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# ASYMPTOTICS FOR MODEL NONLINEAR NONLOCAL EQUATIONS

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**Abstract.** We study the Cauchy problem for the model nonlinear equation

$$\begin{cases} u_t + \mathcal{L}u = \lambda \left| u \right|^{\sigma} u, \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases} \tag{0.1}$$

where $\sigma > 0$, $\lambda \in \mathbf{R}$. We are interested in the critical and subcritical powers of the nonlinearity, especially in the case of large initial data $u_0$ from $\mathbf{L}^{1,a} \cap \mathbf{L}^{\infty}$. We prove that the Cauchy problem (0.1) has a unique global solution $u \in \mathbf{C}\left([0,\infty); \mathbf{L}^{\infty} \cap \mathbf{L}^{1,a}\right)$ and obtain the large time asymptotics.

## 1. Introduction

This paper is devoted to the study of the Cauchy problem for the model nonlinear equation

$$\begin{cases} u_t + \mathcal{L}u = \lambda \left| u \right|^{\sigma} u, \ x \in \mathbf{R}, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}, \end{cases} \tag{1.1}$$

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where $\sigma > 0$, $\lambda \in \mathbf{R}$. The linear part of equation (1.1) is a pseudodifferential operator defined by the Fourier transformation

$$\mathcal{L}u = \overline{\mathcal{F}}_{\xi \to x} \left( L(\xi) \widehat{u}(\xi) \right).$$

Suppose that the linear operator $\mathcal{L}$ satisfies the dissipation condition which in terms of the symbol $L(\xi)$ has the form

$$\operatorname{Re} L(\xi) \geq \alpha \{\xi\}^{\delta} \langle \xi \rangle^{\nu} \tag{1.2}$$

for all $\xi \in \mathbf{R}$, where $\alpha$, $\delta$, $\nu > 0$ and $\{\xi\} = \frac{|\xi|}{\langle \xi \rangle}, \langle \xi \rangle = \sqrt{1 + |\xi|^2}$. Also we suppose that the symbol is sufficiently smooth except at the origin: $L(\xi) \in \mathbf{C}^3(\mathbf{R} \setminus \{0\})$ and has the estimate

$$\left| \partial_{\xi}^{l} L(\xi) \right| \leq C \{\xi\}^{\delta - l} \langle \xi \rangle^{\nu - l} \tag{1.3}$$

for all $\xi \in \mathbf{R} \setminus \{0\}$, $l = 0, 1, 2, 3$.

To find the asymptotic formulas for the solution we assume that the symbol $L(\xi)$ has the following asymptotic representation at the origin

$$L(\xi) = L_0(\xi) + O\left( |\xi|^{\delta + \gamma} \right) \tag{1.4}$$

for $\xi \to 0$ with some $\gamma > 0$. Here the symbol $L_0(\xi) = \alpha_1 |\xi|^{\delta} + i\alpha_2 |\xi|^{\delta - 1} \xi$ is homogeneous of order $\delta$, where $\alpha_1 > 0$, $\alpha_2 \in \mathbf{R}$. Denote $\mathcal{B} = \mathcal{L} - \mathcal{L}_0$, where the operator $\mathcal{L}_0$ is defined by the symbol $L_0(\xi)$. Then the Cauchy problem (1.1) can be written as

$$\begin{cases} u_t + \mathcal{L}_0 u = \lambda |u|^{\sigma} u - \mathcal{B}u, \ x \in \mathbf{R}, \ t > 0, \\ u(0, x) = u_0(x), \ x \in \mathbf{R}. \end{cases} \tag{1.5}$$

We are interested in the global in time existence of solutions to the Cauchy problem (1.1) with supercritical $\sigma > \delta$, critical $\sigma = \delta$ and subcritical $\sigma \in (0, \delta)$ powers of the nonlinearity respectively, especially when the initial data are not small.

We now refer to some typical physical examples which lead to equation (1.1). Equation (1.1) with $L(\xi) = \xi^2$, i.e. $\delta = 2$, is the usual nonlinear heat equation. It was studied in paper [18] in the supercritical case $\sigma > 2$, [8], [9] for the critical case $\sigma = 2$ and papers [5], [6], [10], [19] for the subcritical case $\sigma \in (0, 2)$. In the case $\delta \neq 2$, local in time existence can be easily shown by the contraction mapping principle in the $\mathbf{L}^2$ framework. Nonlinear dissipative equations with a fractional power of the negative Laplacian in the principal part were studied extensively (see, [1], [2], [21], [25], [28] and references cited therein). Blow-up in finite time of positive solutions to the Cauchy problem

$$u_t + |\partial_x|^\delta u - u^{1+\sigma} = 0,\ u(0,x) = u_0(x) > 0 \tag{1.6}$$

was proved in papers [7], [22], [29] for the case $0 < \sigma < 2$, $\delta = 2$, in papers [11], [20] for the case $\sigma = \delta = 2$, and in paper [27] for the case $0 < \sigma \leq \delta$, $0 < \delta \leq 2$. Global in time existence of small solutions to (1.6) was proved in [7] for the supercritical case $\sigma > 2 = \delta$. Global in time existence of solutions to the Cauchy problem (1.6) with "large" initial data with $\delta \neq 2$ in critical and subcritical cases was studied in our previous paper [17]. Some particular cases of the Cauchy problem (1.1) were considered in our papers [15], [12], [13]. As far as we know the general case of the Cauchy problem (1.1) was not studied previously, especially for the case of large initial data. To treat large initial data we follow here the method of paper [16].

Below $\mathcal{F}_{x\to\xi}\phi$ or $\hat{\phi}$ is the Fourier transform of $\phi$ defined by

$$\hat{\phi}(\xi) = (2\pi)^{-\frac{1}{2}} \int_{\mathbf{R}} e^{-ix\xi}\phi(x)\,dx$$

and $\overline{\mathcal{F}}_{\xi\to x}\phi(x)$ is the inverse Fourier transform of $\phi$. By $\mathbf{C}(\mathbf{I};\mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. The usual Lebesgue space is denoted by $\mathbf{L}^p$, $1 \leq p \leq \infty$, the weighted Lebesgue space $\mathbf{L}^{1,a}$ is defined by

$$\mathbf{L}^{p,a} = \left\{\phi \in \mathbf{L}^p; \|\phi\|_{\mathbf{L}^{p,a}} = \|\langle x\rangle^a \phi\|_{\mathbf{L}^p} < \infty\right\},$$

where $\langle x\rangle = \sqrt{1+|x|^2}$, $\{x\} = \frac{|x|}{\langle x\rangle}$, $a \geq 0$. Weighted Sobolev spaces we define as follows

$$\mathbf{W}_p^{k,a} = \Big\{\phi \in \mathbf{L}^p; \|\phi\|_{\mathbf{W}_p^{k,a}} = \sum_{j=0}^{k} \big\|\partial_x^j\phi\big\|_{\mathbf{L}^{p,a}} < \infty\Big\},$$

where $k \geq 0$, $a \geq 0$, $1 \leq p \leq \infty$. In the particular case $p = 2$ we denote $\mathbf{H}^{k,a} = \mathbf{W}_p^{k,a}$. Also we define the norm of the usual Sobolev space $\mathbf{H}^k = \mathbf{H}^{k,0}$ as follows $\|\phi\|_{\mathbf{H}^k} = \sum_{j=0}^{k} \big\|\partial_x^j\phi\big\|_{\mathbf{L}^2}$. We start with a small section, where we obtain preliminary estimates for the linearized Cauchy problem corresponding to (1.1). Also we prove a local existence theorem. In the next three sections we study the supercritical $\sigma > \delta$, critical $\sigma = \delta$ and subcritical $\sigma < \delta$ cases respectively.

## 2. Preliminaries

2.1. **Green operator.** Consider the linear Cauchy problem

$$u_t + \mathcal{L}u = f(t,x),\ \ x \in \mathbf{R}, t>0, u(0,x) = u_0(x),\ \ x \in \mathbf{R}. \tag{2.1}$$

Using the Duhamel principle we rewrite it in the form

$$u(t) = \mathcal{G}(t)\,u_0 + \int_0^t \mathcal{G}(t-\tau)\,f(\tau)\,d\tau,$$

where the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\,\phi = \overline{\mathcal{F}}_{\xi\to x} e^{-tL(\xi)}\hat{\phi}(\xi) = \int_{\mathbf{R}} G(t, x-y)\,\phi(y)\,dy.$$

Also we define

$$\mathcal{G}_0(t)\,\phi = \overline{\mathcal{F}}_{\xi\to x} e^{-tL_0(\xi)}\hat{\phi}(\xi) = t^{-\frac{1}{\delta}} \int_{\mathbf{R}} G_0\left((x-y)\,t^{-\frac{1}{\delta}}\right)\phi(y)\,dy.$$

Let us denote the fractional derivative of order $\beta \in (0,1)$ as follows (see [26])

$$|\partial_x|^{\beta}\,\phi(x) \equiv C\int_{\mathbf{R}} (\phi(x-y) - \phi(x))\,|y|^{-1-\beta}\,dy.$$

We first collect some preliminary estimates of the Green operator $\mathcal{G}(t)$ in the norms $\|\phi\|_{\mathbf{L}^p}$ and $\|\phi\|_{\mathbf{L}^{1,a}}$, where $a \geq 0$, $1 \leq p \leq \infty$.

**Lemma 2.1.** *Let the symbol $L(\xi)$ satisfy estimates (1.2)–(1.4). Suppose that the function $\phi \in \mathbf{L}^{\infty} \cap \mathbf{L}^{1,a}$, where $a \in (0,1)$. Then the estimates*

$$\left\| |\partial_x|^{\beta}\,\mathcal{G}(t)\,\phi \right\|_{\mathbf{L}^p} \leq C\,\{t\}^{-\frac{1}{\nu}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{\beta}{\nu}} \langle t\rangle^{-\frac{1}{\delta}\left(\frac{1}{r}-\frac{1}{p}\right)-\frac{\beta}{\delta}} \|\phi\|_{\mathbf{L}^r},$$

$$\left\| |\cdot|^{b}\left(\mathcal{G}(t)\,\phi - \vartheta G(t)\right)\right\|_{\mathbf{L}^p} \leq \{t\}^{\frac{1}{\nu}\left(\frac{1}{p}-1\right)+\frac{b-a}{\nu}} \langle t\rangle^{\frac{1}{\delta}\left(\frac{1}{p}-1\right)+\frac{b-a}{\delta}} \|\phi\|_{\mathbf{L}^{1,a}},$$

$$\left\|\mathcal{G}(t)\,\phi - \vartheta t^{-\frac{1}{\delta}} G_0(xt^{-\frac{1}{\delta}})\right\|_{\mathbf{L}^{\infty}} \leq C\left(\{t\}^{-\frac{1+a}{\nu}}\langle t\rangle^{-\frac{1+a}{\delta}} + \{t\}^{-\frac{1}{\nu}}\langle t\rangle^{-\frac{1+\gamma}{\delta}}\right)\|\phi\|_{\mathbf{L}^{1,a}}$$

*are valid for all $t>0$, where $1 \leq r \leq p$, $\beta \in [0,1)$, $1 \leq p \leq \infty$, $0 \leq b \leq a$, $\vartheta = \int_{\mathbf{R}} \phi(x)\,dx$.*

**Proof.** The proof is similar to that of Lemma 2.1 from paper [15], so we omit it. □

2.2. **Local existence.** By the contraction mapping principle we easily get the local existence of weak solutions to the Cauchy problem (1.1).

**Proposition 2.2.** *Let* $\sigma > 0$, $\lambda \in \mathbf{R}$. *Suppose that the initial data* $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^{1,a}$, *with* $a \geq 0$. *Then for some* $T > 0$ *there exists a unique solution*

$$u \in \mathbf{C}\left([0,T]\,;\mathbf{L}^\infty \cap \mathbf{L}^{1,a}\right) \cap \mathbf{C}\left((0,T]\,;\mathbf{H}^1\right)$$

*to the Cauchy problem* (1.1).

Note that the property of smoothing of solutions $u \in \mathbf{C}\left((0,T]\,;\mathbf{H}^1\right)$ follows from the dissipativity of the symbol $L(\xi)$ (see, e.g., [23]).

## 3. Supercritical case

3.1. **Small data.** In order to prove the global existence in the case of arbitrary sign of the coefficient $\lambda$ at the nonlinearity we have to assume a smallness condition on the initial data, since in the case $\lambda > 0$ there could be a blow up phenomena (see [7], [22]). Also we need to investigate more carefully the time decay rate of the nonlinear term. For the supercritical case $\sigma > \delta$ the nonlinearity is asymptotically weak, that is it does not affect essentially the main term of large time asymptotics of solutions. Thus we have the following result.

**Theorem 3.1.** *Let* $\sigma > \delta$. *Let* $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^{1,a}$, $a \in (0,1]$. *Suppose that the norm* $\|u_0\|_{\mathbf{L}^{1,a}} + \|u_0\|_{\mathbf{L}^\infty}$ *is sufficiently small. Then there exists a unique solution*

$$u \in \mathbf{Y} \equiv \mathbf{C}\left([0,\infty)\,;\mathbf{L}^\infty \cap \mathbf{L}^{1,a}\right) \cap \mathbf{C}\left((0,\infty)\,;\mathbf{H}^1\right)$$

*to the Cauchy problem* (1.1). *Moreover, the asymptotics*

$$u(t,x) = At^{-\frac{1}{\delta}} G_0(xt^{-\frac{1}{\delta}}) + O(t^{-\frac{1}{\delta}-\gamma}) \tag{3.1}$$

*for large time* $t \to \infty$ *is true uniformly with respect to* $x \in \mathbf{R}$, *where* $A$ *is a constant,* $0 < \gamma < \min\left(\frac{a}{\delta}, \frac{\sigma}{\delta} - 1\right)$.

Theorem 3.1 can be proved by a standard contraction mapping principle (see, e.g., [15]).

3.2. **Large data.** Now we consider the case $\lambda < 0$, then using the method of paper [16] we can remove the smallness condition on the initial data $u_0(x)$.

**Theorem 3.2.** *Let* $\delta \in (1,2]$, $\lambda < 0$, $\sigma > \delta$. *Suppose that the initial data* $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^{1,a}$, $a \in (0,1]$. *Also let* (1.4) *take place with* $\alpha_2 = 0$; $\frac{\delta}{2} \geq \gamma > \frac{\delta}{\sigma} - \frac{1}{2}$ *for* $1 \leq \sigma \leq 2$ *and* $\frac{\delta}{2} \geq \gamma > \frac{\delta}{2} - \frac{\sigma}{4}$ *for* $\sigma > 2$. *Then there exists a unique* *solution* $u \in \mathbf{Y}$ *to the Cauchy problem* (1.1). *Moreover, the asymptotics* (3.1) *take place.*

Before proving Theorem 3.2, we prepare several lemmas. Define the norm

$$\|\phi\|_{p,q} \equiv \|\|\phi(t,\cdot)\|_{\mathbf{L}^q}\|_{\mathbf{L}^p\left(\mathbf{R}_t^+\right)}.$$

**Lemma 3.3.** *Let* $\delta \in (1,2]$, $\sigma \geq 1$. *Suppose that the initial data* $u_0 \in \mathbf{H}^1$. *Let the norms of the solution be bounded*

$$\|u\|_{\infty,2} + \|u\|_{\sigma+2,\sigma+2} + \|u_x\|_{\infty,2} + \left\||\partial_x|^{\frac{\nu}{2}} u_x\right\|_{2,2} + \left\||\partial_x|^{\frac{\delta}{2}} u\right\|_{2,2} \leq C.$$

*Assume that the operator* $\mathcal{B}$ *has a symbol* $B(\xi)$ *such that*

$$|B(\xi)| \leq C|\xi|^{\gamma}\{\xi\}^{\delta}\langle\xi\rangle^{\nu} \tag{3.2}$$

*for all* $\xi \in \mathbf{R}$, *where* $0 < \gamma \leq \frac{\delta}{2}$. *Then the estimate*

$$\int_0^t \|\mathcal{B}u(\tau)\|_{\mathbf{L}^1} d\tau \leq Ct^{\beta}$$

*is true for all* $t > 0$, *where* $\beta > \frac{1}{\sigma} - \frac{\gamma}{\delta}$ *for* $1 \leq \sigma \leq 2$ *and* $\beta > \frac{1}{2} - \frac{\gamma}{\delta} - \frac{\sigma-2}{4\delta}$ *for* $\sigma > 2$.

**Proof.** By the Hölder inequality we get for $2 \leq p \leq \sigma+2$ and $s = \frac{\sigma p}{p-2}$

$$\|u\|_{s,p} \leq \|u\|_{\infty,2}^{1-\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)} \|u\|_{\sigma+2,\sigma+2}^{\left(1-\frac{2}{p}\right)\left(1+\frac{2}{\sigma}\right)} \leq C,$$

and by the Sobolev inequality we have

$$\|u\|_{\mathbf{L}^\infty} \leq C\|u\|_{\mathbf{L}^2}^{1-\frac{1}{\delta}} \left\||\partial_x|^{\frac{\delta}{2}} u\right\|_{\mathbf{L}^2}^{\frac{1}{\delta}}$$

for $\delta > 1$, hence $\|u\|_{s,\infty} \leq C$ for $s \geq 2\delta$. Next we estimate $\mathcal{B}u$. By Lemma 2.1 we have for $\rho_1 \in (0,1)$

$$\begin{aligned}\|\mathcal{B}\mathcal{G}(t-\tau)|u|^{\sigma} u(\tau)\|_{\mathbf{L}^1} &\leq C\{t-\tau\}^{\rho_1-1}(t-\tau)^{-1}\left(\||\partial_x|^{\gamma}\left(|u(\tau)|^{\sigma} u(\tau)\right)\|_{\mathbf{L}^1}\right.\\ &\quad\left.+C\left\||\partial_x|^{\frac{\nu}{2}}\partial_x\left(|u(\tau)|^{\sigma} u(\tau)\right)\right\|_{\mathbf{L}^1}\right).\end{aligned} \tag{3.3}$$

Taking $0 < \rho_2 < \gamma < 1$ we obtain

$$\||\partial_x|^{\gamma}\left(|u|^{\sigma} u\right)\|_{\mathbf{L}^1} \leq C\|u\|_{\mathbf{L}^{2\sigma}}^{\sigma}\left(\||\partial_x|^{\rho_2} u\|_{\mathbf{L}^2} + \|u_x\|_{\mathbf{L}^2}\right). \tag{3.4}$$

On the other hand

$$\|\partial_x\left(|u|^{\sigma} u\right)\|_{\mathbf{L}^1} \leq C\|u\|_{\mathbf{L}^{2\sigma}}^{\sigma}\|u_x\|_{\mathbf{L}^2}. \tag{3.5}$$

Hence we have

$$\begin{aligned}&\left\|\int_0^t\left\|\mathcal{BG}(t-\tau)|u|^{\sigma}u(\tau)\right\|_{\mathbf{L}^1}d\tau\right\|_{\mathbf{L}_t^s(0,\infty)}\\ \leq\ & C\left\|\{t\}^{\rho_1-1}\langle t\rangle^{-1}\right\|_{\mathbf{L}_t^{s_1}(0,\infty)}\Big(\left\||\partial_x|^{\rho_2}u\right\|_{s_2,2}\|u\|_{\sigma s_3,2\sigma}\\ &+\left\||\partial_x|^{\frac{\nu}{2}}\partial_x u\right\|_{s_2,2}\|u\|_{\sigma s_3,2\sigma}+\|u\|_{\infty,\infty}^{\sigma-1}\|\partial_x u\|_{2,2}^2\Big),\end{aligned}$$

where $\frac{1}{s}=\frac{1}{s_1}+\frac{1}{s_2}+\frac{1}{s_3}-1$, and $s_1>1$, $s_2=\frac{\delta}{\rho_2}$, $s_3=\frac{\sigma}{\sigma-1}$ for $1\leq\sigma\leq 2$ and $s_3=\frac{4\delta}{2\delta+\sigma-2}$ for $\sigma>2$. Then by the Hölder inequality we have

$$\int_0^t\|\mathcal{B}u(\tau)\|_{\mathbf{L}^1}d\tau\leq t^{1-\frac{1}{s}}\|\mathcal{B}u\|_{s,1}\leq Ct^{\beta},$$

where $\beta>\frac{1}{\sigma}-\frac{\gamma}{\delta}$ for $1\leq\sigma\leq 2$ and $\beta>\frac{1}{2}-\frac{\gamma}{\delta}-\frac{\sigma-2}{4\delta}$ for $\sigma>2$. Lemma 3.3 is proved. □

Now we estimate the decay rate of the $\mathbf{L}^p$ norms of the solutions.

**Lemma 3.4.** *Let $\delta\in(1,2]$, $\sigma\geq 1$, $\lambda<0$. Suppose that $u_0\in\mathbf{L}^1\cap\mathbf{L}^2$. Also let (1.4) take place with $\alpha_2=0$; $\frac{\delta}{2}\geq\gamma>\frac{\delta}{\sigma}-\frac{1}{2}$ for $1\leq\sigma\leq 2$ and $\frac{\delta}{2}\geq\gamma>\frac{\delta}{2}-\frac{\sigma}{4}$ for $\sigma>2$. Assume that $\mathcal{B}=\mathcal{L}-\mathcal{L}_0$ satisfies (3.2) and*

$$\int_0^t\|\mathcal{B}u(\tau)\|_{\mathbf{L}^1}d\tau\leq C\langle t\rangle^{\beta}\tag{3.6}$$

*for all $t>0$, where $\beta\in\left[0,\frac{1}{2\delta}\right)$. Then the estimates*

$$\|u(t)\|_{\mathbf{L}^p}\leq C\langle t\rangle^{\beta-\frac{1}{\delta}(1-\frac{1}{p})},\ \|u_x(t)\|_{\mathbf{L}^2}\leq C(1+t)^{\beta-\frac{3}{2\delta}}$$

*are valid for all $t>0$, where $1\leq p\leq 2$.*

**Proof.** We estimate the $\mathbf{L}^1$ norm. Denote $S(x)=1$ for all $x>0$ and $S(x)=-1$ for all $x<0$; $S(0)=0$. We multiply equation (1.1) by $S(u(t,x))$ and integrate with respect to $x$ to get

$$\frac{d}{dt}\|u(t)\|_{\mathbf{L}^1}+|\lambda|\|u(t)\|_{\mathbf{L}^{\sigma+1}}^{\sigma+1}=\int_{\mathbf{R}}S(u(t,x))\mathcal{L}udx.$$

By condition (1.4) we represent $\mathcal{L}u=\alpha_1|\partial_x|^{\delta}u+\mathcal{B}u$, where $\alpha_1>0$. Representing the operator $|\partial_x|^{\delta}$ via the Riesz potential (see [26]) let us show that

$$\int_{\mathbf{R}}S(u(t,x))|\partial_x|^{\delta}u(t,x)dx\leq 0.\tag{3.7}$$

We have by [4]

$$\frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}}|x|^{-\nu}e^{ixy}dx=\frac{\pi}{\sqrt{2\pi}\Gamma(\nu)\cos\left(\frac{\pi\nu}{2}\right)}|y|^{\nu-1}$$

for $\nu\in(0,1)$, then

$$|\partial_x|^{-\nu}\phi=\mathcal{F}^{-1}|\xi|^{-\nu}\widehat{\phi}(\xi)=\frac{\pi}{\sqrt{2\pi}\Gamma(\nu)\cos\left(\frac{\pi\nu}{2}\right)}\int_{\mathbf{R}}|x-y|^{\nu-1}\phi(y)\,dy. \quad (3.8)$$

Thus, for $\delta\in(1,2)$ we get with $\nu=2-\delta\in(0,1)$, integrating by parts with respect to $y\in\mathbf{R}$ two times

$$|\partial_x|^{\delta}\phi=\mathcal{F}^{-1}|\xi|^{\delta}\widehat{\phi}(\xi)=\partial_x^2|\partial_x|^{-\nu}\phi=C\partial_x^2\int_{\mathbf{R}}|x-y|^{1-\delta}\phi(y)\,dy.$$

Denote $S(t,x)=\operatorname{sign}(u(t,x))$ and represent $u(t,x)=S(t,x)|u(t,x)|$. We make a regularization

$$K_{\varepsilon}''(x)=\begin{cases}\partial_x^2|x|^{1-\delta}, & \text{for } |x|\geq\varepsilon\\ 0, & \text{for } |x|<\varepsilon, x\neq 0.\end{cases}$$

Note that $K_{\varepsilon}''(x)\geq 0$ for all $x\in\mathbf{R}\backslash 0$. We can easily see that for $u\in\mathbf{C}^2(\mathbf{R})$

$$\partial_x^2\int_{\mathbf{R}}|x-y|^{1-\delta}u(t,y)\,dy=\lim_{\varepsilon\to 0}\partial_x^2\int_{\mathbf{R}}K_{\varepsilon}(x-y)\,u(t,y)\,dy.$$

Then via the identity

$$S(u(t,y))\,S(u(t,x))=1-\frac{1}{2}\left(S(u(t,x))-S(u(t,y))\right)^2$$

we get

$$\begin{aligned}&\int_{\mathbf{R}}dxS(u(t,x))\,\partial_x^2\int_{\mathbf{R}}K_{\varepsilon}(x-y)\,u(t,y)\,dy\\ =\;&-\frac{1}{2}\int_{\mathbf{R}}dy\,|u(t,y)|\int_{\mathbf{R}}dxK_{\varepsilon}''(x-y)\left(S(u(t,x))-S(u(t,y))\right)^2\leq 0,\end{aligned}$$

hence we have

$$\begin{aligned}&\int_{\mathbf{R}}S(u(t,x))|\partial_x|^{\delta}u(t,x)dx=C\int_{\mathbf{R}}dxS(u(t,x))\partial_x^2\int_{\mathbf{R}}dy\,|x-y|^{1-\delta}u(t,y)\\ &=-\frac{C}{2}\lim_{\varepsilon\to 0}\int_{\mathbf{R}}dy\,|u(t,y)|\int_{\mathbf{R}}dxK_{\varepsilon}''(x-y)\left(S(u(t,x))-S(u(t,y))\right)^2\leq 0.\end{aligned}$$

Therefore, we find

$$\frac{d}{dt}\|u(t)\|_{\mathbf{L}^1}+|\lambda|\,\|u(t)\|_{\mathbf{L}^{\sigma+1}}^{\sigma+1}\leq\|\mathcal{B}u(t)\|_{\mathbf{L}^1}. \quad (3.9)$$

Integration of (3.9) in view of estimate (3.6) yields the estimate of the lemma with $p = 1$. Moreover,

$$\sup_{\xi\in\mathbf{R}} |\widehat{u}(t,\xi)| \leq (2\pi)^{-\frac{1}{2}} \|u(t)\|_{\mathbf{L}^1} \leq C\langle t\rangle^{\beta}. \tag{3.10}$$

We now multiply equation (1.1) by $2u$, then integrating with respect to $x \in \mathbf{R}$ we get

$$\frac{d}{dt}\|u(t)\|_{\mathbf{L}^2}^2 = -2\int_{\mathbf{R}} u\mathcal{L}u dx + 2\lambda \|u(t)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2}. \tag{3.11}$$

By the Plancherel theorem using the Fourier splitting method due to [24], we have

$$\begin{aligned}\int_{\mathbf{R}} u\mathcal{L}u dx &= C\int_{|\xi|\leq\chi} |\widehat{u}(t,\xi)|^2 \operatorname{Re} L(\xi)\, d\xi + C\int_{|\xi|\geq\chi} |\widehat{u}(t,\xi)|^2 \operatorname{Re} L(\xi)\, d\xi \\ &\geq \alpha\chi^{\delta}\|u(t)\|_{\mathbf{L}^2}^2 - 4\alpha\chi^{\delta+1}\sup_{|\xi|\leq\chi}|\widehat{u}(t,\xi)|^2,\end{aligned} \tag{3.12}$$

where $\chi > 0$. Thus from (3.11) we have the inequality

$$\frac{d}{dt}\|u(t)\|_{\mathbf{L}^2}^2 \leq -\alpha\chi^{\delta}\|u(t)\|_{\mathbf{L}^2}^2 + 4\alpha\chi^{\delta+1}\sup_{|\xi|\leq\chi}|\widehat{u}(t,\xi)|^2. \tag{3.13}$$

We choose $\alpha\chi^{\delta} = 2(1+t)^{-1}$ and change $\|u(t)\|_{\mathbf{L}^2}^2 = (1+t)^{-2} W(t)$. Then via (3.10) we get from (3.13)

$$\frac{d}{dt}W(t) \leq C(1+t)^{2\beta-\frac{1}{\delta}+1}. \tag{3.14}$$

Integration of (3.14) with respect to time yields a time decay estimate of the $\mathbf{L}^2$ norm

$$\|u(t)\|_{\mathbf{L}^2} \leq C(1+t)^{\beta-\frac{1}{2\delta}} \tag{3.15}$$

for all $t > 0$. Differentiating equation (1.1) with respect to $x$, multiplying the resulting equation by $2u_x$ and integrating with respect to $x \in \mathbf{R}$ and using

$$\int_{\mathbf{R}} u_x\mathcal{L}u_x dx \geq \alpha\chi^{\delta}\|u_x(t)\|_{\mathbf{L}^2}^2 - 4\alpha\chi^{\delta+3}\sup_{|\xi|\leq\chi}|\widehat{u}(t,\xi)|^2,$$

we have the inequality

$$\frac{d}{dt}\|u_x(t)\|_{\mathbf{L}^2}^2 \leq -\alpha\chi^{\delta}\|u_x(t)\|_{\mathbf{L}^2}^2 + 4\alpha\chi^{\delta+3}\sup_{|\xi|\leq\chi}|\widehat{u}(t,\xi)|^2. \tag{3.16}$$

In the same way as in the proof of (3.15) we have the second estimate of the lemma. Lemma 3.4 is proved. □

In the next lemma we improve the estimates of Lemma 3.3.

**Lemma 3.5.** *Let $\delta \in (1,2]$, $\sigma \geq 1$. Suppose that the initial data $u_0 \in \mathbf{H}^1$ and*

$$\sup_{t>0} \left( \langle t \rangle^{\eta} \|u(t)\|_{\mathbf{L}^2} + \langle t \rangle^{\eta+\frac{1}{\delta}} \|u_x(t)\|_{\mathbf{L}^2} \right) + \left\| |\partial_x|^{\frac{\nu}{2}} u_x \right\|_{2,2} \leq C,$$

*where $\eta > 0$. Assume that the operator $\mathcal{B}$ has a symbol $B(\xi)$ such that (3.2) is fulfilled. Then the estimate*

$$\int_0^t \|\mathcal{B}u(\tau)\|_{\mathbf{L}^1} d\tau \leq Ct^{\beta}$$

*is true for all $t > 0$, where $\beta \geq 0$ is such that $\beta > \frac{1}{\sigma} - \frac{\gamma}{\delta} - \eta$ for $1 \leq \sigma \leq 2$ and $\beta > \frac{1}{2} - \frac{\gamma}{\delta} - \frac{\sigma-2}{4\delta} - \eta$ for $\sigma > 2$.*

**Proof.** We have

$$\left\| |\partial_x|^{\frac{\delta}{2}} u \right\|_{\mathbf{L}^2} \leq \|u\|_{\mathbf{L}^2}^{1-\frac{\delta}{2}} \|u_x\|_{\mathbf{L}^2}^{\frac{\delta}{2}} \leq C \langle t \rangle^{-\eta-\frac{1}{2}}$$

and

$$\|u\|_{\mathbf{L}^p} \leq C \|u\|_{\mathbf{L}^\infty}^{1-\frac{2}{p}} \|u\|_{\mathbf{L}^2}^{\frac{2}{p}} \leq C \langle t \rangle^{-\eta-\frac{1}{2\delta}\left(1-\frac{2}{p}\right)}$$

for $2 \leq p \leq \infty$. Applying estimates (3.3)-(3.5) we get

$$\begin{aligned} & \left\| \int_0^t \left\| \mathcal{B}\mathcal{G}(t-\tau) |u|^{\sigma} u(\tau) \right\|_{\mathbf{L}^1} d\tau \right\|_{\mathbf{L}_t^s(0,\infty)} \\ \leq \ & C \left\| \{t\}^{\rho_1 - 1} \langle t \rangle^{-1} \right\|_{\mathbf{L}_t^{s_1}(0,\infty)} \Big( \| |\partial_x|^{\rho_2} u\|_{s_2,2} \|u\|_{\sigma s_3, 2\sigma} \\ & + \left\| |\partial_x|^{\frac{\nu}{2}} \partial_x u \right\|_{s_2,2} \|u\|_{\sigma s_3, 2\sigma} + \|u\|_{\infty,\infty}^{\sigma-1} \|\partial_x u\|_{2,2}^2 \Big), \end{aligned}$$

where $\frac{1}{s} = \frac{1}{s_1} + \frac{1}{s_2} + \frac{1}{s_3} - 1$, and $s_1 > 1$, $s_2 = \left(\frac{\rho_2}{\delta} + \eta\right)^{-1}$, $s_3 = \frac{\sigma}{\sigma-1}$ for $1 \leq \sigma \leq 2$ and $s_3 = \frac{4\delta}{2\delta+\sigma-2}$ for $\sigma > 2$. Then by the Hölder inequality we have the estimate of the lemma. Lemma 3.5 is proved. □

**Proposition 3.6.** *Let $\delta \in (1,2]$, $\sigma \geq 1$, $\lambda < 0$. Suppose that the initial data $u_0 \in \mathbf{H}^1 \cap \mathbf{L}^1$. Also let (1.4) take place with $\alpha_2 = 0$; $\frac{\delta}{2} \geq \gamma > \frac{\delta}{\sigma} - \frac{1}{2}$ for $1 \leq \sigma \leq 2$ and $\frac{\delta}{2} \geq \gamma > \frac{\delta}{2} - \frac{\sigma}{4}$ for $\sigma > 2$. Then the estimates for the solution*

$$\|u(t)\|_{\mathbf{L}^p} \leq C \langle t \rangle^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)}, \|u_x(t)\|_{\mathbf{L}^2} \leq C \langle t \rangle^{-\frac{3}{2\delta}}$$

*are valid for all $t > 0$, where $1 \leq p \leq 2$.*

**Proof.** By the usual energy method we get

$$\frac{d}{dt}\|u(t)\|_{\mathbf{H}^1}^2 + 2\int_{\mathbf{R}} u_x \mathcal{L} u_x dx + 2\int_{\mathbf{R}} u \mathcal{L} u dx$$
$$-2\lambda(\sigma+1)\left\| |u|^\sigma u_x^2 \right\|_{\mathbf{L}^1} - 2\lambda \|u(t)\|_{\mathbf{L}^{\sigma+2}}^{\sigma+2} = 0.$$

Note that

$$2\int_{\mathbf{R}} u_x \mathcal{L} u_x dx + 2\int_{\mathbf{R}} u \mathcal{L} u dx \geq 2\alpha \int_{\mathbf{R}} \{\xi\}^\delta \langle\xi\rangle^{\nu+2} |\widehat{u}(t,\xi)|^2 d\xi$$
$$\geq \alpha \left\| |\partial_x|^{\frac{\delta}{2}} u(t) \right\|_{\mathbf{L}^2}^2 + \alpha \left\| |\partial_x|^{\frac{\nu}{2}} u_x(t) \right\|_{\mathbf{L}^2}^2.$$

Therefore, integrating in time we have the desired estimates by which we can apply Lemma 3.3 to obtain

$$A(t,\beta_0) \equiv t^{-\beta_0} \int_0^t \|\mathcal{B} u(\tau)\|_{\mathbf{L}^1} d\tau \leq C$$

for all $t > 0$, where $\beta_0 > \frac{1}{\sigma} - \frac{\gamma}{\delta}$ for $1 \leq \sigma \leq 2$ and $\beta_0 > \frac{1}{2} - \frac{\gamma}{\delta} - \frac{\sigma-2}{4\delta}$ for $\sigma > 2$.

By the conditions for the parameters $\gamma > \frac{\delta}{\sigma} - \frac{1}{2}$ for $1 \leq \sigma \leq 2$ and $\gamma > \frac{\delta}{2} - \frac{\sigma}{4}$ for $\sigma > 2$ we have $\beta_0 < \frac{1}{2\delta}$. Then from Lemma 3.4 we obtain

$$C_{\eta_0} \equiv \sup_{t>0} \left( \langle t\rangle^{\eta_0} \|u(t)\|_{\mathbf{L}^2} + \langle t\rangle^{\eta_0 + \frac{1}{\delta}} \|u_x(t)\|_{\mathbf{L}^2} \right) \leq C$$

for all $t > 0$, with $\eta_0 = \frac{1}{2\delta} - \beta_0 > 0$. Now we use Lemma 3.5 to improve the estimate $A(t,\beta_1) \leq C$ for all $t > 0$, where $\beta_1 = \max(0, \beta_0 - \eta_0)$. Then by Lemma 3.4 we find a better time decay $C_{\eta_1} \leq C$ for all $t > 0$, where $\eta_1 = \frac{1}{2\delta} - \beta_1 = \min\left(\frac{1}{2\delta}, 2\eta_0\right)$. After repeating these considerations $k$ times we obtain by virtue of Lemma 3.5 $A(t,\beta_k) \leq C$ for all $t > 0$, where $\beta_k = \max(0, \beta_0 - \eta_{k-1})$. And by Lemma 3.4 we find $C_{\eta_k} \leq C$ for all $t > 0$, where $\eta_k = \frac{1}{2\delta} - \beta_k = \min\left(\frac{1}{2\delta}, (k+1)\eta_0\right)$. When $k + 1 > \frac{1}{2\delta\eta_0}$, then $\eta_k = \frac{1}{2\delta}$, and we have the optimal time decay estimates. Proposition 3.6 is proved. □

**Proof of Theorem 3.2.** By the smoothing effect for the parabolic type equations we have $u(t_1) \in \mathbf{H}^1$ for $t_1 > 0$. We take the initial time at $t = t_1$ and apply Proposition 3.6. Then we have the following optimal time decay estimates

$$\|u(t)\|_{\mathbf{L}^p} \leq C \langle t\rangle^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)}, \ \ \|u_x(t)\|_{\mathbf{L}^2} \leq C \langle t\rangle^{-\frac{3}{2\delta}}$$

for all $t > 0$, where $1 \leq p \leq 2$. Then via inequality $\|u\|_{\mathbf{L}^\infty} \leq C \|u\|_{\mathbf{L}^2}^{\frac{1}{2}} \|u_x\|_{\mathbf{L}^2}^{\frac{1}{2}}$ we find that the first estimate of the above holds for $1 \leq p \leq \infty$. By Lemma

2.1 we have

$$\begin{aligned} &\|u(t)\|_{\mathbf{L}^{1,a}} \le \|\mathcal{G}(t)\,u_0\|_{\mathbf{L}^{1,a}} \\ &+C\int_0^t \left( \||u|^{\sigma} u(\tau)\|_{\mathbf{L}^{1,a}} + \langle t-\tau\rangle^{\frac{a}{\delta}} \||u|^{\sigma} u(\tau)\|_{\mathbf{L}^{1}} \right) d\tau \\ \le\; & C\langle t\rangle^{\frac{a}{\delta}} + C\int_0^t \langle\tau\rangle^{-\frac{\sigma}{\delta}} \|u(\tau)\|_{\mathbf{L}^{1,a}}\, d\tau. \end{aligned}$$

Hence by the Gronwall lemma we obtain $\|u(t)\|_{\mathbf{L}^{1,a}} \le C\langle t\rangle^{\frac{a}{\delta}}$ for all $t>0$, since $\sigma > \delta$ in the supercritical case. Therefore using these estimates the local solution given by Proposition 2.2 can be prolonged for all time $t > 0$, hence there exists a unique solution $u \in \mathbf{C}\left([0,\infty);\mathbf{L}^{\infty} \cap \mathbf{L}^{1,a}\right)$ to the Cauchy problem (1.1). Arguing as in the proof of Theorem 3.1 we obtain the asymptotics (3.1). Theorem 3.2 is proved.

## 4. Critical case

This section is devoted to the study of the Cauchy problem (1.1) in the critical case $\sigma = \delta$. Everywhere below we suppose that $\lambda < 0$.

4.1. **Small data.** We assume that

$$\eta = \sigma|\lambda| \int_{\mathbf{R}} |G_0(x)|^{\delta} G_0(x)\,dx > 0. \tag{4.1}$$

Denote $g(t) = 1 + |\theta|^{\delta} \eta \log(1+t)$, $\theta = \int_{\mathbf{R}} u_0(x)dx$. Now we state the results of this section.

**Theorem 4.1.** *Let $\sigma = \delta$, $\lambda < 0$. Suppose that condition* (4.1) *takes place. Assume that the initial data $u_0 \in \mathbf{L}^{\infty} \cap \mathbf{L}^{1,a}$, $a \in (0,1]$ are small, $\|u_0\|_{\mathbf{L}^{\infty}} + \|u_0\|_{\mathbf{L}^{1,a}} \le \varepsilon$, and $|\theta| \ge C\varepsilon > 0$. Then the Cauchy problem* (1.1) *has a unique global solution $u \in \mathbf{Y}$ satisfying the time decay estimate*

$$\left\| u(t) - \theta t^{-\frac{1}{\delta}} G_0\left((\cdot)\,t^{-\frac{1}{\delta}}\right) g^{-\frac{1}{\delta}}(t) \right\|_{\mathbf{L}^{\infty}} \le C\langle t\rangle^{-\frac{1}{\delta}} g^{-1-\frac{1}{\delta}}(t) \log g(t). \tag{4.2}$$

**Proof of Theorem 4.1.** As in paper [12] we make a change of the dependent variable $u = ve^{-\varphi}$. Then we have the Cauchy problem for the new dependent variables $(v,\varphi)$

$$\begin{cases} v_t + \mathcal{L}v = f\left(v, e^{\delta\varphi}\right) \equiv \lambda e^{-\delta\varphi}\left(|v|^{\delta} - \frac{1}{\theta}\int_{\mathbf{R}} |v|^{\delta} v dx\right) v, \\ \varphi'(t) = -\frac{\lambda}{\theta} e^{-\delta\varphi} \int_{\mathbf{R}} |v|^{\delta} v dx, v(0,x) = u_0(x),\ \varphi(0) = 0. \end{cases} \tag{4.3}$$

We denote $h = e^{\delta\varphi}$ and write (4.3) as

$$\begin{cases} v_t + \mathcal{L}v = f(v,h), \ v(0,x) = u_0(x), \\ h' = -\frac{\lambda\delta}{\theta}\int_{\mathbf{R}} |v|^{\delta} v dx, \ h(0) = 1. \end{cases} \tag{4.4}$$

We note that the mean value of the nonlinearity $\int_{\mathbf{R}} f(v,h)\, dx = 0$ for all $t > 0$. We now prove the existence of the solution $(v,h)$ for the Cauchy problem (4.4) by the method of successive approximations. Denote $v_1 = \mathcal{G}(t) u_0$, $h_1 = g(t)$ and define $(v_m, h_m)$, for all $m \geq 2$, as a solution of a linear problem

$$\begin{cases} \partial_t v_m + \mathcal{L}v_m = f(v_{m-1}, h_{m-1}), \ v_m(0,x) = u_0(x), \\ h'_m = -\frac{\lambda\delta}{\theta}\int_{\mathbf{R}} |v_{m-1}|^{\delta} v_{m-1} dx, \ h_m(0) = 1. \end{cases} \tag{4.5}$$

We prove by induction the following estimates

$$\begin{aligned} &\|v_m\|_{\mathbf{X}} \leq C\varepsilon, \quad \|v_m(t) - \mathcal{G}(t) u_0\|_{\mathbf{L}^1} \leq C\varepsilon^{1+\delta} g^{-1}(t), \\ &|h_m(t) - g(t)| \leq C\varepsilon^{\delta}(1 + \log g(t)) \end{aligned} \tag{4.6}$$

for all $m \geq 1$, where the norm $\|\cdot\|_{\mathbf{X}}$ is defined by

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\left(\langle t\rangle^{\frac{1}{\delta}} \|\phi(t)\|_{\mathbf{L}^\infty} + \langle t\rangle^{-\frac{a}{\delta}} \|\phi(t)\|_{\mathbf{L}^{1,a}}\right).$$

By virtue of Lemma 2.1 we see that estimates (4.6) are valid for $m = 1$. We assume that estimates (4.6) are true with $m$ replaced by $m-1$. The integral equations associated with (4.5) are written as

$$\begin{cases} v_m(t) = \mathcal{G}(t) u_0 + \int_0^t \mathcal{G}(t-\tau) f(v_{m-1}(\tau), h_{m-1}(\tau))\, d\tau, \\ h_m(t) = 1 - \frac{\lambda\delta}{\theta}\int_0^t d\tau \int_{\mathbf{R}} |v_{m-1}|^{\delta} v_{m-1} dx. \end{cases}$$

We have

$$\begin{aligned} \|f(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^\infty} &\leq C\varepsilon^{1+\delta}\langle t\rangle^{-1-\frac{1}{\delta}} g^{-1}(t), \\ \|f(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^{1,a}} &\leq C\varepsilon^{1+\delta}\langle t\rangle^{-1+\frac{a}{\delta}} g^{-1}(t) \end{aligned} \tag{4.7}$$

for all $t > 0$, provided that $(v_{m-1}, h_{m-1})$ satisfies (4.6). This yields the estimate

$$\|\langle t\rangle g(t) f(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{X}} \leq C\varepsilon^{1+\delta}.$$

Now in order to prove estimates (4.6) we use the following lemma (see paper [15] for the proof.)

**Lemma 4.2.** *Let the function $f(t,x)$ have a zero mean value $\int_{\mathbf{R}} f(t,x)\,dx = 0$. Then the following inequality*

$$\left\| g^{k}(t) \int_{0}^{t} g^{-k}(\tau)\, \mathcal{G}(t-\tau)\, f(\tau)\, d\tau \right\|_{\mathbf{X}} \le C \left\| \langle t\rangle f \right\|_{\mathbf{X}}$$

*is valid for $k = 0,1,$ provided that the right-hand side is finite.*

Since $f(v_{m-1}(\tau), h_{m-1}(\tau))$ has a zero mean value we get via Lemma 4.2

$$\|v_m\|_{\mathbf{X}} \le C\varepsilon, \|v_m(t) - \mathcal{G}(t)\, u_0\|_{\mathbf{L}^1} \le C\varepsilon^{1+\delta} g^{-1}(t)\,. \tag{4.8}$$

To prove the third estimate in (4.6) we need the following lemma (see [15]).

**Lemma 4.3.** *Assume that $u_0 \in \mathbf{L}^{\infty} \cap \mathbf{L}^{1,a}$, the norm $\|u_0\|_{\mathbf{L}^{\infty}} + \|u_0\|_{\mathbf{L}^{1,a}} = \varepsilon$ is sufficiently small, and $|\theta| \ge C\varepsilon > 0$. Let a function $v$ satisfy the estimates*

$$\langle t\rangle^{\frac{1}{\delta}} \|v\|_{\mathbf{L}^{\infty}} + \|v\|_{\mathbf{L}^1} \le C\varepsilon,\ \ \|v(t) - \mathcal{G}(t)\, u_0\|_{\mathbf{L}^1} \le C\varepsilon^{1+\delta} g^{-1}(t)$$

*for all $t > 0$. Then the inequality*

$$\left| 1 - \frac{\lambda\delta}{\theta} \int_{0}^{t} d\tau \int_{\mathbf{R}} |v|^{\delta}\, v(\tau,x)\, dx - g(t) \right| \le C\varepsilon^{\delta} \left(1 + \log g(t)\right) \tag{4.9}$$

*is valid for all $t > 0$.*

By virtue of (4.8) and applying Lemma 4.3 we find that

$$|h_m(t) - g(t)| \le C\varepsilon^{\delta} \left(1 + \log g(t)\right)$$

for all $t > 0$. Thus by induction we see that estimates (4.6) are valid for all $m \ge 1$. In the same way by induction we can prove that

$$\begin{aligned} &\|v_m - v_{m-1}\|_{\mathbf{X}} + \sup_{t>0} g^{-1}(t)\, |h_m(t) - h_{m-1}(t)| \\ &\le \frac{1}{2} \Big( \|v_{m-1} - v_{m-2}\|_{\mathbf{X}} + \sup_{t>0} g^{-1}(t)\, |h_{m-1}(t) - h_{m-2}(t)| \Big), \end{aligned} \tag{4.10}$$

for all $m > 2$. Therefore we obtain a unique solution $(v,h) \in \mathbf{X} \times \mathbf{C}(0,\infty)$ satisfying the integral equation associated with (4.4) and estimates

$$\|v(t) - \mathcal{G}(t)\, u_0\|_{\mathbf{L}^1} \le C\varepsilon^{1+\delta} g^{-1}(t)\,, |h(t) - g(t)| \le C\varepsilon^{\delta} \left(1 + \log g(t)\right). \tag{4.11}$$

We also have by applying (4.7) to the integral equation associated with (4.4)

$$\|v(t) - \mathcal{G}(t)\, u_0\|_{\mathbf{L}^{\infty}} \le C\varepsilon^{1+\delta} \langle t\rangle^{-\frac{1}{\delta}} g^{-1}(t)\,. \tag{4.12}$$

Then via formulas $u = e^{-\varphi}v = h^{-\frac{1}{\delta}}v$ we find the estimates

$$\left\| u(t) - \theta t^{-\frac{1}{\delta}} G_0\left((\cdot)\, t^{-\frac{1}{\delta}}\right) e^{-\varphi(t)} \right\|_{\mathbf{L}^\infty} \le C\varepsilon^{1+\delta} \langle t\rangle^{-\frac{1}{\delta}} g^{-1-\frac{1}{\delta}}(t), \qquad (4.13)$$

where we used the estimate

$$\left\| \left(\mathcal{G}(t)\, u_0 - \theta t^{-\frac{1}{\delta}} G_0\left((\cdot)\, t^{-\frac{1}{\delta}}\right)\right) e^{-\varphi(t)} \right\|_{\mathbf{L}^\infty} \le Ct^{-\frac{1}{\delta}-\frac{a}{\delta}} \|u_0\|_{\mathbf{L}^{1,a}}$$

and (4.12). By virtue of (4.11) we have

$$|h^{-\frac{1}{\delta}}(t) - g^{-\frac{1}{\delta}}(t)| \le Cg^{-1-\frac{1}{\delta}}(t)\, |h(t) - g(t)|,$$

hence via (4.13) it follows that (4.2) holds. This completes the proof of Theorem 4.1.

4.2. **Large data.** In this section following the method of paper [17] we remove the smallness condition on the initial data $u_0$.

**Theorem 4.4.** *Let* $\delta \in (1,2]$, $\lambda < 0$, $\sigma = \delta$. *Suppose that condition* (4.1) *takes place. Assume that the initial data* $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^{1,a}$, $0 < a \le 1$, *are such that* $\theta \ne 0$. *Also let* (1.4) *take place with* $\alpha_2 = 0$; $\frac{\delta}{2} \ge \gamma > \frac{\delta}{\sigma} - \frac{1}{2}$. *Then the Cauchy problem* (1.1) *has a unique global solution* $u \in \mathbf{Y}$. *Moreover, the asymptotics* (4.2) *takes place.*

Before proving Theorem 4.4 we need a lemma, where we compare the solutions of the following two problems

$$\begin{cases} u_t + \mathcal{L}_0 u - \lambda\, |u|^\sigma u = f,\ x \in \mathbf{R},\ t > 0, \\ u(0,x) = u_0(x),\, x \in \mathbf{R} \end{cases} \qquad (4.14)$$

and

$$\begin{cases} v_t + \mathcal{L}_0 v - \epsilon\lambda v^{\sigma+1} = |f|,\ x \in \mathbf{R},\ t > 0, \\ v(0,x) = |u_0(x)|,\, x \in \mathbf{R}, \end{cases} \qquad (4.15)$$

where the operator $\mathcal{L}_0$ has a symbol $L_0(\xi) = \alpha_1 |\xi|^\delta$.

**Lemma 4.5.** *Let* $0 < \sigma \le \delta$, $\lambda < 0$. *Suppose that* $u_0(x) \in \mathbf{L}^\infty \cap \mathbf{C}(\mathbf{R})$, *and* $0 \le \epsilon \le 1$. *Then* $|u(t,x)| \le v(t,x)$ *for all* $t \ge 0$, $x \in \mathbf{R}$.

**Proof.** Define $r = v - u$. Then we obtain

$$\begin{cases} r_t + \mathcal{L}_0 r - |\lambda| \left(|u|^\sigma u - \epsilon v^{\sigma+1}\right) = |f| - f,\ x \in \mathbf{R},\ t > 0, \\ r(0,x) = |u_0(x)| - u_0(x),\, x \in \mathbf{R}. \end{cases} \qquad (4.16)$$

We need to prove that $r \ge 0$ for all $t \ge 0$, $x \in \mathbf{R}$. Define $R(t) \equiv \inf_{x\in\mathbf{R}} r(t,x)$. By the contrary, suppose that there exists a time $T > 0$ such that $R(T) < 0$. By the continuity we can find an interval $[T_1, T]$ such that $R(t) \le 0$ for all $t \in [T_1, T]$ and $R(T_1) = 0$. By Theorem 2.1 from paper

[3] there exists a point $\zeta(t) \in \mathbf{R}$ such that $R(t) = r(t, \zeta(t))$, moreover $R'(t) = \frac{d}{dt} r(t, \zeta(t))$ almost everywhere on $t \in [T_1, T]$. We have

$$|u|^{\sigma} u - \epsilon v^{\sigma+1} = (v - R)^{\sigma+1} - \epsilon v^{\sigma+1} \geq 0$$

for all $t \in [T_1, T]$. Representing the operator $\mathcal{L}_0$ in the point of minimum $\zeta(t)$ via the Riesz potential (see [26]) let us show that

$$\mathcal{L}_0 r(t, \zeta(t)) = C_\delta \int_{\mathbf{R}} |\zeta(t) - y|^{-\delta-1} (r(t,y) - R(t))\, dy \geq 0,$$

where $C_\delta > 0$ is a constant. By (3.8) we get with $\nu = 2 - \delta \in (0,1)$, integrating by parts with respect to $y \in \mathbf{R}$ two times,

$$\begin{aligned} |\partial_x|^{\delta} \phi &= \mathcal{F}^{-1} |\xi|^{\delta} \widehat{\phi}(\xi) = |\partial_x|^{-\nu} \partial_x^2 \phi \\ &= \frac{(1-\nu)(2-\nu)}{2\Gamma(\nu)\cos\left(\frac{\pi\nu}{2}\right)} \int_{\mathbf{R}} |x-y|^{\nu-3} (\phi(y) - \phi(x))\, dy \\ &= \frac{\delta(\delta-1)}{2\Gamma(2-\delta)\cos\left(\frac{\pi}{2}(2-\delta)\right)} \int_{\mathbf{R}} |x-y|^{-\delta-1} (\phi(y) - \phi(x))\, dy. \end{aligned}$$

In the above calculations the integrals are convergent since in the point of minimum $x = \xi(t)$ we have the estimate $\phi(y) - \phi(x) = O(|x-y|^2)$. Thus for the operator $\mathcal{L}_0$ at the point of minimum $\zeta(t)$ we have a representation

$$\mathcal{L}_0 r(t, \zeta(t)) = C_\delta \int_{\mathbf{R}} |\zeta(t) - y|^{-\delta-1} (r(t,y) - R(t))\, dy \geq 0,$$

where $C_\delta > 0$ is a constant. Therefore by equation (4.16) we get

$$R'(t) \geq 0$$

for all $t \in [T_1, T]$. Integration with respect to time yields $R(t) \geq 0$. This gives a contradiction, hence $u(t,x) \leq |v(t,x)|$ for all $x \in \mathbf{R}$ and $t > T_1$. In the same manner we prove that $v + u \geq 0$ for all $x \in \mathbf{R}$ and $t > T_1$. Lemma 4.5 is proved. □

**Proof of Theorem 4.4.** As in the proof of Theorem 3.2 by Proposition 3.6 we have a rough estimate $\|u(t)\|_{\mathbf{L}^\infty} \leq C \langle t \rangle^{-\frac{1}{\delta}}$ for all $t > 0$. We estimate the $\mathbf{L}^{1,a}$ norm of the solution

$$\|u(t)\|_{\mathbf{L}^{1,a}} \leq C \langle t \rangle^{\frac{a}{\delta}} + C \int_0^t \langle \tau \rangle^{-1} \|u(\tau)\|_{\mathbf{L}^{1,a}}\, d\tau.$$

Hence by the Gronwall inequality we obtain

$$\|u(t)\|_{\mathbf{L}^{1,a}} \leq C \langle t \rangle^{\frac{a}{\delta}} \tag{4.17}$$

for all $t > 0$. By Proposition 3.6 as in the proof of Lemma 3.3 we have the estimates with $\mathcal{B} = \mathcal{L} - \mathcal{L}_0 = \mathcal{L} - \alpha_1 |\partial_x|^\delta$

$$\|\mathcal{B}u(t)\|_{\mathbf{L}^\infty} \le C \langle t\rangle^{-1-\frac{1}{\delta}-\frac{1}{\delta}\left(\gamma-\frac{1}{2}\right)}, \|\mathcal{B}u(t)\|_{\mathbf{L}^{1,a}} \le C \langle t\rangle^{-1+\frac{a}{\delta}-\frac{1}{\delta}\left(\gamma-\frac{1}{2}\right)}$$

for all $t > 0$, where $\gamma > \frac{1}{2}$. Denote $f = -\mathcal{B}u$. We take sufficiently small $\varepsilon > 0$ and consider the following two auxiliary Cauchy problems

$$\begin{cases} U_t + \mathcal{L}_0 U - \lambda U^{\sigma+1} = \varepsilon |f|, \ x \in \mathbf{R}, \ t > 0, \\ U(0,x) = \varepsilon |u_0(x)|, \, x \in \mathbf{R} \end{cases} \tag{4.18}$$

and

$$\begin{cases} V_t + \mathcal{L}_0 V - \varepsilon^\sigma \lambda V^{\sigma+1} = |f|, \ x \in \mathbf{R}, \ t > 0, \\ V(0,x) = |u_0(x)|, \, x \in \mathbf{R}. \end{cases} \tag{4.19}$$

Note that problem (4.19) can be reduced to problem (4.18) by the change $V = \varepsilon^{-1}U$. And the problem (4.18) has a sufficiently small initial data and a small force $\varepsilon |f|$. Therefore we can apply the results of papers [13], [15] to calculate the large time asymptotic behavior of the functions $U$ and $V$. Then by Lemma 4.5 we get an optimal time decay estimate for the solution

$$\|u(t)\|_{\mathbf{L}^\infty} \le C\varepsilon^{-1} \langle t\rangle^{-\frac{1}{\delta}} (\log(2+t))^{-\frac{1}{\delta}} \tag{4.20}$$

for all $t > 0$. Now we make a change of the dependent variable $u = ve^{-\varphi}$. Then we have the Cauchy problem for the new dependent variables $(v, \varphi)$

$$\begin{cases} v_t + \mathcal{L}v = \left(\lambda |u|^\sigma v + fe^\varphi - \frac{1}{\theta}\int_{\mathbf{R}} (\lambda |u|^\sigma v + fe^\varphi)\, dx\right) v \\ \varphi'(t) = -\frac{1}{\theta}\int_{\mathbf{R}} (\lambda |u|^\sigma v + fe^\varphi)\, dx, v(0,x) = u_0(x), \ \varphi(0) = 0. \end{cases} \tag{4.21}$$

We now prove the estimate $\|v(t)\|_{\mathbf{L}^{1,a}} \le C \langle t\rangle^{\frac{a}{\delta}}$ for all $t > 0$. By estimate (4.20) we have since $\delta = \sigma$

$$\left| -\frac{1}{\theta}\int_{\mathbf{R}} (\lambda |u|^\sigma v + fe^\varphi)\, dx\right| \le C\varepsilon^{-\sigma} \langle t\rangle^{-1} (\log(2+t))^{-1},$$

hence $\varphi(t) \le C\varepsilon^{-\sigma} \log(\log(4+t))$. Therefore by (4.20) and (4.17) we get

$$\|v(t)\|_{\mathbf{L}^{1,a}} \le C(\varepsilon, T) \langle t\rangle^{\frac{a}{\delta}}, \|v(t)\|_{\mathbf{L}^p} \le C(\varepsilon, T) \langle t\rangle^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)}$$

for all $0 < t \le T$. Now we consider $t > T$. We apply estimates (4.20) and Lemma 2.1 to the integral equation of (4.21) to get

$$\begin{aligned} &\|v(t) - \mathcal{G}(t-T)\, v(T)\|_{\mathbf{L}^{1,a}} \\ &\le C\varepsilon^{-1} \int_T^t \left( \langle \tau\rangle^{-1} (\log(2+\tau))^{-1} \|v(\tau)\|_{\mathbf{L}^{1,a}} + \langle \tau\rangle^{-1+\frac{a}{\delta}} \right) d\tau \end{aligned}$$

for all $t > T$. Therefore, in view of (4.17) we obtain

$$\|v(t)\|_{\mathbf{L}^{1,a}} \leq C(\varepsilon, T)\langle t\rangle^{\frac{a}{\delta}} + \varepsilon \int_T^t \langle\tau\rangle^{-1} \|v(\tau)\|_{\mathbf{L}^{1,a}} d\tau$$

for all $t > T$. Here $\varepsilon > 0$ is small enough, and $T > 0$ is sufficiently large. Application of Gronwall's inequality yields the estimate $\|v(t)\|_{\mathbf{L}^{1,a}} \leq C\langle t\rangle^{\frac{a}{\delta}}$ for all $t > 0$. In the same manner by virtue of estimates (4.20) and Lemma 2.1 we get

$$\begin{aligned} &\|v(t)\|_{\mathbf{L}^p} \leq C(\varepsilon, T)\langle t\rangle^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)} \\ &+C\varepsilon^{-1}\int_T^{\frac{t}{2}} (t-\tau)^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)-\frac{a}{\delta}} \langle\tau\rangle^{\frac{a}{\delta}-1} (\log(2+\tau))^{-1} d\tau \\ &+C\varepsilon^{-1}\int_{\frac{t}{2}}^t \langle\tau\rangle^{-1} (\log(2+\tau))^{-1} \|v(\tau)\|_{\mathbf{L}^p} d\tau \\ \leq\ & C(\varepsilon, T)\langle t\rangle^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)} + \varepsilon\int_{\frac{t}{2}}^t \langle\tau\rangle^{-1}\|v(\tau)\|_{\mathbf{L}^p} d\tau \end{aligned}$$

for all $t > T$. Then the Gronwall inequality yields the estimate $\|v(t)\|_{\mathbf{L}^p} \leq C\langle t\rangle^{-\frac{1}{\delta}(1-\frac{1}{p})}$ for all $t > 0$. Using the integral equation of (4.21) and Lemma 2.1 we obtain

$$\|v(t) - \mathcal{G}(t) u_0\|_{\mathbf{L}^1} \leq C \log(2+t)^{-1} \tag{4.22}$$

for all $t > 0$. Now we apply Lemma 4.3 to find $|h(t) - g(t)| \leq C \log g(t)$ for all $t > 0$. In the same way as in the end of the proof of Theorem 4.1 we have the asymptotics of solutions (4.2).

## 5. Subcritical case

This section is devoted to the study of the Cauchy problem (1.1) in the subcritical case $\sigma < \delta$. Everywhere below we suppose that $\lambda < 0$.

5.1. **Small data.** We prove global in time existence of small solutions to the Cauchy problem (1.1) in the subcritical case $0 < \sigma < \delta$.

**Theorem 5.1.** *Let $0 < \sigma < \delta$, $\lambda < 0$. Suppose that condition (4.1) takes place. We assume that the initial data $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^{1,a}$, $a \in (0,1)$ are sufficiently small, $\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^{1,a}} \leq \varepsilon$, and $|\theta| \geq C\varepsilon > 0$. Also we suppose that the value $\sigma$ is close to $\delta$, so that $\delta - \sigma \leq C\varepsilon^\sigma$. Then the Cauchy* *problem* (1.1) *has a unique global solution* $u \in \mathbf{Y}$ *satisfying the following time decay estimates*

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{1}{\sigma}} \tag{5.1}$$

*for large* $t > 0$. *Furthermore there exist a number* $A$ *and a function* $V \in \mathbf{L}^\infty \cap \mathbf{L}^{1,a}$ *such that the asymptotic formula*

$$u(t,x) = At^{-\frac{1}{\sigma}}V(xt^{-\frac{1}{\delta}}) + O(t^{-\frac{1}{\sigma}-\gamma}) \tag{5.2}$$

*is valid for* $t \to \infty$ *uniformly with respect to* $x \in \mathbf{R}$, *where* $\gamma = \frac{1}{\delta}\min(a, 1-\frac{\sigma}{\delta})$, *and* $V(\xi)$ *is the solution of the integral equation*

$$\begin{aligned} V(\xi) &= \mathcal{R}(V(\xi)) \\ &\equiv G_0(\xi) - \frac{1}{\Omega}\int_0^1 \frac{dz}{z(1-z)^{\frac{1}{\delta}}}\int_{\mathbf{R}} G_0\Big(\frac{\xi - yz^{\frac{1}{\delta}}}{(1-z)^{\frac{1}{\delta}}}\Big)F(y)\,dy, \end{aligned} \tag{5.3}$$

*with*

$$\begin{aligned} \Omega &= \frac{\sigma}{1-\frac{\sigma}{\delta}}\int_{\mathbf{R}} |V(y)|^\sigma V(y)\,dy, \\ F(y) &= |V(y)|^\sigma V(y) - V(y)\int_{\mathbf{R}} |V(\xi)|^\sigma V(\xi)\,d\xi. \end{aligned}$$

**Remark 5.2.** The condition that the value $\sigma$ should be close to $\delta$, so that $\delta - \sigma \leq C\varepsilon^\sigma$, is rather technical and is caused by the application of the contraction mapping principle for proving global existence of solutions.

**Proof of Theorem 5.1.** As in the proof of Theorem 4.1 we make a change of the dependent variable $u = ve^{-\varphi}$. Thus we have the Cauchy problem (4.3) and (4.4) for the new dependent variables $(v,\varphi)$. We note that the mean value of the nonlinearity $\int_{\mathbf{R}} f(v,h)(t,x)\,dx = 0$ for all $t > 0$. We now prove the existence of the solution $(v,h)$ for the Cauchy problem (4.4) by the successive approximations $(v_m, h_m)$, $m = 1,2,...,$ defined as in (4.5) with $v_1 = \mathcal{G}(t)u_0$ and $h_1(t) = 1 + \frac{\delta|\theta|^\sigma\eta}{\delta-\sigma}t^{1-\frac{\sigma}{\delta}}$, where $\eta$ is the same one defined in (4.1). In the same way as in the proof of (4.6) we have

$$\|v_m\|_{\mathbf{X}} \leq C\varepsilon, \quad \|v_m(t) - \mathcal{G}(t)u_0\|_{\mathbf{L}^1} \leq C\varepsilon^{1+\sigma}, \quad |h_m(t) - h_1(t)| \leq C\varepsilon^\sigma h_1(t) \tag{5.4}$$

for all $m \geq 1$ if we suppose that $\delta - \sigma \leq C\varepsilon^\sigma$. To prove the third estimate in (5.4) we need the following lemma (see [15] for the proof.)

**Lemma 5.3.** *Assume that* $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^{1,a}$*, the norm* $\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^{1,a}} = \varepsilon$ *is sufficiently small and* $|\theta| \geq C\varepsilon > 0$. *Let a function* $v(t,x)$ *satisfy the estimates*

$$\|v\|_{\mathbf{L}^\infty} \leq C\varepsilon \langle t\rangle^{-\frac{1}{\delta}}, \quad \|v\|_{\mathbf{L}^1} \leq C\varepsilon, \ \|v(t) - \mathcal{G}(t) u_0\|_{\mathbf{L}^1} \leq C\varepsilon^{1+\sigma}$$

*for all* $t > 0$. *Then the inequality*

$$\left|1 - \frac{\sigma\lambda}{\theta}\int_0^t d\tau \int_{\mathbf{R}} |v|^\sigma v(\tau,x)dx - h_1(t)\right| \leq C\varepsilon^\sigma h_1(t) + C\varepsilon^\sigma t^{1-\frac{\sigma}{\delta}-\frac{a}{\delta}} \qquad (5.5)$$

*is valid for all* $t > 0$.

In the same way by induction we can prove that $(v_m, h_m) \in \mathbf{X} \times \mathbf{C}(0,\infty)$ is a Cauchy sequence. Therefore taking limits, we obtain a unique solution $v \in \mathbf{X}$, $h = e^{\sigma\varphi} \in \mathbf{C}(0,\infty)$ satisfying the integral equations (4.4) and estimates

$$\|v(t) - \mathcal{G}(t) u_0\|_{\mathbf{L}^1} \leq C\varepsilon^{1+\sigma}, |h(t) - h_1(t)| \leq C\varepsilon^\sigma h_1(t). \qquad (5.6)$$

We also have by (4.4)

$$\|v(t) - \mathcal{G}(t) u_0\|_{\mathbf{L}^\infty} \leq C\varepsilon^{1+\sigma} \langle t\rangle^{-\frac{1}{\delta}}, \qquad (5.7)$$

for large $t > 0$. Thus estimate (5.1) is obtained in the same way as in the last part of the proof of Theorem 4.1 through (5.6) and (5.7).

We now compute the asymptotics of the solution. First we show the existence of solutions to the integral equation (5.3). To prove the existence of the self-similar solutions for equation (5.3) we apply the following lemma (see paper [14] for the proof).

**Lemma 5.4.** *Let the function* $F(x)$ *have the zero mean value* $\int_{\mathbf{R}} F(x)\,dx = 0$, *for some* $0 < a \leq 1$, $1 \leq p \leq \infty$. *Then the following inequalities are valid*

$$\Big\|\int_0^1 \frac{dz}{z(1-z)^{\frac{1}{\delta}}}\int_{\mathbf{R}} G_0\Big(\frac{(\cdot) - yz^{\frac{1}{\delta}}}{(1-z)^{\frac{1}{\delta}}}\Big)F(y)dy\Big\|_{\mathbf{L}^{p,a}} \leq C(\|F\|_{\mathbf{L}^{1,a}} + \|F\|_{\mathbf{L}^{p,a}})$$

*and*

$$\begin{aligned} &\Big\|\int_0^1 \frac{dz}{(1-z)^{\frac{1}{\delta}}}\Big(\frac{1}{z} - \frac{t}{\langle tz\rangle}\Big)\int_{\mathbf{R}} G_0\Big(\frac{(\cdot) - yz^{\frac{1}{\delta}}}{(1-z)^{\frac{1}{\delta}}}\Big)F(y)dy\Big\|_{\mathbf{L}^{p,a}} \\ \leq\ & C\langle t\rangle^{-\frac{a}{\delta}}(\|F\|_{\mathbf{L}^{1,a}} + \|F\|_{\mathbf{L}^{p,a}}) \end{aligned}$$

*for all* $t > 0$.

We define successive approximations $V_{k+1} = \mathcal{R}(V_k)$ for $k = 0, 1, 2, ...,$ where $V_0(\xi) = G_0(\xi)$. By induction via Lemma 5.4 we can prove the estimates

$$\begin{aligned} \sup_{1\le p\le\infty} \|V_{k+1} - V_0\|_{\mathbf{L}^{p,a}} &\le C\varepsilon, \ \sup_{1\le p\le\infty} \|V_k\|_{\mathbf{L}^{p,a}} \le C, \ \Omega_k \ge C\varepsilon^{-1}, \\ \sup_{1\le p\le\infty} \|V_{k+1} - V_k\|_{\mathbf{L}^p} &\le \frac{1}{2} \sup_{1\le p\le\infty} \|V_k - V_{k-1}\|_{\mathbf{L}^p} \end{aligned}$$

for all $k \ge 1$ if $\sigma$ is close to $\delta$. Hence $\mathcal{R}$ is a contraction mapping and there exists a unique solution $V(\xi)$ to integral equation (5.3). We now prove asymptotics of solutions $v$. We prove the estimate

$$\left\| v(t) - t^{-\frac{1}{\delta}} \theta V\left((\cdot) t^{-\frac{1}{\delta}}\right) \right\|_{\mathbf{L}^{p,b}} < C \langle t \rangle^{\frac{b}{2}} t^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)-\gamma} \tag{5.8}$$

for all $t > 0$, $b \in [0, a]$, $1 \le p \le \infty$, where $\gamma = \frac{1}{\delta}\min\left(a, 1 - \frac{\sigma}{\delta}\right)$. Assuming the contrary we suppose that estimate (5.8) is violated for some time $t = T_1$. By the continuity in time we have

$$\left\| v(t) - t^{-\frac{1}{\delta}} \theta V\left((\cdot) t^{-\frac{1}{\delta}}\right) \right\|_{\mathbf{L}^{p,b}} \le C \langle t \rangle^{\frac{b}{2}} t^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)-\gamma} \tag{5.9}$$

for all $t \in (0, T_1]$. By Lemma 2.1 we get

$$\left\| \mathcal{G}(t) u_0 - t^{-\frac{1}{\delta}} \theta G_0\left((\cdot) t^{-\frac{1}{\delta}}\right) \right\|_{\mathbf{L}^{p,b}} \le C \langle t \rangle^{\frac{b}{2}} t^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)} \langle t \rangle^{-\frac{a}{\delta}}. \tag{5.10}$$

Then from (5.9) it follows that

$$\begin{aligned} &\left| h(t) - |\theta|^{\sigma} |\lambda| \Omega t^{1-\frac{\sigma}{\delta}} \right| \\ &\le 1 + \frac{C}{|\theta|} \int_0^t \left( \|v\|_{\mathbf{L}^\infty} + |\theta| \tau^{-\frac{1}{\delta}} \|V\|_{\mathbf{L}^\infty} \right)^{\sigma} \left\| v(\tau, \cdot) - \theta \tau^{-\frac{1}{\delta}} V\left((\cdot)\tau^{-\frac{1}{\delta}}\right) \right\|_{\mathbf{L}^1} d\tau \\ &\le 1 + \frac{C}{|\theta|} \int_0^t \tau^{-\frac{\sigma}{\delta}-\gamma} d\tau \le 1 + C\Omega t^{1-\frac{\sigma}{\delta}-\gamma} \end{aligned} \tag{5.11}$$

for all $t \in (0, T_1]$, $1 \le p \le \infty$. Changing the variables $\tau = zt$ and $\xi \tau^{-\frac{1}{\delta}} = y$ we have

$$\frac{1}{\Omega} \int_0^t \tau^{\frac{\sigma}{\delta}-1} \mathcal{G}_0(t-\tau) \tau^{-\frac{\sigma}{\delta}-\frac{1}{\delta}} F((\cdot)\tau^{-\frac{1}{\delta}}) d\tau = t^{-\frac{1}{\delta}} (V_0(xt^{-\frac{1}{\delta}}) - V(xt^{-\frac{1}{\delta}})),$$

therefore, denoting $f(\tau) = f(v, 1)$, where $f(v, h)$ is defined in (4.3), we get

$$\langle t \rangle^{-\frac{b}{2}} \left\| \theta t^{-\frac{1}{\delta}} V((\cdot) t^{-\frac{1}{\delta}}) - v(t) \right\|_{\mathbf{L}^{p,b}}$$

$$
\begin{aligned}
&\leq C\langle t\rangle^{-\frac{b}{2}}\left\|\theta t^{-\frac{1}{\delta}}V_0((\cdot)t^{-\frac{1}{\delta}})-\mathcal{G}(t)u_0\right\|_{\mathbf{L}^{p,b}}\\
&+C\langle t\rangle^{-\frac{b}{2}}\left\|\int_0^t\left(h^{-1}(\tau)-\frac{1}{\Omega|\theta|^{\sigma}}\tau^{\frac{\sigma}{\delta}}\langle\tau\rangle^{-1}\right)\mathcal{G}(t-\tau)f(\tau)d\tau\right\|_{\mathbf{L}^{p,b}}\\
&+\frac{C\langle t\rangle^{-\frac{b}{2}}}{\Omega|\theta|^{\sigma}}\left\|\int_0^t\left(\mathcal{G}\left(t-\tau\right)-\mathcal{G}_0\left(t-\tau\right)\right)f\left(\tau\right)\frac{\tau^{\frac{\sigma}{\delta}}d\tau}{\langle\tau\rangle}\right\|_{\mathbf{L}^{p,b}}\\
&+\frac{C\langle t\rangle^{-\frac{b}{2}}}{\Omega|\theta|^{\sigma}}\left\|\int_0^t\left(\mathcal{G}_0\left(t-\tau\right)\left(f\left(\tau\right)-\frac{|\theta|^{\sigma}\theta\lambda}{\tau^{\frac{1}{\delta}(\sigma+1)}}F\left((\cdot)\tau^{-\frac{1}{\delta}}\right)\right)\right)\frac{\tau^{\frac{\sigma}{\delta}}d\tau}{\langle\tau\rangle}\right\|_{\mathbf{L}^{p,b}}\\
&+\frac{C|\theta\lambda|\langle t\rangle^{-\frac{b}{2}}}{\Omega}\left\|\int_0^t\mathcal{G}_0\left(t-\tau\right)F\left((\cdot)\tau^{-\frac{1}{\delta}}\right)\left(\frac{1}{\langle\tau\rangle}-\frac{1}{\tau}\right)\tau^{-\frac{1}{\delta}}d\tau\right\|_{\mathbf{L}^{p,b}}\\
&\equiv I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

From (5.10) we have $I_1\leq Ct^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)}\langle t\rangle^{-\frac{a}{\delta}}$. By (5.11) and Lemma 5.4 we obtain

$$
I_2\leq C\langle t\rangle^{-\frac{b}{2}}\left\|\int_0^t\langle\tau\rangle^{-\gamma}h^{-1}\left(\tau\right)\mathcal{G}\left(t-\tau\right)f\left(\tau\right)d\tau\right\|_{\mathbf{L}^{p,b}}\leq Ct^{-\frac{1}{\delta}(1-\frac{1}{p})-\gamma}
$$

for all $t\in(0,T_1]$, $1\leq p\leq\infty$. Via Lemma 2.1 we find

$$
\begin{aligned}
I_3 &\leq C\langle t\rangle^{-\frac{b}{2}}\left\|\int_0^t\left(\mathcal{G}\left(t-\tau\right)-\mathcal{G}_0\left(t-\tau\right)\right)f\left(\tau\right)\frac{\tau^{\frac{\sigma}{\delta}}d\tau}{\langle\tau\rangle}\right\|_{\mathbf{L}^{p,b}}\\
&\leq C\langle t\rangle^{-\frac{b}{2}}\int_0^{\frac{t}{2}}\tau^{\frac{\sigma}{\delta}}\langle\tau\rangle^{-1}\langle t-\tau\rangle^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)-\gamma}\left(\langle\tau\rangle^{\frac{b}{2}}+\langle t-\tau\rangle^{\frac{b}{2}}\right)d\tau\\
&\quad+C\langle t\rangle^{-\frac{b}{2}}\int_{\frac{t}{2}}^t\tau^{\frac{\sigma}{\delta}}\langle\tau\rangle^{-1-\frac{1}{\delta}\left(1-\frac{1}{p}\right)}\langle t-\tau\rangle^{-\gamma}\left(\langle\tau\rangle^{\frac{b}{2}}+\langle t-\tau\rangle^{\frac{b}{2}}\right)d\tau\\
&\leq Ct^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)-\gamma}
\end{aligned}
$$

for all $t\in(0,T_1]$, $1\leq p\leq\infty$. In the same manner we get

$$
\begin{aligned}
I_4 &= \frac{C\langle t\rangle^{-\frac{b}{2}}}{\Omega|\theta|^{\sigma}}\left\|\int_0^t\mathcal{G}_0\left(t-\tau\right)\left(f\left(\tau\right)-\tau^{-\frac{\sigma}{\delta}-\frac{1}{\delta}}|\theta|^{\sigma}\theta\lambda F\left((\cdot)\tau^{-\frac{1}{\delta}}\right)\right)\frac{\tau^{\frac{\sigma}{\delta}}d\tau}{\langle\tau\rangle}\right\|_{\mathbf{L}^{p,b}}\\
&\leq \frac{C}{\Omega|\theta|^{\sigma}}t^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)-\gamma}\left(\sup_{t>0}\left(t^{\frac{1}{\delta}}\|v(t)\|_{\mathbf{L}^\infty}+\theta\|V\|_{\mathbf{L}^\infty}\right)\right)^{\sigma}\\
&\times\sup_{t>0}\sup_{1\leq p\leq\infty}t^{\frac{1}{\delta}\left(1-\frac{1}{p}\right)+\gamma}\langle t\rangle^{-\frac{b}{2}}\left\|v\left(t\right)-t^{-\frac{1}{\delta}}\theta V((\cdot)t^{-\frac{1}{\delta}})\right\|_{\mathbf{L}^{p,b}}\leq Ct^{-\frac{1}{\delta}(1-\frac{1}{p})-\gamma}
\end{aligned}
$$

for all $t \in (0, T_1]$, $1 \le p \le \infty$. Finally, changing independent variables $\tau = zt$ and $\xi \tau^{-\frac{1}{\delta}} = y$ and applying Lemma 5.4 we obtain

$$I_5 = \frac{C\,|\theta\lambda|}{\Omega\,\langle t\rangle^{\frac{b}{2}}} \Big\| \int_0^t \mathcal{G}_0\,(t-\tau)\,F((\cdot t)\tau^{-\frac{1}{\delta}}) \Big(\frac{1}{\langle\tau\rangle} - \frac{1}{\tau}\Big)\tau^{-\frac{1}{\delta}} d\tau \Big\|_{\mathbf{L}^{p,b}}$$

$$= \; Ct^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)} \Big\| \int_0^1 \frac{dz}{(1-z)^{\frac{1}{\delta}}} \Big(\frac{1}{z} - \frac{t}{\langle tz\rangle}\Big) \int_{\mathbf{R}} G_0\Big(\frac{(\cdot) - yz^{\frac{1}{\delta}}}{(1-z)^{\frac{1}{\delta}}}\Big) F(y)dy \Big\|_{\mathbf{L}^{p,b}}$$

$$\le \; Ct^{-\frac{1}{\delta}\left(1-\frac{1}{p}\right)-\gamma}$$

for all $t \in (0, T_1]$, $1 \le p \le \infty$. Hence, (5.8) is true for all $t > 0$. Taking $b = 0$ in (5.8) we get

$$\Big\| v(t) - t^{-\frac{1}{\delta}} \theta V((\cdot)t^{-\frac{1}{\delta}}) \Big\|_{\mathbf{L}^p} \le Ct^{-\frac{1}{\delta}(1-\frac{1}{p})-\gamma}. \tag{5.12}$$

Hence, by virtue of (5.11) and (5.12) we have the asymptotics

$$v\,(t) = t^{-\frac{1}{\delta}} \theta V((\cdot)t^{-\frac{1}{\delta}}) + O(t^{-\frac{1}{\delta}-\gamma}),\ h(t) = |\theta|^{\sigma} \Omega t^{1-\frac{\sigma}{\delta}} (1 + O(t^{-\gamma})) \tag{5.13}$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}$. Therefore via the formula $u = e^{-\varphi} v$ taking into account estimates (5.13) we obtain the asymptotics (5.2) for the solution $u$ with a constant $A = \Omega^{-\frac{1}{\sigma}}$. This completes the proof of Theorem 5.1.

5.2. **Large data.** In this section we remove the smallness condition of the initial data and prove global in time existence of solutions to the Cauchy problem (1.1) with subcritical $\sigma \in (0, \delta)$ powers of the nonlinearity. However we have to suppose that $\sigma \in (0, \delta)$ is sufficiently close to $\delta$.

**Theorem 5.5.** *Let* $\delta \in (1, 2]$, $\lambda < 0$, $\sigma < \delta$. *Assume that condition* (4.1) *takes place. Suppose that the initial data* $u_0 \in \mathbf{L}^{\infty} \cap \mathbf{L}^{1,a}$, $0 < a \le 1$, *are such that* $\theta \neq 0$. *Suppose that* $\delta - \varepsilon < \sigma < \delta$, *where* $\varepsilon > 0$ *is sufficiently small. Also let* (1.4) *take place with* $\alpha_2 = 0$; $\frac{\delta}{2} \ge \gamma > \frac{\delta}{\sigma} - \frac{1}{2}$. *Then there exists a unique global solution* $u \in \mathbf{Y}$ *of the Cauchy problem* (1.1), *satisfying time decay estimate* (5.1) *and asymptotic formulas* (5.2).

**Proof of Theorem 5.5.** As in the proof of Theorem 4.4, by Lemma 2.1 and Proposition 3.6 we have $\|u\,(t)\|_{\mathbf{L}^{\infty}} \le C\,\langle t\rangle^{-\frac{1}{\delta}}$ for all $t > 0$. By Proposition 3.6 as in the proof of Lemma 3.3 we have the estimates $\|\mathcal{B}u\,(t)\|_{\mathbf{L}^{\infty}} \le C\,\langle t\rangle^{-1-\frac{1}{\delta}-\frac{1}{\delta}\left(\gamma-\frac{1}{2}\right)}$ for all $t > 0$, where $\gamma > \frac{1}{2}$, and $\sigma$ is close to $\delta$. Denote $f = -\mathcal{B}u$. As in the proof of Theorem 4.4 we consider two auxiliary Cauchy problems (4.18) and (4.19) with sufficiently small $\varepsilon > 0$. We see that problem

(4.18) has a sufficiently small initial data and a small force $\varepsilon |f|$. Applying the maximum principle to problem (4.18) we obtain $\|U(t)\|_{\mathbf{L}^\infty} \leq C \langle t\rangle^{-\frac{1}{\sigma}}$, hence by Lemma 4.5 we get an optimal time decay estimate $\|u(t)\|_{\mathbf{L}^\infty} \leq C \langle t\rangle^{-\frac{1}{\sigma}}$ and

$$\|u(t)\|_{\mathbf{L}^{1,a}} \leq C \langle t\rangle^{\frac{a}{\delta}} + C \int_0^t \langle \tau\rangle^{-1} \|u(\tau)\|_{\mathbf{L}^{1,a}} \, d\tau.$$

By the Gronwall inequality we obtain the estimate

$$\|u(t)\|_{\mathbf{L}^{1,a}} \leq C \langle t\rangle^{\frac{a}{\delta}} \tag{5.14}$$

for all $t > 0$. By Lemma 2.1 we have

$$\|f(t)\|_{\mathbf{L}^{1,a}} = \|\mathcal{B}u(t)\|_{\mathbf{L}^{1,a}} \leq C \langle t\rangle^{\frac{a}{\delta}-1-\frac{\gamma}{\delta}},$$

since $\sigma \in (1, \delta)$ is sufficiently close to $\delta$. Thus we can apply the results of papers [13], [15] to calculate the large time asymptotic behavior of solutions $U$ and $V$ to problems (4.18) and (4.19) respectively. Then by Lemma 4.5 we get an optimal time decay estimate for the solution

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C \langle t\rangle^{-\frac{1}{\delta}} \left(1 + C\varepsilon (\delta - \sigma)^{-\frac{1}{\sigma}} t^{\frac{1}{\sigma}-\frac{1}{\delta}}\right)^{-1} \tag{5.15}$$

for all $t > 0$, with a large coefficient $(\delta - \sigma)^{-\frac{1}{\sigma}}$. As in the proof of Theorem 4.4, we consider the Cauchy problem (4.21) for the new dependent variables $(v, \varphi)$. By estimate (5.15) we have

$$\left| -\frac{1}{\theta} \int_{\mathbf{R}} \left(\lambda |u|^\sigma v + f e^{\varphi}\right) dx \right| \leq C \langle t\rangle^{-\sigma\frac{1}{\delta}},$$

hence $\varphi(t) \leq C \langle t\rangle^{1-\frac{\sigma}{\delta}}$. Thus by virtue of (5.15) and (5.14) we get

$$\|v(t)\|_{\mathbf{L}^{1,a}} \leq C(T) \langle t\rangle^{\frac{a}{\delta}}, \|v(t)\|_{\mathbf{L}^\infty} \leq C(T) \langle t\rangle^{-\frac{1}{\delta}}$$

for all $0 < t \leq T$. Now we consider $t > T$. We use the integral equation of (4.21). By virtue of estimates (5.15), (5.14) and Lemma 2.1 we get

$$\|v(t)\|_{\mathbf{L}^{1,a}} \leq C(T) \langle t\rangle^{\frac{a}{\delta}} + \epsilon \int_T^t \langle \tau\rangle^{-1} \|v(\tau)\|_{\mathbf{L}^{1,a}} \, d\tau$$

for all $t > T$. Here $\epsilon > 0$ is small enough, and $T > 0$ is sufficiently large (recall that $\sigma < \delta$ is close to $\delta$). Application of the Gronwall lemma yields the estimate

$$\|v(t)\|_{\mathbf{L}^{1,a}} \leq C \langle t\rangle^{\frac{a}{\delta}} \tag{5.16}$$

for all $t > 0$. In the same manner by virtue of estimates (5.15) we get $\|v(t)\|_{\mathbf{L}^\infty} \leq C\langle t\rangle^{-\frac{1}{\delta}}$ for all $t > 0$. By the second equation of system (4.21) and (5.16), using Lemma 5.3 we see that

$$|h(t) - h_1(t)| \leq \epsilon h_1(t),$$

for all $t > 0$. Now the asymptotic formulas are proved in the same manner as at the end of the proof of Theorem 5.1. Theorem 5.5 is proved.

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