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EXISTENCE RESULTS FOR A CLASS OF P-LAPLACIAN PROBLEMS WITH SIGN-CHANGING WEIGHT

Maya Chhetri
Department of Mathematical Sciences
University of North Carolina at Greensboro, NC 27402

Shobha Oruganti
Mathematics, School of Science, Penn State Erie, The Behrend, Erie, PA 16563

R. Shivaji
Department of Mathematics and Statistics, Mississippi State University
Mississippi State, MS 39762

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Abstract. Consider the boundary-value problem

$$\begin{aligned} -\Delta_p u &= \lambda g(x) f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned}$$

where $\lambda > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with sufficiently smooth boundary $\partial\Omega$ and $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator for $p > 1$. Here g is a C^1 sign-changing function that may be negative near the boundary and f is a C^1 nondecreasing function satisfying $f(0) > 0$. We discuss existence results for positive solutions when f satisfies certain additional conditions. We employ the method of sub-super solutions to obtain our results.

1. Introduction

Consider the boundary-value problem

$$\begin{aligned} -\Delta_p u &= \lambda g(x) f(u) \quad \text{in } \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{1.1}$$

where $\lambda > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^N$, $N \geq 1$, with sufficiently smooth boundary $\partial\Omega$ and $\Delta_p u := div(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian operator for $p > 1$. Here g is a C^1 sign-changing function and f is a C^1 function such that $f(0) > 0$. When $p = 2$, some results have been established (see [1], [5], [10] and references therein for $f(0) > 0$; [3], [4] , [8], [13]

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and references therein for $f(0) = 0$; and [2] for $f(0) < 0$). This paper is motivated, in part, by the mathematical difficulty posed by the p-Laplacian operator ($p > 1$) compared to the Laplacian operator ($p = 2$). However, we are restricted to the class of sign-changing functions g that are positive in the interior and may be negative near the boundary. In order to define the class of functions g, we consider the eigenvalue problem

$$\begin{aligned} -\Delta_p v = \lambda |v|^{p-2} v \quad \text{in } \Omega \\ v = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{1.2}$$

Let $\phi_1 \in C^1(\overline{\Omega})$ be the eigenfunction corresponding to the first eigenvalue λ_1 of (1.2) such that $\phi_1 > 0$ in Ω and $\frac{\partial \phi_1}{\partial n} < 0$ on $\partial\Omega$, where n is the unit outer normal derivative (see [7]). Therefore, depending on Ω, there exist positive constants m, δ, σ such that

$$|\nabla \phi_1|^p - \lambda_1 \phi_1^p \geq m \quad \text{on} \quad \Omega_\delta \tag{1.3}$$

and

$$\phi_1 \geq \sigma \quad \text{on} \quad \Omega \setminus \Omega_\delta, \tag{1.4}$$

where $\Omega_\delta := \{x \in \Omega : d(x, \partial\Omega) \leq \delta\}$. Without loss of generality, we will assume that $\|\phi_1\|_\infty = 1$.

Throughout this paper, we make the following assumptions on g: There exist positive constants β and r such that

$$g(x) \geq -\beta \quad \text{in} \quad \Omega_\delta \tag{1.5}$$

and

$$g(x) \geq r \quad \text{in} \quad \Omega \setminus \Omega_\delta. \tag{1.6}$$

Now we are ready to state our results

Theorem 1. *Let $f : [0, \infty) \longrightarrow \mathbb{R}$ be a C^1 nondecreasing function such that $f(0) > 0$. Moreover, suppose there exists $\gamma > 0$ such that*

(F1) $\frac{f(\gamma\alpha)}{f(\gamma)} \geq \frac{\lambda_1}{m} \frac{\beta}{r}$, *where* $\alpha = \sigma^{\frac{p}{p-1}}$ *and*

(F2) $\frac{\gamma^{p-1}}{f(\gamma\alpha)} < \left(\frac{p-1}{p}\right)^{p-1} \frac{r}{\lambda_1} \mu(\gamma)$, *where*

$$\mu(\gamma) = \sup_{s \in [k_2(\gamma), \infty)} \frac{s^{p-1}}{f(s)} \frac{1}{\|e\|_\infty^{p-1} g_0}, \quad k_2(\gamma) = \frac{p}{p-1} \gamma \|e\|_\infty \lambda_1^{p-1}, \quad g_0 = \sup_{x \in \overline{\Omega}} g(x)$$

and $e \in C^1(\overline{\Omega})$ is the positive solution of $-\Delta_p e = 1$ in Ω; $e = 0$ on $\partial\Omega$.

Then there exists a positive solution of (1.1) for every $\lambda \in [\underline{\lambda}(\gamma), \overline{\lambda}(\gamma))$, where

$$\underline{\lambda}(\gamma) = \frac{\lambda_1 (\frac{p}{p-1}\gamma)^{p-1}}{f(\gamma\alpha) r}, \quad \overline{\lambda}(\gamma) = \min\{\lambda^*(\gamma), \mu(\gamma)\} \quad \textit{and} \quad \lambda^*(\gamma) = \frac{m (\frac{p}{p-1}\gamma)^{p-1}}{f(\gamma)\beta}.$$

Remark 1. Note that (F1) implies $\underline{\lambda}(\gamma) \le \lambda^*(\gamma)$ and (F2) implies $\underline{\lambda}(\gamma) < \mu(\gamma)$.

Corollary 1. *Suppose there exists $\gamma_0 > 0$ such that for $\gamma \ge \gamma_0$ (F1) holds and f is p-sublinear at infinity; that is, $\lim_{s\to\infty} \frac{f(s)}{s^{p-1}} = 0$. Then (1.1) has a positive solution for $\lambda \ge \underline{\lambda}(\gamma_0)$.*

Corollary 2. *Let $M > 0$ be such that $\frac{f(0)}{M} \ge \frac{\lambda_1}{m}\frac{\beta}{r}$ and $f(s) \le M$ for $s \ge 0$. Then (1.1) has a positive solution for every $\lambda > 0$.*

Remark 2. In [12], existence of positive solutions for (1.1) is discussed for small $c > 0$ when the right-hand side is of the form

$$F(x,u,c) = g(x)u^{p-1} - u^{q-1} - ch(x);\ q > p,\ h(x) \ge 0$$

and $g(x)$ belongs to the similar class of functions as considered in this paper.

We prove our results by using the method of sub-super solutions. Denoting $h(x,u) = g(x)f(u)$, a function ψ is said to be a subsolution of (1.1) if it is in $W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ such that $\psi \le 0$ on $\partial\Omega$ and

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi \cdot \nabla w \le \int_\Omega \lambda h(x,\psi) w \qquad \text{for all} \quad w \in W,$$

where $W = \{w \in C_0^\infty(\Omega) : w \ge 0 \text{ in } \Omega\}$ (see [9]). A function $\phi \in W^{1,p}(\Omega) \cap C^0(\overline{\Omega})$ is said to be a supersolution if $\phi \ge 0$ on $\partial\Omega$ and satisfies

$$\int_\Omega |\nabla\phi|^{p-2}\nabla\phi \cdot \nabla w \ge \int_\Omega \lambda h(x,\phi) w \qquad \text{for all} \quad w \in W.$$

It is known ([6],[7], and [9]) that if there is a subsolution ψ and a supersolution ϕ of (1.1) such that $\psi \le \phi$ in Ω then (1.1) has a $C^1(\overline{\Omega})$ solution u such that $\psi \le u \le \phi$ in Ω.

In order to construct a crucial positive subsolution, we employ a method similar to that developed in [11] and [12]. We will prove Theorem 1 in Section 2, Corollary 1 in Section 3, Corollary 2 in Section 4, and discuss examples in Section 5.

2. Proof of Theorem 1

Let $\lambda \in [\underline{\lambda}(\gamma), \overline{\lambda}(\gamma))$ and $\psi = \gamma\phi_1^{\frac{p}{p-1}}$. Since $\nabla\psi = \frac{p}{p-1}\gamma\phi_1^{\frac{1}{p-1}}\nabla\phi_1$, we get

$$\int_\Omega |\nabla\psi|^{p-2}\nabla\psi.\nabla w = \Big(\frac{p}{p-1}\gamma\Big)^{p-1}\int_\Omega \phi_1|\nabla\phi_1|^{p-2}\nabla\phi_1 \cdot \nabla w$$
$$= \Big(\frac{p}{p-1}\gamma\Big)^{p-1}\int_\Omega |\nabla\phi_1|^{p-2}\nabla\phi_1[\nabla(\phi_1 w) - w\nabla\phi_1]$$

$$= \Big(\frac{p}{p-1}\gamma\Big)^{p-1}\int_\Omega |\nabla\phi_1|^{p-2}\nabla\phi_1.\nabla(\phi_1 w) - \Big(\frac{p}{p-1}\gamma\Big)^{p-1}\int_\Omega |\nabla\phi_1|^p w$$
$$= \Big(\frac{p}{p-1}\gamma\Big)^{p-1}\int_\Omega \lambda_1|\phi_1|^{p-2}\phi_1(\phi_1 w) - \Big(\frac{p}{p-1}\gamma\Big)^{p-1}\int_\Omega |\nabla\phi_1|^p w \quad \text{(by (1.2))}$$
$$= \Big(\frac{p}{p-1}\gamma\Big)^{p-1}\int_\Omega [\lambda_1\phi_1^p - |\nabla\phi_1|^p] w \quad \text{for all} \quad w\in W. \tag{2.1}$$

Then ψ is a subsolution if

$$\Big(\frac{p}{p-1}\gamma\Big)^{p-1}\int_\Omega [\lambda_1\phi_1^p - |\nabla\phi_1|^p]\, w \le \lambda\int_\Omega g(x) f(\gamma\phi_1^{\frac{p}{p-1}}) w. \tag{2.2}$$

On Ω_δ, using (1.3) and (1.5), we have

$$\begin{aligned}
&\Big(\frac{p}{p-1}\gamma\Big)^{p-1}[\lambda_1\phi_1^p - |\nabla\phi_1|^p] \le -m\Big(\frac{p}{p-1}\gamma\Big)^{p-1}\\
&< -\lambda\beta f(\gamma) \quad \text{since} \quad \lambda < \lambda^*(\gamma) = \frac{m(\frac{p}{p-1}\gamma)^{p-1}}{f(\gamma)\beta}\\
&\le \lambda g(x) f(\gamma\phi_1^{\frac{p}{p-1}}).
\end{aligned} \tag{2.3}$$

On the other hand, on $\Omega\setminus\Omega_\delta$, using (1.4) and (1.6), we get

$$\begin{aligned}
&\Big(\frac{p}{p-1}\gamma\Big)^{p-1}[\lambda_1\phi_1^p - |\nabla\phi_1|^p] \le \lambda_1\Big(\frac{p}{p-1}\gamma\Big)^{p-1}\\
&\le \lambda r f(\gamma\alpha) \quad \text{since} \quad \lambda \ge \underline{\lambda}(\gamma) = \frac{\lambda_1(\frac{p}{p-1}\gamma)^{p-1}}{f(\gamma\alpha)r}\\
&\le \lambda g(x) f(\gamma\phi_1^{\frac{p}{p-1}}).
\end{aligned} \tag{2.4}$$

By (2.3) and (2.4), (2.2) is satisfied and ψ is a positive subsolution.

Next, since $\lambda < \overline{\lambda}(\gamma) \le \mu(\gamma)$ choose $R(\gamma)\in[k_2(\gamma),\infty)$ such that

$$\lambda \le \frac{R[(\gamma)]^{p-1}}{f(R(\gamma))\|e\|_\infty^{p-1} g_0} < \mu(\gamma). \tag{2.5}$$

Let $\phi = \frac{R(\gamma)}{\|e\|_\infty} e$. Then

$$\begin{aligned}
&\int_\Omega |\nabla\phi|^{p-2}\nabla\phi.\nabla w = \int_\Omega \Big(\frac{R(\gamma)}{\|e\|_\infty}\Big)^{p-1} w \qquad \text{for all} \quad w\in W, \quad (2.6)\\
&\ge \int_\Omega \lambda f(R(\gamma)) g_0\, w \ge \int_\Omega \lambda g(x) f\Big(\frac{R(\gamma)}{\|e\|_\infty} e\Big) w
\end{aligned}$$

and hence ϕ is a supersolution. Further, since $R(\gamma) \geq k_2(\gamma)$, using (2.1), (2.6), and the weak comparison principle [7], we see that $\phi \geq \psi$ in $\overline{\Omega}$. Hence the result follows.

3. Proof of Corollary 1

Here (F1) is satisfied for $\gamma \geq \gamma_0$ and hence $\underline{\lambda}(\gamma) \leq \lambda^*(\gamma)$ for all $\gamma \geq \gamma_0$. Also, since f is p-sublinear at ∞, $\mu(\gamma) = \infty$ for $\gamma \geq 0$; i.e., (F2) is satisfied for all $\gamma \geq 0$. Thus, by Theorem 1, there exists a positive solution for all

$$\lambda \in S := \bigcup_{\gamma \geq \gamma_0} [\underline{\lambda}(\gamma), \lambda^*(\gamma) = \overline{\lambda}(\gamma)].$$

But $\underline{\lambda}$ and λ^* are continuous functions and, again since f is p-sublinear at ∞ we have $\lim_{\gamma\to\infty} \underline{\lambda}(\gamma) = \infty = \lim_{\gamma\to\infty} \lambda^*(\gamma)$. Hence $S = [\lambda_0, \infty)$ where $\lambda_0 = \underline{\lambda}(\gamma_0)$ and the result follows.

4. Proof of Corollary 2

Here since $\frac{f(0)}{M} \geq \frac{\lambda_1}{m}\frac{\beta}{r}$ clearly (F1) is satisfied for $\gamma > 0$ and hence $\underline{\lambda}(\gamma) \leq \lambda^*(\gamma)$ for all $\gamma > 0$. Also, since $f(s) \leq M$ for $s \geq 0$, $\mu(\gamma) = \infty$ for all $\gamma > 0$. Thus by Theorem 1, there exists a positive solution for all

$$\lambda \in S := \bigcup_{\gamma > 0} [\underline{\lambda}(\gamma), \lambda^*(\gamma) = \overline{\lambda}(\gamma)].$$

But $\underline{\lambda}$ and λ^* are continuous functions and since $f(s) \leq M$ for $s > 0$ we have $\lim_{\gamma\to\infty} \underline{\lambda}(\gamma) = \infty = \lim_{\gamma\to\infty} \lambda^*(\gamma)$. Also $\underline{\lambda}(0) = 0$. Hence, $S = (0, \infty)$ and the result follows.

5. Examples

Example 1. Consider the problem

$$\begin{aligned} -\Delta_p u &= \lambda g(x)(u+1)^{q-1} \quad \text{in} \quad \Omega \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{5.1}$$

where $1 < q < p$. Here $f(u) = (u+1)^{q-1}$; $1 < q < p$ is clearly a C^1 nondecreasing p-sublinear function with $f(0) = 1$. Further

$$\lim_{\gamma\to\infty} \frac{f(\gamma\alpha)}{f(\gamma)} = \lim_{\gamma\to\infty} \frac{(\gamma\alpha+1)^{q-1}}{(\gamma+1)^{q-1}} = \alpha^{q-1}.$$

Thus if $\frac{\beta}{r}$ is small enough so that $\alpha^{q-1} > \frac{\lambda_1}{m}\frac{\beta}{r}$, then (F1) holds for $\gamma \geq \gamma_0$ for some $\gamma_0 > 0$. Hence all the hypotheses of Corollary 1 are satisfied and there exists $\lambda_0 > 0$ such that (5.1) has a positive solution for all $\lambda \geq \lambda_0$.

Example 2. Consider the problem

$$\begin{aligned} -\Delta_p u &= \lambda g(x) e^{\frac{au}{a+u}} \quad \text{in} \quad \Omega \\ u(x) &= 0 \quad \text{on} \quad \partial\Omega, \end{aligned} \tag{5.2}$$

where $a > 0$ is a constant. Here $f(u) = e^{\frac{au}{a+u}}$ is clearly a C^1 nondecreasing function such that $f(0) = 1$ and $f(u) \leq e^a$ for $u \geq 0$. Hence the hypotheses of Corollary 2 are satisfied if $\frac{\beta}{r}$ is small enough so that $\frac{1}{e^a} \geq \frac{\lambda_1}{m}\frac{\beta}{r}$ and (5.2) will have a positive solution for all $\lambda > 0$.

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