac{2n}{n-2}}} \\ &\leq \frac{C(\sigma)}{(u_i(y_i))^{\frac{2n}{n-2}}} \text{ for } n \geq 6 && (6.14)\\ &\leq \frac{C(\sigma)}{(u_i(y_i))^{\frac{8}{3}}} \text{ for } n = 5. && (6.15) \end{aligned}$$

Now, to get the estimate on the surface integral $\int_{\partial B_\sigma} |\nabla u_i|^2$, choose $\sigma_i \in [\sigma_1, 1/2]$ such that

$$\int_{\partial B_{\sigma_i}} |\nabla u_i|^2 = \inf_{\sigma \in [\sigma_1+\varepsilon_i, 1/2-\varepsilon_i]} \int_{\partial B_\sigma} |\nabla u_i|^2. \tag{6.16}$$

Thus from (6.14) and (6.15)

$$\begin{aligned} \int_{\partial B_{\sigma_i}} |\nabla u_i|^2 &\leq \frac{1}{(1/2-\sigma_1-2\varepsilon_i)} \int_{A_i} |\nabla u_i|^2 \\ &\leq \frac{1}{(1/2-\sigma_1-2\varepsilon_i)} \frac{C(\sigma_i)}{(u_i(y_i))^{\frac{2n}{n-2}}} \text{ for } n \geq 6 \qquad (6.17) \\ &\leq \frac{C(\sigma_i)}{(u_i(y_i))^{\frac{8}{3}}} \text{ for } n = 5. \qquad (6.18) \end{aligned}$$

Observing that

$$|I_1| \leq C \int_{\partial B_\sigma} |\nabla u_i|^2$$

we have proved our claim for the choice of $\sigma = \sigma_i$.

For $n \geq 6$, we follow the arguments as in [12], Proposition 4.4 and Corollary 4.1, to prove

Step 1 : $|y_i| = O(\frac{1}{u_i(y_i)^{\frac{2}{n-2}}})$ so that $y_i u_i(y_i)^{\frac{2}{n-2}} = \xi_i \to \xi$,

Step 2 : Multiplying the Pohozaev identity II by $u_i(y_i)^{\frac{2(\beta-1)}{n-2}}$, and using estimates on I_i and rescaling arguments, we get $\int_{\mathbb{R}^n} \nabla h(z+\xi) \frac{dz}{(1+k^2|z|^2)^n} = 0$ which is a contradiction because this integral is nonzero by our assumption (H3).

Now, let us deal with $n = 5$. In this case,

$$|I_1|, |I_2|, |I_3| \leq \frac{C}{u_i(y_i)^{\frac{8}{3}}}.$$

Again we follow the same arguments as in thr above steps 1 and 2. It works since $\frac{2(\beta-1)}{n-2} < \frac{8}{3}$ for $n = 5$ and if $\beta < n$. This gets the desired contradiction as for $n \geq 6$. □

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