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COMPACTNESS AND QUASILINEAR PROBLEMS WITH CRITICAL EXPONENTS

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Dedicated to Professor Roger Temam for his 65^{th} birthday anniversary

Abstract. A compactness result is revised in order to prove the pointwise convergence of the gradients of a sequence of solutions to a general quasilinear inequality (anisotropic or not, degenerate or not) and for an arbitrary open set. Combining this result with the well-known Brézis-Lieb lemma, we derive simple proofs of Palais-Smale properties in many optimization problems especially on unbounded domains.

1. Introduction.

In his recent works [3, 4], the first author observed that the pointwise convergence obtained by the second author for measure data problems [12, 13, 14, 15], combined with the well-known Brézis-Lieb lemma [2], can be helpful for proving the strong convergence of the gradient in the variational problems with critical exponents even on arbitrary domains. Notice that the concentration-compactness principle due to Lions [11] and the concentration-compactness principle at infinity by Bianchi et al. [1] are widely used to overcome the difficulty due to the lack of compactness. In the case of bounded domains and compact manifolds, the Struwe decomposition is also very useful to recover compactness in nonlinear elliptic problems with critical exponent [17, 6].

For the reader's convenience and to have a self-contained paper, we rephrase, in a more general framework, compactness results established in

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[12, 13, 15] and *reproduce* the proofs given therein (for instance), since they are stated for bounded sets with the usual Leray-Lions operators, and for T-sets.

As examples of applications of our result, we recover in particular recent results in [7] on quasilinear elliptic equations involving critical Sobolev exponents. There is a large literature on critical exponents; see for instance [19, 20, 8, 7] and references therein. Two others examples are given, but our principal result can be applied to a large class of quasilinear elliptic problems where there holds a lack of compactness.

As a corollary of our result, we can state:

Lemma 1. *Let $\widehat{a}$ be a Caratheodory function from $\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}^N$ into $\mathbb{R}^N$ satisfying the usual Leray-Lions growth and monotonicity conditions. Let (u_n) be a bounded sequence in $W^{1,p}_{\text{loc}}(\mathbb{R}^N) = \{v \in L^p_{\text{loc}}(\mathbb{R}^N) : |\nabla v| \in L^p_{\text{loc}}(\mathbb{R}^N)\}$, with $1 < p < +\infty$, (f_n) be a bounded sequence in $L^1_{\text{loc}}(\mathbb{R}^N)$ and (g_n) be a sequence in $W^{-1,p'}_{\text{loc}}(\mathbb{R}^N)$ tending strongly to zero.*

Assume that (u_n) satisfies:

$$(H1) \quad \int_{\mathbb{R}^N} \widehat{a}\big(x, u_n(x), \nabla u_n(x)\big) \cdot \nabla \varphi \, dx = \int_{\mathbb{R}^N} f_n \varphi \, dx + < g_n, \varphi >,$$

$$\forall \varphi \in W^{1,p}_{comp}(\mathbb{R}^N) = \big\{v \in W^{1,p}(\mathbb{R}^N) \text{ with compact support}\big\}, \ \varphi \text{ bounded.}$$

Then

(1) *there exists a function u such that $u_n(x) \to u(x)$ almost everywhere in $\mathbb{R}^N$,*
(2) $u \in W^{1,p}_{\text{loc}}(\mathbb{R}^N)$,
(3) *there exists a subsequence, still denoted (u_n), such that*

$$\nabla u_n(x) \to \nabla u(x) \text{ a.e. } in \ \mathbb{R}^N.$$

This lemma generalizes in particular some results of Boccardo-Murat [18]. Thus, we give an alternate proof for critical exponents equations instead of the concentration-compactness principle of P.L. Lions [11] and the concentration-compactness at infinity of Bianchi and al. [1] for such problems.

The above lemma will be used to show that suitable Palais-Smale sequences are in fact relatively compact. Let us recall the following:

Définition 1 (Palais Smale sequence)**.** *Let X be a real Banach space and $I : X \to \mathbb{R}$ a functional mapping which is Gâteaux differentiable. We say that a sequence $(u_n)_{n\geqslant 0}$ is a Palais-Smale sequence if:*

(a) *there is a real number c such that* $\lim_{n\to+\infty} I(u_n) = c$.
(b) $\lim_{n\to+\infty} I'(u_n) = 0$ *strongly in the dual* X'.

Remark 1. In our application X will be a reflexive Banach space and the conditions (a) and (b) will (often) imply the boundedness of (u_n), from which we will have statement 1. and 2. (with additional compactness). The hypothesis (H1) will often follow from (b).

2. Notations and compactness result

Let Ω be an arbitrary open set in $\mathbb{R}^N$; we shall denote by $\omega \subset\subset \Omega$ any relatively compact open subset ω of Ω (that is, $\overline{\omega} \subset \Omega$, where $\overline{\omega}$ is the closure of ω).

Let $1 < p_i < +\infty,\ i = 1, \dots, N$; we set

$$p = \min\{p_i,\ \ 1 \leqslant i \leqslant N\},$$

$$\mathbb{L}^{\overrightarrow{p}}_{\mathrm{loc}}(\Omega) = \prod_{i=1}^{N} L^{p_i}_{\mathrm{loc}}(\Omega),\ \ \overrightarrow{p} = \big(p_1, \dots, p_N\big),$$

$$W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega) = \Big\{v \in \bigcup_{i=1}^{N} L^{p_i}_{\mathrm{loc}}(\Omega) : \nabla v \in \mathbb{L}^{\overrightarrow{p}}_{\mathrm{loc}}(\Omega)\Big\}.$$

We remark that $\bigcup_{i=1}^{N} L^{p_i}_{\mathrm{loc}}(\Omega) = L^{p}_{\mathrm{loc}}(\Omega)$.

For given $q \in (1, +\infty)$, we denote by $q' := \frac{q}{q-1}$ its conjugate exponent. We shall use the following globally real Lipschitz functions: For $\varepsilon > 0$, $\sigma \in \mathbb{R}$,

$$S_\varepsilon(\sigma) = \begin{cases} \sigma & \text{if } \ |\sigma| \leqslant \varepsilon \\ \varepsilon \mathrm{sign}(\sigma) & \text{otherwise,} \end{cases} \qquad \text{and } \sigma^k := S_k(\sigma) \text{ for } k \geqslant 1.$$

We shall consider a nonlinear map $\widehat{a} : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N$ satisfying:

(L1) $\widehat{a}(x, \cdot, \cdot)$ is a continuous map for almost every x and for all $(\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^N$ and $\widehat{a}(\cdot, \sigma, \xi)$ is measurable (such a property is called Caratheodory property);

(L2) $\widehat{a}$ maps bounded sets of $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)$ into bounded sets of $\prod_{i=1}^{N} L^{p_i'}_{\mathrm{loc}}(\Omega)$, and for almost all $x \in \Omega$, for all (σ, ξ) in $\mathbb{R} \times \mathbb{R}^N$, $\widehat{a}(x, \sigma, \xi) \cdot \xi \geqslant 0$, for almost every $x \in \Omega$ and for all $v \in W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)$; the mapping $u \mapsto \widehat{a}(x, u, \nabla v)$ is continuous from $W^{1,\overrightarrow{p}}(\omega)$-weak into $\prod_{i=1}^{N} L^{p_i'}(\omega)$-strong, for all $\omega \subset\subset \Omega$;

(L3) for almost every $x \in \Omega$, for all $(\sigma, \xi_i) \in \mathbb{R} \times \mathbb{R}^N,\ i = 1, 2,$

$$\big[\widehat{a}(x, \sigma, \xi_1) - \widehat{a}(x, \sigma, \xi_2)\big]\big[\xi_1 - \xi_2\big] > 0,\ \ for\ \xi_1 \neq \xi_2;$$

(L4) if, for some $x \in \Omega$, there is a sequence $(\sigma_n, \xi_{1n}) \in \mathbb{R} \times \mathbb{R}^N$, $\xi_2 \in \mathbb{R}^N$ such that $\left[\widehat{a}\big(x, \sigma_n, \xi_{1n}\big) - \widehat{a}\big(x, \sigma_n, \xi_2\big)\right]\left[\xi_{1n} - \xi_2\right]$ and σ_n are bounded as $n \to +\infty$ then $|\xi_{1n}|$ remains in a bounded set of $\mathbb{R}$ as $n \to +\infty$.

We start with this result concerning the convergence almost everywhere of the gradients:

Theorem 1. *Let (u_n) be a bounded sequence of $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)$. Then*

(i) *There is a subsequence still denoted (u_n) and a function $u \in W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)$ such that $u_n(x) \to u(x)$ almost everywhere in Ω as $n \to +\infty$.*

(ii) *If, furthermore, we assume* (L1)–(L4) *and that for all $\varphi \in C^\infty_c(\Omega)$, for all $k \geqslant k_0 > 0$:*

$$\limsup_{n \to +\infty} \int_\Omega \widehat{a}\big(x, u_n(x), \nabla u_n(x)\big) \cdot \nabla\big(\varphi S_\varepsilon(u_n - u^k)\big) \leqslant o(1)$$

as $\varepsilon \to 0$ then there exists a subsequence still denoted (u_n) such that

$$\nabla u_n(x) \to \nabla u(x) \text{ a.e. in } \Omega.$$

Remark 2.

2.1. The function $o(1)$ in (ii) might depend on k and φ.

2.2. As in [15], one can give a more general framework using T-sets instead of $W^{1,p}_{\mathrm{loc}}(\Omega)$ but in view of our applications to variational problems derived from Euler equations, this framework seems not appropriate.

2.3. The proof of (ii) is the same as in [13, 14, 15]; for convenience we reproduce it here.

2.4. The assumption (L2) is satisfied if $p_1 = \ldots = p_N = p$ and for all $\omega \subset\subset \Omega$, there is a constant $c_\omega > 0$ and a function $a_0 \in L^{p'}(\omega)$ such that for almost every $x \in \omega$, for all $(\sigma, \xi) \in \mathbb{R} \times \mathbb{R}^N$:

$$|\widehat{a}(x, \sigma, \xi)| \leqslant c_\omega\big[\,|\sigma|^{p-1} + |\xi|^{p-1} + a_0(x)\big],$$

and (L4) is true if $\widehat{a}(x, \sigma, \xi) \cdot \xi \geqslant c^1_\omega\,|\xi|^p - c^2_\omega, \quad c^1_\omega > 0$.

2.5. Bounded sets in $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)$ will be bounded in

$$W^{1,\overrightarrow{p}}(\omega) = \Big\{v \in \bigcup_{i=1}^N L^{p_i}(\omega) : \nabla v \in \prod_{j=1}^N L^{p_i}(\omega)\Big\}, \text{ for every } \omega \subset\subset \Omega.$$

2.6. If $p_1 = \ldots = p_N$, we adopt the usual notation for Sobolev spaces, $W^{1,p}(\omega)$, and we use 2.4.

Proof. (i) Let $(\omega_j)_{j \geqslant 0}$ be a sequence of bounded relatively compact subsets in Ω such that $\overline{\omega}_j \subset \omega_{j+1}$ and $\bigcup_{j=0}^{+\infty} \omega_j = \Omega$. Since $(u_n)_{n \geqslant 0}$ is bounded in

$W^{1,\overrightarrow{p}}(\omega_j)$, by the usual embeddings, we deduce that there is a subsequence $u_{n(j)}$ and a function u in $W^{1,\overrightarrow{p}}(\omega_j)$ such that $u_{n(j)}(x) \xrightarrow[n\to+\infty]{} u(x)$. We conclude with the usual diagonal Cantor process.

(*ii*) Let $\varphi \in C_c^\infty(\Omega)$, $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ on ω_j and $\operatorname{supp}(\varphi) \subset \omega_{j+1}$, and set

$$\Delta(u_n,u)(x) = \Big[\widehat{a}\big(x,u_n(x),\nabla u_n(x)\big) - \widehat{a}\big(x,u_n(x),\nabla u(x)\big)\Big]\nabla(u_n-u)(x).$$

Then one has:

(*ii*.1) $\Delta(u_n,u)(x) \geqslant 0$ almost everywhere on Ω (due to (L3)),

(*ii*.2) $\sup_n \int_{\omega_{j+1}} \Delta(u_n,u)dx$ is finite (since (u_n) is in a bounded subset of $W^{1,\overrightarrow{p}}_{\mathrm{loc}}(\Omega)$ and by the growth condition (L2)).

Let us show that $\lim_n \int_\Omega \varphi\Delta(u_n,u)^{\frac{1}{p}}dx = 0$. On one hand, we have

$$\int_\Omega \varphi\Delta(u_n,u)^{\frac{1}{p}}dx = \int_{\{|u|>k\}}\Delta(u_n,u)^{\frac{1}{p}}\varphi dx + \int_{\{|u|\leqslant k\}}\Delta(u_n,u)^{\frac{1}{p}}\varphi dx. \tag{2.1}$$

By the Hölder inequality and noticing that

$$\mathrm{meas}\Big\{x\in\omega_{j+1} : |u| > k\Big\} \leqslant \frac{c(j)}{k^p},$$

one deduces that

$$\int_{\{|u|>k\}}\Delta(u_n,u)^{\frac{1}{p}}\varphi dx \leqslant \frac{c_1(j)}{k^{p-1}}. \tag{2.2}$$

($c_m(j)$ are different constants depending on j and φ but independent of n, ε and k). While for the second integral, we have

$$\int_{\{|u|\leqslant k\}}\Delta(u_n,u)^{\frac{1}{p}}\varphi dx = \int_{\{|u|\leqslant k\}\cap\{|u_n-u|\leqslant\varepsilon\}}\Delta(u_n,u)^{\frac{1}{p}}\varphi dx + \int_{\{|u|\leqslant k\}\cap\{|u_n-u|>\varepsilon\}}\Delta(u_n,u)^{\frac{1}{p}}\varphi dx. \tag{2.3}$$

Moreover, the second term in the right-hand side of the last inequality satisfies

$$\int_{\{|u|\leqslant k\}\cap\{|u_n-u|>\varepsilon\}}\Delta(u_n,u)^{\frac{1}{p}}\varphi dx \leqslant c_2(j)\mathrm{meas}\Big\{x\in\omega_{j+1} : |u_n-u|(x) > \varepsilon\Big\}^{1-\frac{1}{p}}$$

and since (u_n) converges to u in measure, we deduce that, for n sufficiently large, $\mathrm{meas}\Big\{x\in\omega_{j+1} : |u_n-u|(x) > \varepsilon\Big\} \leqslant \varepsilon$. It follows that

$$\limsup_{n\to+\infty}\int_{\{|u|\leqslant k\}\cap\{|u_n-u|>\varepsilon\}}\Delta(u_n,u)^{\frac{1}{p}}\varphi dx \leqslant c_2(j)\varepsilon^{1-\frac{1}{p}}. \tag{2.4}$$

Setting $A^{\varepsilon}_{n,k} = \omega_{j+1} \cap \{|u| \leqslant k\} \cap \{|u_n - u| \leqslant \varepsilon\}$, we obtain from the Hölder inequality:

$$\int_{A^{\varepsilon}_{n,k}} \Delta(u_n, u)^{\frac{1}{p}} \varphi dx \leqslant c_3(j)\Big(I^1_{n,k}(\varepsilon) - I^2_{n,k}(\varepsilon)\Big)^{\frac{1}{p}}, \tag{2.5}$$

with

$$I^1_{n,k}(\varepsilon) = \int_{A^{\varepsilon}_{n,k}} \widehat{a}(x, u_n, \nabla u_n) \cdot \nabla(u_n - u)\varphi dx,$$

$$I^2_{n,k}(\varepsilon) = \int_{\{|u|\leqslant k\}} \widehat{a}(x, u_n, \nabla u) \cdot \nabla S_{\varepsilon}(u_n - u)\varphi dx.$$

Since $\widehat{a}(x, u_n, \nabla u) \to \widehat{a}(x, u, \nabla u)$ strongly in $\prod_{i=1}^{N} L^{p'_i}(\omega_{j+1})$ (by the last statement of (L2)) and $\nabla S_{\varepsilon}(u_n - u) \rightharpoonup 0$ in $\prod_{i=1}^{N} L^{p_i}(\omega_{j+1})$-weak, we deduce that

$$\lim_{n\to+\infty} I^2_{n,k}(\varepsilon) = 0, \tag{2.6}$$

while for the term $I^1_{n,k}(\varepsilon)$, we get:

$$\begin{aligned} I^1_{n,k}(\varepsilon) \quad \leqslant \quad & \int_{\Omega} \widehat{a}(x, u_n, \nabla u_n) \cdot \nabla\big(\varphi S_{\varepsilon}(u_n - u^k)\big) \\ & - \int_{\Omega} \widehat{a}(x, u_n, \nabla u_n) \cdot \nabla\varphi S_{\varepsilon}(u_n - u^k) dx. \end{aligned} \tag{2.7}$$

Since

$$\left|\int_{\Omega} \widehat{a}(x, u_n, \nabla u_n) \cdot \nabla\varphi S_{\varepsilon}(u_n - u^k) dx\right| \leqslant c_4(j)\varepsilon, \tag{2.8}$$

the assumption (ii) implies then

$$\limsup_{n\to+\infty} I^1_{n,k}(\varepsilon) \leqslant c_4(j)\varepsilon + o(1) \text{ as } \varepsilon \to 0. \tag{2.9}$$

Combining relations (2.5), (2.6), and (2.9), it follows:

$$\limsup_{n\to+\infty} \int_{A^{\varepsilon}_{n,k}} \Delta(u_n, u)^{\frac{1}{p}} \varphi dx \leqslant o(1) \text{ as } \varepsilon \to 0. \tag{2.10}$$

From relation (2.1), (2.2), (2.3), (2.4), and (2.10), we deduce:

$$\limsup_{n\to+\infty} \int_{\Omega} \Delta(u_n, u)^{\frac{1}{p}} \varphi dx \leqslant o(1) \text{ (as } \varepsilon \to 0) + O\Big(\frac{1}{k^{p-1}}\Big). \tag{2.11}$$

Letting first $\varepsilon \to 0$ and then k to infinity, we then obtain:

$$\lim_{n\to+\infty}\int_\Omega \Delta(u_n,u)^{\frac{1}{p}}\varphi dx = 0.$$

From this we deduce that for a subsequence $(u_{j_n})_{n\geqslant 0}$,

$$\Delta(u_{j_n},u)(x) \to 0 \text{ a.e. on } \omega_j.$$

Arguing as in Leray-Lions [10, 9], we deduce from (L4) that

$$\nabla u_{j_n}(x) \to \nabla u(x) \text{ a.e. in } \omega_j.$$

The proof is achieved by the diagonal process of Cantor. □

Proof of Lemma 1 (Corollary of Theorem 1). Here, we have $p_1 = \ldots = p_N = p$. Since (u_n) belongs to a bounded subset of $W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N)$, statement (i) of Theorem 1 implies that there is a function u and a subsequence still denoted by (u_n) such that

$$u_n(x) \xrightarrow[n\to+\infty]{} u(x) \text{ a.e. in } \mathbb{R}^N,$$

and

$$u \in W^{1,p}_{\mathrm{loc}}(\mathbb{R}^N).$$

Then for all $\varphi \in C_c^\infty(\mathbb{R}^N)$, $\varphi S_\varepsilon(u_n - u^k)$ is an element of $W^{1,p}_{comp}(\mathbb{R}^N)$ and

$$\left|\int_{\mathbb{R}^N} f_n\varphi S_\varepsilon(u_n-u^k)dx\right| \leqslant \varepsilon\,|\varphi|_\infty\,|f_n|_{L^1(\omega)} \leqslant c(\varphi)\varepsilon, \tag{2.12}$$

(for every φ such that $\operatorname{supp}(\varphi)\subset\omega$, $\overline{\omega}$ is a compact of $\mathbb{R}^N$), and

$$\left|< g_n, \varphi S_\varepsilon(u_n-u^k) >\right| \leqslant |g_n|_{W^{-1,p'}(\omega)}\left|\varphi S_\varepsilon(u_n-u^k)\right|_{W^{1,p}(\mathbb{R}^N)}.$$

Using the fact that $\left|\varphi S_\varepsilon(u_n-u^k)\right|_{W^{1,p}(\mathbb{R}^N)}$ is bounded independently of ε, n, k and that $|g_n|_{W^{-1,p'}(\omega)} \xrightarrow[n\to\infty]{} 0$, it holds:

$$\limsup_n \int_{\mathbb{R}^N}\widehat{a}(x,u_n,\nabla u_n)\cdot\nabla\big(\varphi S_\varepsilon(u_n-u^k)\big)dx \leqslant O(\varepsilon).$$

Finally, Theorem 1 ends the proof. □

Remark 3. Many extensions of Theorem 1 can be made (for instance on manifolds or on measure spaces). Here, we choose the above framework for the applications we made here. Nevertheless, one can use Theorem 1 for weighted spaces choosing correctly the open set Ω and the map $\widehat{a}$.

Some results in that direction have been already made by Marchi [16], and also by Fengquan Li, Zhao Huixiu [5] using the method of [12, 13].

3. Some examples of applications.

3.1. **Example 1.** We start by recovering a recent result of H. Ohya [7] using this alternate proof (without concentration-compactness principles) to show that we simplify the author's proof.

For this, we recall a part of the author's framework. Let $1 < p < N$, $p^* = \frac{Np}{N-p}$ and let Ω be an unbounded open set with smooth boundary if $\partial\Omega \neq \emptyset$, $\theta(x)$, $a(x)$, $K(x)$ be three nonnegative functions with the additional regularity that $\theta \in C^2(\Omega)$, $a \in L^r(\Omega)$ for some $r \in [\frac{N}{p}, +\infty]$, and K is such that $e^{(p-p^*)\theta(x)}K(x) = V(x)$, being bounded.

The author defined the following sets and quantities

$$L^p(\theta,\Omega) = \Big\{u \in L^p(\Omega); \int_\Omega e^{p\theta(x)}\,|u|^p\,dx < +\infty\Big\},$$

$$W^{1,p}(\theta,\Omega) = \Big\{u \in W_0^{1,p}(\Omega) : \int_\Omega e^{p\theta}\big(\,|u|^p + |\nabla u|^p\,\big) < +\infty\Big\},$$

$$\lambda_1 = \inf_{u\in W^{1,p}(\theta,\Omega)\setminus\{0\}} \left\{\int_\Omega e^{p\theta(x)}\,|\nabla u|^p\,dx \Big/ \int_\Omega e^{p\theta(x)} a(x)\,|u|^p\,dx\right\} > 0,$$

$$I_\theta(u) = \frac{1}{p}\int_\Omega e^{p\theta(x)}\Big(\,|\nabla u|^p - \lambda a(x)\,|u|^p\Big)dx - \frac{1}{p^*}\int_\Omega e^{p\theta(x)}K(x)\,|u|^{p^*}\,dx,$$

for $u \in W^{1,p}(\theta,\Omega)$. The author showed, under some hypotheses on θ and Ω, that $W^{1,p}(\theta,\Omega)$ is embedded continuously (respectively compactly) in $L^q(\theta,\Omega)$ provided that $q \in [p,p^*]$ (respectively $q \in [p,p^*)$). Moreover, in this situation the Poincaré inequality holds true, which implies that the semi-norm $\big(\int_\Omega e^{p\theta}\,|\nabla u|^p\,dx\big)^{\frac{1}{p}}$ is in fact a norm on $W^{1,p}(\theta,\Omega)$. We shall prove the following Palais-Smale property:

Theorem 2. *Under the above property, for every $\lambda < \lambda_1$, any Palais-Smale sequence $(u_m)_m$ of I_θ on $X = W^{1,p}(\theta,\Omega)$ satisfying*

(a) $I_\theta(u_m) \to b_\theta$

(b) $I'_\theta(u_m) \to 0$ *in* $\Big(W^{1,p}(\theta,\Omega)\Big)'$ *(dual of X)*

contains a convergent subsequence in $W^{1,p}(\theta,\Omega)$ provided that

$$0 < b_\theta < b_\theta^* = \frac{1}{N} S^{\frac{N}{p}}\,|V|_\infty^{-\frac{N-p}{p}},$$

where

$$S = \inf_{u\in W_0^{1,p}(\Omega)\setminus\{0\}} \Big\{\int_\Omega |\nabla u|^p\,dx \Big/ \Big(\int_\Omega |u|^{p^*}\,dx\Big)^{\frac{p}{p^*}}\Big\}.$$

Remark 4. Theorem 2 corresponds to Theorem 4.1 in [7].

Proof of Theorem 2. The sequence (u_m) is bounded in X as was observed in [7]. On one hand one has

$$I_\theta(u_m) - \frac{1}{p^*} < I'_\theta(u_m), u_m >= \frac{1}{N}\left\|u_m\right\|_X^p - \frac{\lambda}{N}\int_\Omega a(x)\left|u_m\right|^p e^{p\theta}dx$$

and applying the Poincaré-Sobolev inequality, there is a positive constant c such that

$$\frac{1}{N}\Big(1-\frac{\lambda}{\lambda_1}\Big)\left\|u_m\right\|_X^p \leqslant I_\theta(u_m) - \frac{1}{p^*} < I'_\theta(u_m), u_m >\leqslant c.$$

Thus, $\int_\Omega e^{p\theta(x)}K(x)\left|u_m\right|^{p^*}dx$ and $\int_\Omega e^{p\theta(x)}a(x)\left|u_m\right|^p dx$ are bounded independently of m. If we set

$$\widehat{a}(x,\xi) = e^{p\theta(x)}\left|\xi\right|^{p-2}\xi,\ x\in\Omega,\ \xi\in\mathbb{R}^N,$$

then $\widehat{a}$ satisfies conditions (L1)-(L4) (since $\theta \in C(\Omega)$). With the definition of X, we may then apply the first statement of Theorem 1 to conclude that there is an element $u \in X$ and a subsequence still denoted (u_m) such that

$$u_m(x) \to u(x) \text{ a.e. in } \Omega.$$

Furthermore, since $S_\varepsilon(u_m - u^k) \in X$ and $\left\|S_\varepsilon(u_m-u^k)\right\|_X \leqslant \left\|u_m\right\|_X + \left\|u\right\|_X$, we get for all $\varphi \in C_c^\infty(\Omega)$:

$$\begin{aligned}
&\int_\Omega \widehat{a}(x,\nabla u_m)\cdot\nabla\Big(\varphi S_\varepsilon(u_m-u^k)\Big)dx\\
&\leqslant \lambda\varepsilon\Big(\int_\Omega e^{p\theta(x)}\left|u_m\right|^p a(x)dx\Big)^{\frac{1}{p'}}\Big(\int_\Omega \varphi^p e^{p\theta(x)}a(x)dx\Big)^{\frac{1}{p}}\\
&+\varepsilon\Big(\int_\Omega e^{p\theta(x)}\left|u_m\right|^{p^*}K(x)dx\Big)^{\frac{1}{p^{*\prime}}}\Big(\int_\Omega \varphi^{p^*}e^{p\theta(x)}K(x)dx\Big)^{\frac{1}{p^*}}\\
&+c\left\|I'_\theta(u_m)\right\|_{X'}\Big(\left\|u_m\right\|_X + \left\|u\right\|_X\Big).
\end{aligned}$$

Thus,

$$\limsup_{m\to+\infty}\int_\Omega \widehat{a}(x,\nabla u_m)\cdot\nabla\big(\varphi S_\varepsilon(u_m-u^k)\big)dx \leqslant O(\varepsilon).$$

Thus, from Theorem 1, passing if necessary to a subsequence,

$$\nabla u_m(x) \xrightarrow[m\to+\infty]{} \nabla u(x) \text{ a.e. in } \Omega.$$

From Vitali's theorem, we then deduce that u is a critical point of I_θ; that is,

$$\forall \phi \in X, \ < I'_\theta(u), \phi >= 0.$$

At this stage, we apply the Brézis-Lieb lemma [2]; that is, from the equations,

$$< I'_\theta(u, u >= 0, \quad < I'_\theta(u_m), u_m >= o(1),$$

we then have

$$\lim_{m\to+\infty} \int_\Omega e^{p\theta(x)} \left|\nabla(u-u_m)\right|^p dx = \lim_{m\to+\infty} \int_\Omega e^{p\theta(x)} \left|u_m - u\right|^{p^*}(x) K(x) dx =: \ell.$$

(Note that from the integrability of a and the boundedness of (u_m) in X, we deduce that

$$\lim_{m\to+\infty} \int_\Omega a(x) \left|u_m - u\right|^p e^{p\theta(x)} dx = 0$$

(see Proposition 1 below).)

Since $V(x) = e^{(p-p^*)\theta(x)} K(x)$ is bounded, we deduce

$$\ell \leqslant |V|_\infty \int_\Omega e^{p^*\theta(x)} \left|u_m - u\right|^{p^*}(x) dx + o(1).$$

From the definition of S, one obtains $\ell \leqslant |V|_\infty S^{\frac{p^*}{p}} \ell^{\frac{p^*}{p}}$.

If $\ell = 0$, the proof is done. If $\ell \neq 0$, then $\ell \geqslant S^{\frac{N}{p}} |V|_\infty^{-\frac{N-p}{p}}$. Since $\lim_m I_\theta(u_m) = b_\theta$, thus,

$$b_\theta = \frac{1}{N} \int_\Omega |u|^{p^*} K(x) e^{p\theta(x)} dx + \frac{1}{N}\ell + o(1)$$

(still using Brézis-Lieb's lemma).

Then $b_\theta \geqslant \frac{1}{N}\ell \geqslant \frac{1}{N} S^{\frac{N}{p}} |V|_\infty^{-\frac{N-p}{p}} = b_\theta^*$, which gives a contradiction. Thus necessarily $\ell = 0$. □

In the next example, we give a classical existence result with a new proof based on our approach.

3.2. **Example 2.** In this paragraph, we are concerned with the existence of (at least) one positive solution to the elliptic problem

$$-\Delta_p u = \lambda a(x) |u|^{p-2} u + |u|^{p^*-2} u \ \text{ in } \mathbb{R}^N, \tag{3.1}$$

where the function a satisfies the following conditions: $a \geqslant 0$ on $\mathbb{R}^N$, $a \not\equiv 0$ and $a \in L^{\frac{N}{p}}(\mathbb{R}^N)$. The parameter λ is assumed to be positive, $1 < p < N$ and $N \geqslant 3$.

Consider the Euler-Lagrange functional associated to Problem (3.1) defined by

$$J_\lambda(u) := \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p \, dx - \frac{\lambda}{p}\int_{\mathbb{R}^N} a(x)\,|u|^p \, dx - \frac{1}{p^*}\int_{\mathbb{R}^N} |u|^{p^*} \, dx$$

which is of class $C^1(\mathcal{D}^{1,p}(\mathbb{R}^N))$. We recall that $\mathcal{D}^{1,p}(\mathbb{R}^N)$ is the completion of the space $\mathcal{D}(\mathbb{R}^N)$ with respect to the norm

$$\|\varphi\| := \Big(\int_{\mathbb{R}^N} |\nabla \varphi|^p \, dx\Big)^{\frac{1}{p}}.$$

The space $\mathcal{D}^{1,p}(\mathbb{R}^N)$ can also be seen as

$$\mathcal{D}^{1,p}(\mathbb{R}^N) = \Big\{\varphi \in L^{p^*}(\mathbb{R}^N) \;:\; |\nabla\varphi| \in L^p(\mathbb{R}^N)\Big\}.$$

By solutions of Problem (3.1) we understand critical points of the functional J_λ. Note that the functional J_λ is bounded neither above nor below on $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Then, to find possible critical points of J_λ, we limit the study to the corresponding Nehari manifold which contains all critical points of J_λ. We recall that the Nehari manifold associated to J_λ, denoted by $\mathcal{N}_{J_\lambda}$, is defined by

$$\mathcal{N}_{J_\lambda} := \big\{\varphi \in \mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\} \;:\; J_\lambda'(\varphi)(\varphi) = 0\big\}.$$

In the sequel, we will set $\|u\|_{a,p} := (\int_{\mathbb{R}^N} a(x)\,|u|^p \, dx)^{\frac{1}{p}}$.

Lemma 2. *For every $\lambda > 0$, the functional J_λ is bounded below on the Nehari manifold $\mathcal{N}_{J_\lambda}$.*

Proof. For every $u \in \mathcal{N}_{J_\lambda}$, it holds

$$J_\lambda(u) = \Big(\frac{1}{p} - \frac{1}{p^*}\Big)\|u\|_{p^*}^{p^*} = \frac{1}{N}\|u\|_{p^*}^{p^*},$$

and this ends the proof. □

As before, due to the integrability of a and the compact embedding $D^{1,p}(\mathbb{R}^N) \subset L^p_{\mathrm{loc}}(\mathbb{R}^N)$, Vitali's theorem implies the following statement:

Lemma 3. *The functional*

$$\mathcal{D}^{1,p}(\mathbb{R}^N) \longrightarrow \mathbb{R}, \quad u \longmapsto \int_{\mathbb{R}^N} a(x)\,|u|^p \, dx$$

is weakly continuous.

We recall that the Nehari manifold can be characterized more explicitly by

$$\mathcal{N}_{J_\lambda} := \Big\{ t\varphi \, ; \; (t, \varphi) \in (\mathbb{R} \setminus \{0\}) \times (\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}), \frac{d}{dt} J_\lambda(t\varphi) = 0 \Big\}.$$

For this reason, we introduce the modified functional

$$\widetilde{J}_\lambda(t, u) := J_\lambda(tu), \; \text{ on } \; \mathbb{R} \times \mathcal{D}^{1,p}(\mathbb{R}^N).$$

Since we are interested in positive solutions to Problem (3.1), we restrict ourselves in what follows to $t > 0$. A direct computation shows that for every $u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}$, there is a unique value $\lambda(u)$ of λ defined by

$$\lambda(u) = \frac{\|\nabla u\|_p^p}{\|u\|_{a,p}^p},$$

such that for every $\lambda \in (0, \lambda(u))$, one has $t(u, \lambda)u \in \mathcal{N}_{J_\lambda}$, where

$$t(u, \lambda) = \Big(\frac{\|\nabla u\|_p^p - \lambda \, \|u\|_{a,p}^p}{\|u\|_{p^*}^{p^*}} \Big)^{\frac{1}{p^*-p}}. \tag{3.2}$$

We introduce

$$\lambda_1 := \inf_{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}} \lambda(u)$$

which is not other than the first eigenvalue to the nonlinear eigenvalue problem

$$-\Delta_p u = \lambda a(x) \, |u|^{p-2} \, u \; \text{ in } \; \mathbb{R}^N.$$

From Lemma (3), we get clearly that the characteristic value λ_1 is positive. Now, for every $\lambda \in (0, \lambda_1)$ one has more precisely

$$\mathcal{N}_{J_\lambda} = \left\{ t(u, \lambda)u \; : \; u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\} \right\}.$$

For every $\lambda \in (0, \lambda_1)$, we introduce

$$\alpha(\lambda) := \inf_{u \in \mathcal{N}_{J_\lambda}} J_\lambda(u) = \inf_{u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}} J_\lambda(t(u, \lambda)u).$$

It is not difficult to see that the functional

$$\mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\} \longrightarrow \mathbb{R}, \quad u \longmapsto J_\lambda(t(u, \lambda)u)$$

is $0-$homogeneous. Then we get

$$\alpha(\lambda) = \inf_{u \in \mathbb{S}} J_\lambda(t(u, \lambda)u), \tag{3.3}$$

where $\mathbb{S}$ is the unit sphere in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. In this particular case, we have in fact

$$\alpha(\lambda) := \inf_{u\in\mathbb{S}} \Big(\frac{1}{p} - \frac{1}{p^*}\Big)\Big(\frac{\|\nabla u\|_p^p - \lambda \|u\|_{a,p}^p}{\|u\|_{p^*}^p}\Big)^{\frac{p^*}{p^*-p}}.$$

In what follows, we will write $(PS)_c$ to denote a Palais-Smale sequence in J_λ with the level $c \in \mathbb{R}$.

Lemma 4. *Let $\lambda \in (0, \lambda_1)$. There exists a minimizing sequence of* (3.3) *denoted by $(u_n) \subset \mathbb{S}$ such that*

(i) $0 < \liminf_{n\to\infty} t(u_n, \lambda) \leqslant \limsup_{n\to\infty} t(u_n, \lambda) < +\infty$.

(ii) $(t(u_n, \lambda)u_n)$ *is a bounded Palais-Smale sequence for J_λ.*

Proof. Let us denote $U_n := t(u_n, \lambda)u_n$; it holds obviously that $t(u_n, \lambda) = \|\nabla U_n\|_p$. On one hand, one has

$$\Big(\frac{1}{p} - \frac{1}{p^*}\Big)\Big(\|\nabla U_n\|_p^p - \lambda \|U_n\|_{a,p}^p \Big) = \alpha(\lambda) + \mathrm{o}_n(1).$$

Then,

$$\|\nabla U_n\|_p^p = \lambda \|U_n\|_{a,p}^p + N\alpha(\lambda) + \mathrm{o}_n(1).$$

Applying the Hölder and Young inequalities to $\|U_n\|_{a,p}^p$, we conclude that $\|\nabla U_n\|_p$ is bounded.

On the other hand, for every $u \in \mathbb{S}$, one gets

$$J_\lambda(tu) \geqslant \frac{t^p}{p}\Big(1 - \frac{\lambda}{\lambda_1}\Big) - \frac{t^{p^*}}{p^* S^{\frac{p^*}{p}}},$$

where S is the best Sobolev constant in the embedding $\mathcal{D}^{1,p}(\mathbb{R}^N) \subset L^{p^*}(\mathbb{R}^N)$. It follows that

$$\exists\, \delta(\lambda) > 0 \ \text{ such that } \ \alpha(\lambda) > \delta(\lambda) > 0, \tag{3.4}$$

and consequently $\liminf_{n\to\infty} \|\nabla U_n\|_p > 0$. This ends the claim (i).

For every $u \in \mathcal{D}^{1,p}(\mathbb{R}^N) \setminus \{0\}$ and $\lambda \in (0, \lambda_1)$, we have $\partial_t \widetilde{J}_\lambda(t(u, \lambda), u) = 0$ and $\partial_{tt} \widetilde{J}_\lambda(t(u, \lambda), u) < 0$. The implicit function theorem implies that $t(u, \lambda)$ is C^1 with respect to u since J_λ is. Let us introduce the C^1 functional $\mathcal{J}_\lambda$ defined on $\mathbb{S}$ by

$$\mathcal{J}_\lambda(u) = \widetilde{J}_\lambda(t(u, \lambda), u) \equiv J_\lambda(t(u, \lambda)u).$$

Then

$$\alpha(\lambda) = \inf_{u\in\mathbb{S}} \mathcal{J}_\lambda(u).$$

Using the Ekeland variational principle on the complete manifold $(\mathbb{S}, \|\ \|)$ to the functional $\mathcal{J}_\lambda$, there exists a minimizing sequence of (3.3) denoted by $(u_n) \subset \mathbb{S}$ such that:

$$\left|\mathcal{J}'_\lambda(u_n)(\varphi_n)\right| \leqslant \frac{1}{n}||\varphi_n||, \text{ for every } \varphi_n \in \mathrm{T}_{u_n}\mathbb{S},$$

where $\mathrm{T}_{u_n}\mathbb{S}$ is the tangent space to $\mathbb{S}$ at the point u_n. Moreover, for every $\varphi_n \in \mathrm{T}_{u_n}\mathbb{S}$, one has

$$\begin{aligned} \mathcal{J}'_\lambda(u_n)(\varphi_n) &= \partial_t\widetilde{J}_\lambda(t(u_n,\lambda),u_n)t'(u_n,\lambda)(\varphi_n) + \partial_u\widetilde{J}_\lambda(t(u_n,\lambda),u_n)(\varphi_n), \\ &= \partial_u\widetilde{J}_\lambda(t(u_n,\lambda),u_n)(\varphi_n), \end{aligned}$$

since $\partial_t\widetilde{J}_\lambda(t(u_n,\lambda),u_n) \equiv 0$, where $t'(u_n,\lambda)$ denotes the derivative of $t(.,\lambda)$ with respect to its first variable at the point (u_n,λ).
Furthermore, let

$$\begin{aligned} \pi\ :\ \ \mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\} &\longrightarrow \mathbb{R}\times\mathbb{S} \\ u &\longmapsto \left(||u||, \tfrac{u}{||u||}\right) := (\pi_1(u),\pi_2(u)). \end{aligned}$$

Applying Hölder's inequality, we get for every $(u,\varphi) \in \left(\mathcal{D}^{1,p}(\mathbb{R}^N)\setminus\{0\}\right) \times \mathcal{D}^{1,p}(\mathbb{R}^N)$:

$$|\pi'_1(u)(\varphi)| \leqslant ||\varphi||, \quad ||\pi'_2(u)(\varphi)|| \leqslant 2\frac{||\varphi||}{||u||}.$$

From (i), there is a positive constant C such that

$$t(u_n,\lambda) \geqslant C, \ \ \forall\ n \in \mathbb{N}.$$

Then, for every $\varphi \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, there are $\varphi_n^1 \in \mathbb{R}$ and $\varphi_n^2 \in \mathrm{T}_{u_n}\mathbb{S}$ such that $|\varphi_n^1| \leqslant ||\varphi||$, $||\varphi_n^2|| \leqslant \frac{2}{C}||\varphi||$ and

$$\begin{aligned} J'_\lambda(t(u_n,\lambda)u_n)(\varphi) &= \partial_t\widetilde{J}_\lambda(t(u_n,\lambda),u_n)(\varphi_n^1) + \partial_u\widetilde{J}_\lambda(t(u_n,\lambda),u_n)(\varphi_n^2) \\ &= \partial_u\widetilde{J}_\lambda(t(u_n,\lambda),u_n)(\varphi_n^2) = \mathcal{J}'_\lambda(u_n)(\varphi_n^2). \end{aligned}$$

Therefore,

$$J'_\lambda(t(u_n,\lambda)u_n)(\varphi) \leqslant \frac{1}{n}||\varphi_n^2|| \leqslant \frac{2}{nC}||\varphi||.$$

We easily conclude that

$$\lim_{n\to\infty} J'_\lambda(U_n) = 0 \ \text{ in } \ \mathcal{D}^{-1,p'}(\mathbb{R}^N),$$

which achieves the proof. □

As a consequence of Theorem 4 (see Example 3 with $\rho = \rho_1 = 1 = K$, $p_c = p^* = \frac{Np}{N-p}$), one has:

Lemma 5. *Let $\lambda \in (0,\lambda_1)$ and $(U_n) \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a Palais-Smale sequence for J_λ such that $U_n \rightharpoonup U$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$. Then, passing if necessary to a subsequence, we get*

$$\nabla U_n \longrightarrow \nabla U \quad a.e.\ in\ \mathbb{R}^N. \tag{3.5}$$

Lemma 6. *Let $\lambda \in (0,\lambda_1)$ and $(U_n) \subset \mathcal{D}^{1,p}(\mathbb{R}^N)$ be a (PS)$_c$ sequence for J_λ. If*

$$0 < c < \frac{1}{N} S^{\frac{N}{p}} \tag{3.6}$$

then Problem (3.1) *has a nontrivial solution.*

Proof. From Theorem 4, J_λ satisfies the Palais-Smale conditions if $0 < c < \frac{1}{N} S^{\frac{N}{p}}$. Thus, any weak limit U of U_n in $D^{1,p}(\mathbb{R}^N)$ satisfies then $J'_\lambda(U) = 0$, $J_\lambda(U) = c > 0$. Consequently, U is a nontrivial critical point of J_λ. □

Lemma 7. *For every $\lambda \in (0,\lambda_1)$ we have $0 < \alpha(\lambda) < \frac{1}{N} S^{\frac{N}{p}}$.*

Proof. Let

$$\Psi_\varepsilon(x) := \frac{\left(N\varepsilon \frac{N-p}{p-1}\right)^{\frac{N-p}{p^2}}}{\left(\varepsilon + |x|^{\frac{p}{p-1}}\right)^{\frac{N-p}{p}}}, \quad x \in \mathbb{R}^N, \quad \varepsilon > 0.$$

It is well known that

$$||\nabla \Psi_\varepsilon||_p^p = ||\Psi_\varepsilon||_{p^*}^{p^*} = S^{N/p}.$$

Moreover,

$$J_\lambda(t(\Psi_\varepsilon,\lambda)\Psi_\varepsilon) := \left(\frac{1}{p} - \frac{1}{p^*}\right)\left(\frac{||\nabla\Psi_\varepsilon||_p^p - \lambda||\Psi_\varepsilon||_{a,p}^p}{||\Psi_\varepsilon||_{p^*}^p}\right)^{\frac{p^*}{p^*-p}} < \frac{1}{N} S^{N/p}.$$

Then, using (3.4) we obtain

$$0 < \alpha(\lambda) < \frac{1}{N} S^{N/p}.$$ □

Theorem 3. *For every $\lambda \in (0,\lambda_1)$, Problem* (3.1) *admits at least one positive solution.*

Proof. From the preceding lemmas, it is clear that Problem (3.1) possesses a solution u which is nontrivial, since $\alpha(\lambda) \neq 0$. Since J_λ is even in $u, u \in \mathcal{D}^{1,p}(\mathbb{R}^N)$ implies that $|u| \in \mathcal{D}^{1,p}(\mathbb{R}^N)$, and $t(u,\lambda) = t(|u|,\lambda)$, we conclude that Problem (3.1) has a nontrivial nonnegative solution. The maximum principle achieves the proof. □

3.3. **Example 3.** Similar examples on weighted spaces can be made. Let $\rho,\ \rho_1$ be two nonnegative continuous functions on $\mathbb{R}^N$ and assume that the closed set $F=\left\{x\in\mathbb{R}^N:\rho(x)=0\right\}\cup\left\{x\in\mathbb{R}^N:\rho_1(x)=0\right\}$ is of measure zero. We set $\Omega=\mathbb{R}^N\backslash F$.

For every $1<p<+\infty$, we define

$$Y:=D^{1,p}(\mathbb{R}^N,\rho)=\left\{u\in L^p_{\mathrm{loc}}(\Omega):\int_{\mathbb{R}^N}|\nabla u|^p\,\rho(x)dx<+\infty\right\}.$$

Let $p_c>p$ so that

$$S_\rho=\operatorname*{Inf}_{Y\backslash\{0\}}\left\{\int_{\mathbb{R}^N}|\nabla u|^p\,\rho(x)dx\Big/\Big(\int_{\mathbb{R}^N}|u|^{p_c}\,\rho_1(x)dx\Big)^{\frac{p}{p_c}}\right\}>0.$$

We set $\dfrac{1}{N_c}=\dfrac{1}{p}-\dfrac{1}{p_c}$ and let

$$a\in L_+^{\frac{N_c}{p}}(\mathbb{R}^N,\rho_1)=\left\{f\geqslant 0:\int_{\mathbb{R}^N}f(x)^{\frac{N_c}{p}}\rho_1(x)dx<+\infty\right\},$$

and the Euler-Lagrange functional defined on Y by

$$\begin{aligned}I_\rho(u)=\int_{\mathbb{R}^N}|\nabla u|^p\,\rho(x)dx-\frac{\lambda}{p}\int_{\mathbb{R}^N}|u|^p\,a(x)\rho_1(x)dx\\-\frac{1}{p_c}\int_{\mathbb{R}^N}K(x)\,|u|^{p_c}\,\rho_1(x)dx,\end{aligned}$$

where K is a positive function in $L^\infty(\mathbb{R}^N)$. As previously, we have

$$\lambda_{1\rho}=\inf\Big(\int_{\mathbb{R}^N}|\nabla u|^p\,\rho(x)\Big/|u|^p\,a(x)\rho_1(x)dx\Big)>0$$

(see Proposition 1).

Theorem 4. *Let $\lambda<\lambda_{1\rho}$, and (u_m) be a Palais-Smale sequence of I_ρ satisfying*

(a) $I_\rho(u_m)\to b_\rho$,
(b) $I'_p(u_m)\to 0$ *in* Y'.

Then

(1) (u_m) *contains a weakly convergent subsequence to a function* $u\in Y$ *satisfying*

$$I'_\rho(u)=0.$$

(2) *This subsequence is strongly convergent in Y provided that*

$$0<b_\rho<b^*_\rho=\frac{1}{N_c}S_\rho^{\frac{N_c}{p}}\,|K|_\infty^{-\frac{N_c-p}{p}}\,.$$

We first need the following:

Proposition 1. *Under the above assumptions, if $(u_m)_m$ is weakly convergent to u in Y, then*

$$\lim_m \int_{\mathbb{R}^N} a(x)\,|u_m-u|^p\,\rho_1(x)dx=0.$$

Proof. Since $\int_{\mathbb{R}^N}|u_m|^{p_c}\,\rho_1(x)dx$ is bounded uniformly with respect to m, then for every subset A of $\mathbb{R}^N$

$$\int_A a(x)\,|u_m-u|^p\,\rho_1(x)\,dx \leqslant c\Big(\int_A a(x)^{\frac{N_c}{p}}\rho_1(x)dx\Big)^{\frac{p}{N_c}}<+\infty.$$

But $Y\subset W^{1,p}_{\mathrm{loc}}(\Omega)$ and then Y is compactly embedded in $L^p_{\mathrm{loc}}(\Omega)$.

Thus, using Vitali's theorem one has, for every bounded set ω in $\mathbb{R}^N$,

$$\lim_{m\to+\infty}\int_\omega a(x)\,|u_m-u|^p\,\rho_1(x)dx=0.$$

(Recall that $\Omega=\mathbb{R}^N\backslash F$ and $\mathrm{meas}\,(F)=0$). Combining the two relations and the integrability of a, we conclude

$$\lim_m \int_{\mathbb{R}^N} a(x)\,|u_m-u|^p\,\rho_1(x)dx=0. \qquad \square$$

Notice that the previous proposition implies that $\lambda_{1\rho}>0$.

The proof of Theorem 4 is similar to that of Theorem 2; we sketch it as the following:

Sketch of the proof of Theorem 4. Setting $\|u_m\|^p_Y=\int_{\mathbb{R}^N}|\nabla u_m|^p\,\rho(x)dx$, it follows that

$$\|u_m\|^p_Y\left(1-\frac{\lambda}{\lambda_{1\rho}}\right)\frac{1}{N_c}\leqslant I_\rho(u_m)-\frac{1}{p_c}<I'_\rho(u_m),u_m>.$$

Thus, $\|u_m\|_Y$ remains in a bounded subset of $\mathbb{R}$ which implies that

$$\int_{\mathbb{R}^N}a(x)\,|u_m|^p\,\rho_1(x)dx \text{ and } \int_{\mathbb{R}^N}|u_m|^{p_c}\,K(x)\rho_1(x)dx$$

are in a bounded subset of $\mathbb{R}$. Thus $\int_{\mathbb{R}^N}|u_m|^{p_c}\,\rho_1(x)dx$ is bounded. We deduce in particular that (u_m) remains in a bounded subset of $W^{1,p}_{\mathrm{loc}}(\Omega)$ (since for every $\omega\subset\subset\Omega$, $\rho_\omega=\inf_\omega\rho(x)>0$ and $\rho_{1\omega}=\inf_\omega\rho_1(x)>0$). We may appeal to the first statement (i) of Theorem 1: There exist $u\in D^{1,p}(\mathbb{R}^N,\rho)$ and a subsequence still denoted by $(u_m)_m$ such that $u_m(x)\to u(x)$ almost everywhere in $\mathbb{R}^N$. Moreover, since $S_\varepsilon(u_m-u^k)\in Y$, if we set $\widehat{a}(x,\xi)=\rho(x)\,|\xi|^{p-2}\,\xi$, for all $x\in\Omega$ and $\xi\in\mathbb{R}^N$, then the conditions (L1)-(L4) are satisfied. Furthermore, for all $\varphi \in C_c^\infty(\Omega)$, choosing $\omega \subset\subset \Omega$ such that $\operatorname{supp}\varphi \subset \omega$, $\sup_\omega \rho(x)$ and $\sup_\omega \rho_1(x)$ are finite, it holds then

$$\int_\Omega \widehat{a}\big(x, \nabla u_m(x)\big) \cdot \nabla\big(\varphi S_\varepsilon(u_m - u^k)\big) dx \leqslant c(\varphi)\varepsilon + \big\| I'_\rho(u_m \big\|_{Y'} \left(\|u_m\|_Y + \|u\|_Y \right).$$

Consequently, statement (ii) of Theorem 1 is also satisfied, so we conclude that (for a subsequence) $\nabla u_m(x) \to \nabla u(x)$ almost everywhere on $\mathbb{R}^N$. From Vitali's theorem, we deduce that

$$\forall \varphi \in D^{1,p}(\mathbb{R}^N) < I'_\rho(u), \varphi >= 0.$$

To conclude that $\|u_m - u\|_Y \to 0$, we use the Brézis-Lieb lemma to get

$$\int_{\mathbb{R}^N} |\nabla u_m - \nabla u|^p \rho(x) dx = -\int_{\mathbb{R}^N} |\nabla u|^p \rho(x) dx + \int_{\mathbb{R}^N} |\nabla u_m|^p \rho(x) dx + o(1),$$

$$\begin{aligned}\int_{\mathbb{R}^N} |u_m - u|^{p_c} K(x)\rho_1(x) dx \ &= \ -\int_{\mathbb{R}^N} |u|^{p_c} K(x)\rho_1(x) dx \\ &\quad + \int_{\mathbb{R}^N} |\nabla u_m|^{p_c} K(x)\rho_1(x) dx + o(1).\end{aligned}$$

Since $< I'_\rho(u), u >= 0$ and $< I'_\rho(u_m), u_m >= o(1)$, the above equalities imply that

$$\lim_{m\to\infty} \|u_m - u\|_Y^p = \lim_m \int_{\mathbb{R}^N} |u_m - u|^{p_c} K(x)\rho_1(x) dx =: \ell_0,$$

noticing that

$$\lim_m \int_{\mathbb{R}^N} a(x) |u_m - u|^p \rho_1(x) dx = 0$$

(see Proposition 1). Since $K \in L^\infty(\mathbb{R}^N)$

$$\int_{\mathbb{R}^N} |u_m - u|^{p_c} K(x)\rho_1(x) dx \leqslant |K|_\infty S_p^{\frac{p_c}{p}} \left(\int_{\mathbb{R}^N} |\nabla(u_m - u)|^p \rho(x) dx \right)^{\frac{p_c}{p}}.$$

Thus,

$$\ell_0 \leqslant |K|_\infty S_\rho^{\frac{p_c}{p}} \ell_0^{\frac{p_c}{p}}.$$

If $\ell_0 = 0$ the proof is done. If $\ell_0 \neq 0$, then $\ell_0 \geqslant S_\rho^{\frac{Nc}{p}} |K|_\infty^{-\frac{N_c - p}{p}}$. Since

$$\lim_m I_\rho(u_m) = b_\rho, b_\rho \geqslant \frac{1}{N_c}\ell_0 \geqslant \frac{1}{N_c} S_\rho^{-\frac{N_c - p}{p}} |K|_\infty^{-\frac{N_c - p}{p}} = b_\rho^*,$$

which leads to a contradiction. □

Theorem 5. *There exists $\lambda^* \in (0, \lambda_{1\rho})$ such that the problem*

$$-\mathrm{div}\left(\rho(x)|\nabla u|^{p-2}\nabla u\right) = \rho_1(x)\left(\lambda a(x)|u|^{p-2}u + K(x)|u|^{p_c-2}u\right) \quad \text{in } \mathbb{R}^N \tag{3.7}$$

has a nontrivial positive solution for every $\lambda \in (\lambda^*, \lambda_{1\rho})$.

Sketch of the proof. Let $\lambda \in (0, \lambda_{1\rho})$ and Φ be a positive eigenfunction associated to the positive eigenvalue $\lambda_{1\rho}$. It follows that

$$I_\rho(t(\Phi,\lambda)\Phi) = \frac{C}{N_c}\Big(1 - \frac{\lambda}{\lambda_{1\rho}}\Big)^{N_c/p},$$

where C is a positive constant depending on the data of Problem 3.7 and $t(.,.)$ is defined in the same manner as (3.2) with minor changes. It is clear that there exists $\lambda^* \in (0, \lambda_{1\rho})$ such that for all $\lambda \in (\lambda^*, \lambda_{1\rho})$:

$$0 < \frac{C}{N_c}\Big(1 - \frac{\lambda}{\lambda_{1\rho}}\Big)^{N_c/p} < \frac{1}{N_c} S_\rho^{\frac{N_c}{p}} \, |K|_\infty^{-\frac{N_c-p}{p}} .$$

Following the different steps of the previous example with slight changes, we get the result. □

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REFERENCES

[1] G. Bianchi, J. Chabrowski, and A. Szulkin, *On symmetric solutions of an elliptic equation with a nonlinearity involving critical Sobolev exponent,* Nonlinear Anal., 25 (1995), 41-59.

[2] H. Brézis and E. Lieb, *A relation between pointwise convergence of functions and convergence of functional,* Proceedings of the AMS, vol 88, N°3 (July 1983).

[3] A. EL Hamidi, "Sur les conditions nécessaires et suffisantes d'existence de solutions pour des problèmes nonlinéaires," Habilitation à Diriger des Recherches, Université de La Rochelle, (2004).

[4] A. EL Hamidi, *Multiple solutions with changing sign energy in a nonlinear equation*, Commun. Pure Appl. Anal., 3 (2004), 253–260.

[5] Fengquan Li and Zhao Huixiu, *Anisotropic parabolic equations with measure data*, J. Partial Diff. Eqs, 14 (2001), 21–30.

[6] E. Hebey and F. Robert, *Coercivity and Struwe's compactness for Paneitz type operators with constant coefficients*, Calc. Var. Partial Differential Equations, 13 (2001), 491–517

[7] H. Ohya, *Existence results for some quasilinear elliptic equations involving critical Sobolev exponents*, Adv. Diff. Equ., 9 (2004), 1339–1368.

[8] P. De Napoli and M.C. Mariani, *Mountain pass solutions to equations of p-Laplacian type*, Nonlinear Analysis, 54 (2003), 1205–1219.
[9] J.L. Lions, "Quelques méthodes de résolution de problèmes aux limites nonlinéaires," Dunod Paris, (1969).
[10] J. Leray and J.L. Lions, *Quelques résultats de Visik sur les problèmes elliptiques non linéaires par les méthodes de Minty-Browder*, Bull. Soc. Math. France, 93 (1965), 97–107.
[11] P.L. Lions, *The concentration-compactness principle in the calculus of variations. The limit case, I, II,* Rev. Mat. Iberoamericana, 1 (1985), 145–201 and 45–121.
[12] J.M. Rakotoson, *Compactness results for quasilinear elliptic problems*, Applied Math. Letters, 4 (1991), 33–35.
[13] J.M. Rakotoson, *Quasilinear elliptic problems with measure as data*, Diff. Int. Equa., 4 (1991), 449–457.
[14] J.M. Rakotoson, *Equations et inéquations avec données mesures*, C.R.A.S, t 314 (1992), 105–107.
[15] J.M. Rakotoson, *Generalized solution in a new type sets for problems with measures as data*, Diff. Int. Equ., 6 (1993), 27–36.
[16] S. Marchi, *Note on a paper of J.M. Rakotoson*, Equationi a derivate parziali, Instituto Lombardo (Rend. Sc.), A 127 (1991), 25–31.
[17] M. Struwe, *A global compactness result for elliptic boundary value problems involving limiting nonlinearities*, Math. Z., 187 (1984), 511–517.
[18] L. Boccardo and F. Murat, *Almost everywhere convergence of the gradients of solutions to elliptic and parabolic equations*, Nonlinear Analysis, 19 (1992), 581–597.
[19] F. Demengel and E. Hebey, *On some nonlinear equations involving the p-Laplacian with critical Sobolev growth*, Adv. Differential Equations, 3 (1998), 533–574.
[20] F. Demengel and E. Hebey, *On some nonlinear equations involving the p-Laplacian with critical Sobolev growth and perturbation terms*, Appl. Anal., 72 (1999), 75–109.