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HALF-LINEAR DIFFERENTIAL EQUATIONS WITH OSCILLATING COEFFICIENT

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Abstract. We study asymptotic properties of solutions of the nonoscillatory half-linear differential equation

$$(a(t)\Phi(x'))' + b(t)\Phi(x) = 0$$

where the functions a, b are continuous for $t \geq 0$, $a(t) > 0$ and $\Phi(u) = |u|^{p-2}u$, $p > 1$. In particular, the existence and uniqueness of the zero-convergent solutions and the limit characterization of principal solutions are proved when the function b changes sign. An integral characterization of the principal solutions, the boundedness of all solutions, and applications to the Riccati equation are considered as well.

1. Introduction

Consider the half-linear equation

$$(a(t)\Phi(x'))' + b(t)\Phi(x) = 0 \tag{1.1}$$

where the functions a, b are continuous for $t \geq 0$, $a(t) > 0$ and $\Phi(u) = |u|^{p-2}u$, $p > 1$.

A prototype of (1.1) is the linear second-order differential equation

$$(a(t)x')' + b(t)x = 0. \tag{1.2}$$

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Equation (1.1) has been extensively studied in the last few years, especially in virtue of the striking similarity between the qualitative behavior of (1.1) and (1.2): see, e.g., the recent books [1, 6, 14] and references therein. In particular, concerning the asymptotic theory, a considerable effort has been devoted to the study of principal solutions of (1.1); see e.g. [3, 4, 5, 7, 8, 9, 12]. In the linear case this concept was introduced in 1936 by W. Leighton and M. Morse and later on analyzed by P. Hartman and A. Wintner; see [11, Chapter 11]. If (1.2) is nonoscillatory, then there exists a solution u of (1.2), called a principal solution, which is uniquely determined up to a constant factor, such that

$$\lim_{t\to\infty} \frac{u(t)}{x(t)} = 0, \tag{1.3}$$

where x denotes an arbitrary solution of (1.2), linearly independent of u. Principal solutions of (1.2) can be equivalently characterized by the following integral condition:

$$\int^{\infty} \frac{dt}{a(t)u^2(t)} = \infty. \tag{1.4}$$

Property (1.3) is the simplest and most typical property characterizing principal solutions, because, roughly speaking, it means that in the linear case the principal solution is the "smallest one" in a neighborhood of infinity. Property (1.3) can be easily proved by the Wronskian identity. However, in the half-linear case the Wronskian identity is missing (see [7]) and this fact has been the main difficulty in extending such a property to (1.1).

It is well known (see, e.g., [11, Chapter 11]) that in the linear case properties (1.3) and (1.4) are equivalent to the property

$$\frac{u'(t)}{u(t)} < \frac{x'(t)}{x(t)} \quad \text{for large } t, \tag{1.5}$$

where x denotes any nontrivial solution of (1.2), $x \neq \lambda u$, $\lambda \in R$. Following property (1.5), independently J.D. Mirzov and A.Elbert-T.Kusano have extended the notion of principal solution to the half-linear equation. More precisely, in [9, 13] it is proved that there exists a nontrivial solution u of (1.1) such that for every nontrivial solution x of (1.1) with $x \neq \lambda u$, $\lambda \in R$, inequality (1.5) is satisfied.

Such a solution u is said to be *a principal solution* of (1.1) and nontrivial solutions of (1.1), which are not principal, are called *nonprincipal solutions.* In addition for any $\mu \neq 0$ there exists a unique principal solution u such that $u(0) = \mu$; i.e., principal solutions are determined uniquely up to a constant factor ([9, 13]).

The question concerning the equivalence between this definition and the limit property (1.3) has been posed in [5] and later solved by the authors in [3] and [4] for the case $b(t) < 0$ and $b(t) > 0$, respectively. In addition in both cases, two extensions of the integral characterization (1.4) have been also presented. Our approach was based on some monotone and asymptotic properties of solutions of (1.1) and on the so-called reciprocity principle. However, this approach cannot be applied when b changes sign.

The aim of this paper is to study asymptotic properties of solutions of (1.1), in particular zero-convergent solutions, and to prove the limit characterization of principal solutions (1.3), when the function b changes sign. An integral characterization of these solutions, the boundedness and some applications will be considered as well.

In the paper we assume

$$\int_0^\infty |b(t)|\,\Phi\Big(\int_t^\infty \frac{1}{\Phi^*(a(s))} ds\Big) dt < \infty, \tag{1.6}$$

where Φ^* is the inverse of the map Φ; i.e. $\Phi^*(u) = |u|^{p^*-2}u$, $p^* = p/(p-1)$. Clearly, (1.6) yields

$$\int_0^\infty \frac{1}{\Phi^*(a(s))}\, ds < \infty, \tag{1.7}$$

and the function

$$A(t) = \int_t^\infty \frac{1}{\Phi^*(a(s))} ds \tag{1.8}$$

will be used in our later consideration. In the last section, the stronger assumption than (1.6), namely

$$\int_0^\infty \frac{1}{\Phi^*(a(t))}\, dt < \infty, \quad \int_0^\infty |b(t)| dt < \infty \tag{1.9}$$

will be also considered.

2. Convergence to Zero

We recall that every nontrivial solution x is defined on the whole interval $[0, \infty)$ and $x(t) \not\equiv 0$ for large t. In addition, the *homogeneity property* holds; that is, if x is a solution of (1.1) then so is λx for any constant λ (see, e.g., [6]). We start with the following uniqueness result.

Theorem 1. *Assume* (1.6). *Then* (1.1) *is nonoscillatory and there exists a unique eventually positive solution x of* (1.1) *such that*

$$\lim_{t\to\infty} x(t) = 0, \quad \lim_{t\to\infty} x^{[1]}(t) = -1. \tag{2.1}$$

To prove Theorem 1 we use the following local version of the Banach contraction theorem.

Theorem A. *Let $\mathcal{X}$ be a complete metric space with distance δ and let Ω be a closed subset of $\mathcal{X}$. Let T be an operator such that*

(i) $T : \Omega \longrightarrow \Omega$;

(ii) *T is a contraction in Ω; i.e., there exists $k < 1$ such that for any $u, v \in \Omega$ we have*

$$\delta(T(u), T(v)) \le k\, \delta(u, v).$$

Then T has a unique fixed point in Ω.

Proof of Theorem 1. Choose $t_0 \ge 0$ such that

$$\int_{t_0}^{\infty} |b(t)|\Phi(A(t))dt \le H, \tag{2.2}$$

where $A(t)$ is defined by (1.8) and H is a suitable positive constant such that

$$H \le 1/4. \tag{2.3}$$

Set

$$\Omega = \big\{u \in C[t_0, \infty) : \ \Phi^*(1/2)A(t) \le u(t) \le \Phi^*(2)A(t)\big\}$$

and consider the operator T given by

$$T(u)(t) = \int_t^{\infty} \frac{1}{\Phi^*(a(s))} \Phi^*\Big(1 - \int_s^{\infty} b(r)\Phi(u(r))dr\Big)ds.$$

In view of (2.2) and (2.3), for any $w \in \Omega$ and $s \ge t_0$ we have

$$\frac{1}{2} \le 1 - 2H \le 1 - \int_s^{\infty} b(r)\Phi(w(r))dr \le 1 + 2H < 2. \tag{2.4}$$

Hence,

$$\Phi^*(1 - 2H)A(t) \le T(u)(t) \le \Phi^*(1 + 2H)A(t)$$

and $T(\Omega) \subset \Omega$. Let $\mathcal{X}$ be the metric space of continuous functions f on $[t_0, \infty)$ such that $\sup_{t\in[t_0,\infty)} |f(t)|/A(t) < \infty$, with distance d given by

$$d(f, g) = \sup_{t\in[t_0,\infty)} \frac{|f(t) - g(t)|}{A(t)}. \tag{2.5}$$

Clearly d is a metric on $\mathcal{X}$ and $\mathcal{X}$ is complete with this distance. Now we prove that T is a contraction in Ω with respect to d. From the mean value theorem we have for $M, N > 0$ and $q > 0$

$$|M^q - N^q| \le q|M - N| \max\big\{N^{q-1}, M^{q-1}\big\}. \tag{2.6}$$

Then for $u, v \in \Omega$ and $r \ge t_0$ we obtain

$$|\Phi(u(r)) - \Phi(v(r))| \le (p - 1)H_1 A^{p-2}(r)|u(r) - v(r)| \tag{2.7}$$

where, noting that $(p^*-1)(p-2)=2-p^*$,

$$H_1=\begin{cases}(1/2)^{2-p^*} & \text{if } \ 1<p\le 2,\\ 2^{2-p^*} & \text{if } \ p>2.\end{cases}$$

By (2.4) and (2.6) we obtain

$$\Big|\Phi^*\Big(1-\int_s^\infty b(r)\Phi(u(r))dr\Big)-\Phi^*\Big(1-\int_s^\infty b(r)\Phi(v(r))dr\Big)\Big| \le (p^*-1)H_2\Big|\int_s^\infty b(r)[\Phi(v(r))-\Phi(u(r))]dr\Big|,$$

where

$$H_2=\begin{cases}(1+2H)^{p^*-2} & \text{if } \ 1<p\le 2,\\ (1-2H)^{p^*-2} & \text{if } \ p>2.\end{cases}$$

In view of (2.7) and taking into account that $(p-1)(p^*-1)=1$, we have

$$\begin{aligned}&|T(u)(t)-T(v)(t)|\\ &\le \int_t^\infty \frac{1}{\Phi^*(a(s))}\Big|\Phi^*\Big(1-\int_s^\infty b(r)\Phi(v(r))dr\Big)-\Phi^*\Big(1-\int_s^\infty b(r)\Phi(v(r))dr\Big)\Big|ds\\ &\le (p^*-1)H_2\int_t^\infty \frac{1}{\Phi^*(a(s))}\int_s^\infty |b(r)|\,|\Phi(v(r))-\Phi(u(r))|\,dr\ ds\\ &\le H_1H_2\int_t^\infty \frac{1}{\Phi^*(a(s))}\int_s^\infty |b(r)|A^{p-1}(r)\frac{|u(r)-v(r)|}{A(r)}dr\ ds,\end{aligned}$$

or

$$|T(u)(t)-T(v)(t)|\le H_3A(t)\sup_{t\in[t_0,\infty)}\frac{|u(t)-v(t)|}{A(t)}$$

where

$$H_3=HH_1H_2=\begin{cases}H[2(1+2H)]^{p^*-2} & \text{if } \ 1<p\le 2,\\ H[2/(1-2H)]^{2-p^*} & \text{if } \ p>2.\end{cases}$$

Hence for any $u,v\in\Omega$ we have

$$|T(u)(t)-T(v)(t)|\le d(T(u),T(v))\le H_3\,d(u,v).$$

Setting $F(H)=H[2(1+2H)]^{p^*-2}$ or $F(H)=H[2/(1+2H)]^{2-p^*}$, according to $1<p\le 2$ or $p>2$, we have, in both cases, $F(0)=0, F'(0)>0$ and so there exists an open right neighborhood of zero such that $F(H)>0$; i.e., there exists $H>0$ (and $H<1/4$) such that $0<H_3<1$. Hence the operator T is a contraction in Ω. By applying Theorem A, we obtain the existence of a unique fixed point of T in Ω. It is easy to verify that every eventually positive solution x of (1.1) satisfying (2.1) belongs to Ω (by choosing a suitable large t_0) and the proof is complete. $\square$

In virtue of Theorem 1, equation (1.1) is nonoscillatory and it is possible to divide all nontrivial solutions of (1.1) into the following three classes:

$$\mathbb{M}^+ = \{x \text{ a solution of (1.1)} : \exists t_x \geq 0 : \ x(t)x'(t) > 0 \text{ for } t > t_x\}$$
$$\mathbb{M}^- = \{x \text{ a solution of (1.1)} : \exists t_x \geq 0 : \ x(t)x'(t) < 0 \text{ for } t > t_x\}$$
$$\mathbb{W} = \{x \text{ a solution of (1.1)} : \exists \text{ a positive sequence } \{t_n\}, t_n \to \infty, \text{ such that } x(t) \neq 0 \text{ for } t \geq t_1, \ x'(t_n) = 0\}.$$

Solutions from $\mathbb{W}$ are called *weakly oscillatory solutions*. For instance, for equation (1.1) with $b \equiv 0$ for $t \geq t_0$, and a satisfying (1.7), solutions in $\mathbb{M}^-$ and in $\mathbb{M}^+$ are

$$c + \int_t^\infty \frac{ds}{\Phi^*(a(s))} \quad \text{and} \qquad c + \int_{t_0}^t \frac{ds}{\Phi^*(a(s))},$$

respectively, where c is a real constant, while solutions in $\mathbb{W}$ are $x(t) = x(t_0) \neq 0$.

If either $b(t) \leq 0$, $b \not\equiv 0$, or $b(t) > 0$ for large t, then $\mathbb{W}$ is empty (see, e.g. [2, Lemma 1], [10, Lemma 1]). When b changes sign, solutions in $\mathbb{W}$ can exist, as the following example shows.

Example. Consider the equation (1.1) with ($t \geq 1$)

$$a(t) = e^t, \qquad b(t) = \frac{1 - \sin 2t + 2\cos 2t}{4(e^t + \sin t) + e^{-t}\sin^2 t}\chi(t), \qquad \Phi(u) = u^2 \operatorname{sgn} u,$$

where for $k \in \mathbb{N}$

$$\chi(t) = \begin{cases} -1 & \text{if } t \in (\frac{\pi}{4} + 2k\pi, \frac{5\pi}{4} + 2k\pi), \\ 0 & \text{if } t = \frac{\pi}{4} + k\pi, \\ 1 & \text{otherwise.} \end{cases}$$

The assumption (1.6) is satisfied and $x(t) = 2 + e^{-t}\sin t$ is a weakly oscillatory solution.

The following result describes the asymptotic behavior of the quasiderivative

$$x^{[1]}(t) = a(t)\Phi(x'(t)) \tag{2.8}$$

of any zero-convergent solution x of (1.1).

Theorem 2. *Assume* (1.6). *Then the unique solution x of* (1.1) *such that*

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x^{[1]}(t) = 0 \tag{2.9}$$

is the trivial solution.

Proof. Let x be a nontrivial solution of (1.1) satisfying (2.9). Without loss of generality assume $x(t) > 0$ for $t \ge t_0 \ge 0$ and

$$\int_{t_0}^{\infty} |b(t)|\Phi(A(t))dt < \frac{1}{\Phi(2)}. \tag{2.10}$$

Integrating (1.1) on $(t, \infty), t \ge t_0$, we obtain

$$|x^{[1]}(t)| \le \int_t^{\infty} |b(s)|\Phi(x(s))ds. \tag{2.11}$$

In view of (2.9) we have $\lim_{t\to\infty} x(t)/A(t) = 0$ and so there exists a constant $k > 0$ such that $x(t) < kA(t)$ for $t \ge t_0$. Then from (2.11) the function $\Phi^*\left(|x^{[1]}|\right)$is bounded on $[t_0, \infty)$. In view of the continuity of Φ^* and (2.9), let $T, t_0 \le T < \infty$, be such that

$$\sup_{t\in[t_0,\infty)} \Phi^*\left(|x^{[1]}(t)|\right) = \Phi^*\left(|x^{[1]}(T)|\right). \tag{2.12}$$

Clearly, $\Phi^*\left(|x^{[1]}(T)|\right) > 0$. From (2.8) we have

$$x(s) = \int_s^{\infty} \frac{1}{\Phi^*(a(r))}\Phi^*\big(-x^{[1]}(r)\big)\, dr \le \int_s^{\infty} \frac{1}{\Phi^*(a(r))}\Phi^*\big(|x^{[1]}(r)|\big)\, dr.$$

In view of (2.11) we obtain

$$\begin{aligned} |x^{[1]}(t)| &\le \int_t^{\infty} |b(s)|\Phi\Big(\int_s^{\infty} \frac{1}{\Phi^*(a(r))}\Phi^*\big(|x^{[1]}(r)|\big)\, dr\Big)ds \\ &\le \Phi\Big(\sup_{r\in[t,\infty)} \Phi^*\big(|x^{[1]}(r)|\big)\Big) \int_t^{\infty} |b(s)|\Phi\Big(\int_s^{\infty} \frac{dr}{\Phi^*(a(r))}\Big)\, ds, \end{aligned}$$

or, in view of (2.10) and (2.12),

$$\begin{aligned} &\Phi^*(|x^{[1]}(t)|) \le \Big(\sup\nolimits_{r\ge t} \Phi^*(|x^{[1]}(r)|)\Big)\, \Phi^*\Big(\int_t^{\infty} |b(s)|\Phi\Big(\int_s^{\infty} \frac{dr}{\Phi^*(a(r))}\Big)ds\Big) \\ &\le \frac{1}{2}\Big(\sup\nolimits_{r\ge t} \Phi^*(|x^{[1]}(r)|)\Big) \le \frac{1}{2}\Big(\sup\nolimits_{r\ge t_0} \Phi^*(|x^{[1]}(r)|)\Big) = \frac{1}{2}\Phi^*(|x^{[1]}(T)|), \end{aligned}$$

which is a contradiction for $t = T$. □

3. Limit properties of principal solutions

In this section we will show that the typical limit property (1.3) of the principal solutions continues to hold also in the half-linear case.

Lemma 1. *Assume* (1.6). *Any principal solution* u *of* (1.1) *is in the class* $\mathbb{M}^-$ *and* $\lim_{t\to\infty} u(t) = 0$.

Proof. Without loss of generality assume that u is eventually positive. If $u \in \mathbb{W} \cup \mathbb{M}^+$, then, by considering the solution x of (1.1) defined in Theorem 1, the inequality (1.5) gives a contradiction for large t. Hence $u \in \mathbb{M}^-$. If $\lim_{t\to\infty} u(t) > 0$, by considering again the solution x of (1.1) defined in Theorem 1, we obtain

$$\lim_{t\to\infty} \frac{u(t)}{x(t)} = \infty.$$

In view of (1.5) the function u/x is eventually positive decreasing, and this gives a contradiction. □

Theorem 3. *Assume* (1.6). *A solution* u *of* (1.1) *is a principal solution if and only if*

$$u \in \mathbb{M}^-, \ \lim_{t\to\infty} u(t) = 0, \ \lim_{t\to\infty} u^{[1]}(t) = d_x \neq 0. \tag{3.1}$$

Proof. Suppose that u satisfies (3.1) and let us show that u is a principal solution of (1.1). By contradiction assume that u is a nonprincipal solution of (1.1) and let z be a principal solution of (1.1). Without loss of generality, suppose $u(t) > 0, z(t) > 0$ for large t, and $\lim_{t\to\infty} u^{[1]}(t) = -1$.

Since z is a principal solution of (1.1), from Lemma 1 we have $z \in \mathbb{M}^-$ and $\lim_{t\to\infty} z(t) = 0$. In view of (1.5) the function z/u is eventually positive decreasing and from (3.1) we have $\lim_{t\to\infty} u(t)/A(t) = 1$. Hence there exists $H > 0$ such that for large t

$$\frac{z(t)}{A(t)} = \frac{z(t)}{u(t)} \frac{u(t)}{A(t)} \le H;$$

i.e., the function z/A is bounded on $[T, \infty), T$ large. Integrating (1.1) on $(t_1, t_2), T < t_1 < t_2$, we have

$$\left| z^{[1]}(t_2) - z^{[1]}(t_1) \right| \le \int_{t_1}^{t_2} |b(s)| \Phi(A(s)) \ \frac{\Phi(z(s))}{\Phi(A(s))} ds \le \Phi(H) \int_{t_1}^{t_2} |b(s)| \Phi(A(s)) \ ds$$

and so $\lim_{t\to\infty} z^{[1]}(t) = -d_z$, where $0 \le d_z < \infty$. In view of Theorem 2 we have $d_z > 0$. From Theorem 1, taking into account the homogeneity property, we obtain a contradiction.

Conversely, let u be a principal solution of (1.1). Since, in virtue of the above argument, any solution u of (1.1) satisfying (3.1) is a principal solution, taking into account that principal solutions of (1.1) are unique up to a constant factor, the assertion easily follows. □

Theorem 4. *Assume* (1.6). *A nontrivial solution* u *of* (1.1) *is a principal solution if and only if*

$$\lim_{t\to\infty} \frac{u(t)}{x(t)} = 0 \tag{3.2}$$

for any nontrivial solution x *of* (1.1) *such that* $x \neq \lambda u$, $\lambda \in R$.

Proof. Assume that (3.2) holds for any solution x of (1.1) such that $x \neq \lambda u$, $\lambda \in R$. Suppose u is a nonprincipal solution of (1.1) and let z be a principal solution of (1.1). Without loss of generality, assume $u(t) > 0$, $z(t) > 0$ for $t \geq t_1 \geq 0$. Then for $t \geq t_1$ we obtain

$$[z'(t)/z(t)] < [u'(t)/u(t)] \tag{3.3}$$

and, because $u \neq \mu z$ for any $\mu \in R$, we have from (3.2)

$$\lim_{t\to\infty} u(t)/z(t) = 0. \tag{3.4}$$

In view of (3.3), the ratio $u(t)/z(t)$ is positive increasing; that gives a contradiction with (3.4).

Conversely, let u be a principal solution of (1.1) and x any positive solution such that $x \neq \lambda u$. Without loss of generality, in view of Theorem 3, suppose $u(t) > 0$ for $t \geq T \geq 0$, $\lim_{t\to\infty} u(t) = 0$, $\lim_{t\to\infty} u^{[1]}(t) = -1$. Then

$$\lim_{t\to\infty} \frac{u(t)}{A(t)} = 1. \tag{3.5}$$

From (1.5) the function u/x is eventually decreasing and so

$$\lim_{t\to\infty} [u(t)/x(t)] = c,$$

where $0 \leq c < \infty$. Assume, by contradiction, that there exists an eventually positive solution z, $z \neq \lambda u$ such that

$$\lim_{t\to\infty} \frac{u(t)}{z(t)} = 1. \tag{3.6}$$

Then from (3.5) and (3.6) we obtain

$$\lim_{t\to\infty} \frac{z(t)}{A(t)} = \lim_{t\to\infty} \frac{z(t)}{u(t)} \frac{u(t)}{A(t)} = 1,$$

and so $\lim_{t\to\infty} z(t) = 0$. Again from (3.5) and (3.6) there exists a positive constant h such that for large t

$$\frac{z(t)}{A(t)} = \frac{z(t)}{u(t)} \frac{u(t)}{A(t)} \leq h,$$

and, using the same argument as in the final part of the proof of Theorem 3, we obtain that $\lim_{t\to\infty} z^{[1]}(t)$ exists finitely. In virtue of Theorem 2 we have $\lim_{t\to\infty} z^{[1]}(t) = d_z \neq 0$. Since z is an eventually positive function approaching zero as $t \to \infty$, necessarily $d_z < 0$. Then $z \in \mathbb{M}^-$ and from Theorem 3 a contradiction follows. □

4. Boundedness of solutions and applications

In this section we present further asymptotic properties of solutions of (1.1) under the assumption (1.9). As a consequence, an extension of the property (1.4) is also given. Consider the associated Riccati equation

$$w' + (p-1)\frac{|w|^{p^*}}{\Phi^*(a(t))} + b(t) = 0. \tag{4.1}$$

Equation (4.1) is related with (1.1) because for any solution x of (1.1) such that $x(t) \neq 0$ for $t \geq T \geq 0$ the function

$$w_x(t) = \frac{a(t)\Phi(x'(t))}{\Phi(x(t))} \tag{4.2}$$

is a solution of (4.1). The following holds.

Theorem 5. *If* (1.9) *holds, then every solution of* (1.1) *is bounded.*

Proof. Let x be a solution of (1.1). Clearly, if $x \in \mathbb{M}^-$, then x is bounded. Now assume $x \in \mathbb{M}^+ \cup \mathbb{W}$ and, without loss of generality, suppose $x(t) > 0$ for $t \geq t_x, 0 \leq t_x$. Consider the corresponding Riccati solution given by (4.2). From (4.1) we have $(t_x \leq t)$,

$$w_x(t) - w_x(t_x) \ + (p-1)\int_{t_x}^{t} \frac{|w_x(s)|^{p^*}}{\Phi^*(a(s))}ds + \int_{t_x}^{t} b(s)ds = 0. \tag{4.3}$$

Since $|w_x(s)|^{p^*}/\Phi^*(a(s))$ is nonnegative, we obtain

$$w_x(t) \leq w_x(t_x) + \int_{t_x}^{t} |b(s)|ds. \tag{4.4}$$

Assume $x \in \mathbb{M}^+$ and, without loss of generality, suppose $x(t) > 0, x'(t) > 0$ for $t \geq t_x$. In view of (4.2) and (4.4), there exists a constant $K_x > 0$ such that $0 < w_x(t) < K_x$ for $t \geq t_x$. Integrating on (t_x, t) we obtain

$$0 < \log\frac{x(t)}{x(t_x)} < \Phi^*(K_x)\int_{t_x}^{t} \frac{1}{\Phi^*(a(s))}ds,$$

which gives the boundedness of x.

Finally, assume $x \in \mathbb{W}$ and let $\{\sigma_n\}$ be an increasing sequence such that $\sigma_n \geq t_x$, $x'(\sigma_n) = 0, n \in \mathbb{N}$, and $\lim_n \sigma_n = \infty$. From (4.3) we obtain

$$(p-1)\int_{\sigma_1}^{\sigma_n} \frac{|w_x(s)|^{p^*}}{\Phi^*(a(s))}ds \leq \int_{\sigma_1}^{\sigma_n} |b(s)|ds.$$

Since, as claimed, $|w_x(s)|^{p^*}/\Phi^*(a(s))$ is nonnegative, the integral

$$\int_{t_x}^{\infty} \frac{|w_x(s)|^{p^*}}{\Phi^*(a(s))} ds$$

is convergent. Therefore, in view of (4.3) and (4.4), there exists a positive constant M_x such that $-M_x < w_x(t) < M_x$ for $t \geq t_x$. Integrating on (t_x, t) we obtain

$$-\Phi^*(M_x) \int_{t_x}^{t} \frac{1}{\Phi^*(a(s))} ds < \log \frac{x(t)}{x(t_x)} < \Phi^*(K_x) \int_{t_x}^{t} \frac{1}{\Phi^*(a(s))} ds, \qquad (4.5)$$

which gives the boundedness of x. □

Theorem 6. *Assume* (1.9) *and let x be a nontrivial solution of* (1.1). *Then*

$$\lim_{t\to\infty} x(t) = c_x, \qquad \lim_{t\to\infty} x^{[1]}(t) = d_x, \qquad (4.6)$$

where $|c_x| < \infty$, $|d_x| < \infty$ *and* $|c_x| + |d_x| > 0$.

Proof. Without loss of generality, suppose $x(t) > 0$ for $t \geq t_x, 0 \leq t_x$. Integrating twice equation (1.1) we obtain $(t_x < t_1 < t_2)$

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} \frac{1}{\Phi^*(a(s))} \Phi^*\Big(x^{[1]}(t_x) - \int_{t_x}^{s} b(r)\Phi(x(r))dr\Big) ds.$$

In view of Theorem 5, there exists a positive constant N such that

$$|x(t_2) - x(t_1)| \leq N \int_{t_1}^{t_2} \frac{ds}{\Phi^*(a(s))};$$

i.e., the first statement in (4.6) holds. The second one follows in a similar way.

In order to complete the proof, let x be a positive solution of (1.1) for $t \geq t_x \geq 0$, satisfying (4.6). Clearly, if $x \in \mathbb{M}^+$, then $c_x > 0$. If $x \in \mathbb{W}$, then $d_x = 0$ and, from (4.5), we have $c_x > 0$. Finally, if $x \in \mathbb{M}^-$, $x'(t) < 0$ for $t \geq t_x$ and $d_x = 0$, integrating (1.1) on $(t, \infty), t \geq t_x$, we obtain

$$\begin{aligned} x^{[1]}(t) &= \int_{t}^{\infty} b(s)\Phi(x(s))ds \geq -\int_{t}^{\infty} |b(s)|\Phi(x(s))ds \\ &\geq -\Phi(x(t)) \int_{t}^{\infty} |b(s)|ds. \end{aligned}$$

Then there exists a positive constant k_x such that

$$\frac{x'(t)}{x(t)} \geq -\Phi^*\Big(\frac{k_x}{a(t)}\Big).$$

Integrating this inequality, the assertion easily follows. □

From Theorems 1 and 6, we obtain the following result concerning the asymptotic behavior of weakly oscillatory solutions.

Corollary 1. *Assume* (1.9). *Any weakly oscillatory solution* x *of* (1.1) *satisfies*

$$\lim_{t\to\infty} x(t) = \ell_x, \quad 0 < |\ell_x| < \infty, \quad \lim_{t\to\infty} x^{[1]}(t) = 0.$$

This fact is "surprising", in some sense. Indeed, in spite of the possible existence of weakly oscillatory solutions, if (1.9) is satisfied, then any solution of (1.1) and its quasiderivative have the finite limit as $t \to \infty$.

Another consequence of Theorem 6 concerns the asymptotic behavior of nonprincipal solutions of (1.1), as the following result shows.

Corollary 2. *Assume* (1.9). *Any nonprincipal solution* x *of* (1.1) *satisfies* $\lim_{t\to\infty} x(t) = \ell_x, 0 < |\ell_x| < \infty$.

Proof. If $x \in \mathbb{M}^+ \cup \mathbb{W}$ the assertion follows. Let $x \in \mathbb{M}^-$: in view of Theorems 1 and 6, and the homogeneity property, the assertion again follows. □

From Theorem 2 and Corollary 2, we obtain the following extension to the half-linear case of the integral characterization (1.4).

Theorem 7. *(i) If* (1.6) *holds, then a principal solution* u *of* (1.1) *satisfies*

$$\int^{\infty} \frac{1}{\Phi^*(a(t))u^2(t)}\, dt = \infty. \tag{4.7}$$

(ii) If (1.9) *holds, then any nonprincipal solution* x *of* (1.1) *satisfies*

$$\int^{\infty} \frac{1}{\Phi^*(a(t))x^2(t)}\, dt < \infty.$$

Proof. Claim (i). Let u be a principal solution of (1.1) and, without loss of generality, suppose $u(t) > 0$, $u'(t) < 0$ for $t \geq T \geq 0$. In view of Theorem 2 we have $u(\infty) = 0, u^{[1]}(\infty) = \ell_u < 0$. By using the l'Hopital rule we obtain

$$\lim_{t\to\infty} \frac{u(t)}{\int_t^\infty \Phi^*(a^{-1}(s))ds} = \Phi^*(\ell_u),$$

and so there exists a positive constant k such that for $t \geq T$

$$u(t) < k \int_t^\infty \frac{ds}{\Phi^*(a(s))}.$$

Then

$$\int_T^\infty \frac{1}{\Phi^*(a(s))} \frac{1}{u^2(s)}\, ds > \frac{1}{k^2} \int_T^\infty \frac{1}{\Phi^*(a(s))} \Big(\int_s^\infty \frac{dr}{\Phi^*(a(r))} \Big)^{-2} ds$$

$$= \frac{1}{k^2}\Big(\int_s^\infty \frac{dr}{\Phi^*(a(r))}\Big)^{-1}\Big|_T^\infty = \infty;$$

i.e., condition (4.7) is satisfied.

Claim (ii). By Corollary 2 $\lim_{t\to\infty} x(t) \neq 0$ and therefore, in view of (1.9), the assertion immediately follows. □

Remark. Since (1.9) implies (1.6), by Theorem 7, if (1.9) holds, then a nontrivial solution of (1.1) is a principal solution if and only if (4.7) holds. Hence, when (1.9) is satisfied, the integral characterization (4.7) can be considered as the natural extension of the property (1.4), stated in the linear case. When (1.6) holds, but (1.9) is not satisfied, a natural question which arises is whether (4.7) holds only for principal solutions. If $b(t) < 0$ eventually, the answer is positive, as is proved in [3, Theorem 3.2]. If $b(t) > 0$ eventually, the answer is negative, as is shown in [4, Theorems 3,4] and [5, Theorem 3.3].

If u is a principal solution of (1.1), then the corresponding solution of (4.1) is called the *minimal solution* of (4.1) because, in view of (1.5), for any other continuable solution w_x of (4.1) we have $w_u(t) < w_x(t)$ for large t.

Applying Theorems 3, 6, and Corollary 2, we obtain the following:

Corollary 3. *Assume* (1.9). *Then the minimal solution* w_u *of* (4.1) *satisfies*

$$\lim_{t\to\infty} w_u(t) = -\infty$$

and every continuable solution w_x *of (4.1), which is not minimal, satisfies*

$$\lim_{t\to\infty} w_u(t) = d_x, \ |d_x| < \infty.$$

Finally, the following application can be viewed as an extension of the well-known relation between principal and nonprincipal solutions stated in the linear case by the Wronskian identity.

Corollary 4. *Assume* (1.9). *Then for any nonprincipal solution* x *of* (1.1) *there exists a principal solution* u *of* (1.1), $u(t) \neq 0$ *for* $t \geq T$, *such that*

$$\lim_{t\to\infty}\Big(\int_T^t \frac{1}{\Phi^*(a(s))u^2(s)}\,ds\Big)\Big(\int_t^\infty \frac{1}{\Phi^*(a(s))x^2(s)}\,ds\Big) = 1. \tag{4.8}$$

Proof. Let u be a principal solution and x a nonprincipal solution of (1.1). Assume $u(t) \neq 0$ and $x(t) \neq 0$ for $t \geq T$. Denote

$$I(t) = \int_T^t \frac{ds}{\Phi^*(a(s))u^2(s)}, \quad J(t) = \int_t^\infty \frac{ds}{\Phi^*(a(s))x^2(s)}.$$

From Theorems 3 and 6 and Corollary 2 we obtain

$$\lim_{t\to\infty}\Big(\Phi^*(x^{[1]}(t))u(t) - x(t)\Phi^*(u^{[1]}(t))\Big) = c \neq 0.$$

By Theorem 4 and the l'Hopital rule we have

$$\lim_{t\to\infty}\frac{x(t)/u(t)}{I(t)} = \lim_{t\to\infty}\Big(\Phi^*(x^{[1]}(t))u(t) - x(t)\Phi^*(u^{[1]}(t))\Big) = c,$$

and similarly, $\lim_{t\to\infty}\frac{u(t)/x(t)}{J(t)} = c$. From here and the homogeneity property, the assertion follows. □

REFERENCES

[1] R.P. Agarwal, S.R. Grace, and D. O'Regan, "Oscillation Theory of Second Order Linear, Half-Linear, Superlinear and Sublinear Dynamic Equations," Kluwer Academic Publishers, Dordrecht/Boston/London, 2002.

[2] M. Cecchi, Z. Došlá, and M. Marini, *On nonoscillatory solutions of differential equations with p-Laplacian*, Adv. Math. Sci. Appl., 11 (2001), 419–436.

[3] M. Cecchi, Z. Došlá, and M. Marini, *Principal solutions and minimal sets of quasilinear differential equations*, Dynamic Systems and Appl., 13 (2004), 223–234.

[4] M. Cecchi, Z. Došlá, and M. Marini, *Half-linear equations and characteristic properties of the principal solution*, J. Differential Eqs., 208 (2005), 494–507.

[5] O. Došlý and A. Elbert, *Integral characterization of the principal solution of half-linear second order differential equations*, Studia Sci. Math. Hungarica, 36 (2000), 455–469.

[6] O. Došlý, *Half-linear differential equations*, In "Handbook of Differential Equations, Ordinary Differential Equations I.," Eds. A. Cañada, P. Drabek, A. Fonda, Elsevier, Amsterdam, 2004.

[7] A. Elbert, *The wronskian and the half-linear differential equations*, Studia Sci. Math. Hungarica, 15 (1980), 101–105.

[8] A. Elbert, *On the half-linear second order differential equations*, Acta Math. Hung., 49 (1987), 487–508.

[9] A. Elbert and T. Kusano, *Principal solutions of non-oscillatory half-linear differential equations*, Adv. Math. Sci. Appl., 8 (1998), 745–759.

[10] X. Fan, W.T. Li, and C. Zhong, *A Classification scheme for positive solutions of second order iterative differential equations*, El. J. Diff. Equat., 25 (2000), 1–14.

[11] P. Hartman, "Ordinary Differential Equations," 2 Ed., Birkäuser, Boston-Basel-Stuttgart, 1982.

[12] J. Jaroš, T. Kusano, and T. Tanigawa, *Nonoscillation theory for second order half-linear differential equations in the framework of regular variation*, Result. Math., 43 (2003), 129–149.

[13] J.D. Mirzov, *Principal and nonprincipal solutions of a nonoscillatory system*, Tbilisi Inst. Prikl. Mat. Trudy, 31 (1988), 100–117.

[14] J.D. Mirzov, "Asymptotic Properties of Solutions of Systems of Nonlinear Nonautonomous Ordinary Differential Equations," (Russian), Maikop, Adygeja Publ. (1993), the english version: Folia Fac. Sci. Nat. Univ. Masaryk. Brun., Mathematica, 14 (2004).