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POSITIVE SOLUTIONS FOR AN INTEGRO-DIFFERENTIAL EQUATION WITH SINGULAR NONLINEAR TERM

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Abstract. The existence of a positive solution in a weighted Sobolev space for a homogeneous semilinear elliptic integro-differential Dirichlet problem is proved. The integral operator of the equation depends on a nonlinear function with a singularity at the origin.

1. Introduction

In this paper we establish an existence result for the following integro-differential problem

$$\begin{cases} -\Delta u(y) = \displaystyle\int_\Omega K(y,z) g\big(z,u(z)\big)dz, & \text{for } y \in \Omega, \\ u(y) = 0, & \text{for } y \in \partial\Omega, \end{cases} \tag{1.1}$$

with $\Omega \subset \mathbb{R}^N$, $N \geq 3$, open, bounded and sufficiently smooth and $g(z,s)$, $z \in \Omega, s > 0$, bounded in a neighborhood of $+\infty$ and possibly nonsmooth as $s \to 0^+$; in particular we do not exclude that

$$\varliminf_{s\to 0^+} g(y,s) = 0; \qquad \varlimsup_{s\to 0^+} g(y,s) = +\infty.$$

Our arguments work also for $N = 1$. Indeed, the Green's function of $-\Delta$ on $[0,1]$ belongs to L^∞ and is bounded from below by the product of the distances from $\{0,1\}$ (see (3.5) in the following pages). These facts produce a significant simplification of the proofs.

The interest of nonlinear integral equations of the second type with integral part depending on the reciprocal of the solution is both theoretical and practical (see [9]). They are based on the comparison between the local

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expression of the solutions and the integral term that describes the global behavior. This permits us to model several physical phenomena. For example, in the papers [2; 13] the authors study some aspects of the filtration theory regarding the Boussinesq's equation that are described by a Hammerstein equation with nonlinear term depending on the reciprocal of the solution. Moreover, in signal theory, as Nowosad pointed out in the papers [15; 16], a basic signal f is used for the transmission of messages. Obviously, it is desired that the transmitted and the received messages coincide. This happens if the Fourier transform of f satisfies a Hammerstein equation with the integrand depending on the reciprocal of the solution. In this paper we consider a particular model in which the local expression of the solution is given by its Laplacian.

This paper seems to be the first one regarding nonlinear integro-differential equations with integral term depending on the reciprocal of the solutions. However, these ones play a fundamental role in the analysis of systems of PDEs. Due to the few assumptions on the kernel K, (1.1) covers partly the stationary states of the following elliptic-parabolic problem

$$\begin{cases} u_t + f(t,x,u)_x + g(t,x,u) + P_x = (a(t,x)u_x)_x, & t>0,\ 0<x<1, \\ -P_{xx} + P = h(t,x,u,u_x) + k(t,x,u), & t>0,\ 0<x<1, \end{cases}$$

where $0 < a_* \le a(\cdot,\cdot) < a^*$, that guarantees the uniform parabolicity of the first equation. That system describes the motion of viscous nonlinear shallow water waves and the small amplitude viscous radial deformation waves in hyperelastic rods (see [5] and the references therein). If f, g are linear in u, using the Green's function, the first equation gives u in terms of P. Substituting in the second one (that is elliptic in P) and assuming that $h = 0, k(\cdot,\cdot,u) = 1/u$, we get an integro-differential problem of the same type as (1.1) in the unknown P.

One of the key points of this paper is that we do not assume anything about the existence of super or sub solutions. More precisely, denoting $\delta(x) := \mathrm{dist}(x, \partial\Omega)$, $x \in \mathbb{R}^N$, we shall assume

($\mathcal{A}_1$) *$g : \Omega \times \mathbb{R}^*_+ \to \mathbb{R}$ is a Carathèodory function (namely $g(\cdot, s)$ is measurable in Ω for all $s > 0$; $g(z,\cdot)$ is continuous in $\mathbb{R}^*_+$ for almost all $z \in \Omega$) such that*

$$0 \le g(z,s) \le \frac{\varphi_0(z)}{s^p}, \quad z \in \Omega, \quad 0 < s \le \frac{1}{2}, \quad p \ge \frac{N}{N-1},$$

where $\varphi_0 \in L^p(\Omega)$ *is a nonnegative map such that*

$$\frac{\varphi_0}{\delta^{p-1}} \in L^p(\Omega).$$

Moreover, $g^*(\cdot, s) \in L^p(\Omega), s > 0,$ *where*

$$g^*(z,s) := \sup_{s \le t} g(z,t), \quad (z,s) \in \Omega \times \mathbb{R}_+^*,$$

(*that is a Carathéodory function*).

($\mathcal{A}_2$) $K \in L^q(\Omega \times \Omega),\ q > N,$ *is a nonnegative kernel such that*

$$\frac{\delta(z)}{c_0} \le \int_\Omega K(y,z)\delta(y)dy \le c_0\delta(z), \quad z \in \Omega,$$

for some positive constant c_0.

($\mathcal{A}_3$) *There exist* $\mu_0 > 0$ *and* $\Omega_0 \subset \Omega, |\Omega_0| > 0,$ *such that*

$$\varliminf_{s\to 0} \frac{g(z,s)}{s} \ge \mu_0, \quad uniformly\ with\ respect\ to\ z \in \Omega_0.$$

Due to the assumption ($\mathcal{A}_1$), assuming the existence of a subsolution, the existence of solutions to (1.1) is trivial.

We prove that if μ_0 is bigger than the smallest characteristic value of the operator

$$\varphi \mapsto \int_{\Omega_0} H(x,\cdot)\varphi(x)dx, \quad \varphi \in L^1(\Omega_0),$$

there exists a weak solution $u_0 \in L^1(\Omega)$ to (1.1), that is positive almost everywhere in Ω, with $\delta|\nabla u| \in L^1(\Omega)$ and with trivial trace on $\partial\Omega$.

Our arguments use the properties of the Green's function $G(x,y)$ associated to $-\Delta$ in Ω with homogeneous conditions on $\partial\Omega$ and the properties of the kernel

$$H(x,z) := \int_\Omega G(x,y)K(y,z)dy.$$

In the first part of the paper we look for an existence result for the integral equation of Hammerstein type

$$u(x) = \int_\Omega H(x,z)g(z,u(z))dz. \tag{1.2}$$

The argument is based on the results of the papers [7; 8; 9], where references for this type of equation can be found.

The paper is organized as follows: Section 2: Notation and results; Section 3: Properties of the kernels G, K, H; Section 4: Proofs of Theorems 1 and 2; Section 5: On the integral equation (1.2); Section 6: Proof of Theorem 3.

2. Notation and results

Let us list the notation mostly used in this paper.

$\mathbb{R}_+ := [0, +\infty)$; $\mathbb{R}_+^* := (0, +\infty)$; $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. For $E \subset \mathbb{R}^N$ a measurable set, $|E|$ is the measure of E, $|\cdot|_{q,E}$ is the $L^q(E)$ norm and $L^q_+(E)$ is the cone of all $\varphi \in L^q(E), \varphi \geq 0$ almost everywhere in E. $L^1(\delta, E)$ is the set of all φ such that $\delta\varphi \in L^1(E)$, $L^1_+(\delta, E)$ is the cone of all $\varphi \geq 0$ almost everywhere in E such that $\delta\varphi \in L^1(E)$ and $W^{1,1}(\delta, E)$ is the space of all $\varphi \in L^1(E)$ with the modulus of the gradient (in the sense of distributions) belonging to $L^1_+(\delta, E)$. $W^{1,1}_0(\delta, E)$ is the subspace of all $\varphi \in W^{1,1}(\delta, E)$ with trivial trace on ∂E.

Let u, v be two maps; $u \leq v$ is the set of all points $x \in \Omega$ such that $u(x) \leq v(x)$. Analogously, we define $u < v,\ u \geq v,\ u > v$.

Finally, $D = \operatorname{diam}(\Omega), B_R(x)(\subset \mathbb{R}^N)$ is the ball centered at x with radius R and σ_N is the $(N-1)-$dimensional measure of $\partial B_1(0)$.

Let $E \subset \Omega$ be a measurable set; define

$$\lambda(E) := \inf\left\{\lambda(E, \varphi) : \varphi \in L^1_+(\Omega), \varphi \neq 0\right\},$$

where

$$\lambda(E, \varphi) = \sup_{z \in E^*(\varphi)} \frac{\varphi(z)\chi_E(z)}{\int_E H(x, z)\varphi(x)dx};$$

$$E^*(\varphi) = \left\{z \in E : \int_E H(x, z)\varphi(x)dx \neq 0\right\}.$$

The main results of this paper are the following:

Theorem 1. *Let $E \subset \Omega$ be a measurable set, with $|E| > 0$. $\lambda(E)$ is the smallest positive characteristic value of the operator*

$$\varphi \mapsto \int_E H(x, \cdot)\varphi(x)dx, \quad \varphi \in L^1(\Omega).$$

Useful for the following Theorem 3 is the left continuity of $\lambda(E)$.

Theorem 2. *$\lambda(\cdot)$ is left continuous; more precisely, for each measurable set $E \subset \Omega$, $|E| > 0$, and $\alpha > 0$ there exists $\sigma > 0$ such that for every measurable set $F \subset E, |E \setminus F| < \sigma$, we have*

$$\lambda(E) \leq \lambda(F) \leq \lambda(E) + \alpha.$$

Other properties of $\lambda(E)$ are proved in Section 4. Finally, as said in the Introduction, the following result holds.

Theorem 3. *Assume $(\mathcal{A}_i)$, $i = 1, 2, 3$, and $\mu_0 > \lambda(\Omega_0)$. There exists $u_0 \in W^{1,1}_0(\delta, \Omega), u_0 > 0$ almost everywhere in Ω, a weak solution to* (1.1).

3. Properties of the kernels G, K, H.

In this section we prove some properties of the kernels G, K, H and of the associated integral operators that are crucial in the proofs of Theorems 1, 2, and 3.

The exponent q present in the following statements is the one of $(\mathcal{A}_2)$ and q' is the conjugate one.

Lemma 3.1. *There exists $c_1 > 0$ such that, for each $x, y \in \Omega, x \neq y$, we have*

$$\frac{1}{c_1|x-y|^{N-2}} \leq G(x,y) \leq \frac{c_1}{|x-y|^{N-2}}, \quad |x-y| \to 0. \tag{3.1}$$

$$|\nabla_x G(x,y)| \leq \frac{c_1}{|x-y|^{N-1}}. \tag{3.2}$$

$$|\nabla_x G(x,y)| \leq \frac{c_1 \delta(y)}{|x-y|^{N}}. \tag{3.3}$$

$$|\delta(x)\nabla_x G(x,y)| \leq \frac{c_1 \delta(y)}{|x-y|^{N-1}}. \tag{3.4}$$

$$\frac{\delta(x)\delta(y)}{c_1} \leq G(x,y); \qquad \int_\Omega G(x,y)dx \leq c_1\delta(y). \tag{3.5}$$

$$\Big(\int_\Omega G(x,y)^r dy\Big)^{\frac{1}{r}} \leq c_1 \int_\Omega G(x,y)dy, \quad 1 \leq r < \frac{N}{N-1}. \tag{3.6}$$

$$\Big(\int_\Omega G(x,y)^{\frac{N}{N-1}} dy\Big)^{\frac{N-1}{N}} \leq c_1\delta(x)|\log\delta(x)|. \tag{3.7}$$

Proof. (3.1) is well known (see for example [1, Chapter 4]). (3.2) and (3.3) are proved in [12; 17]. (3.4) is a consequence of (3.1) through (3.3). (3.5) is proved in [4, Lemma 3.2; 6, Theorem 9; 18, Theorem 1]. Finally, (3.6) and (3.7) are shown in [3, Theorem 1 and (1.9)]. □

Lemma 3.2. *We have*

$$K(y,\cdot)g^*(\cdot,s) \in L^1(\Omega), \qquad \int_\Omega K(\cdot,z)g^*(z,s)dz \in L^q(\Omega),$$

for each $s > 0$ and almost every $y \in \Omega$.

Proof. The claim follows from $(\mathcal{A}_1)$ and $(\mathcal{A}_2)$. □

Lemma 3.3. *The following statements are equivalent*

i) there exists $c_0 > 0$ such that

$$\frac{\delta(z)}{c_0} \leq \int_\Omega K(y,z)\delta(y)dy \leq c_0\delta(z), \quad z \in \Omega.$$

ii) there exists $c_2 > 0$ *such that*

$$\frac{\delta(x)\delta(z)}{c_2} \le H(x,z); \quad \int_\Omega H(x,z)dx \le c_2\delta(z), \quad x,z \in \Omega.$$

Proof. $i) \Rightarrow ii)$ This is a trivial consequence of the definition of $H(x,z)$ and (3.5).

Proof. $ii) \Rightarrow i)$ Let $\varphi_1(x)$ be a positive eigenfunction and λ_1 the first eigenvalue of the Dirichlet problem for $-\Delta$ on Ω. We have that

$$\lambda_1 \int_\Omega H(x,z)\varphi_1(x)dx = \lambda_1 \int_\Omega K(y,z)dy \int_\Omega G(x,y)\varphi_1(x)dx$$
$$= \int_\Omega K(y,z)\varphi_1(y)dy.$$

By Theorem 9 in [6], there exists $c_3 > 0$ such that

$$\frac{\delta(x)}{c_3} \le \varphi_1(x) \le c_3\delta(x).$$

Therefore, using $ii)$,

$$\frac{\lambda_1\delta(z)}{c_2}\int_\Omega \delta(x)\varphi_1(x)dx \le c_3 \int_\Omega K(y,z)\delta(y)dy$$

and

$$\frac{1}{c_3}\int_\Omega K(y,z)\delta(y)dy \le \lambda_1|\varphi_1|_{\infty,\Omega}\int_\Omega H(x,z)dx \le \lambda_1 c_2|\varphi_1|_{\infty,\Omega}\delta(z).$$

Then $i)$ is proved. □

Theorem 3.4. *The following statements hold*

$$H : \varphi \mapsto \int_\Omega H(\cdot,z)\varphi(z)dz \tag{3.8}$$

is bounded from $L^1(\delta,\Omega)$ *into* $L^1(\Omega)$.

$$\tilde{H} : \varphi \mapsto \int_\Omega H(x,\cdot)\varphi(x)dx \tag{3.9}$$

is bounded from $L^1(\delta,\Omega)$ *in* $L^q(\Omega)$.

$$\text{For each } s > 0 : \ (x,z) \mapsto H(x,z)g^*(z,s) \text{ belongs to } L^1(\Omega\times\Omega). \tag{3.10}$$

Proof of (3.8). Let $\varphi \in L^1(\delta,\Omega)$. From $ii)$ of Lemma 3.3,

$$|H(\varphi)|_{1,\Omega} \le \int_\Omega |\varphi(z)|dz \int_\Omega H(x,z)dx \le c_2 \int_\Omega |\varphi(z)|\delta(z)dz = c_2|\delta\varphi|_{1,\Omega}.$$

Proof of (3.9). Let $\varphi \in L^1(\delta,\Omega)$. Since $q' < \frac{N}{N-1}$, by (3.6) and (3.5),

$$\begin{aligned}
|\tilde{H}(\varphi))|_{q,\Omega}^q &= \int_\Omega |\tilde{H}(\varphi)(z)|^q dz = \int_\Omega dz \Big| \int_\Omega \varphi(x)dx \int_\Omega G(x,y)K(y,z)dy\Big|^q \\
&\leq \int_\Omega dz \left\{ \int_\Omega |\varphi(x)|dx \Big(\int_\Omega G(x,y)^{q'}dy\Big)^{\frac{1}{q'}} \Big(\int_\Omega K(y,z)^q dy\Big)^{\frac{1}{q}} \right\}^q \qquad (3.11)\\
&\leq c_1^{2q} \int_\Omega dz \left\{ \int_\Omega |\varphi(x)\delta(x)|dx \Big(\int_\Omega |K(y,z)|^q dy\Big)^{\frac{1}{q}} \right\}^q = c_1^{2q} |K|_{q,\Omega\times\Omega}^q |\varphi\delta|_{1,\Omega}^q.
\end{aligned}$$

Proof of (3.10). From $(\mathcal{A}_1)$ we have $g^*(\cdot,s) \in L^1(\Omega)$, hence, using (3.8), (3.10) is a consequence of the Tonelli theorem. □

Theorem 3.5. *H is compact from $L^1(\delta,\Omega)$ into $L^1(\Omega)$.*

Proof. We claim that H is the limit of a sequence of linear compact operators from $L^1(\delta,\Omega)$ into $L^1(\Omega)$. Let $\widetilde{D} := \{(x,x) : x \in \mathbb{R}^N\}$ be the diagonal set of $\mathbb{R}^N \times \mathbb{R}^N$. Recall that the Green's function $G(x,y)$ is strictly positive in $\Omega\times\Omega$, continuous in $(\bar{\Omega}\times\bar{\Omega}) \setminus \widetilde{D}$, vanishes on $\partial(\Omega\times\Omega)\setminus\widetilde{D}$ and, since $N > 1$,

$$\lim_{|x-y|\to 0} G(x,y) = +\infty$$

(see [1, Chapter 4]). Let $n \in \mathbb{N}$ and define

$$G_n(x,y) := \begin{cases} \dfrac{nG(x,y)}{n+G(x,y)}, & \text{for } x \neq y, \\ n, & \text{for } x = y. \end{cases}$$

Clearly $G_n \leq G, G_n \in C(\bar{\Omega}\times\bar{\Omega}), G_n$ is strictly positive in $\Omega\times\Omega$, and vanishes on $\partial(\Omega\times\Omega)$. Consider the linear operator

$$H_n(\varphi) := \chi_{\Omega_n}(\cdot) \int_\Omega H_n(\cdot,z)\varphi(z)dz, \quad \varphi \in L^1(\delta,\Omega),$$

where

$$\Omega_n = \{x \in \Omega : \delta(x) \geq \frac{1}{n}\}, \qquad H_n(x,z) := \int_\Omega G_n(x,y)K(y,z)dy.$$

Since $G_n \leq G_{n+1} \leq G, H_n$ is continuous from $L^1(\delta,\Omega)$ into $L^1(\Omega)$ and

$$\|H_n\| \leq \|H_{n+1}\| \leq \|H\|. \qquad (3.12)$$

The claim is a consequence of the following lemmas.

Lemma 3.6. *H_n is compact from $L^1(\delta,\Omega)$ into $L^1(\Omega)$.*

Proof. Let $\mathcal{F} \subset L^1(\delta, \Omega)$ be bounded; by (3.8) and (3.12), $H_n(\mathcal{F})$ is bounded in $L^1(\Omega)$. We prove the equicontinuity of $H_n(\varphi), \varphi \in \mathcal{F}$, in $L^1(\Omega)$,

$$\begin{aligned}\Delta(h,\varphi) &= |H_n(\varphi)(\cdot + h) - H_n(\varphi)|_{1,\Omega} \\ &\le \int_\Omega \chi_{\Omega_n}(x+h)dx \int_\Omega |\big(H_n(x+h,z) - H_n(x,z)\big)\varphi(z)|dz \\ &+ \int_\Omega |\chi_{\Omega_n}(x+h) - \chi_{\Omega_n}(x)|dx \int_\Omega H_n(x,z)|\varphi(z)|dz.\end{aligned}$$

Let $h \in \mathbb{R}, x \in \Omega$ be such that

$$|h| \le \frac{1}{2n} \quad \text{and} \quad (x+h) \in \Omega_n;$$

we have

$$\frac{1}{n} \le \delta(x+h) \le \delta(x) + |h|,$$

hence, $\frac{1}{2n} \le \delta(x)$. Observe that if $|\chi_{\Omega_n}(x+h) - \chi_{\Omega_n}(x)| = 1$ and $x \in \Omega$, we have only two possibilities

$$\big((x+h) \in \Omega_n \text{ and } x \notin \Omega_n\big) \text{ or } \big(x \in \Omega_n \text{ and } (x+h) \notin \Omega_n\big),$$

thus

$$\Big(\delta(x) < \frac{1}{n} \le \delta(x+h)\Big) \text{ or } \Big(\delta(x+h) < \frac{1}{n} \le \delta(x)\Big);$$

that implies

$$\Big(\frac{1}{n} - |h| \le \delta(x) < \frac{1}{n}\Big) \text{ or } \Big(\frac{1}{n} \le \delta(x) < \frac{1}{n} + |h|\Big),$$

and finally

$$\frac{1}{n} - |h| \le \delta(x) < \frac{1}{n} + |h|.$$

Denoting

$$E_h = \Big\{x \in \Omega : \frac{1}{n} - |h| \le \delta(x) < \frac{1}{n} + |h|\Big\},$$

we have

$$\lim_{h \to 0} |E_h| = 0,$$

and

$$\begin{aligned}\Delta(h,\varphi) &\le \int_{\Omega_{2n}} dx \int_\Omega |H_n(x+h,z) - H_n(x,z)| \cdot |\varphi(z)|dz \\ &+ \int_{E_h} dx \int_\Omega H_n(x,z)|\varphi(z)|dz = \Delta_1(h,\varphi) + \Delta_2(h,\varphi).\end{aligned}$$

We estimate $\Delta_1(h,\varphi)$ and $\Delta_2(h,\varphi)$. Since

$$\Delta_1(h,\varphi) \le \int_{\Omega\times\Omega} K(y,z)|\varphi(z)|dy\,dz \int_{\Omega_{2n}} |G_n(x+h,y)-G_n(x,y)|dx$$

and

$$x+th\in\Omega, \qquad \text{for all } \ 0\le t\le 1,\ \ x\in\Omega_{2n},\ \ |h|<\frac{1}{2n},$$

we have

$$\gamma(h,y) := \int_{\Omega_{2n}} |G_n(x+h,y)-G_n(x,y)|dx$$

$$= \int_{\Omega_{2n}} dx\Big|\int_0^1 \frac{d}{dt}G_n(x+th,y)dt\Big| = \int_{\Omega_{2n}} dx\Big|\int_0^1 \frac{n^2\nabla_x G(x+th,y)\cdot h}{(n+G(x+th,y))^2}dt\Big|,$$

for every $|h|<1/2n$. From (3.1) and (3.3),

$$\gamma(h,y) \le n^2|h|\int_{\Omega_{2n}} dx\int_0^1 \frac{\frac{c_1\delta(y)}{|x+th-y|^N}}{\big(n+\frac{1}{c_1|x+th-y|^{N-2}}\big)^2}dt$$

$$\le n^2c_1^3|h|\delta(y)\int_0^1 dt\int_\Omega \frac{|x+th-y|^{N-4}}{(nc_1|x+th-y|^{N-2}+1)^2}dx$$

$$\le n^2c_1^3|h|\delta(y)\int_0^1 dt\int_{B_D(y-th)} \frac{|x+th-y|^{N-4}}{(nc_1|x+th-y|^{N-2}+1)^2}dx$$

$$= n^2c_1^3|h|\delta(y)\sigma_N\int_0^D \frac{\rho^{N-4}\cdot\rho^{N-1}}{(nc_1\rho^{N-2}+1)^2}d\rho \le n^2c_1^3\sigma_N|h|\delta(y)\frac{D^{2N-4}}{2N-4},$$

for each $|h|<1/2n$. Then, there exists $c>0$, independent of h and y, such that

$$\int_{\Omega_{2n}} |G_n(x+h,y)-G_n(x,y)|dx \le c|h|\delta(y), \qquad |h|<\frac{1}{2n}.$$

Due to $(\mathcal{A}_2)$,

$$\Delta_1(h,\varphi) \le c|h|\int_{\Omega\times\Omega}\delta(y)K(y,z)|\varphi(z)|dydz \le cc_0|h|\int_\Omega \delta(z)|\varphi(z)|dz.$$

Let $|h|<1/(2n)$; using the Hölder inequality, $(\mathcal{A}_2)$, (3.6), and (3.5),

$$\Delta_2(h,\varphi) \le \int_{\Omega\times\Omega} K(y,z)|\varphi(z)|dydz\int_{E_h} G(x,y)dx$$

$$\le \int_{\Omega\times\Omega} K(y,z)|\varphi(z)|dydz\Big(\int_\Omega G(x,y)^{q'}dx\Big)^{\frac{1}{q'}}|E_h|^{\frac{1}{q}}$$

$$\le c_1^2|E_h|^{\frac{1}{q}}\int_{\Omega\times\Omega}\delta(y)K(y,z)|\varphi(z)|dy\,dz \le c_0c_1^2|E_h|^{\frac{1}{q}}\int_\Omega |\varphi(z)|\delta(z)dz.$$

Thanks to the estimates on $\Delta_1(h,\varphi), \Delta_2(h,\varphi)$,

$$|H_n(\varphi)(\cdot+h)-H_n(\varphi)|_{1,\Omega} \leq \left(cc_0|h|+c_0c_1^2|E_h|^{\frac{1}{q}}\right)\int_\Omega |\varphi(z)|\delta(z)dz, \quad |h|\leq \frac{1}{2n}.$$

Then, $H_n(\varphi), \varphi \in \mathcal{F}$, is equicontinuous in $L^1(\Omega)$. Due to the Frechet-Kolmogorov theorem $H_n(\mathcal{F})$ is relatively compact in $L^1(\Omega)$; this proves the compactness of H_n. □

Lemma 3.7. *$H_n \to H$ in the operator norm.*

Proof. Let $\varphi \in L^1(\delta,\Omega), |\delta\varphi|_{1,\Omega} = 1$. We have

$$\Lambda_n(\varphi) = |H(\varphi)-H_n(\varphi)|_{1,\Omega} = \int_\Omega dx \Big|\int_\Omega \big(H(x,z)-\chi_{\Omega_n}(x)H_n(x,z)\big)\varphi(z)dz\Big|$$
$$\leq \int_{\Omega\setminus\Omega_n} dx\int_\Omega H(x,z)|\varphi(z)|dz + \int_{\Omega_n} dx\int_\Omega |H(x,z)-H_n(x,z)||\varphi(z)|dz$$
$$= \int_{\Omega\times\Omega} K(y,z)|\varphi(z)|dydz\Big(\int_{\Omega\setminus\Omega_n} G(x,y)dx + \int_{\Omega_n}\Big|G(x,y)-\frac{nG(x,y)}{n+G(x,y)}\Big|dx\Big)$$
$$= \Lambda_n'(\varphi)+\Lambda_n''(\varphi).$$

Using the Hölder inequality, (3.5), (3.6), and $(\mathcal{A}_2)$, we get

$$\Lambda_n'(\varphi) \leq \int_{\Omega\times\Omega} K(y,z)|\varphi(z)|dydz\Big(\int_{\Omega\setminus\Omega_n} G(x,y)^{q'}dx\Big)^{\frac{1}{q'}}|\Omega\setminus\Omega_n|^{\frac{1}{q}}$$
$$\leq c_1^2|\Omega\setminus\Omega_n|^{\frac{1}{q}}\int_\Omega|\varphi(z)|dz\int_\Omega \delta(y)K(y,z)dy$$
$$\leq c_0c_1^2|\Omega\setminus\Omega_n|^{\frac{1}{q}}\int_\Omega |\varphi(z)|\delta(z)dz \leq c_0c_1^2|\Omega\setminus\Omega_n|^{\frac{1}{q}}.$$

Again using the Hölder inequality, (3.7), and (3.5),

$$\Lambda_n''(\varphi) \leq \int_{\Omega\times\Omega} K(y,z)|\varphi(z)|dydz\Big(\int_{\Omega_n} G(x,y)^{\frac{N}{N-1}}dx\Big)^{\frac{N-1}{N}}$$
$$\times\Big(\int_{\Omega_n}\Big(\frac{G(x,y)}{n+G(x,y)}\Big)^N dx\Big)^{\frac{1}{N}}$$
$$\leq c_1\int_{\Omega\times\Omega} K(y,z)|\varphi(z)|\delta(y)|\ln\delta(y)|dydz\Big(\int_\Omega \frac{G(x,y)}{n+G(x,y)}dx\Big)^{\frac{1}{N}}$$
$$\leq \frac{c_1}{\sqrt[N]{n}}\int_{\Omega\times\Omega} K(y,z)|\varphi(z)|\delta(y)|\ln\delta(y)|dydz\Big(\int_\Omega G(x,y)dx\Big)^{\frac{1}{N}}$$
$$\leq \frac{c_1^{\frac{1+N}{N}}}{\sqrt[N]{n}}\int_{\Omega\times\Omega} K(y,z)|\varphi(z)|\delta(y)^{\frac{1+N}{N}}|\ln\delta(y)|dydz.$$

Hence there exists $c > 0$, independent of n, φ, such that, by $(\mathcal{A}_2)$,

$$\Lambda_n''(\varphi) \leq \frac{c}{\sqrt[N]{n}} \int_{\Omega\times\Omega} K(y,z)|\varphi(z)|\delta(y)dydz \leq \frac{c_0 c}{\sqrt[N]{n}} \int_\Omega |\varphi(z)|\delta(z)dz \leq \frac{c_0 c}{\sqrt[N]{n}}.$$

Finally, from the estimates on $\Lambda_n'(\varphi), \Lambda_n''(\varphi)$, there exists $c > 0$, independent of n, φ, such that

$$|H(\varphi) - H_n(\varphi)|_{1,\Omega} \leq c\big(|\Omega \setminus \Omega_n|^{\frac{1}{q}} + \frac{1}{\sqrt[N]{n}}\big),$$

for every $\varphi \in L^1(\delta,\Omega)$, $|\delta\varphi|_{1,\Omega} = 1$. This proves the claim. □

Theorem 3.8. *$\tilde{H}$ is compact from $L^1(\delta,\Omega)$ into $L^q(\Omega)$.*

Proof. Let $\mathcal{F} \subset L^1(\delta,\Omega)$ be bounded; by (3.9), $\tilde{H}(\mathcal{F})$ is bounded in $L^q(\Omega)$. We prove the equicontinuity of $\tilde{H}(\varphi), \varphi \in \mathcal{F}$, in $L^q(\Omega)$. Arguing as in (3.11)

$$|\tilde{H}(\varphi)(\cdot + h) - \tilde{H}(\varphi)|_{q,\Omega}^q \leq c_1^{2q}|\varphi\delta|_{1,\Omega}^q \cdot \int_{\Omega\times\Omega} |K(y,z+h) - K(y,z)|^q dydz.$$

Therefore, the equicontinuity of $\tilde{H}(\varphi), \varphi \in \mathcal{F}$, is a consequence of the boundedness of $\mathcal{F}$ in $L^1(\delta,\Omega)$ and of the $L^q(\Omega\times\Omega)$ mean continuity of K. Finally, the compactness of $\tilde{H}$ is a consequence of the Frechet-Kolmogorov theorem. □

Corollary 3.9. *Let $E \subset \Omega$ be a measurable set, $|E| > 0$. The operator*

$$\tilde{H}_E(\varphi) := \int_E H(x,\cdot)\varphi(x)dx, \quad \varphi \in L^1(\delta, E)$$

is compact from $L^1(\delta,E)$ into $L^q(E)$.

4. Proofs of Theorems 1 and 2

Let $E \subset \Omega$ be measurable, $|E| > 0$. The following lemmas are needed.

Lemma 4.1. *For every $\varphi \in L^1_+(E)$ we have*

$$\int_E H(x,z)\varphi(x)dx \geq \frac{\delta(z)}{c_2} \int_E \delta(x)\varphi(x)dx, \tag{4.1}$$

$$E^*(\varphi) = E \setminus \partial\Omega, \quad \varphi \neq 0, \tag{4.2}$$

$$\frac{1}{c_1^2|E|^{\frac{1}{q'}}|K|_{q,\Omega\times\Omega} \cdot \sup_E \delta} \leq \lambda(E) \leq \frac{c_2}{|\delta|_{2,E}^2}. \tag{4.3}$$

Proof. (4.1) is a direct consequence of Lemma 3.3 *ii*). Let $\varphi \in L^1_+(E), \varphi \neq 0$; clearly

$$\int_E \delta(x)\varphi(x)dx > 0.$$

Hence, (4.2) follows from (4.1).

We prove (4.3). Since $\delta > 0$, by the definition of $\lambda(E)$, (4.1), and (4.2),

$$\begin{aligned}\lambda(E) \leq \lambda(E,\delta) &= \operatorname*{esssup}_{z\in E^*(\delta)} \frac{\delta(z)\chi_{E(z)}}{\int_E H(x,z)\delta(x)dx} \\ &\leq \operatorname*{esssup}_{z\in E^*(\delta)} \frac{c_2\delta(z)}{\delta(z)\int_E \delta(x)^2dx} = \frac{c_2}{|\delta|_{2,E}^2}.\end{aligned}$$

Moreover, for each $\varphi \in L^1_+(E), \varphi \neq 0$, using the definition of $\lambda(E,\varphi)$,

$$\begin{aligned}&\int_E \varphi(z)dz \leq \lambda(E,\varphi)\int_E \varphi(x)dx \int_E H(x,z)dz \\ &\leq \lambda(E,\varphi)\int_E \varphi(x)dx\int_E dz\Big(\int_\Omega G(x,y)^{q'}dy\Big)^{\frac{1}{q'}}\Big(\int_\Omega K(y,z)^q dy\Big)^{\frac{1}{q}}.\end{aligned}$$

By (3.5) and (3.6),

$$\begin{aligned}\int_E \varphi(z)dz &\leq \lambda(E,\varphi)c_1^2\int_E \varphi(x)\delta(x)dx\int_E dz\Big(\int_\Omega K(y,z)^q dy\Big)^{\frac{1}{q}} \\ &\leq \lambda(E,\varphi)c_1^2|E|^{\frac{1}{q'}}|K|_{q,\Omega\times\Omega}\cdot\sup_E \delta\cdot\int_E \varphi(z)dz.\end{aligned}$$

Since $0 < \int_E \varphi(z)dz$,

$$\frac{1}{c_1^2|E|^{\frac{1}{q'}}|K|_{q,\Omega\times\Omega}\cdot\sup_E\delta} \leq \lambda(E,\varphi), \qquad \text{for all } \varphi\in L^1_+(E),\ \varphi\neq 0.$$

Again from the definition of $\lambda(E)$, we have the lower bound for $\lambda(E)$ stated in (4.3). □

Lemma 4.2. *There exists $\Phi \in L^q_+(E)$, $\Phi > 0$ almost everywhere in E, such that*

$$\lambda(E) = \operatorname*{esssup}_{z\in E\setminus\partial\Omega} \frac{\Phi(z)}{\int_E H(x,z)\Phi(x)dx}.$$

Proof. Due to the definition of $\lambda(E)$, there exists $(\varphi_n)_{n\in\mathbb{N}^*}, |\delta\varphi_n|_{1,E} = 1$, such that

$$\operatorname*{esssup}_{z\in E\setminus\partial\Omega} \frac{\varphi_n(z)}{\tilde{H}_E(\varphi_n)(z)} < \lambda(E) + \frac{1}{n}. \tag{4.4}$$

Denoting $\tilde{H}_E(\varphi_n) = \Phi_n$, due to the compactness of $\tilde{H}_E$ from $L^1(\delta,E)$ in $L^q(E)$ (see Corollary 3.6) there exists $\Phi \in L^q(E)$, such that, passing to a subsequence, $\Phi_n \to \Phi$, in $L^q(E)$. From (4.1),

$$\Phi_n(z) \geq \frac{\delta(z)}{c_2}|\delta\varphi_n|_{1,E},$$

then $\Phi > 0$ almost everywhere in E. Moreover, since $H(\cdot,\cdot) \geq 0$, from (4.4),

$$\Phi_n = \tilde{H}_E(\varphi_n) \leq \big(\lambda(E) + \frac{1}{n}\big)\tilde{H}_E(\Phi_n) \quad \text{in } E.$$

By the continuity of $\tilde{H}_E$ (see Corollary 3.9),

$$\Phi \leq \lambda(E)\tilde{H}_E(\Phi), \quad \text{in } E,$$

hence, from the definition of $\lambda(E)$,

$$\operatorname*{esssup}_{z\in E\setminus\partial\Omega} \frac{\Phi(z)}{\tilde{H}_E(\Phi)(z)} = \lambda(E);$$

then, the proof is done. □

Proof of Theorem 1. We argue as in [14, Theorem 2.5]. Let $\Phi \in L^1_+(E), |\Phi|_{1,E} = 1$, be a minimum point for $\varphi \mapsto \lambda(E,\varphi)$ (see Lemma 4.2). Consider the set

$$\mathcal{E} := \big\{\psi \in L^1(E) : |\psi|_{1,E} \leq 1, \psi \geq 0 \ a.e.\big\};$$

it is closed, bounded and convex. Moreover, the operator

$$A_n(\psi) := \frac{\tilde{H}_E(\psi + \frac{\Phi}{n})}{\big|\tilde{H}_E(\psi + \frac{\Phi}{n})\big|_{1,E}}, \quad n \in \mathbb{N}^*,$$

maps $\mathcal{E}$ into itself. The compactness of $\tilde{H}_E$ from $L^1(E)(\subset L^1(\delta,\Omega))$ into itself and the fact that

$$\big|\tilde{H}_E(\psi + \frac{\Phi}{n})\big|_{1,E} \geq \frac{1}{n}\big|\tilde{H}_E(\Phi)\big|_{1,E} > 0, \qquad \psi \in \mathcal{E},$$

imply that $A_n(\mathcal{E})$ is compact. Due to the Schauder fixed-point theorem, there exists $\psi_n \in \mathcal{E}$ such that $A_n(\psi_n) = \psi_n$. Clearly, $|\psi_n|_{1,E} = 1$. Denoting

$$\mu_n = \frac{1}{\big|\tilde{H}_E(\psi_n + \frac{\Phi}{n})\big|_{1,E}},$$

we can rewrite the previous identity on ψ_n in the following way

$$\mu_n \tilde{H}_E(\psi_n + \frac{\Phi}{n}) = \psi_n. \tag{4.5}$$

Due to the positivity of $H(x,z)$ and Lemma 4.2,

$$\psi_n \geq \frac{\mu_n}{n}\tilde{H}_E(\Phi) \geq \frac{\mu_n}{\lambda(E)n}\Phi. \tag{4.6}$$

We claim that

$$\frac{\rho}{n}(1 + \rho + \cdots + \rho^k)\Phi \leq \psi_n, \qquad k \in \mathbb{N}, \tag{4.7}$$

where $\rho = \frac{\mu_n}{\lambda(E)}$. The estimate for $k = 0$ is the one stated in (4.6). For $k \geq 1$, observe that

$$\psi_n = \mu_n \tilde{H}_E(\psi_n + \frac{\Phi}{n}) \geq \mu_n \tilde{H}_E(\frac{\rho}{n}(1 + \rho + \cdots + \rho^k)\Phi + \frac{\Phi}{n})$$

$$= \mu_n\big(\frac{\rho}{n}(1 + \rho + \cdots + \rho^k) + \frac{1}{n}\big)\tilde{H}_E(\Phi) \geq \frac{\mu_n}{\lambda(E)n}(1 + \rho + \cdots + \rho^{k+1})\Phi.$$

Arguing by induction we get (4.7). From (4.7), integrating on E,

$$\frac{\rho}{n}(1 + \rho + \cdots + \rho^k) \leq 1, \quad k \in \mathbb{N}.$$

Then, $\rho < 1$, namely

$$\mu_n < \lambda(E), \qquad n \in \mathbb{N}^*. \tag{4.8}$$

By the compactness of $\tilde{H}_E$ (see Corollary 3.9) and the boundedness of $(\psi_n)_{n\in\mathbb{N}^*}$, there exist $(n_i)_{i\in\mathbb{N}}, n_i \to \infty$, $\Psi \in L^1(E)$, $\mu_0 \geq 0$, such that

$$\tilde{H}_E(\psi_{n_i} + \frac{\Phi}{n_i}) \longrightarrow \Psi \text{ in } L^1(E), \qquad \mu_0 = \lim_i \mu_{n_i}.$$

From (4.5), $\mu_0|\Psi|_{1,E} = 1$, hence, $\mu_0 > 0$ and $\Psi \neq 0$. Again by (4.5), $(\psi_{n_i})_{i\in\mathbb{N}}$ converges to $\mu_0\Psi$, due to the continuity of $\tilde{H}_E$ (see Theorem 3.4),

$$\mu_0 \tilde{H}_E(\Psi) = \Psi.$$

Using Lemma 3.3 *ii*), we get $\Psi \in L^1_+(E)$ and $\mu_0 = \lambda(E, \Psi)$. From the definition of $\lambda(E)$, $\lambda(E) \leq \mu_0$, and, by (4.8), we can conclude that: $\lambda(E) = \mu_0$. Finally, using again the definition of $\lambda(E)$, we have that μ_0 is the smallest characteristic value of $\tilde{H}_E$. □

Lemma 4.3. *For each $\alpha > 0$ and $\varphi \in L^q_+(E)$ there exists $\sigma > 0$ such that for every measurable $F \subset E, |E \setminus F| < \sigma$, we have*

$$\int_{E\setminus F} H(x,z)\varphi(x)dx < \alpha \int_E H(x,z)\varphi(x)dx, \quad z \in \Omega.$$

Proof. We begin by observing that for each measurable $S \subset \Omega$

$$\int_S G(x,y)\varphi(x)dx \leq \Big(\int_\Omega G(x,y)^{q'}dx\Big)^{\frac{1}{q'}}\Big(\int_S \varphi(x)^q dx\Big)^{\frac{1}{q}}.$$

Since $q' < \dfrac{N}{N-1}$, due to the symmetry of G and (3.5), (3.6), we get

$$\int_S G(x,y)\varphi(x)dx \leq c_1^2|\varphi|_{q,S}\delta(y), \quad y \in \Omega. \tag{4.9}$$

Moreover, again using (3.5),

$$\int_S G(x,y)\varphi(x)dx \geq \frac{\delta(y)}{c_1}\int_S \varphi(x)\delta(x)dx, \quad y\in\Omega. \tag{4.10}$$

Let $\alpha > 0$. Due to the absolute continuity of the integral of $\varphi^q\chi_E$, there exists $\sigma > 0$ such that for each measurable set $F\subset E, |E\setminus F| < \sigma$:

$$\Big(\int_{E\setminus F}\varphi(x)^q dx\Big)^{\frac{1}{q}} < \frac{\alpha}{c_1^3}|\varphi\delta|_{1,E},$$

hence

$$c_1^2|\varphi|_{q,(E\setminus F)}\delta(y) < \frac{\alpha}{c_1}|\varphi\delta|_{1,E}\delta(y), \quad y\in\Omega.$$

Using (4.9) and (4.10),

$$\int_{E\setminus F} G(x,y)\varphi(x)dx < \alpha\int_E G(x,y)\varphi(x)dx, \quad y\in\Omega.$$

Multiplying by $K(y,z)$ and integrating on Ω with respect to y we get the claim. □

Proof of Theorem 2. Let $F\subset E$ and $\varphi\in L^1_+(E)$. Since $\varphi\chi_F\in L^1_+(E)$, if $\varphi\chi_F\neq 0$, from the definition of $\lambda(E)$, we get

$$\begin{aligned}\lambda(E) &\leq \operatorname*{esssup}_{z\in E\setminus\partial\Omega}\frac{(\varphi\chi_F)(z)}{\int_E H(x,z)(\varphi\chi_F)(x)dx}\\ &= \operatorname*{esssup}_{z\in F\setminus\partial\Omega}\frac{(\varphi\chi_F)(z)}{\int_F H(x,z)(\varphi\chi_F)(x)dx} = \lambda(F,\varphi\chi_F),\end{aligned}$$

then

$$\begin{aligned}\lambda(E) &\leq \inf\big\{\lambda(E,\varphi\chi_F) : \varphi\in L^1_+(E), \varphi\chi_F\neq 0\big\}\\ &= \inf\big\{\lambda(F,\varphi) : \varphi\in L^1_+(F), \varphi\neq 0\big\} = \lambda(F).\end{aligned}$$

We continue by proving the other estimate stated in the claim.

Let $\alpha > 0$ (since $\lambda(E) < +\infty$, see (4.3)), denote $\beta = \frac{\alpha}{1+\lambda(E)+\alpha}$. Let $\Phi\in L^q_+(\Omega)$ be such that (see Lemma 4.2)

$$\lambda(E) = \operatorname*{esssup}_{z\in E\setminus\partial\Omega}\frac{\Phi(z)}{\int_E H(x,z)\Phi(x)dx}.$$

By the previous lemma, there exists $\sigma > 0$ such that for each measurable $F\subset E, |E\setminus F| < \sigma$:

$$\int_{E\setminus F} H(x,z)\Phi(x)dx < \beta\int_E H(x,z)\Phi(x)dx, \quad z\in\Omega.$$

Therefore, using the fact $|E \setminus F| < \sigma$,

$$\begin{aligned}\lambda(F) \le \lambda(F, \Phi\chi_F) &= \operatorname*{esssup}_{z \in F \setminus \partial\Omega} \frac{(\Phi\chi_F)(z)}{\int_F H(x,z)(\Phi\chi_F)(x)dx} \\ &= \operatorname*{esssup}_{z \in F \setminus \partial\Omega} \frac{(\Phi\chi_F)(z)}{\int_E H(x,z)\Phi(x)dx} \cdot \frac{\int_E H(x,z)\Phi(x)dx}{\int_F H(x,z)\Phi(x)dx} \\ &\le \operatorname*{esssup}_{z \in F \setminus \partial\Omega} \lambda(E) \frac{\int_E H(x,z)\Phi(x)dx}{\int_E H(x,z)\Phi(x)dx - \int_{E\setminus F} H(x,z)\Phi(x)dx} \\ &= \lambda(E) \operatorname*{esssup}_{z \in F \setminus \partial\Omega} \frac{1}{1 - \frac{\int_{E\setminus F} H(x,z)\Phi(x)dx}{\int_E H(x,z)\Phi(x)dx}} \le \frac{\lambda(E)}{1-\beta}.\end{aligned}$$

Due to the definition of β,

$$\begin{aligned}\lambda(F) &\le \frac{\lambda(E)}{1 - \frac{\alpha}{1+\lambda(E)+\alpha}} = \frac{\lambda(E)(1+\lambda(E)+\alpha)}{1+\lambda(E)} \\ &\le \lambda(E)\Big(1 + \frac{\alpha}{1+\lambda(E)}\Big) \le \lambda(E) + \alpha.\end{aligned}$$

Then the proof is done. □

5. On the integral equation (1.2)

Since $g(z, \cdot)$ is not defined in 0, we search for a solution in the limit points of the set of solutions of the approximate integral equations

$$u(x) = \int_\Omega H(x,z) g(z, \varepsilon + u(z))dz, \quad \varepsilon > 0. \tag{5.1}$$

Thanks to $(\mathcal{A}_1)$ and (3.8), there exists a solution $u_\varepsilon \in L^1_+(\Omega), \varepsilon > 0$, to (5.1), (see [8, Appendix 2]).

Denoting $g_\varepsilon = g(\cdot, \varepsilon + u_\varepsilon)$, the following statements are consequences of $(\mathcal{A}_1), (\mathcal{A}_2)$, and Lemma 3.3.

Lemma 5.1 (boundedness of $(\delta g_\varepsilon)_{\varepsilon>0}$) (see [9, Lemma 5.1].) *Let $E \subset \Omega$ be a measurable set and $0 < \varepsilon \le \frac{1}{4}$. We have*

$$|\delta g_\varepsilon|_{1,E} \le T(E)^{\frac{p}{p+1}} + T(E),$$

where

$$T(E) = |\delta g^*(\cdot, 1/4)|_{1,E} + c_2 |\delta^{1-p}\varphi_0|^{\frac{1}{p}}_{1,E},$$

and c_2 is the constant of Lemma 3.3 ii).

Corollary 5.2 (see [9, Lemma 5.2].) *For each* $\lambda > 0$, *there exists* $\sigma > 0$ *such that for every measurable set* $E \subset \Omega, |E| < \sigma$, *and any* $0 < \varepsilon \leq 1/4$, *we have*

$$|\delta g_\varepsilon|_{1,E} < \lambda.$$

Lemma 5.3. *Let* $\varepsilon > 0$. *We have*

$$g_\varepsilon \in L^p(\Omega), \tag{5.2}$$

$$\int_\Omega K(\cdot, z) g_\varepsilon(z) dz \in L^q(\Omega). \tag{5.3}$$

Proof. Since $g_\varepsilon \leq g^*(\cdot, \varepsilon)$, (5.2) and (5.3) are consequence of $(\mathcal{A}_1)$ and Lemma 3.2, respectively. □

For the sake of simplicity we fix an increasing sequence

$$(\Omega_n)_{n \in \mathbb{N}^*}, \quad \frac{1}{n} \leq \operatorname{dist}(\Omega_n, \partial\Omega)$$

that covers Ω.

The proof of the following lemma is similar to the one of [9, Lemma 5.4]; we simply sketch and improve it. Observe that this statement can be deduced by Theorem 3.5 and Lemma 5.1, however in the proof we show a constructive method to approximate the solution that is useful in the other proofs.

Lemma 5.4 (convergence). *There exists* $(\varepsilon_k)_{k \in \mathbb{N}}, \varepsilon_k \to 0$, *such that, for each* $n \in \mathbb{N}^*$

$$\Big(\int_{\Omega_n} H(\cdot, z) g_{\varepsilon_k}(z) dz \Big)_{k \in \mathbb{N}}$$

is converging in $L^1(\Omega)$. *Denoting*

$$v_n := \lim_k \int_{\Omega_n} H(\cdot, z) g_{\varepsilon_k}(z) dz,$$

$(v_n)_{n \in \mathbb{N}^*}$ *is increasing and* $v_n \in L^1(\Omega), n \in \mathbb{N}$. *Denoting also*

$$u_0 := \sup_n v_n = \lim_n v_n,$$

we have $u_0 \in L^1_+(\Omega)$ *and* $u_{\varepsilon_k} \to u_0$ *in* $L^1(\Omega)$.

Proof. Due to the boundedness of $(\delta g_\varepsilon)_{\varepsilon > 0}$ in $L^1(\Omega)$, each family $(\chi_{\Omega_n} g_\varepsilon)_{\varepsilon > 0}$, $n \in \mathbb{N}$, is bounded in $L^1(\Omega)$. Moreover, due to the compactness of H (see Theorem 3.5), there exists $(\varepsilon^1_k)_{k \in \mathbb{N}^*}, \varepsilon^1_k \to 0$, such that

$$\Big(\int_{\Omega_1} H(\cdot, z) g_{\varepsilon^1_k}(z) dz \Big)_{k \in \mathbb{N}^*}$$

is converging in $L^1(\Omega)$ to some function v_1. There exists $(\varepsilon_k^n)_{k\in\mathbb{N}^*}, \varepsilon_k^n \to 0$, a subsequence of $(\varepsilon_k^1)_{k\in\mathbb{N}^*}, \cdots, (\varepsilon_k^{n-1})_{k\in\mathbb{N}^*}$, such that

$$\Big(\int_{\Omega_i} H(\cdot,z) g_{\varepsilon_k^n}(z)dz\Big)_{k\in\mathbb{N}^*}, \quad 1\le i\le n,$$

is converging in $L^1(\Omega)$ to some function v_n. Clearly, $v_1 \le v_2 \le \cdots \le v_n$.

Let $(\varepsilon_k)_{k\in\mathbb{N}}$, be the diagonal sequence; it is an extract of each $(\varepsilon_k^n)_{k\in\mathbb{N}}$, it is infinitesimal and

$$v_n = \lim_k \int_{\Omega_n} H(\cdot,z) g_{\varepsilon_k}(z)dz, \quad \text{in } L^1(\Omega), \quad n\in\mathbb{N}^*.$$

$(v_n)_{n\in\mathbb{N}^*}$ is increasing and $v_n \in L^1(\Omega)$. There exists a measurable nonnegative map $u_0:\Omega\to\mathbb{R}$, such that

$$u_0 = \operatorname*{esssup}_n v_n = \lim_n v_n, \quad \text{a.e. in } \Omega.$$

Consider

$$u'_{k,n} = \int_{\Omega_n} H(\cdot,z) g_{\varepsilon_k}(z)dz, \quad u''_{k,n} = u_{\varepsilon_k} - u'_{k,n};$$

since

$$\int_\Omega u'_{k,n}(x)dx \le \int_{\Omega_n} g_{\varepsilon_k}(z)dz \int_\Omega H(x,z)dx,$$

using Lemmas 3.3.*ii*) and Lemma 5.1,

$$\int_\Omega u'_{k,n}(x)dx \le c_2 \int_{\Omega_n} \delta(z) g_{\varepsilon_k}(z)dz \le c_2\big(T(\Omega)^{\frac{p}{p+1}} + T(\Omega)\big).$$

Due to the definition of v_n, and the Fatou lemma,

$$\int_\Omega v_n(x)dx \le c_2\big(T(\Omega)^{\frac{p}{p+1}} + T(\Omega)\big).$$

By the Beppo Levi theorem,

$$\int_\Omega u_0(x)dx \le c_2\big(T(\Omega)^{\frac{p}{p+1}} + T(\Omega)\big).$$

Hence $u_0 \in L^1(\Omega)$. We continue by proving that

$$\lim_k |u_{\varepsilon_k} - u_0|_{1,\Omega} = 0.$$

From the Fubini and Tonelli theorems and Lemma 3.3.*ii*),

$$\int_\Omega u''_{k,n}(x)dx \le c_2 \int_{\Omega\setminus\Omega_n} \delta(z) g_{\varepsilon_k}(z)dz.$$

Therefore, by Corollary 5.2,

$$\lim_n \int_\Omega u''_{k,n}(x)dx = 0,$$

uniformly with respect to k. Let $\sigma > 0$. There exists $M_0 \in \mathbb{N}$ such that

$$\int_\Omega u''_{k,n}(x)dx < \sigma, \qquad n > M_0, \;\; k \in \mathbb{N}. \tag{5.4}$$

Observe that

$$\int_\Omega |u_{\varepsilon_k} - u_0|dx \le \int_\Omega |u'_{k,n} - v_n|dx + \int_\Omega \big(u_0 - v_n\big)dx + \int_\Omega u''_{k,n}dx.$$

Since $\lim_k |u'_{k,n} - v_n|_{1,\Omega} = 0$,

$$\overline{\lim_k} |u_{\varepsilon_k} - u_0|_{1,\Omega} \le \int_\Omega \big(u_0 - v_n\big)dx + \sigma, \qquad n > M_0.$$

Finally, since $u_0 \in L^1(\Omega)$, using the dominated convergence theorem,

$$\overline{\lim_k} |u_{\varepsilon_k} - u_0|_{1,\Omega} \le \sigma,$$

thus $u_{\varepsilon_k} \to u_0$ in $L^1(\Omega)$. □

In addition to the upper bound stated in Lemma 5.1, the following statements hold (see [9, (5.6)]).

Lemma 5.5. *We have*

$$\overline{\lim_k} |g_{\varepsilon_k}|_{1,\Omega_n \cap X} \le c_2 L n^2, \qquad |g(\cdot, u_0)|_{1,\Omega_n\cap X} \le c_2 L n^2,$$

for each $n \in \mathbb{N}^*$, *where* $X = \{x \in \Omega : u_0(x) \le L\}, L > 0$.

Proof. Let $u'_{k,n} u''_{k,n}$ be as in the proof of the previous lemma. From Lemma 3.3,

$$u'_{k,n}(x) \ge \frac{\delta(x)}{c_2} \int_{\Omega_n} \delta(z) g_{\varepsilon_k}(z)dz \ge \frac{1}{c_2 n^2} |g_{\varepsilon_k}|_{1,\Omega_n}, \quad x \in \Omega_n.$$

Multiplying by $\dfrac{g_{\varepsilon_k}}{1 + u'_{k,n}}$ and integrating on $\Omega_n \cap X$,

$$\frac{1}{c_2 n^2} |g_{\varepsilon_k}|_{1,\Omega_n} \int_{\Omega_n \cap X} \frac{g_{\varepsilon_k}}{1 + u'_{k,n}} dx \le \int_{\Omega_n\cap X} \frac{u'_{k,n}}{1 + u'_{k,n}} g_{\varepsilon_k} dx. \tag{5.5}$$

Due to the boundedness of $\big(|g_{\varepsilon_k}|_{1,\Omega_n}\big)_{k\in\mathbb{N}^*}$ and Lemma 5.5 in [9],

$$\lim_k |g_{\varepsilon_k}|_{1,\Omega_n} \int_{\Omega_n \cap X} |\frac{1}{1 + u'_{k,n}} - \frac{1}{1 + v_n}| g_{\varepsilon_k} dx = 0$$

and

$$\lim_k \int_{\Omega_n \cap X} |\frac{u'_{k,n}}{1+u'_{k,n}} - \frac{v_n}{1+v_n}| g_{\varepsilon_k} dx = 0.$$

Hence, from (5.5),

$$\overline{\lim_k} \Big(\frac{1}{c_2 n^2} |g_{\varepsilon_k}|_{1,\Omega_n} \int_{\Omega_n \cap X} \frac{g_{\varepsilon_k}}{1+v_n} dx \Big) \le \overline{\lim_k} \int_{\Omega_n \cap X} \frac{v_n}{1+v_n} g_{\varepsilon_k} dx.$$

Recalling that $u_0 = \sup_n v_n$,

$$\frac{1}{1+L} \overline{\lim_k} |g_{\varepsilon_k}|^2_{1,\Omega_n \cap X} \le \frac{c_2 n^2 L}{1+L} \overline{\lim_k} |g_{\varepsilon_k}|_{1,\Omega_n \cap X}.$$

This implies the first estimate of the statement; the second one is a consequence of the Fatou lemma. $\square$

A consequence of these lemmas, as in [9, Theorem 4], is the following fundamental result.

Theorem 5.6 (see [9, Theorem 4]). *Assume $\mu_0 > \lambda(\Omega_0)$. We have*

$$u_0 > 0 \text{ a.e. in } \Omega \text{ and } u_0(x) = \int_\Omega H(x,z) g(z, u_0(z)) dz.$$

The last result of this section is the following, that is useful for the next one.

Lemma 5.7. *The following statements hold*

$$g(\cdot, u_0) \in L^1(\delta, \Omega), \tag{5.6}$$

$$g_{\varepsilon_k}(\cdot) \to g(\cdot, u_0) \quad \text{in } L^1(\delta, \Omega), \tag{5.7}$$

$$\int_\Omega K(\cdot, z) g(z, u_0(z)) dz \in L^1_+(\delta, \Omega), \tag{5.8}$$

$$\int_\Omega K(\cdot, z) g_{\varepsilon_k}(z) dz \to \int_\Omega K(\cdot, z) g(z, u_0(z)) dz \quad \text{in } L^1(\delta, \Omega). \tag{5.9}$$

Proof (5.6). Since $u_{\varepsilon_k} \to u_0$ almost everywhere in Ω (see Lemma 5.4) and $u_0 > 0$ almost everywhere in Ω (see the previous theorem), we have

$$g_{\varepsilon_k} \to g(\cdot, u_0), \quad \text{a.e. in } \Omega.$$

Using Lemma 5.1 and the Fatou lemma,

$$\int_\Omega \delta(z) g(z, u_0(z)) dz \le \underline{\lim_k} \int_\Omega \delta(z) g_{\varepsilon_k}(z) dz \le T(\Omega)^{\frac{p}{p+1}} + T(\Omega),$$

hence (5.6) is done.

Proof of (5.7). If $\operatorname{essinf} u_0 > 0$, due to [8, Lemma 3], (5.7) is trivial. If $\operatorname{essinf} u_0 = 0$, there exists a decreasing family of measurable sets $(X_l)_{l>0}$, $|X_l| > 0$, such that

$$u_0(x) \le \frac{1}{1+l}, \qquad x \in X_l.$$

Observe that

$$\int_\Omega \delta(z)|g_{\varepsilon_k}(z) - g(z,u_0(z))|dz \tag{5.10}$$
$$\le \int_{\Omega\setminus X_l} \delta(z)|g_{\varepsilon_k}(z) - g(z,u_0(z))|dz + \int_{\Omega\setminus\Omega_n} \big(\delta(z)g_{\varepsilon_k}(z) + \delta(z)g(z,u_0(z))\big)dz$$
$$+ \int_{\Omega_n\cap X_l} \big(\delta(z)g_{\varepsilon_k}(z) + \delta(z)g(z,u_0(z))\big)dz.$$

Let $\sigma > 0$. By Corollary 5.2 and the absolute continuity of the integral of $\delta g(\cdot, u_0)$, there exists $n \in \mathbb{N}$ such that

$$\int_{\Omega\setminus\Omega_n} \big(\delta(z)g_{\varepsilon_k}(z) + \delta(z)g(z,u_0(z))\big)dz < \frac{\sigma}{3}, \qquad k \in \mathbb{N}.$$

From Lemma 5.5, there exists $l \in \mathbb{N}$ such that

$$\overline{\lim_k} \int_{\Omega_n\cap X_l} \big(\delta(z)g_{\varepsilon_k}(z) + \delta(z)g(z,u_0(z))\big)dz \le \frac{2c_2 n^2}{1+l} < \frac{\sigma}{3}.$$

Hence, from [8, Lemma 3], there exists k_0 such that

$$\int_{\Omega\setminus X_l} \delta(z)|g_{\varepsilon_k}(z) - g(z,u_0(z))|dz < \frac{\sigma}{3}, \qquad k > k_0.$$

Therefore, by (5.10),

$$\overline{\lim_k} \int_\Omega \delta(z)|g_{\varepsilon_k}(z) - g(z,u_0(z))|dz < \sigma.$$

This implies (5.7).

Proof of (5.8). It is a consequence of $(\mathcal{A}_2)$ and (5.6).

Proof of (5.9). Since, from $(\mathcal{A}_2)$,

$$\int_\Omega \delta(y)dy\Big|\int_\Omega K(y,z)\big(g_{\varepsilon_k}(z) - g(z,u_0(z))\big)dz\Big|$$
$$\le c_0 \int_\Omega \delta(z)|g_{\varepsilon_k}(z) - g(z,u_0(z))|dz,$$

the claim follows by (5.7). □

6. Proof of Theorem 3

We begin by observing that

$$\nabla_x H(x,z) = \int_\Omega \nabla_x G(x,y)K(y,z)dy. \tag{6.1}$$

Let $x_0 \in \Omega$ and denote $x = (x_i, x'), 1 \leq i \leq n$. There exists $\theta \in]0,1[$ such that

$$\frac{H(x_{0,i}+h, x_0', z) - H(x_0,z)}{h} = \int_\Omega G_{x_i}(x_{0,i}+\theta h, x_0', y)K(y,z)dy.$$

Since, for each $E \subset \Omega$, by (3.2), we get

$$\begin{aligned}
&\int_E |G_{x_i}(x_i, x_0', y)K(y,z)|dy \\
&\leq c_1 \Big(\int_E \frac{1}{\sqrt{(x_i-y_i)^2+|x_0'-y'|^2}^{(N-1)q'}} dy\Big)^{\frac{1}{q'}} \Big(\int_E K(y,z)^q dy\Big)^{\frac{1}{q}} \\
&\leq c_1 |E|^{\frac{1}{rq'}} \Big(\int_{B_D(x_i,x_0')} \frac{1}{\sqrt{(x_i-y_i)^2+|x_0'-y'|^2}^{(N-1)q'r'}} dy\Big)^{\frac{1}{q'r'}} \Big(\int_E K(y,z)^q dy\Big)^{\frac{1}{q}} \\
&\leq c_1 |E|^{\frac{1}{rq'}} \Big(\int_{B_D(0)} \frac{1}{|y|^{(N-1)q'r'}} dy\Big)^{\frac{1}{q'r'}} \Big(\int_E K(y,z)^q dy\Big)^{\frac{1}{q}},
\end{aligned}$$

where $\frac{N(q-1)}{q-N} < r$ and r' is the conjugate exponent of r. The integral

$$E \mapsto \int_E |G_{x_i}(x_i, x_0', y)K(y,z)|dy$$

is absolutely continuous uniformly with respect to x_i. Using the Vitali theorem, passing to the limit as $h \to 0$ we get (6.1).

Lemma 6.1. *The following statements hold*

$$\int_\Omega \nabla_x H(\cdot,z) g(z,u_0(z))dz \in L^1(\delta,\Omega)^N, \tag{6.2}$$

$$\nabla u_\varepsilon = \int_\Omega \nabla_x H(\cdot,z) g_\varepsilon(z)dz \in L^\infty(\Omega)^N, \tag{6.3}$$

$$\nabla u_{\varepsilon_k} \to \int_\Omega \nabla_x H(\cdot,z) g(z,u_0(z))dz \quad in\ L^1(\delta,\Omega)^N, \tag{6.4}$$

$$\int_\Omega \nabla_x H(\cdot,z) g(z,u_0(z))dz = \nabla u_0 \tag{6.5}$$

in the sense of distributions.

Proof of (6.2). By (3.4) and (6.1),

$$I = \int_\Omega \delta(x)dx|\int_\Omega \nabla_x H(x,z)g(z,u_0(z))dz|$$

$$\leq \int_{\Omega\times\Omega\times\Omega} \delta(x)|\nabla_x G(x,y)|K(y,z)g(z,u_0(z))dxdydz$$

$$\leq c_1 \int_{\Omega\times\Omega\times\Omega} \frac{\delta(y)}{|x-y|^{N-1}} K(y,z)g(z,u_0(z))dxdydz.$$

Observe that

$$\int_\Omega \frac{dx}{|x-y|^{N-1}} \leq \int_{B_D(y)} \frac{dx}{|x-y|^{N-1}} = \int_{B_D(0)} \frac{dx}{|x|^{N-1}} = \sigma_N D. \tag{6.6}$$

Hence, from $(\mathcal{A}_2)$,

$$I \leq c_1 \int_\Omega g(z,u_0(z))dz \int_\Omega \delta(y)K(y,z)dy \int_\Omega \frac{dx}{|x-y|^{N-1}}$$

$$\leq c_0 c_1 \sigma_N D \int_\Omega \delta(z)g(z,u_0(z))dz.$$

Therefore, using (5.6), we get (6.2).

Proof (6.3). Since $g_\varepsilon(z) \leq g^*(z,\varepsilon)$, by Lemma 3.2,

$$k_\varepsilon := \int_\Omega K(\cdot,z)g_\varepsilon(z)dz \in L^q(\Omega).$$

Arguing as for (6.1),

$$\nabla u_\varepsilon(x) = \int_\Omega \nabla_x G(x,y)k_\varepsilon(y)dy = \int_\Omega \nabla_x H(x,z)g_\varepsilon(z)dz.$$

Moreover, by (3.2),

$$|\nabla u_\varepsilon(x)| \leq \int_\Omega |\nabla_x G(x,y)|k_\varepsilon(y)dy \leq c_1 \int_\Omega \frac{k_\varepsilon(y)}{|x-y|^{N-1}}dy$$

$$\leq c_1|k_\varepsilon|_{q,\Omega}\Big(\int_\Omega \frac{dy}{|x-y|^{(N-1)q'}}\Big)^{\frac{1}{q'}} \leq c_1|k_\varepsilon|_{q,\Omega}\Big(\int_{B_D(x)} \frac{dy}{|x-y|^{(N-1)q'}}\Big)^{\frac{1}{q'}}$$

$$\leq c_1|k_\varepsilon|_{q,\Omega}\Big(\int_{B_D(0)} \frac{dy}{|y|^{(N-1)q'}}\Big)^{\frac{1}{q'}}.$$

Since $(N-1)q' < N$, we have that $\nabla u_\varepsilon \in L^\infty(\Omega)^N$. □

Proof (6.4). By (6.1), (6.2), (6.3),

$$J = \int_\Omega \delta(x)\big|\nabla u_{\varepsilon_k}(x) - \int_\Omega \nabla_x H(x,z)g(z,u_0(z))dz\big|dx$$

$$= \int_\Omega \delta(x) \Big| \int_\Omega \nabla_x H(x,z)\big(g_{\varepsilon_k}(z) - g(z,u_0(z))\big)dz \Big| dx$$

$$\leq \int_{\Omega\times\Omega\times\Omega} \delta(x)\big|\nabla_x G(x,z)\big|K(y,z)\big|g_{\varepsilon_k}(z) - g(z,u_0(z))\big|dxdydz.$$

Hence, from (3.4) and $(\mathcal{A}_2)$,

$$J \leq c_1 \int_\Omega \big|g_{\varepsilon_k}(z) - g(z,u_0(z))\big|dz \int_\Omega \delta(y)K(y,z)dy \int_\Omega \frac{dx}{|x-y|^{N-1}}$$

$$\leq c_0 c_1 D\sigma_N \int_\Omega \delta(z)\big|g_{\varepsilon_k}(z) - g(z,u_0(z))\big|dz.$$

Then, (5.7) implies (6.4).

Proof (6.5). Let $\varphi \in \mathcal{D}(\Omega)$. (6.4) implies

$$\int_\Omega \varphi(x)dx \int_\Omega \nabla_x H(x,z)g(z,u_0(z))dz$$

$$= \lim_k \int_\Omega \varphi(x)\nabla u_{\varepsilon_k}(x)dx = -\lim_k \int_\Omega \big(\nabla\varphi(x)\big)u_{\varepsilon_k}(x)dx.$$

Since $u_{\varepsilon_k} \to u_0$ in $L^1(\Omega)$ (see Lemma 5.4),

$$\int_\Omega \varphi(x)dx \int_\Omega \nabla_x H(x,z)g(z,u_0(z))dz = -\int_\Omega \nabla\varphi(x)u_0(x)dx,$$

namely (6.5).

Lemma 6.2. *Let $\varepsilon > 0$. We have $u_\varepsilon \in W^{2,q}(\Omega)(\subset C^1(\bar{\Omega}))$ and*

$$\begin{cases} -\Delta u_\varepsilon(y) = \int_\Omega K(y,z)g\big(z,\varepsilon + u_\varepsilon(z)\big)dz, & \text{for } y \in \Omega, \\ u_\varepsilon(y) = 0, & \text{for } y \in \partial\Omega. \end{cases}$$

Proof. As in the previous lemma denote

$$k_\varepsilon = \int_\Omega K(\cdot,z)g_\varepsilon(z)dz.$$

Since $k_\varepsilon \in L^q(\Omega)$ (see Lemma 5.3), there exists $k_{\varepsilon,n} \in C^\infty(\bar{\Omega})$ such that $k_{\varepsilon,n} \to k_\varepsilon$ in $L^q(\Omega)$. Denoting

$$u_{\varepsilon,n}(x) = \int_\Omega G(x,y)k_{\varepsilon,n}(y)dy,$$

due to the regularity of $k_{\varepsilon,n}$,

$$-\Delta u_{\varepsilon,n} = k_{\varepsilon,n}, \quad \text{in } \Omega; \quad u_{\varepsilon,n}\big|_{\partial\Omega} = 0.$$

By [11, Theorem 9.15 and Lemma 9.17], $u_{\varepsilon,n} \in W^{2,q}(\Omega)$ and

$$\|u_{\varepsilon,n} - u_{\varepsilon,m}\|_{W^{2,q}(\Omega)} \le c|k_{\varepsilon,n} - k_{\varepsilon,m}|_{q,\Omega},$$

with c independent of n and m. Hence,

$$u_\varepsilon \in W^{2,q}(\Omega) \quad \text{e} \quad -\Delta u_\varepsilon = k_\varepsilon.$$

Due to the Sobolev embedding theorem (see [10, Theorem 5.6]), $u_\varepsilon \in C^1(\bar{\Omega})$. Finally, since

$$u_\varepsilon(x) = \int_\Omega G(x,y)k_\varepsilon(y)dy,$$

we have that $u_\varepsilon\big|_{\partial\Omega} = 0$. □

Proof of Theorem 3. From Lemma 6.1, $u_0 \in W^{1,1}(\delta,\Omega)$. Since $u_{\varepsilon_k} \to u_0$ in $L^1(\Omega)$, by the trace theorem (see [10, page 258]), we have that $u_0 \in W_0^{1,1}(\delta,\Omega)$. We prove that u_0 is a weak solution to (1.1). Let $\varphi \in \mathcal{D}(\Omega)$. By Lemma 5.4,

$$-\int_\Omega \big(\Delta\varphi(y)\big)u_0(y)dy = -\lim_k \int_\Omega \big(\Delta\varphi(y)\big)u_{\varepsilon_k}(y)dy = -\lim_k \int_\Omega \varphi(y)\Delta u_{\varepsilon_k}(y)dy$$
$$= \lim_k \int_\Omega \varphi(y)dy \int_\Omega K(y,z)g_{\varepsilon_k}(z)dz.$$

Since $\operatorname{dist}(\operatorname{supp}\varphi, \partial\Omega) > 0$, by virtue of (5.9),

$$-\int_\Omega \big(\Delta\varphi(y)\big)u_0(y)dy = \int_\Omega \varphi(y)dy \int_\Omega K(y,z)g(z,u_0(z))dz.$$

The proof is done. □

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