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ON L^1-SPECTRAL THEORY OF NEUTRON TRANSPORT

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Abstract. This paper provides a systematic analysis of (weak) compactness problems in connection with L^1-spectral theory of general neutron transport equations.

1. Introduction

Let

$$T : \varphi \in D(T) \to -v\frac{\partial \varphi}{\partial x} - \sigma(x,v)\varphi$$

be the advection operator (with *no incoming particles* boundary condition) arising in nuclear reactor theory and let K be the collision operator which describes the interactions of neutrons with the host medium $\Omega \subset R^n$. The main spectral feature of the stationary neutron transport operator $T+K$ relies on the compactness (or weak compactness in L^1) of some *power* of $K(\lambda - T)^{-1}$. Indeed, according to Gohberg-Schmulyan's theorem [13],

$$\sigma(T+K) \cap \{\operatorname{Re}\lambda > s(T)\},$$

the so-called *asymptotic spectrum* of T, consists at most of isolated eigenvalues with finite algebraic multiplicities where $s(T)$ denotes the spectral bound of T; i.e., $s(T) = \sup\{\operatorname{Re}\lambda;\ \lambda \in \sigma(T)\}$. On the other hand, the time asymptotic behavior ($t \to \infty$) of the c_0-semigroup $\{V(t);\ t \geq 0\}$ generated by $T+K$, which governs the Cauchy problem

$$\frac{\partial \varphi}{\partial t} + v\frac{\partial \varphi}{\partial x} + \sigma(x,v)\varphi - K\varphi = 0, \quad \varphi(0) = \varphi_0,$$

depends heavily on the nature of the spectrum of $\{V(t);\ t \geq 0\}$ outside of the disc $\{\nu; |\nu| \leq e^{s(T)t}\}$ (see [14]). Of course,

$$e^{t\{\sigma(T+K)\cap\{\operatorname{Re}\lambda > s(T)\}\}} \subset \sigma(e^{t(T+K)}) \cap \Big\{\nu; |\nu| > e^{s(T)t}\Big\}. \qquad (1)$$

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However, this inclusion is a priori strict because of the lack, in general, of a spectral mapping theorem. Thus a direct spectral analysis of $e^{t(T+K)}$ is necessary. To this end, we expand $V(t)$ into a Dyson-Phillips expansion

$$V(t) = \sum_{0}^{\infty} U_j(t)$$

where

$$U_0(t) = e^{tT}, \;\; U_{j+1}(t) = \int_0^t U_0(s) K U_j(t-s) ds \;\; (j \geq 0).$$

A basic result is that (1) is an equality provided that some *remainder term* $R_m(t) := \sum_{j=m}^{\infty} U_j(t)$ is compact (or weakly compact in L^1); see [14], [17], [18], [20], [11], [6] Chapter 2 and [7] for more details. In this case,

$$\sigma(e^{t(T+K)}) \cap \left\{ \nu; |\nu| > e^{s(T)t} \right\},$$

the so-called *asymptotic spectrum* of $V(t)$, consists at most of isolated eigenvalues with finite algebraic multiplicities. Thus, a discrete asymptotic spectrum of the stationary transport operator $T+K$ relies on the compactness of some power of $K(\lambda - T)^{-1}$ while a discrete asymptotic spectrum of the corresponding semigroup relies on the compactness of some remainder term $R_m(t)$. These are two basic compactness problems in neutron transport theory. Of course, there exists a great deal of work on this topic since the fifties already covering the usual models (see [6] Chapter 4 and references therein). The present work and the companion one [8] provide a systematic analysis of compactness problems which are at the very core of spectral theory of general neutron transport equations covering, in particular, the classical continuous or multigroup models. In particular, in [8] it is shown, for spatial domains Ω with finite Lebesgue measure, that $R_1(t) = V(t) - U(t)$ is compact in L^p $(1 < p < \infty)$ if and only if the affine hyperplanes have zero μ measure where μ is the velocity measure. The mathematical analysis relies on interpolation and "Fourier integral" type arguments which do not cover the (physical) L^1 space. The L^1 theory we deal with here relies on different mathematical tools and provides us with *different* results. In particular, we show that in L^1 spaces, regardless of the choice of the velocity measure μ, $V(t) - U(t)$ is never weakly compact when $n \geq 3$. On the other hand, for $n = 1$, $V(t) - U(t)$ is weakly compact if and only if the velocity measure μ is continuous; i.e., the points have zero μ measure. However, for arbitrary dimension n, we show the weak compactness of the remainder terms $R_m(t)$ for m large enough, typically under the following assumption on the velocity measure : There exists $\alpha > 0$ such that for all $c > 0$ there exists $c' > 0$ such that

$$\sup_{e \in S^{n-1}} \mu \otimes \mu \left\{ (v, v') \in R^n \times R^n;\ |v| \leq c,\ \left|v'\right| \leq c,\ \left|(v - v').e\right| < \varepsilon \right\} \leq c' \varepsilon^{\alpha}.$$

Other weak compactness results concerning the iterates of $K(\lambda - T)^{-1}$ are also given. Mathematically speaking, some relevant operators are shown to be convolution operators with suitable Radon measures. The Fourier analysis of such measures enables us to derive smoothing properties of their convolution iterates from which various weak compactness results are obtained. Our paper is organized as follows: Section 2 (respectively Section 3) is devoted to a thorough analysis of the different aspects of weak compactness of the powers of $K(\lambda - T)^{-1}$ (respectively of the remainder terms $R_m(t)$) for constant scattering kernels and finite velocity measure μ. In Section 4 we give much more precise results in one dimension and show that these results are no longer true in n dimensions with $n \geq 3$. In Section 5 we show how the above compactness results provide a complete foundation of the L^1-spectral theory of general neutron transport models with weakly compact (with respect to *velocities*) collision operators. A similar treatment of neutron transport equations on the torus as well as additional compactness results are also given in [9].

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2. Stationary equations. A model case

Let $\Omega \subset R^n$ be an open set with *finite* volume (not necessarily bounded) and let μ be a *finite* and positive Radon measure on R^n with support Ξ. We denote by $\{U(t); t \geq 0\}$ the classical advection c_0-semigroup (without collision frequency) [19]

$$U(t) : \varphi \in L^1(\Omega \times \Xi) \to \varphi(x - tv, v)\chi_{\{(x,v);\, t<\tau(x,v)\}} \ \in L^1(\Omega \times \Xi),$$

where $\tau(x, v) = \inf\{s > 0 : x - sv \notin \Omega\}$ and χ_F denotes the indicator function of a subset $F \subset \Omega \times \Xi$. Let T be its generator and

$$(\lambda - T)^{-1} : \varphi \in L^1(\Omega \times \Xi) \to \int_0^{\tau(x,v)} e^{-\lambda t} \varphi(x - tv, v) dt \ \ (\lambda > 0)$$

be the resolvent of T. Finally, let

$$M : \varphi \in L^1(\Omega \times \Xi) \to \widetilde{\varphi}(.) = \int \varphi(., v)\ d\mu(v) \in L^1(\Omega) \tag{2}$$

be the (velocity) averaging operator. We are concerned in this section with weak compactness properties of the *iterates* of $M(\lambda - T)^{-1}$ in $L^1(\Omega \times \Xi)$. We start with the following results which were first pointed out in ([2] Proposition 3 and example 1) for the whole space.

Proposition 1. *(i) If* $n \geq 2$ *or if* $n = 1$ *and* $0 \in \Xi$, *then the operator* $M(\lambda - T)^{-1} : L^1(\Omega \times \Xi) \to L^1(\Omega)$ *is not weakly compact.*

(ii) If the hyperplanes through the origin have zero μ *measure, then* $M(\lambda - T)^{-1}$ *maps weakly compact sets into compact sets.*

Proof. We adapt the arguments of [2] to our framework. Without loss of generality, we may assume that $0 \in \Omega$.

(i) Let $\{f_j\}_j \subset C_c(\Omega \times \Xi)$ (the continuous functions with compact support in $\Omega \times \Xi$) be normalized in $L^1(\Omega \times \Xi)$ and converging in the weak star topology of measures on $\Omega \times \Xi$ to the Dirac mass $\delta_{(0,\overline{v})} = \delta_{x=0} \otimes \delta_{v=\overline{v}}$, where $\overline{v} \in \Xi$. Then, for a $\psi \in C_c(\Omega)$,

$$\begin{aligned}\langle M(\lambda - T)^{-1} f_j, \psi\rangle &= \int_\Omega \psi(x)dx \int d\mu(v) \int_0^{\tau(x,v)} e^{-\lambda t} f_j(x - tv, v)dt \\ &= \int d\mu(v) \int_\Omega \psi(x)dx \int_0^{\tau(x,v)} e^{-\lambda t} f_j(x - tv, v)dt \qquad (3) \\ &= \int d\mu(v) \int_\Omega f_j(y, v)dy \int_0^{\tau(y,-v)} e^{-\lambda t} \psi(y + tv)dt \\ &= \int_{\Omega \times \Xi} \Big[\int_0^{\tau(y,-v)} e^{-\lambda t} \psi(y + tv)dt\Big] f_j(y, v)dy d\mu(v)\end{aligned}$$

and

$$\langle M(\lambda - T)^{-1} f_j, \psi\rangle \to \int_0^{\tau(0,-v)} e^{-\lambda t} \psi(t\overline{v})dt \quad \text{as } j \to \infty;$$

i.e., $M(\lambda - T)^{-1} f_j$ converges in the weak star topology of measures to the Radon measure

$$\psi \in C_c(\Omega) \to \int_0^{\tau(0,-v)} e^{-\lambda t} \psi(t\overline{v})dt$$

supported by the *line* $R\overline{v}$ and consequently $M(\lambda - T)^{-1} f$ is *not* weakly compact if $n > 1$. If $n = 1$, using a domination argument if necessary, we may assume that Ω is an *interval.* In this case, (3) reduces to

$$\langle M(\lambda - T)^{-1} f_j, \psi\rangle = \int_{\Omega \times \Xi} \Big[\int_0^\infty e^{-\lambda t} \psi(y + tv)dt\Big] f_j(y, v)dy \, d\mu(v),$$

where ψ is extended by zero outside of Ω. Hence

$$\langle M(\lambda - T)^{-1} f_j, \psi\rangle \to \int_0^\infty e^{-\lambda t} \psi(t\overline{v})dt \quad \text{as } j \to \infty.$$

If $0 \in \Xi$, then the choice $\overline{v} = 0$ shows that $M(\lambda - T)^{-1} f_j$ converges to the Dirac measure $\frac{1}{\lambda}\delta_{x=0}$. (We note however that if $n = 1$ and *if* $0 \notin \Xi$, then it is easy to see that $M(\lambda - T)^{-1}$ is always a *compact* operator).

(ii) Let $\Lambda \subset L^1(\Omega \times \Xi)$ be relatively weakly compact. Let $g = M(\lambda - T)^{-1} f \quad (f \in \Lambda)$. We decompose g as $g = g_1 + g_2$ where

$$g_1 = M(\lambda - T)^{-1}(f\chi_{\{f>\alpha\}}), \;\; g_2 = M(\lambda - T)^{-1}(f\chi_{\{f<\alpha\}})$$

and $\{f > \alpha\}$ denotes the set $\{(x, v);\; f(x, v) > \alpha\}$. We note that

$$dx \otimes d\mu \{f > \alpha\} \leq \frac{\|f\|}{\alpha} \leq \frac{c}{\alpha} \to 0 \text{ as } \alpha \to \infty$$

so that, by the *equi-integrability* of Λ,

$$\int_{\{f>\alpha\}} |f(x, v)|\, dxd\mu(v) \to 0 \text{ uniformly in } f \in \Lambda$$

as $\alpha \to \infty$. Thus,

$$\|g_1\| = \left\|M(\lambda - T)^{-1}(f\chi_{\{f>\alpha\}})\right\| \to 0$$

uniformly in $f \in \Lambda$ as $\alpha \to \infty$. Let $\varepsilon > 0$ be *fixed* and let α be large enough such that

$$\left\|M(\lambda - T)^{-1}(f\chi_{\{f>\alpha\}})\right\| \leq \varepsilon \text{ uniformly in } f \in \Lambda.$$

On the other hand, $\left\{f\chi_{\{f<\alpha\}};\; f \in \Lambda\right\}$ is a bounded subset of $L^2(\Omega \times \Xi)$ and consequently $\{g_2;\; f \in \Lambda\}$ is relatively compact in $L^2(\Omega)$ (see [8] Theorem 6) and consequently relatively compact in $L^1(\Omega)$ so $\{g = M(\lambda - T)^{-1} f : f \in \Lambda\}$ is relatively compact since, for *all* $\varepsilon > 0$, this set is contained in a compact set plus an ε-ball. This ends the proof of *(ii)*. □

Before giving our compactness results we derive a *necessary* condition on the velocity measure.

Proposition 2. *We assume that μ is invariant under the symmetry about the origin $v \to -v$. If some power of $M(\lambda - T)^{-1}$ is weakly compact, then the hyperplanes through the origin have zero μ measure.*

Proof. Since the square of a weakly compact operator in L^1 is compact [1], we may assume that some power of $M(\lambda - T)^{-1}$ is compact in L^1. Since $M^2 = \mu(\Xi)M$, then some power of $M(\lambda - T)^{-1}M$ is also compact. On the other hand, since $M(\lambda - T)^{-1}M$ maps also $L^p(\Omega \times \Xi)$ into $L^p(\Omega)$ for all $p \in [1, \infty]$, then by interpolation, some power of $M(\lambda - T)^{-1}M$ is compact in L^2. Without loss of generality, we may assume that $\left[M(\lambda - T)^{-1}M\right]^{2^m}$ is compact in L^2 for some integer m. By using the invariance of μ under the symmetry about the origin and arguing as in ([6] Theorem 4. 6, page 64), one shows that $M(\lambda - T)^{-1}M$ is *selfadjoint* for λ *real.* Hence the compactness of $\left[M(\lambda - T)^{-1}M\right]^{2^m}$ implies that $\left[M(\lambda - T)^{-1}M\right]^{2^{m-1}}$ is compact by the fact that the square of a selfadjoint operator O is compact if and only if O is. It follows, by induction, that $M(\lambda - T)^{-1}M$ is compact. We use now Vladimirov's argument [15] as in [5] to prove that $(\lambda - T)^{-1}M$ is compact. Hence $M(\lambda - T^*)^{-1}$ is compact and this implies ([6] Remark 3.1, page 35) that the hyperplanes through the origin have *zero* μ-measure. □

In view of Proposition 2, we deal with a velocity measure μ such that:

$$\text{The hyperplanes through the origin have zero } \mu \text{ measure.} \tag{4}$$

If we except the one-dimensional case (see Section 4), Assumption (4) alone does not seem to be sufficient to derive compactness results, in contrast with the L^p theory $(1 < p < \infty)$ [8] ; (see however [7] for *Dunford-Pettis* results). We will need a stronger assumption on μ. Note first a characterization of (4) :

Lemma 3. ([6] Lemma 3.1, page 32) *The hyperplanes through the origin have zero μ measure if and only if* $\sup_{e \in S^{n-1}} \mu\{v;\ |v.e| \le \varepsilon\} \to 0$ *as* $\varepsilon \to 0$.

We will give below (Proposition 6) another characterization of (4) . Consider the advection c_0-semigroup in the *whole* space

$$U_\infty(t) : \varphi \in L^1(R^n \times \Xi) \to \varphi(x - tv, v) \in L^1(R^n \times \Xi).$$

We need the following useful domination result.

Lemma 4. *Let $m \ge 1$ be an integer. Then*

$$\left[M(\lambda - T)^{-1}M\right]^m \varphi \le R\left[M(\lambda - T_\infty)^{-1}M\right]^m E\varphi; \quad \varphi \in L^1_+(\Omega \times \Xi), \tag{5}$$

where $E : L^1(\Omega \times \Xi) \to L^1(R^n \times \Xi)$ *is the extension operator (by zero) to* $R^n \times \Xi$ *and* $R : L^1(R^n \times \Xi) \to L^1(\Omega \times \Xi)$ *is the restriction operator.*

Proof. We note that $U(t)\varphi \le RU_\infty(t)E\varphi;\ \ \varphi \in L^1_+(\Omega \times \Xi)$. It follows that for $\lambda > 0$

$$(\lambda - T)^{-1}\varphi \le R(\lambda - T_\infty)^{-1}E\varphi;\ \ \varphi \in L^1_+(\Omega \times \Xi),$$

where

$$(\lambda - T_\infty)^{-1} : \varphi \in L^1(R^n \times \Xi) \to \int_0^\infty e^{-\lambda t}\varphi(x - tv, v)dt \in L^1(R^n \times \Xi).$$

Since E and R commute with the averaging operator M, it follows that

$$M(\lambda - T)^{-1}M\varphi \le RM(\lambda - T_\infty)^{-1}ME\varphi$$

and, by induction,

$$\left[M(\lambda-T)^{-1}M\right]^m\varphi\le R\left[M(\lambda-T_\infty)^{-1}M\right]^m E\varphi;\quad \varphi\in L^1_+(\Omega\times\Xi).$$

This ends the proof. □

We start with a basic observation.

Proposition 5. *Let $\lambda>0$. There exists a finite Radon measure β on R^n such that*

$$M(\lambda-T_\infty)^{-1}M\varphi=\beta * M\varphi,\ \ \varphi\in L^1(R^n\times\Xi).$$

Moreover, the Fourier transform of β is given by

$$\widehat{\beta}(\zeta)=(2\pi)^{-\frac{n}{2}}\int_{R^n}\frac{d\mu(v)}{\lambda+i\zeta.v}.$$

Proof. Let $\varphi\in L^1(R^n\times\Xi)$. Then

$$\begin{aligned}M(\lambda-T_\infty)^{-1}M\varphi&=\int_{R^n}d\mu(v)\int_0^\infty e^{-\lambda t}(M\varphi)(x-tv)dt\\&=\int_0^\infty e^{-\lambda t}dt\int_{R^n}(M\varphi)(x-tv)d\mu(v)\\&=\int_0^\infty e^{-\lambda t}dt\int_{R^n}(M\varphi)(x-z)\ d\mu_t(z)=\int_0^\infty e^{-\lambda t}\left[\mu_t * M\varphi\right]dt,\end{aligned}$$

where μ_t is the image of μ under the dilation $v\to tv$. Hence

$$M(\lambda-T_\infty)^{-1}M\varphi=\int(M\varphi)(x-z)\ d\beta(z)=\beta * M\varphi,$$

where $\beta=\int_0^\infty e^{-\lambda t}\mu_t dt$ is the Radon measure acting as

$$\langle\beta,\psi\rangle=\int_0^\infty e^{-\lambda t}\langle\mu_t,\psi\rangle dt,\ \ \psi\in C_c(R^n).$$

Finally,

$$\begin{aligned}\widehat{\beta}(\zeta)&=(2\pi)^{-\frac{n}{2}}\int_{R^n}e^{-i\zeta.v}d\beta(v)=\int_0^\infty e^{-\lambda t}\widehat{\mu_t}(\zeta)dt\\&=(2\pi)^{-\frac{n}{2}}\int_0^\infty e^{-\lambda t}\Big[\int_{R^n}e^{-i\zeta.v}d\mu_t(v)\Big]dt\\&=(2\pi)^{-\frac{n}{2}}\int_0^\infty e^{-\lambda t}\Big[\int_{R^n}e^{-it\zeta.v}d\mu(v)\Big]dt\\&=(2\pi)^{-\frac{n}{2}}\int_{R^n}\Big[\int_0^\infty e^{-\lambda t}e^{-it\zeta.v}dt\Big]d\mu(v)=(2\pi)^{-\frac{n}{2}}\int_{R^n}\frac{d\mu(v)}{\lambda+i\zeta.v},\end{aligned}$$

which ends the proof. □

Proposition 6. *The hyperplanes through the origin have zero μ measure if and only if*

$$\int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v} \to 0 \ \ as \ |\zeta| \to \infty. \tag{6}$$

Proof. The fact that (4) implies (6) is known (see [6] page 34). Conversely, let there exist $e \in S^{n-1}$ such that $\mu\{v; v.e = 0\} > 0$. Then

$$(2\pi)^{\frac{n}{2}} \operatorname{Re} \widehat{\beta}(\zeta) = \operatorname{Re} \int_{R^n} \frac{(\lambda - i\zeta.v) d\mu(v)}{\lambda^2 + (\zeta.v)^2} = \lambda \int_{R^n} \frac{d\mu(v)}{\lambda^2 + (\zeta.v)^2}$$

so, for *all* $t \in R$,

$$\begin{aligned} (2\pi)^{\frac{n}{2}} \operatorname{Re} \widehat{\beta}(te) &= \lambda \int_{R^n} \frac{d\mu(v)}{\lambda^2 + t^2 (e.v)^2} \\ &\geq \lambda \int_{\{v; v.e=0\}} \frac{d\mu(v)}{\lambda^2 + t^2(e.v)^2} = \frac{1}{\lambda} \mu\{v; v.e = 0\} \end{aligned}$$

and consequently $\widehat{\beta}(.)$ does not go to zero in the *direction* $e \in S^{n-1}$. □

We are now ready to state the main result of this section under the assumption that the convergence in (6) is "sufficiently fast".

Theorem 7. *We assume that* Ω *has finite volume and there exists an integer* m *such that*

$$\int_{R^n} d\zeta \left| \int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v} \right|^{2m} < \infty. \tag{7}$$

Then $[M(\lambda - T)^{-1}]^{m+1}$ *is weakly compact and* $[M(\lambda - T)^{-1}]^{m+2}$ *is compact.*

Proof. It is easy to see that the weak compactness of $\left[M(\lambda - T)^{-1}M\right]^m$ implies the the weak compactness of $\left[M(\lambda - T)^{-1}\right]^{m+1}$. In view of the domination (5), it suffices that $R\left[M(\lambda - T_\infty)^{-1}M\right]^m E$ be weakly compact. To this end, it suffices that

$$R\left[M(\lambda - T_\infty)^{-1}M\right]^m : L^1(R^n \times \Xi) \to L^1(\Omega)$$

be weakly compact. We are going to show that $R\left[M(\lambda - T_\infty)^{-1}M\right]^m$ has the *smoothing* effect

$$R\left[M(\lambda - T_\infty)^{-1}M\right]^m : L^1(R^n \times \Xi) \to L^2(\Omega) \tag{8}$$

so the weak compactness will follow from the fact that bounded sets in $L^2(\Omega)$ are relatively weakly compact in $L^1(\Omega)$ since Ω has *finite* volume. To prove (8), it suffices to show that

$$\left[M(\lambda - T_\infty)^{-1}M\right]^m : L^1(R^n \times \Xi) \to L^2(R^n). \tag{9}$$

According to Proposition 5,

$$M(\lambda - T_\infty)^{-1} M\varphi = \beta * M\varphi, \ \ \varphi \in L^1(R^n \times \Xi),$$

where $\beta = \int_0^\infty e^{-\lambda t}\mu_t dt$. It follows that

$$\left[M(\lambda - T_\infty)^{-1} M\right]^m \varphi = \|\mu\|^{m-1}\, \nu * M\varphi,$$

where $\nu = \beta * \cdots * \beta$ (m times). The Fourier transform of $[M(\lambda - T_\infty)^{-1}M]^m \varphi$ is then equal to

$$\|\mu\|^{m-1} (2\pi)^{\frac{n}{2}} \widehat{\nu}(\zeta)\widehat{M\varphi}(\zeta) = \|\mu\|^{m-1} \Big[\int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v}\Big]^m \widehat{M\varphi}(\zeta).$$

By (7), $\widehat{\nu}(.) \in L^2(R^n)$ and so $\left[M(\lambda - T_\infty)^{-1} M\right]^m \varphi \in L^2(R^n)$. By the Parseval theorem, $\left\|\left[M(\lambda - T_\infty)^{-1} M\right]^m \varphi\right\|^2_{L^2(R^n)}$ is equal to

$$\begin{aligned} & \|\mu\|^{2(m-1)} \int_{R^n} d\zeta \left|\widehat{M\varphi}(\zeta)\right|^2 \Big| \int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v}\Big|^{2m} \\ \leq \ & \|\mu\|^{2(m-1)} (2\pi)^{-\frac{n}{2}} \|\varphi\|^2_{L^1(R^n\times\Xi)} \int_{R^n} d\zeta \Big| \int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v}\Big|^{2m}, \end{aligned}$$

which ends the proof of (9). Finally, the compactness of $[M(\lambda - T)^{-1}]^{m+2}$ is a consequence of the weak compactness of $[M(\lambda - T)^{-1}]^{m+1}$ and the fact that $M(\lambda - T)^{-1}$ maps weakly compact sets into compact sets (Proposition 1). □

We end this section with a sufficient geometrical condition on μ ensuring (7).

Corollary 8. *We assume there exist* $\alpha > 0$ *and* $c > 0$ *such that*

$$\sup_{e \in S^{n-1}} \mu\{v;\ |v.e| \leq \varepsilon\} \leq c\varepsilon^\alpha.$$

Then (7) *is satisfied by all integers* $m > \frac{n(\alpha+1)}{2\alpha}$.

Proof. We note that

$$|\widehat{\beta}(\zeta)| = (2\pi)^{-\frac{n}{2}} \Big| \int_{R^n} \frac{d\mu(v)}{\lambda + i\zeta.v}\Big| \leq (2\pi)^{-\frac{n}{2}} \int_{R^n} \frac{d\mu(v)}{\sqrt{\lambda^2 + |\zeta|^2 |e.v|^2}},$$

where $e = \frac{\zeta}{|\zeta|}$. Hence, for every $\varepsilon > 0$, $\left|\widehat{\beta}(\zeta)\right|$ is less than

$$(2\pi)^{-\frac{n}{2}} \Big[\frac{1}{\lambda}\mu\{v;\ |e.v| < \varepsilon\} + \frac{\|\mu\|}{|\zeta|\,\varepsilon}\Big] \leq (2\pi)^{-\frac{n}{2}} \Big[(\frac{c}{\lambda} + \|\mu\|)(\varepsilon^\alpha + \frac{1}{|\zeta|\,\varepsilon})\Big].$$

The choice $\varepsilon = \frac{1}{|\zeta|^{\frac{1}{\alpha+1}}}$ leads to $\left|\widehat{\beta}(\zeta)\right| \le (2\pi)^{-\frac{n}{2}} \frac{\frac{c}{\lambda}+\|\mu\|}{|\zeta|^{\frac{\alpha}{\alpha+1}}}$ and to

$$\left|\widehat{\beta}(\zeta)\right|^{2m} \le \frac{(2\pi)^{-mn}(\frac{c}{\lambda}+\|\mu\|)^{2m}}{|\zeta|^{\frac{2m\alpha}{\alpha+1}}}.$$

Hence it suffices that $\frac{2m\alpha}{\alpha+1} > n$; i.e., $m > \frac{n(\alpha+1)}{2\alpha}$. □

3. Evolution equations. A model case

We deal now with the c_0-semigroup $\{V(t); t \in R\}$ generated by the operator $T + M$ defined in Section 2. We recall that this perturbed semigroup is given by a Dyson-Phillips expansion $V(t) = \sum_{j=0}^{\infty} U_j(t)$ where

$$U_0(t) = U(t) \text{ and } U_j(t) = \int_0^t U(t-s) M U_{j-1}(s) ds \quad (j \ge 1).$$

Let $R_m(t) = \sum_{j=m}^{\infty} U_j(t)$ $(m \ge 1)$ be the remainder terms of the Dyson-Phillips expansion. We are concerned in this section with conditions on the velocity measure μ under which *some* remainder term $R_m(t)$ is weakly compact. We observe that $U_j = [UM]^j * U \quad (j \ge 1)$ where $*$ is the convolution operator which associates to strongly continuous (operator valued) mappings $f, g : [0, \infty[\to L(L^1(\Omega \times \Xi))$ the strongly continuous mapping

$$f * g : t \in [0, \infty) \to \int_0^t f(t-s) g(s) ds \in L(L^1(\Omega \times \Xi))$$

and $[UM]^j = [UM] * \cdots * [UM]$ (j times) where $[UM]$ denotes the mapping $t \ge 0 \to U(t)M$. We note that: $(f, g) \to f * g$ is associative. We recall (see [6] Chapter 2, Theorem 2.6, page 16) that $R_m(t)$ is weakly compact for all $t \ge 0$ *if and only if* $U_m(t)$ is. According to the convex compactness property of the space of weakly compact operators provided with the *strong* operator topology ([12] or [7]), the weak compactness of $[UM]^m(t)$ for all $t \ge 0$ implies the weak compactness of $U_m(t) = \int_0^t [UM]^m(s) U(t-s) ds$. Thus, we may deal with

$$[UM]^m(t) = [UM] * [UM] \cdots * [UM](t) \ (m \textit{ times}).$$

On the other hand, since $M^2 = \|\mu\| M$, one sees that

$$[UM]^m(t) = \frac{1}{\|\mu\|^{m-2}} U * [MUM] \cdots * [MUM](t),$$

where the term $[MUM]$ appears $m-1$ times and $[MUM]$ denotes the mapping $t \ge 0 \to MU(t)M$. By appealing again to the convex compactness property of the space of weakly compact operators provided with the strong operator topology, we may deal with the weak compactness of $[MUM]^{m-1}$.

The domination $U(t)\varphi \leq RU_\infty(t)E\varphi$ for $\varphi \in L^1_+(\Omega \times \Xi)$ implies

$$MU(t)M\varphi \leq MRU_\infty(t)EM\varphi = RMU_\infty(t)ME\varphi, \quad \varphi \in L^1_+(\Omega \times \Xi),$$

from which we derive the domination in the lattice sense

$$[MUM]^m(t) \leq R\,[MU_\infty M]^m(t)E. \tag{10}$$

Proposition 9. *For all $m \in N$ $(m \geq 1)$*

$$[MU_\infty M]^m(t)\varphi = \beta^m(t) * M\varphi \tag{11}$$

where $\beta^m(t)$ are finite Radon measures on R^n defined inductively by: $\beta^1(t) = \mu_t$ the image of μ under the dilation $v \to tv$ and

$$\beta^{j+1}(t) := \|\mu\| \int_0^t \mu_{t-s} * \beta^j(s)ds \ \ (j \geq 1).$$

Proof. Let $\varphi \in L^1(R^n \times \Xi)$. We have

$$\begin{aligned} [MU_\infty(t)M]\,\varphi &= \int (M\varphi)(x - tv)d\mu(v) \\ &= \int (M\varphi)(x-y)d\mu_t(y) = \mu_t * M\varphi, \end{aligned}$$

where μ_t is the image of μ under the dilation $v \to tv$. Let $\beta^1(t) := \mu_t$. It follows that $[MU_\infty M]^2(t)\varphi$ is equal to

$$\begin{aligned} &\int_0^t MU_\infty(t-s)MMU_\infty(s)M\varphi ds \\ &= \int_0^t \mu_{t-s} * M(\mu_s * M\varphi)ds = \|\mu\| \int_0^t \mu_{t-s} * \mu_s * M\varphi ds = \beta^2(t) * M\varphi, \end{aligned}$$

where $\beta^2(t) := \|\mu\| \int_0^t \mu_{t-s} * \mu_s ds$. More generally, $[MU_\infty M]^j(t)\varphi = \beta^j(t) * M\varphi$ where $\beta^j(t)$ is defined inductively by

$$\beta^{j+1}(t) := \|\mu\| \int_0^t \mu_{t-s} * \beta^j(s)ds \ \ (j \geq 1)$$

and this ends the proof. □

Before stating the main result of this section we recall ([8] Lemma 3) that the *affine* (i.e., translated) hyperplanes have zero μ measure if and only if

$$\sup_{e \in S^{n-1}} \mu \otimes \mu \left\{(v, v') \in \Xi \times \Xi;\ \left|(v - v').e\right| < \varepsilon\right\} \to 0 \ \text{ as } \ \varepsilon \to 0. \tag{12}$$

If we except the one-dimensional case (see Section 4), Assumption (12) alone does not seem to be sufficient to derive compactness results, in contrast with the L^p theory $(1 < p < \infty)$ [8] ; (see however [7] for *Dunford-Pettis* results). Our result is based on the assumption that the convergence in (12) is "sufficiently fast."

Theorem 10. *We assume there exist* $0 < \tau < 1$ *and an even integer* $m > \frac{n}{(1-\tau)}$ *such that*

$$\int_{|\zeta|\geq 1} d\zeta \Big[\sup_{e\in S^{n-1}} \mu\otimes\mu \Big\{ (v,v') \in \Xi\times\Xi;\ \big|(v-v').e\big| < \frac{1}{|\zeta|^{\tau}} \Big\} \Big]^m < \infty. \quad (13)$$

Then the remainder terms $R_j(t)$ *are weakly compact for all* $t \geq 0$ *and* $j \geq m+1$.

Proof. According to the domination (10), the weak compactness of

$$R\,[MU_\infty M]^m\,(t) : L^1(R^n\times\Xi) \to L^1(\Omega)$$

would imply the weak compactness of $[MUM]^m\,(t)$. The set Ω has *finite* volume so it suffices to show that

$$R\,[MU_\infty M]^m\,(t) : L^1(R^n\times\Xi) \to L^2(\Omega)$$

since bounded subsets of $L^2(\Omega)$ are relatively weakly compact in $L^1(\Omega)$. This is the case if we can prove

$$[MU_\infty M]^m\,(t) : L^1(R^n\times\Xi) \to L^2(R^n). \quad (14)$$

We choose an *even* integer $m = 2p$ $(p \in N)$. Then

$$[MU_\infty M]^{2p}\,(t)\varphi = \Big[[MU_\infty M]^2\Big]^p\,(t)\varphi = \beta^{2p}(t) * M\varphi.$$

We note first that $\beta^2(t) = \|\mu\| \int_0^t \mu_{t-s} * \mu_s ds$ so

$$\begin{aligned}
\widehat{\beta^2(t)}(\zeta) &= \|\mu\|\,(2\pi)^{\frac{n}{2}} \int_0^t \widehat{\mu_s}(\zeta)\widehat{\mu_{t-s}}(\zeta)ds \\
&= \|\mu\|\,(2\pi)^{-\frac{n}{2}} \int_0^t \Big[\int e^{-i\zeta.v} d\mu_s(v)\Big]\Big[\int e^{-i\zeta.v'} d\mu_{t-s}(v')\Big]ds \\
&= \|\mu\|\,(2\pi)^{-\frac{n}{2}} \int_0^t \Big[\int e^{-is\zeta.v} d\mu(v)\Big]\Big[\int e^{-i(t-s)\zeta.v'} d\mu(v')\Big]ds \\
&= \|\mu\|\,(2\pi)^{-\frac{n}{2}} \int\int \Big[\int_0^t e^{-is\zeta.v}e^{-i(t-s)\zeta.v'}ds\Big] d\mu(v)d\mu(v') \\
&= \|\mu\|\,(2\pi)^{-\frac{n}{2}} \int\int \frac{e^{-itv'.\zeta} - e^{-itv.\zeta}}{i(v-v').\zeta} d\mu(v)d\mu(v'). \quad (15)
\end{aligned}$$

Introduce the polar coordinates $\zeta = |\zeta|\, e$ and decompose the double integral in (15) as

$$\int\int_{\{(v,v');\ |(v-v').e|\le\varepsilon\}} \frac{e^{-itv'.\zeta}-e^{-itv.\zeta}}{i(v-v').\zeta} d\mu(v)d\mu(v')$$
$$+\int\int_{\{(v,v');\ |(v-v').e|>\varepsilon\}} \frac{e^{-itv'.\zeta}-e^{-itv.\zeta}}{i(v-v').\zeta} d\mu(v)d\mu(v'),$$

where $\varepsilon>0$ is arbitrary. Hence $\left|\widehat{\beta^2(t)}(\zeta)\right| \|\mu\|^{-1} (2\pi)^{\frac{n}{2}}$ is less than or equal to

$$t\int\int_{\{(v,v');\ |(v-v').e|\le\varepsilon\}} d\mu(v)d\mu(v') + \frac{2}{|\zeta|\,\varepsilon}\int\int d\mu(v)d\mu(v').$$

By the choice $\varepsilon = |\zeta|^{\tau}$, it follows that $\|\mu\|^{-1}(2\pi)^{\frac{n}{2}}\left|\widehat{\beta^2(t)}(\zeta)\right|$ is majorized by

$$t\sup_{e\in S^{n-1}}\mu\otimes\mu\Big\{(v,v')\in\Xi\times\Xi;\ \big|(v-v').e\big|<\frac{1}{|\zeta|^{\tau}}\Big\}+\frac{2\,\|\mu\|^2}{|\zeta|^{1-\tau}}$$

so

$$\left|\widehat{\beta^2(t)}(\zeta)\right| \le C_1(t)(a(\zeta)+b(\zeta))$$

where $b(\zeta) := \frac{1}{|\zeta|^{1-\tau}}$, $C_1(t) := \|\mu\|\,(2\pi)^{-\frac{n}{2}}\max\{t, 2\,\|\mu\|^2\}$ and

$$a(\zeta) := \sup_{e\in S^{n-1}}\mu\otimes\mu\Big\{(v,v')\in\Xi\times\Xi;\ \big|(v-v').e\big|<\frac{1}{|\zeta|^{\tau}}\Big\}.$$

Now $\beta^4(t) = \|\mu\|\int_0^t \beta^2(s)*\beta^2(t-s)ds$ and

$$\left|\widehat{\beta^4(t)}(\zeta)\right| = \|\mu\|\,(2\pi)^{\frac{n}{2}}\left|\int_0^t \widehat{\beta^2(s)}(\zeta)\widehat{\beta^2(t-s)}(\zeta)ds\right|$$
$$\le \|\mu\|\,(2\pi)^{\frac{n}{2}}\int_0^t C_1(s)C_1(t-s)ds(a(\zeta)+b(\zeta))^2 =: C_2(t)(a(\zeta)+b(\zeta))^2.$$

It follows by induction that there exists $C_p(t)$ locally bounded in t such that

$$\left|\widehat{\beta^{2p}(t)}(\zeta)\right| \le C_p(t)(a(\zeta)+b(\zeta))^p.$$

The Fourier transform of $[MU_\infty M]^{2p}\varphi$ is equal to

$$(2\pi)^{\frac{n}{2}}\widehat{\beta^{2p}(t)}(\zeta)\widehat{M\varphi}(\zeta) \tag{16}$$

whose modulus is majorized by $C_p(t)(a(\zeta)+b(\zeta))^p\,\|\varphi\|_{L^1}$. Hence, knowing that $m=2p$, $[MU_\infty M]^m(t)\varphi$ belongs to $L^2(R^n)$ provided that

$$\int_{|\zeta|\ge1}(a(\zeta)+b(\zeta))^m d\zeta<\infty.$$

Since $b(\zeta) = \frac{1}{|\zeta|^{1-\tau}}$, this is the case if

$$\int_{|\zeta|\geq 1} a(\zeta)^m d\zeta < \infty \text{ with } m > \frac{n}{1-\tau}$$

which amounts to Assumption (13). By Parseval's identity, (16) shows that $[MU_\infty M]^m(t)$ is a bounded operator from $L^1(R^n \times \Xi)$ into $L^2(R^n)$. Finally $[MUM]^m(t)$ is weakly compact and so is $R_{m+1}(t)$. By the convex compactness property of the space of weakly compact operators provided with the strong operator topology, it follows that $R_j(t)$ is weakly compact for all $j \geq m+1$. □

We give a sufficient geometrical condition on μ ensuring (13).

Corollary 11. *We assume there exist $c > 0$ and $\alpha > 0$ such that*

$$\sup_{e\in S^{n-1}} \mu\otimes\mu\left\{(v,v')\in \Xi\times\Xi;\ \left|(v-v').e\right| < \varepsilon\right\} \leq c\varepsilon^\alpha; \tag{17}$$

then $R_j(t)$ is weakly compact for all $t \geq 0$ and $j \geq j_0$ where j_0 is the smallest odd integer greater than $\frac{n(\alpha+1)}{\alpha} + 1$.

Proof. Under (17)

$$a(\zeta) + b(\zeta) \leq c\frac{1}{|\zeta|^{\alpha\tau}} + \frac{1}{|\zeta|^{1-\tau}}.$$

The choice $\alpha\tau = 1-\tau$ (i.e. $\tau = \frac{1}{\alpha+1}$) leads to $a(\zeta)+b(\zeta) \leq \frac{c'}{|\zeta|^{\frac{\alpha}{\alpha+1}}}$ and (13) amounts to $m > \frac{n(\alpha+1)}{\alpha}$. □

Remark 12. We point out that the weak compactness of some remainder term $R_m(t)$ for all $t \geq 0$ implies the compactness of $R_{m+2}(t)$ (see [7]).

4. Complementary results

In the present section, we complement the preceding results with some negative results. We give also very precise one-dimensional results. Here T, M, $U(t)$, and $V(t)$ are defined as in Sections 2 and 3.

Theorem 13. *Let $n \geq 3$ and $\lambda > 0$. Then, regardless of the choice of μ,*
(i) $(\lambda - T + M)^{-1} - (\lambda - T)^{-1}$ is not weakly compact.
(ii) For all $t > 0$, $V(t) - U(t)$ is not weakly compact.

Proof. Consider *(i).* It is easy to see that

$$(\lambda - T + M)^{-1} - (\lambda - T)^{-1} = \sum_{m=1}^{\infty} (\lambda - T)^{-1}\left[M(\lambda - T)^{-1}\right]^m \tag{18}$$

so that $(\lambda-T+M)^{-1}\geq(\lambda-T)^{-1}M(\lambda-T)^{-1}$ in the lattice sense. Hence the weak compactness of $(\lambda-T+M)^{-1}-(\lambda-T)^{-1}$ would imply that $(\lambda-T)^{-1}M(\lambda-T)^{-1}$ is also weakly compact. Let us show that the latter is *not* weakly compact if $n\geq 3$. Assume momentarily that Ω is convex. It is easy to see that

$$\begin{aligned}&(\lambda-T)^{-1}M(\lambda-T)^{-1}f\\ &=\int_0^{\tau(x,v)}e^{-\lambda t}dt\int_{\Xi}d\mu(v')\int_0^{\tau(x,v')}e^{-\lambda s}f(x-tv-sv',v')ds\\ &=\int_{\Xi}d\mu(v')\int_0^{\tau(x,v)}\int_0^{\tau(x,v')}e^{-\lambda t}e^{-\lambda s}f(x-tv-sv',v')dsdt\\ &=\int_{\Xi}d\mu(v')\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}e^{-\lambda s}f(x-tv-sv',v')dsdt\end{aligned}$$

thanks to the convexity of Ω where f has been extended by zero outside of Ω. Let $\{f_j\}_j\subset L^1(\Omega\times\Xi)$ be a normalized sequence converging in the weak star topology of measures on $\Omega\times\Xi$ to the Dirac mass $\delta_{(0,\overline{v})}=\delta_{x=0}\otimes\delta_{v=\overline{v}}$ where $\overline{v}\in\Xi$. Let $\psi\in C_c(\Omega\times\Xi)$, then

$$\begin{aligned}&\int_{\Omega\times\Xi}((\lambda-T)^{-1}M(\lambda-T)^{-1}f_j)\psi\\ &=\int_{\Xi}d\mu(v)\int_{\Omega}\psi(x,v)dx\int_{\Xi}d\mu(v')\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}e^{-\lambda s}f_j(x-tv-sv',v')dsdt\\ &=\int_{\Xi}d\mu(v)\int_{\Xi}d\mu(v')\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}e^{-\lambda s}dsdt\int_{\Omega}\psi(y+tv+sv',v)f_j(y,v')dy\\ &=\int_{\Omega\times\Xi}dyd\mu(v')f_j(y,v')\Big[\int_{\Xi}d\mu(v)\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}e^{-\lambda s}\psi(y+tv+sv',v)dsdt\Big]\end{aligned}$$

which tends to

$$\int_{\Xi}d\mu(v)\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}e^{-\lambda s}\psi(tv+s\overline{v},v)dsdt,$$

where ψ has been extended by zero outside of Ω. Thus $(\lambda-T)^{-1}M(\lambda-T)^{-1}f_j$ converges, in the weak star topology, to the finite Radon measure β on $\Omega\times\Xi$:

$$\psi\in C_c(\Omega\times V)\to\int_{\Xi}d\mu(v)\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}e^{-\lambda s}\psi(tv+s\overline{v},v)dsdt.$$

We claim that β is *not* a function. Indeed, if there exists $f\in L^1(\Omega\times\Xi)$ such that for all $\psi\in C_c(\Omega\times\Xi)$

$$\int_{\Xi}d\mu(v)\int_0^{\infty}\int_0^{\infty}e^{-\lambda t}e^{-\lambda s}\psi(tv+s\overline{v},v)dsdt=\int_{\Omega\times\Xi}f(x,v)\psi(x,v)dxd\mu(v)$$

then, for μ-*almost all* $v \in \Xi$, the measure on Ω

$$\varphi \in C_c(\Omega) \to \int_0^\infty \int_0^\infty e^{-\lambda t} e^{-\lambda s} \varphi(tv + s\overline{v}) ds dt \tag{19}$$

would have a density $x \to f(x, v)$ (with respect to the Euclidean n-dimensional Lebesgue measure dx) and, because of $n > 2$, this is inconsistent with the fact that (19) is supported by a *two*-dimensional linear space. Thus $(\lambda - T)^{-1} M (\lambda - T)^{-1}$ is not weakly compact and this ends the proof of *(i)* when Ω is convex. If Ω is not convex, we choose a convex open subset $\widehat{\Omega} \subset \Omega$. Let T_Ω (respectively $T_{\widehat{\Omega}}$) be the streaming operator corresponding to Ω (respectively $\widehat{\Omega}$). If $R_{\widehat{\Omega}} : L^1(\Omega \times \Xi) \to L^1(\widehat{\Omega} \times \Xi)$ is the restriction operator, a simple calculation shows that the restriction of $R_{\widehat{\Omega}}(\lambda - T_\Omega)^{-1} M (\lambda - T_\Omega)^{-1}$ to $L^1(\widehat{\Omega} \times \Xi)$ dominates $(\lambda - T_{\widehat{\Omega}})^{-1} M (\lambda - T_{\widehat{\Omega}})^{-1}$ in the lattice sense so the latter would be weakly compact and then *(i)* follows from the previous arguments.

Consider *(ii)*. The Dyson-Phillips expansion $V(t) - U(t) = \sum_{j=1}^\infty U_j(t)$ shows that $V(t) - U(t) \geq U_1(t)$ in the lattice sense so that the weak compactness of $V(t) - U(t)$ for *some* $t > 0$ would imply that $U_1(t)$ is also weakly compact. By arguing as in the proof of *(i)*, one shows that $U_1(t)$ is not weakly compact. □

Remark 14. It is not difficult to show that $(\lambda - T + M)^{-1} - (\lambda - T)^{-1}$ is weakly compact if and only if $(\lambda - T)^{-1} M (\lambda - T)^{-1}$ is. Thus, Theorem 13 shows that for $n \geq 3$ we cannot hope to avoid the hypothesis that some iterate of $M(\lambda - T)^{-1}$ be weakly compact. Similarly, $V(t) - U(t)$ is weakly compact if and only if $U_1(t)$ is, and Theorem 13 shows that we cannot avoid appealing to remainder terms $R_j(t)$ with $j \geq 2$. This justifies, a posteriori, Vidav's ideas [13] [14] but *only* for the L^1 context, the situation being completely different in L^p $(1 < p < \infty)$ [8].

Remark 15. We can show that if the hyperplanes have zero μ measure, then $(\lambda - T + M)^{-1} - (\lambda - T)^{-1}$ maps weakly compact sets into compact ones. The same result holds for $V(t) - U(t)$ if the affine hyperplanes have zero μ measure [7].

The case $n = 1$ is quite surprising. Indeed, we have:

Theorem 16. *Let $n = 1$ and $\Omega = (-a, a)$. Let μ be a finite positive Radon measure on R with support $\Xi \ni 0$.*

(i) $M(\lambda - T)^{-1}$ is an integral operator but is not weakly compact.

(ii) $(\lambda - T)^{-1} M$ is compact if and only if $\mu\{0\} = 0$. In particular, if $\mu\{0\} = 0$, then $(\lambda - T + M)^{-1} - (\lambda - T)^{-1}$ is compact.

(iii) $V(t) - U(t)$ is weakly compact for all $t \geq 0$ if and only if the measure μ is continuous; i.e., the points have zero μ-measure.

Proof. *(i)* The fact that $M(\lambda - T)^{-1}$ is not weakly compact has been noted in Proposition 1. It is also easy to see that it is an *integral* operator. To prove *(ii)*, note that

$$O\varphi = (\lambda - T)^{-1} M\varphi = \begin{cases} \frac{1}{|v|} \int_{-a}^{x} e^{-\lambda \frac{|x-y|}{|v|}} M\varphi(y) dy \text{ if } v > 0 \\ \frac{1}{|v|} \int_{x}^{a} e^{-\lambda \frac{|x-y|}{|v|}} M\varphi(y) dy \text{ if } v < 0. \end{cases}$$

Under the condition $\mu\{0\} = 0$, there exists a sequence $\{h_k\}_k$ of continuous functions with compact support such that, for each k, h_k vanishes in some neighborhood of $v = 0$ and $h_k \to 1$ in $L^1(\Xi)$. We approximate O by

$$O_k : \varphi \to \begin{cases} \frac{h_k(v)}{|v|} \int_{-a}^{x} e^{-\lambda \frac{|x-y|}{|v|}} M\varphi(y) dy \text{ if } v > 0 \\ \frac{h_k(v)}{|v|} \int_{x}^{a} e^{-\lambda \frac{|x-y|}{|v|}} M\varphi(y) dy \text{ if } v < 0. \end{cases}$$

It is not difficult to prove that O_k is a compact operator in $L^1(\Omega \times \Xi)$. A computation shows that

$$\|O - O_k\| \leq \frac{\|M\|_{L(L^1, L^1)}}{\lambda} \|1 - h_k\|_{L^1(\Xi)}$$

so $\|O - O_k\| \to 0$ as $k \to \infty$ and O is compact. Conversely, if $(\lambda - T)^{-1} M$ is compact in L^1, then by a standard interpolation argument $(\lambda - T)^{-1} M$ would be compact in L^p for $1 < p < \infty$ and this implies (see [6] Remark 3.1, page 35) that $\mu\{0\} = 0$.

Before showing *(iii)*, we note that μ is continuous if and only if

$$\sup_{v' \in \Xi} \mu\left\{\left[v' - \varepsilon, v' + \varepsilon\right]\right\} \to 0 \text{ as } \varepsilon \to 0.$$

Indeed, since μ is *finite*, $\mu\{v';\ |v'| \geq c\} \to 0$ as $c \to \infty$ and a simple compactness argument shows that $\mu\{[v' - \varepsilon, v' + \varepsilon]\} \to 0$ as $\varepsilon \to 0$ uniformly in $v' \in C$ with C bounded.

Suppose that μ is continuous. We recall that $V(t) - U(t)$ is weakly compact for all $t \geq 0$ if and only if $U_1(t)$ is weakly compact for all $t \geq 0$ [6] Chapter 2, Theorem 2.6. Let us show that $U_1(t)$ is weakly compact. We note that $U_1(t)\varphi$ is equal to

$$\int_0^t ds \int \varphi(x - (t-s)v - sv', v') \chi_{\{(s,t); x - (t-s)v - sv' \in \Omega\}} d\mu(v')$$
$$= \int d\mu(v') \int_0^t \varphi(x - (t-s)v - sv', v') \chi_{\{(s,t); x - (t-s)v - sv' \in \Omega\}} ds.$$

On the other hand,

$$\int_0^t \varphi(x-(t-s)v-sv',v')\chi_{\{(s,t);x-(t-s)v-sv'\in\Omega\}}ds$$
$$=\begin{cases}\int_{(x-tv)\vee(-a)}^{(x-tv')\wedge a}\varphi(y,v')\frac{dy}{|v-v'|} & \text{if } v'<v\\ \int_{(x-tv')\vee(-a)}^{(x-tv)\wedge a}\varphi(y,v')\frac{dy}{|v-v'|} & \text{if } v'>v\end{cases}$$

so

$$U_1(t)\varphi=\int_{-\infty}^{v}d\mu(v')\int_{(x-tv)\vee(-a)}^{(x-tv')\wedge a}\varphi(y,v')\frac{dy}{|v-v'|}$$
$$+\int_{v}^{\infty}d\mu(v')\int_{(x-tv')\vee(-a)}^{(x-tv)\wedge a}\varphi(y,v')\frac{dy}{|v-v'|}=O_1\varphi+O_2\varphi.$$

Let us show that both O_1 and O_2 are weakly compact. We restrict ourselves for instance to O_1 since the same argument holds for O_2. Note that O_1 is an integral operator

$$O_1\varphi=\int_\Xi\int_{-a}^{+a}\varphi(y,v')E(v,v',x,y)dyd\mu(v')$$

with kernel

$$E(v,v',x,y):=\chi_{\{v'<v\}}\chi_{\{y+tv'\leq x\leq y+tv\}}\left|v-v'\right|^{-1}. \tag{20}$$

Let

$$O_1^\varepsilon:\varphi\rightarrow\int_\Xi\int_{-a}^{+a}\varphi(y,v')E_\varepsilon(v,v',x,y)dyd\mu(v')$$

with kernel $E_\varepsilon(v,v',x,y)=E(v,v',x,y)\chi_{\{|v-v'|\geq\varepsilon\}}$. One sees that O_1^ε is weakly compact since $E_\varepsilon(.,.,.,.)$ is bounded and $[-a,a]\times\Xi$ has finite measure. It suffices to show that $O_1^\varepsilon\rightarrow O_1$ as $\varepsilon\rightarrow 0$ in the norm operator topology. We note that

$$\begin{aligned}\|O_1-O_1^\varepsilon\| &= \sup_{y,v'}\int_{\{(x,v);\,0\leq v-v'<\varepsilon,\,y+tv'\leq x\leq y+tv\}}(v-v')^{-1}dxd\mu(v)\\ &= t\sup_{v'}\mu\left\{\left[v',v'+\varepsilon\right]\right\}\rightarrow 0 \text{ as } \varepsilon\rightarrow 0\end{aligned}$$

which proves the first part of *(iii)*. Conversely, suppose that $V(t)-U(t)$ is weakly compact for all $t\geq 0$ or equivalently that $U_1(t)$ is weakly compact for all $t\geq 0$. Let us show first that $U_1(t)$ is compact in $L^p(\Omega\times R)$ $(1<p<\infty)$. To this end, we use an argument from [20], page 19: By the weak compactness of $U_1(t)$ in $L^1(\Omega\times R)$, for each decreasing sequence of measurable sets $A_m\subset\Omega\times R$ such that $\cap A_m=\emptyset$ we have $\|\chi_{A_m}U_1(t)\|_{L(L^1(\Omega\times R))}\rightarrow 0$ as $m\rightarrow\infty$. The Riesz interpolation theorem implies that $\|\chi_{A_m} U_1(t)\|_{L(L^p\ (\Omega\times R))} \to 0$ as $m \to \infty$ $(1 < p < \infty)$ which *characterizes* the compactness of the *integral* operator $U_1(t)$ in $L^p(\Omega \times R)$ $(1 < p < \infty)$ since R has finite μ measure; see [21] Corollary 4.2. It follows (see [8]), that the points should have zero μ measure. □

Remark 17. (i) The weak compactness of $(\lambda - T)^{-1}M$ in one dimension is known [10].

(ii) The case $n = 2$ is a borderline case between the two different situations described in Theorem 13 and 16. However we conjecture that Theorem 13 is still true for $n = 2$.

5. Spectral theory of neutron transport

In this section, we show how the above compactness results provide a sound foundation to L^1-spectral theory of neutron transport. Let $\Omega \subset R^n$ be an arbitrary open set with *finite* volume and let μ be a positive (not necessarily finite) Radon measure on R^n with support Ξ. We introduce the general advection c_0-semigroup $\{U(t); t \geq 0\}$ with no incoming particles boundary condition [19]

$$U(t) : \varphi \in L^1(\Omega\times\Xi) \to e^{-\int_0^t \sigma(x-sv,v)ds}\varphi(x-tv, v)\chi_{\{(x,v);\ t<\tau(x,v)\}} \in L^1(\Omega\times\Xi)$$

where $\sigma(.,.)$, the collision frequency, is real and bounded from *below*. We denote by T the generator of $\{U(t); t \geq 0\}$. Let K be a collision operator

$$K : \varphi \in L^1(\Omega \times \Xi) \to \int_\Xi k(x, v, v')\varphi(x, v')d\mu(v') \in L^1(\Omega \times \Xi)$$

with the assumption

$$\int_\Xi |k(., v, .)|\, d\mu(v) \in L^\infty(\Omega \times \Xi)$$

ensuring the boundedness of K in $L^1(\Omega \times \Xi)$. Let $\left\{V^K(t); t \geq 0\right\}$ be the c_0-semigroup generated by $T + K$. Following B. Lods [4], we assume that K is *regular* in L^1 in the sense that the family of operators (indexed by $x \in \Omega$)

$$\psi \in L^1(\Xi) \to \int_\Xi k(x, ., v')\psi(v')d\mu(v') \in L^1(\Xi)$$

is *collectively weakly compact*. This amounts to

$$\left\{\left|k(x, ., v')\right|;\ (x, v') \in \Omega \times \Xi\right\} \text{ is relatively weakly compact} \tag{21}$$

in $L^1(\Xi)$. This assumption can be checked by the well-known Dunford-Pettis criterion (see [1]). We note that the positive collision operator

$$|K| : \varphi \in L^1(\Omega \times \Xi) \to \int_\Xi \left|k(x,v,v')\right| \varphi(x,v')d\mu(v') \in L^1(\Omega \times \Xi)$$

is also regular. On the other hand,

$$\left|\left[K(\lambda - T)^{-1}\right]^m \varphi\right| \le \left[|K|\,(\lambda - T)^{-1}\right]^m |\varphi|$$

and $|U_j^K(t)\varphi| \le U_j^{|K|}(t)|\varphi|$, where $\{U_j^K\}$ denotes the terms of the Dyson-Phillips expansion of $V^K(t)$ and $\{U_j^{|K|}\}$ those of the semigroup $V^{|K|}(t)$ generated by $T + |K|$. Thus, as far as the weak compactness is concerned, by using domination arguments, it suffices to assume that the collision operator K is positive. On the other hand, if $k_i = k \wedge i\chi_{\{v\in\Xi;\ |v|\le i\}}$ and

$$K_i\varphi = \int_\Xi k_i(x,v,v')\varphi(x,v')d\mu(v'),$$

then

$$\|K - K_i\| \le \sup_{(x,v')\in\Omega\times\Xi} \int_{\{v\in\Xi;|v|>i\}} k(x,v,v')d\mu(v) \to 0 \ \text{ as } \ i \to \infty.$$

Thus, we may replace K by some truncation K_i since $[K(\lambda - T)^{-1}]^m$ and $U_j^K(t)$ depend continuously on K in the norm operator topology. This means that without loss of generality we may suppose in the sequel that Ξ is bounded and consequently that μ is *finite*. A basic property of a positive regular collision operator is that it can be approximated in the norm operator topology by collision operators *dominated* by collision operators of the form

$$\varphi \in L^1(\Omega \times \Xi) \to f(v) \int_\Xi \varphi(x,v')d\mu(v'), \tag{22}$$

where $f \in L^1(\Xi)$ [4]. Thus we may assume that K has the form (22). By approximation again we may suppose that $f \in L^1(\Xi) \cap L^\infty(\Xi)$ and finally, by a domination argument, we may even assume that f is a constant c. In this case, the collision operator K is nothing but the velocity averaging operator

$$M : \varphi \in L^1(\Omega \times \Xi) \to c \int_\Xi \varphi(x,v')d\mu(v').$$

Finally, we note that *all* the weak compactness results given in the previous sections, proved for $\sigma(.,.) = 0$, remain true, by a domination argument, for a general collision frequency $\sigma(.,.)$ bounded from below. Hence, the following results are consequences of Theorem 7, Corollary 8, Theorem 10, and Corollary 11.

Theorem 18. *Let $\Omega \subset R^n$ $(n \geq 2)$ be an arbitrary open set with finite volume. Let μ be a positive (not necessarily finite) Radon measure on R^n with support Ξ and let K be a regular collision operator in the sense* (21).

(i) We assume there exists $\alpha > 0$ such that for all $c > 0$ there exists $c' > 0$ such that

$$\sup_{e \in S^{n-1}} \mu\{v; |v| \leq c,\ |v.e| \leq \varepsilon\} \leq c' \varepsilon^{\alpha}. \tag{23}$$

Then some power of $K(\lambda - T)^{-1}$ is weakly compact $(\lambda > 0)$.

(ii) We assume there exists $\alpha > 0$ such that for all $c > 0$ there exists $c' > 0$ such that

$$\sup_{e \in S^{n-1}} \mu \otimes \mu \left\{(v, v'); |v| \leq c, \left|v'\right| \leq c,\ \left|(v - v').e\right| < \varepsilon\right\} \leq c' \varepsilon^{\alpha}. \tag{24}$$

Then some remainder term of the Dyson-Phillips expansion is weakly compact.

We are ready to summarize the main *spectral results.* The cases $n \geq 2$ and $n = 1$ are dealt with separately.

Theorem 19. *Let $n \geq 2$ and let $\Omega \subset R^n$ be an arbitrary open set with finite volume. Let μ be a positive (not necessarily finite) Radon measure on R^n with support Ξ. We assume that the collision operator K is regular in the sense* (21).

(i) If (23) *is satisfied, then $\sigma(T + K) \cap \{\operatorname{Re}\lambda > s(T)\}$ consists at most of isolated eigenvalues with finite algebraic multiplicities where $s(T)$ is the spectral bound of T.*

(ii) If (24) *is satisfied, then $\{U(t); t \geq 0\}$ and $\{V(t); t \geq 0\}$ have the same essential type and consequently $\sigma(V(t)) \cap \left\{\nu; |\nu| > e^{s(T)t}\right\}$ consists at most of isolated eigenvalues with finite algebraic multiplicities.*

In one dimension, Theorem 16 implies much more precise spectral results. Indeed, the essential spectrum in L^1 spaces being stable by weakly compact perturbations (see, for instance, [20] or [3]), we have:

Theorem 20. *Let $n = 1$ and $\Omega = (-a, a)$. Let μ be a positive (not necessarily finite) Radon measure on R with support $\Xi \ni 0$. We assume that the collision operator K is regular in the sense* (21).

(i) If $\mu\{0\} = 0$, then $T + K$ and T have the same essential spectrum. In particular, $\sigma(T+K) \cap \{\operatorname{Re}\lambda > s(T)\}$ consists at most of isolated eigenvalues with finite algebraic multiplicities where $s(T)$ is the spectral bound of T.

(ii) If the measure μ is continuous, then $V(t)$ and $U(t)$ have the same essential spectrum. In particular, $\sigma(V(t)) \cap \left\{\nu; |\nu| > e^{s(T)t}\right\}$ consists at most of isolated eigenvalues with finite algebraic multiplicities.

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