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ON FIRST-ORDER PERTURBATIONS OF THE SCHRÖDINGER EQUATION WITH CONJUGATION

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Abstract. We show that first-order perturbations of the Schrödinger equation with conjugation are globally and uniquely solvable for L^2 initial data. Global Strichartz estimates are established under minimal assumptions on the "conjugate magnetic" potential.

1. Introduction

In this paper, we deal with the question of obtaining (global) Strichartz estimates for the "conjugate magnetic" Schrödinger equation

$$\begin{aligned} &\partial_t u - i\Delta u + \vec{V}(t,x)\cdot\nabla\bar{u} = F \quad (t,x)\in \mathbf{R}^+\times\mathbf{R}^n \\ &u(0,x) = f(x). \end{aligned} \tag{1.1}$$

In a recent paper [10], the author has established similar *a priori* estimates for the solutions of the magnetic Schrödinger equation:

$$\begin{aligned} &\partial_t u - i\Delta u + \vec{V}(t,x)\cdot\nabla u = F \quad (t,x)\in \mathbf{R}^+\times\mathbf{R}^n \\ &u(0,x) = f(x). \end{aligned} \tag{1.2}$$

The gradient Schrödinger equation (1.2) was studies also by I. Bejenaru in his thesis, [1].

Such models appear in geometric equations involving the covariant Laplacian, with the natural interpretation of $\vec{V}$ as the connection. It turns out that solutions to (1.2) exist globally under appropriate assumptions on V. However, (1.1) exhibits far better behavior due to the presence of the complex conjugation.

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Let us also recall the set of the Strichartz exponents for the regular Schrödinger equation. We say that a pair of indices is Strichartz admissible if $2 \leq q, r \leq \infty$, $2/q + n/r = n/2$ and $(q, r, n) \neq (2, \infty, 2)$. Then

$$\left\|e^{it\Delta} f\right\|_{L^q L^r} \leq C\|f\|_{L^2} \tag{1.3}$$

$$\left\| \int_0^t e^{i(t-s)\Delta} F(s, \cdot) ds L^q L^r \right\| \leq C\|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}}. \tag{1.4}$$

Our goal is to achieve the same estimates under some integrability conditions on the potential V.

The existence and uniqueness problem for (1.2), (1.1) has been studied extensively by many authors in the physics as well as in the mathematics literature. We should first point to the pioneering work of Doi, [4, 5], who has devised a method to obtain solutions via energy estimates. The approach is to cleverly exploit the properties of pseudodiferential operators of order 0 to obtain *a priori* control of $\|u(t, \cdot)\|_{L^2}$ in terms of $\|f\|_{L^2}$ and $\|F\|_{L^1_T L^2}$.

We also mention the work of Kenig-Ponce-Vega, [9]. These authors have been able to derive *a priori* estimates for the L^2 norms of the solution as well as local smoothing effects, which are known to imply (at least local) Strichartz estimates.

Note that both (1.2) and (1.1) have the important scaling invariance $u \to u^\lambda(t, x) = u(\lambda^2 t, \lambda x)$, $V \to V^\lambda(t, x) = \lambda V(\lambda^2 t, \lambda x)$. That is, whenever (u, V) satisfy (1.1), (1.2), so does (u^α, V^α) with initial data $f^\alpha(x) = f(\alpha x)$.

1.1. **Strichartz estimates for the conjugate magnetic Schrödinger operator in dimension $n \geq 3$.** To state our results, we need some definitions. For a function u, let $u_k = P_k u$ be its k^{th} Littlewood-Paley piece, as in Section 3. Define the Besov spaces $B^q_{p,s}$ by the norm

$$\|u\|_{B^q_{p,s}} = \Big(\sum_k 2^{ksq} \|u_k\|^q_{L^p} \Big)^{1/q},$$

and let $W^{p,s}$ be the homogeneous Sobolev space with s derivatives in L^p; that is $\|f\|_{W^{p,s}} = \||\nabla|^s f\|_{L^p}$.

Theorem 1. *Let $n \geq 3$ and $0 < h < 1$. There exists an $\varepsilon = \varepsilon(h, n) > 0$, so that whenever*

$$\|V\|_{L^\infty W^{n/(2+h), 1+h}} + \|V\|_{L^\infty B^1_{n/2,2}} + \left\||\nabla|^{-1} \partial_t V\right\|_{L^\infty L^{n/2}} \leq \varepsilon,$$

the equation (1.1) has a global solution u, provided the data $f \in L^2(\mathbf{R}^n)$ and the forcing term $F \in L^{\tilde{q}'} L^{\tilde{r}'}$. In addition, for all admissible pairs (q, r),

$(\tilde{q}, \tilde{r})$

$$\|u\|_{L^q L^r} \lesssim \Big(\sum_k \|u_k\|_{L^q L^r}^2\Big)^{1/2} \lesssim \|f\|_{L^2} + \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}}, \tag{1.5}$$

More generally, let $s > 0$ *and* p_1, p_2, q_1, q_2 *satisfy* $1/2 = 1/p_1 + 1/p_2$; $1/2 + 1/n = 1/q_1 + 1/q_2$ *and assume only*

$$\|\partial V\|_{L^\infty L^{n/2}} + \|V\|_{L^\infty B^1_{n/2,2}} + \big\||\nabla|^{-1}\partial_t V\big\|_{L^\infty L^{n/2}} \le \varepsilon_s.$$

Then the a priori estimate

$$\begin{aligned}&\Big(\sum_k 2^{2ks}\|u_k\|_{L^q L^r}^2\Big)^{1/2}\\ &\lesssim \|f\|_{\dot{H}^s} + \|F\|_{L^{\tilde{q}'}\dot{W}^{\tilde{r}',s}} + \Big(\sum_k 2^{2ks}\|V_k\|^2_{L^{p_1}_t L^{q_1}_x}\Big)^{1/2}\|\nabla u\|_{L^{p_2}_t L^{q_2}_x}.\end{aligned}$$

holds true.

Remarks. (1) Theorem 1 covers the important case of small *time-independent* potentials V.

(2) The constant ε_s has the behavior $\varepsilon_s \sim \varepsilon s$ as $s \to 0$.

(3) The conclusion in Theorem 1 holds under smallness assumptions for

$$\begin{aligned}&\|V\|_{L^\infty L^n} + \|\nabla V\|_{L^p_t L^m_x} + \big\||\nabla|^{-1}\partial_t V\big\|_{L^p L^m}\\ &+ \|V\|_{L^p_t W_x^{mn/(mh+n),1+h}} + \Big(\sum_k 2^{2k}\|V_k\|^2_{L^p L^m}\Big)^{1/2},\end{aligned}$$

for some $p, m \ge 1$ and $2/p + n/m = 2$, but we shall omit the proof.

1.2. **Two-dimensional case.** In dimension two, one needs to slightly change the assumptions, mainly due to the presence of the false endpoint Strichartz estimate for $(2, \infty)$.

Theorem 2. *For every* $1 >> \delta > 0$, *there exists* $\varepsilon = \varepsilon(\delta)$, *so that whenever* $V : \mathbf{R}^2 \to C^2$ *is a vector-valued potential satisfying*

$$\|V\|_{L^{1/\varepsilon} W^{1/(1-\delta/2),1+\delta}} + \big\||\nabla|^{-1}\partial_t V\big\|_{L^{1/\delta} L^{1/(1-\delta)}} + \|V\|_{L^{1/\delta} B^1_{1/(1-\delta),2}} \le \varepsilon,$$

the two-dimensional Schrödinger equation (1.1) *has a global solution* u, *provided the data* $f \in L^2(\mathbf{R}^n)$ *and the forcing term* $F \in L^{\tilde{q}'} L^{\tilde{r}'}$. *Moreover, for all admissible pairs* (q, r), $(\tilde{q}, \tilde{r})$,

$$\|u\|_{L^q L^r} \lesssim \Big(\sum_k \|u_k\|_{L^q L^r}^2\Big)^{1/2} \lesssim \|f\|_{L^2} + \|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}}. \tag{1.6}$$

Remark. Note that the case of small *time-independent* potentials is not covered by our statement. This actually may fail. More precisely, we conjecture that it is possible to construct (small) time-independent decaying potential V, so that $\|u\|_{L^q L^r} = \infty$, with finite L^2 data and right-hand side.

2. Applications

In this section, we give applications to small global solutions of some quasilinear Schrödinger equations. Our first example is a class of equations in the form

$$\partial_t u - i\Delta u + u \cdot \partial \bar{u} = 0, \quad u(0,x) = f(x). \tag{2.1}$$

Here, we may allow ∂ to be any vector-valued multiplier operator, so that its symbol satisfies $|m(\xi)| \sim |\xi|$.

Proposition 1. *Let $s \geq n/2 - 1$ and $n \geq 5$. Then there exists $\varepsilon > 0$, so that whenever the initial data $f \in H^s$, with $\|f\|_{\dot{H}^{n/2-1}} \leq \varepsilon$, (2.1) has a global solution, satisfying*

$$\|u\|_{L^q \dot{W}^{r,s}} \lesssim (\sum_k \|\nabla^s u_k\|^2_{L^q L^r})^{1/2} \leq C\|f\|_{\dot{H}^s}$$

and $\|u\|_{L^\infty_t \dot{H}^{n/2-1}} \leq 2\varepsilon$.

Remark. The scale-invariant space for the problem is clearly $\dot{H}^{n/2-1}$, which means that the results presented here are optimal in the sense of smoothness of the data.

Our next example is the equation

$$\partial_t u - i\Delta u + \partial \bar{u} \cdot \partial \bar{u} = 0, \quad u(0,x) = f(x), \tag{2.2}$$

where again ∂ is allowed to be any PDO of order one. We have the following

Proposition 2. *Let $n \geq 4$ and $s \geq n/2$. Then there exists an $\varepsilon > 0$, so that (2.2) has a unique global solution, whenever $f \in H^s(\mathbf{R}^n)$ with $\|f\|_{\dot{H}^{n/2}} \leq \varepsilon$. Moreover, the solution satisfies*

$$\sup_{(q,r)-Strichartz} \|u\|_{L^q \dot{W}^{r,s}} \leq C\|f\|_{\dot{H}^s}.$$

Remark. This equation was considered in two spatial dimensions by S. Cohn, [2], [3] by using the method of normal forms. In one dimension, small solutions of this equation were studied by Hayashi-Naumkin, [7], in weighted Sobolev spaces. Our methods give the optimal results in dimensions $n \geq 4$ (note that the scale-invariant space is $\dot{H}^{n/2}$). The result in dimension three holds for data (small) in Besov one spaces, while one can prove global well posedness of (2.2) in the two-dimensional case with data in H^2, which is small in $\dot{H}^1$. We do not pursue these results here.

3. Preliminaries

3.1. **Littlewood-Paley projections.** We define the Littlewood-Paley projections, which will be used frequently throughout the paper. Introduce a positive, smooth, and even function ψ, supported in $\{\xi : |\xi| \leq 2\}$ with $\psi(\xi) = 1$ for all $|\xi| \leq 1$. Define $\varphi(\xi) = \psi(\xi) - \psi(2\xi)$, which is supported in the annulus $1/2 \leq |\xi| \leq 2$. Clearly $\sum_{k\in\mathcal{Z}} \varphi(2^{-k}\xi) = 1$ for all $\xi \neq 0$.

The k^{th} Littlewood-Paley projection is defined as a multiplier-type operator by $\widehat{P_k f}(\xi) = \varphi(2^{-k}\xi)\hat{f}(\xi)$. Note that the kernel of P_k is integrable, smooth and real valued. In particular, it commutes with differential operators and the complex conjugation.

Also of interest will be the properties of products under the action of P_k. We have that for any two (Schwartz) functions f, g

$$\begin{aligned} P_k(fg) &= \sum_{l\geq k-2} P_k(f_l g_{l-2\leq\cdot\leq l+2}) + \text{symmetric term} + \\ &+ P_k(f_{\leq k-4} g_{k-1\leq\cdot\leq k+1}) + \text{symmetric term} = \\ &= f_{\leq k-4} g_k + [P_k, f_{\leq k-4}] g_{k-1\leq\cdot k+1} + \text{symmetric terms} \\ &+ \sum_{l\geq k-2} P_k(f_l g_{l-2\leq\cdot\leq l+2}) + \text{symmetric term.} \end{aligned}$$

We need the following technical lemma.

Lemma 1. *Let $\{a_l\}$, $\{b_l\}$ are two sequences and $\varepsilon > 0$. Then*

$$\Big(\sum_k 2^{2\varepsilon k}\Big(\sum_{l\geq k-2} 2^{-\varepsilon l} a_l b_l\Big)^2\Big)^{1/2} \leq C_\varepsilon \|a\|_{l^\infty} \|b\|_{l^2}.$$

Proof. Fix the sequence $\{a_l\}$ and consider the linear operator (mapping a sequence into a sequence)

$$(Tb)_k := 2^{\varepsilon(k-l)} \sum_{l\geq k-2} a_l b_l.$$

We will show that $T : l^1 \to l^1$ and $T : l^\infty \to l^\infty$. Indeed,

$$\|Tb\|_{l^1} \leq \sum_l |a_l||b_l| \sum_{k\leq l+2} 2^{\varepsilon(k-l)} \lesssim \varepsilon^{-1} \|a\|_{l^\infty} \|b\|_{l^1},$$

$$\|Tb\|_{l^\infty} \leq \sup_k \sup_l |a_l| \sup_l |b_l| \sum_{l\geq k-2} 2^{\varepsilon(k-l)} \lesssim \varepsilon^{-1}\|a\|_{l^\infty}\|b\|_{l^\infty}.$$

It follows that for $1 \leq p \leq \infty, T : l^p \to l^p$ has norm no bigger than $C_\varepsilon \|a\|_{l^\infty}$, hence the statement of the lemma. □

4. Proof of Theorem 1 and Theorem 2

Our approach is based on the observation that (1.1) is equivalent (at least for classical solutions) to a wave equation, which *gains a derivative*.

Our considerations are for the case $n \geq 3$, with some mild changes needed for the two-dimensional case.

We first perform some technical reductions, which will facilitate our estimates in the sequel. Take $0 < \delta << 1$ and define $V^\delta(t,x) := V(t,x)(1-\psi(t/\delta)), F^\delta(t,x) := F(t,x)(1-\psi(t/\delta))$. That is, V^δ is a function coinciding with V for all $t \geq 2\delta$, $V^\delta(t) = 0 = F^\delta(t)$ for all $t \leq \delta$.

For the rest of this section, we consider (1.1) with V and F replaced by V^δ and F^δ respectively and V and F Schwartz functions. In the end, (1.5) will follow from the corresponding estimate for u^δ by a limiting argument as $\delta \to 0$, since all of our estimates will be uniform in δ and independent of the smoothness constants of V and F.

Next, we make some standard frequency localizations of the problem. Take a Littlewood-Paley projection on (1.1) to obtain

$$\partial_t u_k - i\Delta u_k + V^\delta_{\leq k-4} \cdot \nabla \bar{u}_k = F^\delta_k + E_k, \tag{4.1}$$

where E_k is the error term arising in the process. According to our considerations above

$$\begin{aligned} E_k &= [P_k, V^\delta_{\leq k-4}]\nabla \bar{u}_{k-1\leq \cdot k+1} + \sum_{l\geq k-2} P_k(V^\delta_l \cdot \nabla \bar{u}_{l-2\leq\cdot\leq l+2}) \qquad (4.2) \\ &+ \sum_{l\geq k-2} P_k(V^\delta_{l-2\leq\cdot\leq l+2} \cdot \nabla \bar{u}_l) + P_k(V^\delta_{k-1\leq\cdot\leq k+1} \cdot \nabla \bar{u}_{\leq k-4}). \end{aligned}$$

Note that in terms of L^p behavior and Littlewood-Paley theory one informally treats these error terms as in the form $(\partial_x V^\delta)u$.

Since, for all $q, r \geq 2$, $\|u\|_{L^qL^r} \lesssim (\sum_k \|u_k\|^2_{L^qL^r})^{1/2}$ and $\|F\|_{L^{q'}L^{r'}} \gtrsim (\sum_k \|F_k\|^2_{L^{q'}L^{r'}})^{1/2}$, it will suffice to show that

$$(\sum_k \sup_{q,r-Strichartz} \|u_k\|^2_{L^qL^r})^{1/2} \lesssim \|f\|_{L^2} + (\sum_k \left\|F^\delta_k\right\|^2_{L^{\tilde{q}'}L^{\tilde{r}'}})^{1/2}. \tag{4.3}$$

When estimating the terms arising from the left-hand side of (4.3), we will often get terms in the form $C\varepsilon \sup_{q,r}(\sum_k \|u_k\|^2_{L^qL^r})^{1/2}$. Those are terms that can be absorbed on the left-hand side.

Apply the operator $\partial_t + i\Delta$ to (4.1). We get

$$(\partial_t^2 + \Delta^2)u_k + [(\partial_t + i\Delta)V^\delta_{\le k-4}] \cdot \nabla \bar{u}_k + V^\delta_{\le k-4} \cdot [(\partial_t + i\Delta)\nabla \bar{u}_k]$$
$$+ i \sum_j \partial_j V^\delta_{\le k-4} \cdot \partial_j \nabla \bar{u}_k = (\partial_t + i\Delta)(F^\delta_k + E_k).$$

Note that by the equation (4.1), the term

$$[(\partial_t + i\Delta)\nabla \bar{u}_k] = \nabla \overline{(\partial_t - i\Delta)u_k} = \nabla(\overline{-V^\delta_{\le k-4} \cdot \nabla \bar{u}_k - F^\delta_k - E_k}).$$

We also have initial data $u_k(0,x) = f_k(x)$ and

$$\begin{aligned} \partial_t u_k(0,x) &= i\Delta u_k(0,x) - V^\delta_{\le k-4}(0,x)\nabla \bar{u}_k(0,x) + F^\delta_k(0,x) + E_k(0,x) \\ &= i\Delta f_k, \end{aligned}$$

taking into account that $V^\delta(0,x) = F^\delta(0,x) = E(0,x) = 0$.

One might write the corresponding Duhamel's formula for the newly obtained equation satisfied by u_k as follows

$$\begin{aligned} & u_k(x,t) \\ &= \cos(t\Delta)f_k + \frac{\sin(t\Delta)}{\Delta} i\Delta f_k + \int_0^t \frac{\sin((t-s)\Delta)}{\Delta}[-(\partial_s + i\Delta)V^\delta_{\le k-4} \cdot \nabla \bar{u}_k] \\ &- \int_0^t \frac{\sin((t-s)\Delta)}{\Delta}[V^\delta_{\le k-4} \cdot \nabla(\overline{-V^\delta_{\le k-4} \cdot \nabla \bar{u}_k - F_k - E_k}) - \nabla V^\delta_{\le k-4} \cdot \nabla^2 \bar{u}_k] \\ &+ \int_0^t \frac{\sin((t-s)\Delta)}{\Delta}[(\partial_s + i\Delta)(F_k + E_k)]ds. \end{aligned}$$

We refer to this integral equation as the smoothing version of the conjugate magnetic Schrödinger operator. Next, we show that (1.5) holds.

4.1. **Strichartz estimates with L^2 data.** Clearly, there are various terms coming from the smoothing version of the conjugate magnetic Schrödinger operator, which we need to handle. When controlling the forcing terms in this proof, we employ the dual endpoint $q = 2, r = 2n/(n+2)$, which is of course unavailable for the two dimensional case. For that case, one needs to slightly readjust the proof and use instead $q' = 2 + \delta, r' = n(4 + 2\delta)/(2n + n\delta + 4\delta + 4)$.

4.1.1. *Terms coming from initial data.* By Euler's formula one expresses $\cos(t\Delta)$ and $\sin(t\Delta)$ in terms of $e^{it\Delta}$ (i.e., the generator for the Schrödinger equation). By the standard Strichartz estimates (1.3), one has the same Strichartz estimates for $\cos(t\Delta), \sin(t\Delta)$, whence

$$\|\cos(t\Delta) f_k\|_{L^q L^r} \lesssim \|f_k\|_{L^2}, \qquad \|i \sin(t\Delta) f_k\|_{L^q L^r} \lesssim \|f_k\|_{L^2}.$$

Summing in k yields

$$(\sum_k \|\cos(t\Delta) f_k\|^2_{L^q L^r} + \sum_k \|\sin(t\Delta) f_k\|^2_{L^q L^r})^{1/2} \lesssim \|f\|_{L^2},$$

which is part of the right-hand side of (1.5).

4.1.2. *Forcing terms.*

- By (1.4) and Hölder's inequality and Sobolev embedding

$$\begin{aligned}
&\left\|\int_0^t \frac{\sin((t-s)\Delta)}{\Delta}[-(\partial_s + i\Delta) V^\delta_{\leq k-4} \cdot \nabla \bar{u}_k] ds\right\|_{L^q L^r} \\
&\lesssim 2^{-2k} \|[(1-\psi(\cdot/\delta))\partial_t V_{\leq k-4} \nabla \bar{u}_k\|_{L^2 L^{2n/(n+2)}} + \\
&+ 2^{-2k} \left\|\Delta V^\delta_{\leq k-4} \nabla \bar{u}_k\right\|_{L^2 L^{2n/(n+2)}} + 2^{-2k}\delta^{-1} \left\|\psi'(\cdot/\delta) V_{\leq k-4} \nabla \bar{u}_k\right\|_{L^1 L^2} \\
&\lesssim 2^{-k} (\|\partial_t V_{\leq k-4}\|_{L^\infty L^{n/2}} + \left\|\Delta V^\delta_{\leq k-4}\right\|_{L^\infty L^{n/2}}) \|u_k\|_{L^2 L^{2n/(n-2)}} \\
&+ 2^{-k}\delta^{-1} \left\|\psi'(\cdot/\delta)\right\|_{L^1_t} \|V_{\leq k-4}\|_{L^\infty_{tx}} \|u_k\|_{L^\infty L^2} \\
&\lesssim (\left\||\nabla|^{-1}\partial_t V\right\|_{L^\infty L^{n/2}} + \|\nabla V\|_{L^\infty L^{n/2}}) \|u_k\|_{L^2 L^{2n/(n-2)}} + \|V\|_{L^\infty L^n} \|u_k\|_{L^\infty L^2}.
\end{aligned}$$

Taking squares and summing in k yields an estimate

$$\begin{aligned}
&\sum_k \left\|\int_0^t \frac{\sin((t-s)\Delta)}{\Delta}[-(\partial_s + i\Delta) V^\delta_{\leq k-4} \cdot \nabla \bar{u}_k] ds\right\|^2_{L^q L^r} \\
&\lesssim \varepsilon \sum_k \sup_{(q,r)-Str.} \|u_k\|^2_{L^q L^r} \lesssim \varepsilon^2 \sum_k \sup_{(q,r)-Str.} \|u_k\|^2_{L^q L^r}.
\end{aligned}$$

- By (1.4), $\|\nabla g_k\|_{L^r} \sim 2^k \|g_k\|_{L^r}$ and Hölder's inequality

$$\begin{aligned}
&\left\|\int_0^t \frac{\sin((t-s)\Delta)}{\Delta} V^\delta_{\leq k-4} \cdot V^\delta_{\leq k-4} \nabla^2 \bar{u}_k ds\right\|_{L^q L^r} \\
&\lesssim \left\|\Delta^{-1}[V^\delta_{\leq k-4} \cdot V^\delta_{\leq k-4} \nabla^2 \bar{u}_k]\right\|_{L^2 L^{2n/(n+2)}} \\
&\lesssim 2^{-2k} \|V\|^2_{L^\infty L^n} \left\|\nabla^2 u_k\right\|_{L^2 L^{2n/(n-2)}} \lesssim \|V\|^2_{L^\infty L^n} \|u_k\|_{L^2 L^{2n/(n-2)}}.
\end{aligned}$$

Squaring and summation in k shows the term is bounded by $C\varepsilon(\sum_k \|u_k\|^2_{L^2 L^{2n/(n-2)}})$.

• Another similar term is treated as above

$$\left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} V^\delta_{\le k-4} \cdot \nabla V^\delta_{\le k-4} \nabla \bar{u}_k ds \right\|_{L^q L^r}$$
$$\lesssim \left\| \Delta^{-1} [V^\delta_{\le k-4} \cdot \nabla V^\delta_{\le k-4} \nabla \bar{u}_k] \right\|_{L^2 L^{2n/(n+2)}} \lesssim \|V\|^2_{L^\infty L^n} \|u_k\|_{L^2 L^{2n/(n-2)}}.$$

Squaring and summation in k gives a manageable term.

•

$$\left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} \nabla V^\delta_{\le k-4} \nabla^2 \bar{u}_k ds \right\|_{L^q L^r} \lesssim \left\| \Delta^{-1} [\nabla V^\delta_{\le k-4} \nabla^2 \bar{u}_k] \right\|_{L^2 L^{2n/(n+2)}}$$
$$\lesssim \|\nabla V\|_{L^\infty L^{n/2}} \|u_k\|_{L^2 L^{2n/(n-2)}}.$$

Squaring and summation in k shows a bound by $C\varepsilon \sum_k \|u_k\|^2_{L^2 L^{2n/(n-2)}}$

• For the term involving the right-hand side we have, by (1.4), Hölder's and Sobolev embedding

$$\left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} V^\delta_{\le k-4} \nabla F^\delta_k ds \right\|_{L^q L^r} \lesssim \left\| \Delta^{-1} [V^\delta_{\le k-4} \nabla F^\delta_k] \right\|_{L^{\tilde{q}'} L^{\tilde{r}'}}$$
$$\lesssim 2^{-2k} \|V_{\le k-4}\|_{L^\infty_{tx}} \|\nabla F_k\|_{L^{\tilde{q}'} L^{\tilde{r}'}} \lesssim \|V\|_{L^\infty L^n} \|F_k\|_{L^{\tilde{q}'} L^{\tilde{r}'}}.$$

• Another very easy term is

$$\left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} \Delta F^\delta_k ds \right\|_{L^q L^r} \lesssim \|F_k\|_{L^{\tilde{q}'} L^{\tilde{r}'}}.$$

• The last term involving F_k is

$$\int_0^t \frac{\sin((t-s)\Delta)}{\Delta} \partial_s F^\delta_k(s,\cdot) ds = \int_0^t \cos((t-s)\Delta) F^\delta_k(s,\cdot) ds,$$

where, in the last identity, we have used integration by parts and the fact that $F^\delta(0,x) = 0$. An application of the Strichartz estimates to the last formula yields

$$\sum_k \left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} \partial_s F^\delta_k ds \right\|^2_{L^q L^r} \lesssim \sum_k \|F_k\|^2_{L^{\tilde{q}'} L^{\tilde{r}'}}.$$

4.1.3. *Error terms.* We have two types of error terms. For the first one

$$\left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} V^\delta_{\le k-4} \nabla E_k ds \right\|_{L^q L^r} \lesssim \left\| \Delta^{-1} [V^\delta_{\le k-4} \nabla E_k] \right\|_{L^2 L^{2n/(n+2)}}$$
$$\lesssim 2^{-k} \|V_{\le k-4}\|_{L^\infty} \|E_k\|_{L^2 L^{2n/(n+2)}} \lesssim \|V\|_{L^\infty L^n} \|E_k\|_{L^2 L^{2n/(n+2)}}.$$

The second error term is identical to the last two cases for F considered above (recall $E_k(0,x) = 0$, because of the time cutoffs introduced in the definition). We have

$$\begin{aligned}
&\left\| \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} (\partial_s + i\Delta) E_k ds \right\|_{L^q L^r} \\
&\leq \left\| \int_0^t \sin((t-s)\Delta) E_k ds \right\|_{L^q L^r} + \left\| \int_0^t \cos((t-s)\Delta) E_k ds \right\|_{L^q L^r} \\
&\lesssim \|E_k\|_{L^2 L^{2n/(n+2)}}.
\end{aligned}$$

In both error terms, it clearly suffices to show

$$\sum_k \|E_k\|^2_{L^2 L^{2n/(n+2)}} \lesssim \varepsilon^2 \sum_k \|u_k\|^2_{L^2 L^{2n/(n-2)}}. \tag{4.4}$$

We estimate on a term-by-term basis in formula (4.2).

For the first term, recall the Calderòn commutator result, which states that for all $1 \leq p, q, r \leq \infty$ such that $1/p = 1/q + 1/r$, one has

$$\|[P_k, f]\nabla g\|_{L^p} \lesssim \|\nabla f\|_{L^q} \|g\|_{L^r}.$$

Therefore,

$$\sum_k \left\| [P_k, V^\delta_{\leq k-4}] \nabla \bar{u}_k \right\|^2_{L^2 L^{2n/(n+2)}} \lesssim \|\nabla V\|^2_{L^\infty L^{n/2}} \sum_k \|u_k\|^2_{L^2 L^{2n/(n-2)}}.$$

The second and third terms in (4.2) are treated in a similar fashion, so we concentrate on the second one. For any positive $h \leq 1$, we have

$$\begin{aligned}
&\sum_k \Big(\sum_{l \geq k-2} \left\| P_k (V^\delta_l \nabla \bar{u}_{l-2 \leq \cdot \leq l+2}) \right\|_{L^2 L^{2n/(n+2)}} \Big)^2 \\
&\lesssim \sum_k \Big(\sum_{l \geq k-2} 2^{hk} \left\| P_k (V^\delta_l \nabla \bar{u}_{l-2 \leq \cdot \leq l+2}) \right\|_{L^2 L^{2n/(n+2+2h)}} \Big)^2 \\
&\lesssim \sum_k 2^{2hk} \Big(\sum_{l \geq k-2} 2^l \|V_l\|_{L^\infty L^{n/(2+h)}} \|u_l\|_{L^2 L^{2n/(n-2)}} \Big)^2 \\
&\lesssim \sum_k 2^{2hk} \Big(\sum_{l \geq k-2} 2^{-hl} \|V_l\|_{L^\infty W^{n/(2+h), 1+h}} \|u_l\|_{L^2 L^{2n/(n-2)}} \Big)^2.
\end{aligned}$$

One obtains by Lemma 1

$$\sum_k 2^{2hk} \Big(\sum_{l \geq k-2} 2^{-hl} \|V_l\|_{W^{n/(2+h), 1+h}} \|u_l\|_{L^2 L^{2n/(n-2)}} \Big)^2$$

$$\lesssim \|V\|^2_{W^{n/(2+h),1+h}} \sum_l \|u_l\|^2_{L^2 L^{2n/(n-2)}} \lesssim \varepsilon^2 \sum_l \|u_l\|^2_{L^2 L^{2n/(n-2)}}.$$

The last fourth term in (4.2) is handled as follows

$$\begin{aligned}
&\sum_k \Big\| P_k(V^\delta_{k-1\le\cdot\le k+1} \nabla \bar{u}_{\le k-4}) \Big\|^2_{L^2 L^{2n/(n+2)}} \\
&= \Big\| \Big(\sum_k \Big\| P_k(V^\delta_{k-1\le\cdot\le k+1} \nabla \bar{u}_{\le k-4}) \Big\|^2_{L^{2n/(n+2)}} \Big)^{1/2} \Big\|^2_{L^2_t} \\
&\lesssim \Big\| \Big(\sum_k 2^{2k} \|V_{k-1\le\cdot\le k+1}\|^2_{L^{n/2}} \|u\|^2_{L^{2n/(n-2)}} \Big)^{1/2} \Big\|^2_{L^2} \\
&\lesssim \|V\|^2_{L^\infty B^1_{n/2,2}} \|u\|^2_{L^2 L^{2n/(n-2)}} \lesssim \varepsilon^2 \|u\|^2_{L^2 L^{2n/(n-2)}}.
\end{aligned}$$

4.2. **Strichartz estimates for $\dot{H}^s$ data.** This section shall largely refer to the previous one. Indeed, for all forcing terms but the error terms, the dominant frequency is on the u, rather than on the potential V. Denote these terms by U. Therefore, we have estimates of the form

$$\begin{aligned}
&\Big(\sum_k 2^{2ks} \|U_k\|^2_{L^2 L^{2n/(n+2)}} \Big)^{1/2} \\
&\lesssim (\|\partial V\|_{L^\infty L^{n/2}} + \|V\|_{L^\infty B^1_{n/2,2}} + \big\| |\nabla|^{-1} \partial_t V \big\|_{L^\infty L^{n/2}}) \\
&\times \Big(\sum_k 2^{2ks} \|u_k\|^2_{L^2 L^{2n/(n-2)}} \Big)^{1/2} + \|F_k\|_{L^{\tilde{q}'} L^{\tilde{r}'}}.
\end{aligned}$$

For the error terms, we proceed as follows. According to (4.4), the relevant estimate to prove is

$$\Big(\sum_k 2^{2ks} \|E_k\|^2_{L^2 L^{2n/(n+2)}} \Big) \lesssim \varepsilon^2 \Big(\sum_k 2^{2ks} \|u_k\|^2_{L^2 L^{2n/(n-2)}} \Big)^{1/2}. \tag{4.5}$$

We have

$$\sum_k 2^{2ks} \Big\| [P_k, V^\delta_{\le k-4} \Big\|^2_{L^2 L^{2n/(n+2)}} \lesssim \|\partial V\|^2_{L^\infty L^{n/2}} \sum_k 2^{2ks} \|u_k\|^2_{L^2 L^{2n/(n-2)}},$$

which takes care of the first term.

For the next term, we have by the inclusion $l^1 \hookrightarrow l^2$

$$\sum_k 2^{2ks} \Big(\sum_{l\ge k-2} \Big\| P_k(V^\delta_l \nabla u_{l-2\le\cdot\le l+2}) \Big\|^2_{L^2 L^{2n/(n+2)}} \Big)^{1/2}$$

$$\lesssim \sum_k 2^{ks} \sum_{l\geq k-2} \left\| V_l^\delta \nabla u_{l-2\leq\cdot\leq l+2} \right\|_{L^2 L^{2n/(n+2)}}$$

$$\lesssim \sum_l 2^{ls} \|u_{l-2\leq\cdot\leq l+2}\|_{L^2 L^{2n/(n-2)}} \|V_l\|_{L^\infty L^{n/2}}$$

$$\leq C_s \Big(\sum_l 2^{2ls} \|u_l\|^2_{L^2 L^{2n/(n-2)}} \Big)^{1/2} \|V\|_{L^\infty B^1_{n/2,2}}.$$

The last error term is estimated by

$$\sum_k 2^{2ks} \left\| P_k (V^\delta_{k-1\leq\cdot\leq k+1} \nabla \bar{u}_{\leq k-4}) \right\|^2_{L^2 L^{2n/(n+2)}}$$

$$\lesssim \left\| \Big(\sum_k 2^{2ks} \|V_{k-1\leq\cdot\leq k+1}\|^2_{L^{q_1}} \Big)^{1/2} \|\partial u\|_{L^{q_2}_x} \right\|^2_{L^2_t}$$

$$\lesssim \Big(\sum_k 2^{2ks} \|V_k\|^2_{L^{p_1}_t L^{q_1}_x} \Big) \|\partial u\|^2_{L^{p_2}_t L^{q_2}_x}.$$

All in all, we get

$$\Big(\sum_k 2^{2ks} \|u_k\|^2_{L^q L^r} \Big)^{1/2} \leq C\|f\|_{\dot{H}^s} + C\|F\|_{L^{\tilde{q}'} L^{\tilde{r}'}}$$

$$+ C_s \varepsilon \Big(\sum_k 2^{2ks} \|u_k\|^2_{L^q L^r} \Big)^{1/2} + \Big(\sum_k 2^{2ks} \|V_k\|^2_{L^{p_1}_t L^{q_1}_x} \Big)^{1/2} \|\partial u\|_{L^{p_2}_t L^{q_2}_x},$$

hence the Strichartz estimates for data $f \in \dot{H}^s$.

5. Applications to quasilinear Schrödinger equations

Proof of Proposition 1. Start with data $f \in \mathcal{S}$. Take $u^0 = 0$ and solve the equation

$$u^{j+1}_t - i\Delta u^{j+1} + u^j \cdot \partial \overline{u^{j+1}} = 0, \qquad u(0,x) = f(x) \tag{5.1}$$

for $j \geq 0$. This can be done as long as

$$\left\| u^j \right\|_{L^\infty W^{n/(2+h),1+h}} + \left\| u^j \right\|_{L^\infty B^1_{n/2,2}} + \left\| |\nabla|^{-1} \partial_t u^j \right\|_{L^\infty L^{n/2}} \lesssim \varepsilon.$$

We will show by induction that u^j is a smooth function, a solution of (5.1), and

$$\Big(\sum_k 2^{2k(n/2-1)} \left\| u^j_k \right\|^2_{L^q L^r} \Big)^{1/2} \leq 2C_n \varepsilon, \tag{5.2}$$

where C_n is the constant in Theorem 1. Take $j \geq 1$. By Sobolev embedding

$$\|u^j\|_{L^\infty W^{n/(2+h),1+h}} + \|u^j\|_{L^\infty B^1_{n/2,2}} \leq C_n \Big(\sum_k 2^{2k(n/2-1)} \Big\|u^j_k\Big\|^2_{L^q L^r}\Big)^{1/2} \leq C_n \varepsilon.$$

Next, use the fact that u^j is a classical solution of $\partial_t u^j - i\Delta u^j + u^{j-1}\partial \overline{u^j} = 0$. We get

$$|\nabla|^{-1} u^j_t = |\nabla|^{-1}(i\Delta u^j - u^{j-1}\partial \overline{u^j}).$$

We get by Sobolev embedding

$$\begin{aligned} \Big\| |\nabla|^{-1} u^j_t \Big\|_{L^\infty L^{n/2}} &\leq C\|\partial u^j\|_{L^\infty L^{n/2}} + C_n \|u^{j-1}\|_{L^\infty L^n} \|\partial u^j\|_{L^{n/2}} \\ &\leq C_n \|u^j\|_{L^\infty W^{n/(2+h),1+h}} (1 + C_n \|u^{j-1}\|_{L^\infty W^{n/(2+h),1+h}}) \leq D_n \varepsilon. \end{aligned}$$

It follows that

$$\|u^j\|_{L^\infty W^{n/(2+h),1+h}} + \|u^j\|_{L^\infty B^1_{n/2,2}} + \||\nabla|^{-1}\partial_t u^j\|_{L^\infty L^{n/2}} \lesssim \varepsilon.$$

Invoking Theorem 1 with $s = n/2-1$, $p_1 = 2$, $p_2 = \infty$, $q_1 = 2n/(n-2)$, $q_2 = n/2$, yields existence of (smooth) u^{j+1} as well as the estimates

$$\begin{aligned} &\Big(\sum_k 2^{2k(n/2-1)} \Big\|u^{j+1}_k\Big\|^2_{L^q L^r}\Big)^{1/2} \\ &\leq C_n \Big(\|f\|_{\dot H^{n/2-1}} + \Big(\sum_k 2^{2k(n/2-1)} \Big\|u^j_k\Big\|^2_{L^2_t L_x^{2n/(n-2)}}\Big)^{1/2} \|\nabla u^{j+1}\|_{L^\infty_t L^{n/2}_x}\Big) \\ &\leq C_n \|f\|_{\dot H^{n/2-1}} + D_n \varepsilon \|u^{j+1}\|_{L^\infty \dot H^{n/2-1}}. \end{aligned}$$

Having ε such that $D_n \varepsilon \leq 1/2$ allows one to hide the second term on the right-hand side above, whence we deduce (5.2).

Forming the difference of the j^{th} and the $(j+1)^{st}$ equations and using the smoothing version of the conjugate magnetic operator (see Section 4), we obtain the estimates

$$\Big(\sum_k 2^{2ks} \Big\|u^{j+1}_k - u^j_k\Big\|^2_{L^q L^r}\Big)^{1/2} \lesssim \varepsilon \Big(\sum_k 2^{2ks} \Big\|u^j_k - u^{j-1}_k\Big\|^2_{L^q L^r}\Big)^{1/2},$$

for any $s \geq n/2 - 1$. This shows that $u = \lim_j u^j$ exists and

$$\Big(\sum_k 2^{2k(n/2-1)} \|u_k\|^2_{L^q L^r}\Big)^{1/2} \lesssim \varepsilon.$$

Moreover, applying the estimates of Theorem 1 once more for $s > 0$ yields

$$\begin{aligned}&\Big(\sum_k 2^{2ks}\|u_k\|^2_{L^qL^r}\Big)^{1/2}\\ &\leq C_n\|f\|_{\dot{H}^s} + C_n\Big(\sum_k 2^{2ks}\|u_k\|^2_{L^2_tL^{2n/(n-2)}_x}\Big)^{1/2}\|\nabla u\|_{L^\infty_tL^{n/2}_x}\\ &\leq C_n\|f\|_{\dot{H}^s} + D_n\varepsilon\Big(\sum_k 2^{2ks}\|u_k\|^2_{L^2_tL^{2n/(n-2)}_x}\Big)^{1/2}.\end{aligned}$$

We get

$$\|u\|_{L^q\dot{W}^{r,s}} \lesssim \Big(\sum_k 2^{2ks}\|u_k\|^2_{L^qL^r}\Big)^{1/2} \lesssim \|f\|_{\dot{H}^s}.$$

□

Next, we show the global existence and uniqueness result for (2.2).

Proof of Proposition 2. One could pursue the approach of the proof of Proposition 1. However, it seems more convenient to set up the smoothing version directly for (2.2) and then argue by a fixed-point argument.

To make matters simpler, place a time cutoff close to $t = 0$, that is we consider

$$\partial_t u - i\Delta u + (1 - \varphi(t/\delta))\partial\bar{u}\partial\bar{u} = 0. \tag{5.3}$$

Obviously, (smooth) solutions to (5.3) will coincide with (smooth) solutions to (2.2) for all $t > 2\delta$. Thus, if we show existence and uniqueness for (5.3) for all $\delta > 0$, it is clear that will imply existence and uniqueness estimates for the original equation (2.2). Moreover, our estimates will be uniform in $\delta > 0$, and one would be able to pass to a limit $\delta > 0$, to get the corresponding estimates for (2.2).

Apply $\partial_t + i\Delta$ to both sides of (2.2). For smooth solutions, we get

$$\begin{aligned}&(\partial_t^2 + \Delta^2)u + 2\varphi_{>\delta}(t)\partial(\overline{\partial_t u - i\Delta u})\partial\bar{u}\\ &\quad - \frac{2}{\delta}\varphi'(t/\delta)(\partial\bar{u})\partial\bar{u} + i\varphi_{>\delta}(t)(\partial^2\bar{u})\partial^2\bar{u} = 0,\end{aligned}$$

which implies

$$(\partial_t^2 + \Delta^2)u - 2\varphi^2_{>\delta}(t)[\partial(\partial u)^2](\partial\bar{u}) - \frac{2}{\delta}\varphi'(t/\delta)(\partial\bar{u})\partial\bar{u} + i\varphi_{>\delta}(t)(\partial^2\bar{u})\partial^2\bar{u} = 0.$$

For the initial data, we have from the time cutoff $\partial_t u(x,0) = i\Delta f$, so we get the corresponding integral equation

$$\begin{aligned} u(x,t) &= \cos(t\Delta)f + \frac{\sin(t\Delta)}{\Delta} i\Delta f + \frac{2}{\delta}\int_0^t \frac{\sin((t-s)\Delta)}{\Delta}\varphi'(s/\delta)(\partial\bar{u})(\partial\bar{u})ds \\ &\quad - 2\int_0^t \frac{\sin((t-s)\Delta)}{\Delta}[\varphi^2_{>\delta}(s)\partial[(\partial u)^2](\partial\bar{u})]ds \\ &\quad - i\int_0^t \frac{\sin((t-s)\Delta)}{\Delta}\varphi_{>\delta}(s)(\partial^2\bar{u})(\partial^2\bar{u}). \end{aligned}$$

Set a fixed-point problem in the form $u = \Lambda u$, with an underlying metric space

$$X = \left\{ u : \left\| |\nabla|^{n/2} u \right\|_{L^q L^r} \leq 2\varepsilon; \quad \| |\nabla|^s u\|_{L^q L^r} \leq 4C_n \|f\|_{\dot{H}^s} \right\},$$

where C_n is the constant in the Strichartz inequalities.

We show that $\Lambda : X \to X$. For the terms $\|\cos(t\Delta)f\|_{\dot{H}^s}$, $\|\sin(t\Delta)f\|_{\dot{H}^s}$, use the Strichartz estimates to get

$$\|\cos(t\Delta)f\|_{\dot{H}^s} + \|\sin(t\Delta)f\|_{\dot{H}^s} \leq 2C_n\|f\|_{\dot{H}^s}.$$

Next, recall the fractional differentiation estimates of Kato-Ponce,

$$\| |\nabla|^s (fg)\|_{L^p} \leq C\| |\nabla|^s f\|_{L^{p_1}} \|g\|_{L^{p_2}} + C\| |\nabla|^s g\|_{L^{p_1}} \|f\|_{L^{p_2}},$$

whenever $1/p = 1/p_1 + 1/p_2$. We get by the Strichartz estimates and the fractional differentiation estimates for all $s \geq n/2$

$$\begin{aligned} &\delta^{-1}\left\| |\nabla|^s \int_0^t \frac{\sin((t-s)\Delta)}{\Delta}\varphi'(s/\delta)(\partial\bar{u})(\partial\bar{u})ds \right\|_{L^q L^r} \\ &\leq C\left\| |\nabla|^{s-2}[(\partial u)(\partial u)] \right\|_{L^\infty L^2} \\ &\lesssim C\left\| |\nabla|^{s-1} u \right\|_{L^\infty L^{2n/(n-2)}} \|\partial u\|_{L^\infty L^n} \leq C\| |\nabla|^s u\|_{L^\infty L^2} \left\| |\nabla|^{n/2} u \right\|_{L^\infty L^2}. \end{aligned}$$

Similarly,

$$\begin{aligned} &\left\| |\nabla|^s \int_0^t \frac{\sin((t-s)\Delta)}{\Delta}[\varphi^2_{>\delta}\partial[(\partial u)^2](\partial\bar{u})]ds \right\|_{L^q L^r} \\ &\lesssim \left\| |\nabla|^{s-2}(\partial^2 u \partial u \partial\bar{u}) \right\|_{L^2 L^{2n/(n+2)}} \\ &\lesssim \| |\nabla|^s u\|_{L^\infty L^2} \|\partial u\|^2_{L^4_t L^{2n}} + \left\| |\nabla|^{s-1} u \right\|_{L^4 L^{2n/(n-3)}} \left\| \partial^2 u \right\|_{L^\infty_t L^{n/2}} \|\partial u\|_{L^4_t L^{2n}} \\ &\lesssim \| |\nabla|^s u\|_{L^\infty L^2} \left\| |\nabla|^{n/2} u \right\|^2_{L^4 L^{2n/(n-1)}} \end{aligned}$$

$$+ \||\nabla|^s u\|_{L^4 L^{2n/(n-1)}} \left\||\nabla|^{n/2} u\right\|_{L^\infty L^2} \left\||\nabla|^{n/2} u\right\|_{L^4 L^{2n/(n-1)}}.$$

For the last term in the nonlinearity, we obtain

$$\left\||\nabla|^s \int_0^t \frac{\sin((t-s)\Delta)}{\Delta} \varphi_{>\delta}(\partial^2 \bar{u})(\partial^2 \bar{u}) ds\right\|_{L^q L^r}$$
$$\lesssim \left\||\nabla|^{s-2}(\partial^2 \bar{u})(\partial^2 \bar{u})\right\|_{L^2 L^{2n/(n+2)}}$$
$$\lesssim \||\nabla|^s u\|_{L^2 L^{2n/(n-2)}} \left\|\partial^2 u\right\|_{L_t^\infty L^{n/2}} \lesssim \||\nabla|^s u\|_{L^2 L^{2n/(n-2)}} \left\||\nabla|^{n/2} u\right\|_{L^\infty L^2}.$$

Taking into account that $(\infty, 2)$, $(4, 2n/(n-1))$, and $(2, 2n/(n-2)$ are all Strichartz pairs, we conclude that for small enough ε, $\Lambda u \in X$, whenever $u \in X$ and moreover

$$\sup_{(q,r)-Strichartz} \|\Lambda u\|_X \leq 4C_n \|f\|_{\dot{H}^s}.$$

Similarly, one proves (by using the smallness of ε) that $\Lambda : X \to X$ is a contraction This implies that for a small $\|f\|_{\dot{H}^{n/2}}$ and $\|f\|_{\dot{H}^s} < \infty$, one has a unique global solution for (2.2). $\square$

REFERENCES

[1] I. Bejenaru, Ph.D. thesis, University of California, Berkeley, 2004.

[2] Steve Cohn, *Resonance and long time existence for the quadratic semilinear Schrödinger equation*, Comm. Pure Appl. Math., 45 (1992), 973–1001.

[3] Steve Cohn, *Global existence for the nonresonant Schrödinger equation in two space dimensions,* Canad. Appl. Math. Quart., 2 (1994), 257–282.

[4] Shin-ichi Doi, *On the Cauchy problem for Schrödinger type equations and the regularity of solutions*, J. Math. Kyoto Univ., 34 (1994), 319–328.

[5] Shin-ichi Doi, *Remarks on the Cauchy problem for Schrödinger-type equations*, Comm. Partial Differential Equations, 21 (1996), 163–178.

[6] J. Ginibre and G. Velo, *Smoothing properties and retarded estimates for some dispersive evolution equations*, Comm. Math. Phys., 123 (1989), 535–573.

[7] Nakao Hayashi and Pavel I. Naumkin, *A quadratic nonlinear Schrödinger equation in one space dimension,* J. Differential Equations, 186 (2002), 165–185.

[8] M. Keel and T. Tao, *Endpoint Strichartz estimates*, Amer. J. Math., 120 (1998), 955–980.

[9] Carlos E. Kenig, Gustavo Ponce, and Luis Vega, *Smoothing effects and local existence theory for the generalized nonlinear Schrödinger equations*, Invent. Math., 134 (1998), 489–545.

[10] Atanas Stefanov, *Strichartz estimates for the magnetic Schrödinger equation*, preprint.