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ON BLOW-UP RESULTS FOR SOLUTIONS OF INHOMOGENEOUS EVOLUTION EQUATIONS AND INEQUALITIES. II

A.G. Kartsatos
Department of Mathematics, University of South Florida
Tampa, Florida 33620-5700

V.V. Kurta
Mathematical Reviews, 416 Fourth Street, P.O. Box 8604
Ann Arbor, Michigan 48107-8604

(Submitted by: Reza Aftabizadeh)

Abstract. The main purpose of this work is to further develop ideas and methods from a recent paper by the authors. In particular, we obtain new blow-up results for solutions of the inequality

$$|u|_t \geq \Delta[|u|^\sigma u] + |u|^q + \omega(x)$$

on the half-space $\mathbb{R}^1_+ \times \mathbb{R}^n$, where $n \geq 1$, $\sigma \geq 0$, $q > 1+\sigma$, and $\omega : \mathbb{R}^n \to \mathbb{R}^1$ is a globally integrable function such that $\int_{\mathbb{R}^n} \omega(x)dx > 0$, and establish that for $n > 2$ the critical blow-up exponent $q^* = n(1+\sigma)/(n-2)$ is of the blow-up type.

1. Introduction and preliminaries

In what follows, $n \geq 1$ is a natural number, $q > 1$ and $\sigma \geq 0$ are real numbers, $\omega : \mathbb{R}^n \to \mathbb{R}^1$ is a measurable and globally integrable function, $\Delta := \sum_{i=1}^n \partial^2/\partial x_i^2$, and $\mathbb{S} := (0, +\infty) \times \mathbb{R}^n$.

We obtain new blow-up results for solutions of the inequality

$$|u|_t \geq \Delta[|u|^\sigma u] + |u|^q + \omega(x) \tag{1}$$

and the corresponding equation

$$|u|_t = \Delta[|u|^\sigma u] + |u|^q + \omega(x) \tag{2}$$

on the half-space $\mathbb{S}$. As a consequence, we obtain new blow-up results for nonnegative solutions of the inequality

$$u_t \geq \Delta[|u|^\sigma u] + |u|^{q-1}u + \omega(x) \tag{3}$$

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and the corresponding equation

$$u_t = \Delta[|u|^\sigma u] + |u|^{q-1}u + \omega(x) \tag{4}$$

on the half-space $\mathbb{S}$. As another consequence of our results, we have blow-up results for global solutions of the Cauchy problem for (1) and (2), and, respectively, for global nonnegative solutions of the Cauchy problem for (3) and (4), with arbitrary initial data. Especially, we would like to stress that using the approach developed here, which in turn is a further development of ideas and methods from [3], one can easily prove that the Cauchy problem for the inequality

$$u_t \geq \Delta[|u|^\sigma u] + |u|^q + \omega(x) \tag{5}$$

and the corresponding equation

$$u_t = \Delta[|u|^\sigma u] + |u|^q + \omega(x), \tag{6}$$

on the half-space $\mathbb{S}$ with nonnegative initial data $u_0(x)$ on $\mathbb{R}^n$, has no global solutions if $\int_{\mathbb{R}^n} \omega(x)dx > 0$ and $1 + \sigma < q \leq n(1+\sigma)/(n-2)$ for $n > 2$, or if $\int_{\mathbb{R}^n} \omega(x)dx > 0$ and $1 + \sigma < q < \infty$ for $n = 1$ or $n = 2$. Therefore, using the approach developed in this paper, one can improve the main result of [1], which has been an inspiration for our investigations, by showing that for $n > 2$ and $\sigma = 0$ the critical blow-up exponent $q^* = n/(n-2)$ is of the blow-up type for solutions of the Cauchy problem for the equation

$$u_t = \Delta u + |u|^q + \omega(x), \tag{7}$$

on the half-space $\mathbb{S}$ with nonnegative initial data $u_0(x)$ on $\mathbb{R}^n$, if $\int_{\mathbb{R}^n} \omega(x)dx > 0$. In this connection, the reader is referred to the abstract of [1] and Theorem 2.1 (b) therein.

Finally, we would like to note that the approach developed here, as well as in our earlier papers (see, e.g., [3, 4, 5, 6]), for model problems is directly applicable to situations involving wide classes of equations, inequalities and systems of equations and inequalities, with linear and nonlinear differential operators, considered on the half-space $\mathbb{S}$ as well as on other unbounded domains in $\mathbb{R}^1 \times \mathbb{R}^n$ and on Riemannian manifolds. For a survey of blow-up results for solutions of the Cauchy problem we refer to [2, 8, 9] and the references therein.

2. The results

In what follows, $q^* = n(1+\sigma)/(n-2)$ if $n > 2$, and $q^* = \infty$ if $n = 1$ or 2. Moreover, $q > 1$ is a real number and the inequality $q \leq q^*$ means that $q < \infty$ if $q^* = \infty$ and $q \leq q^*$ if $q^* < \infty$.

Definition 1. By a solution of (1) on $\mathbb{S}$ we understand a function $u : \mathbb{S} \to \mathbb{R}^1$ which belongs to the space $L_{q,\text{loc}}(\mathbb{S}) \cap L_{1+\sigma,\text{loc}}(\mathbb{S})$ and satisfies the integral inequality

$$\int_{\mathbb{S}} \left[-|u|\varphi_t - u|u|^\sigma \Delta\varphi\right] dtdx \geq \int_{\mathbb{S}} \left[|u|^q + \omega(x)\right] \varphi dtdx \tag{1'}$$

for every nonnegative function $\varphi \in C^\infty(\mathbb{S})$ with compact support.

Theorem 1. *Let* $1 + \sigma < q \leq q^*$ *and* $\int_{\mathbb{R}^n} \omega(x)dx > 0$. *Then* (1) *has no solutions on* $\mathbb{S}$.

Definition 2. By a solution of (2) on $\mathbb{S}$ we understand a function $u : \mathbb{S} \to \mathbb{R}^1$ which belongs to the space $L_{q,\text{loc}}(\mathbb{S}) \cap L_{1+\sigma,\text{loc}}(\mathbb{S})$ and satisfies the integral identity

$$\int_{\mathbb{S}} \left[-|u|\varphi_t - u|u|^\sigma \Delta\varphi\right] dtdx = \int_{\mathbb{S}} \left[|u|^q + \omega(x)\right] \varphi dtdx \tag{2'}$$

for every function $\varphi \in C^\infty(\mathbb{S})$ with compact support.

Theorem 2. *Let* $1 + \sigma < q \leq q^*$ *and* $\int_{\mathbb{R}^n} \omega(x)dx > 0$. *Then* (2) *has no solutions on* $\mathbb{S}$.

Definition 3. By a solution of (3) on $\mathbb{S}$ we understand a function $u : \mathbb{S} \to \mathbb{R}^1$ which belongs to the space $L_{q,\text{loc}}(\mathbb{S}) \cap L_{1+\sigma,\text{loc}}(\mathbb{S})$ and satisfies the integral inequality

$$\int_{\mathbb{S}} \left[-u\varphi_t - u|u|^\sigma \Delta\varphi\right] dtdx \geq \int_{\mathbb{S}} \left[u|u|^{q-1} + \omega(x)\right] \varphi dtdx \tag{3'}$$

for every nonnegative function $\varphi \in C^\infty(\mathbb{S})$ with compact support.

Theorem 3. *Let* $1 + \sigma < q \leq q^*$ *and* $\int_{\mathbb{R}^n} \omega(x)dx > 0$. *Then* (3) *has no nonnegative solutions on* $\mathbb{S}$.

Definition 4. By a solution of (4) on $\mathbb{S}$ we understand a function $u : \mathbb{S} \to \mathbb{R}^1$ which belongs to the space $L_{q,\text{loc}}(\mathbb{S}) \cap L_{1+\sigma,\text{loc}}(\mathbb{S})$ and satisfies the integral identity

$$\int_{\mathbb{S}} \left[-u\varphi_t - u|u|^\sigma \Delta\varphi\right] dtdx = \int_{\mathbb{S}} \left[u|u|^{q-1} + \omega(x)\right] \varphi dtdx \tag{4'}$$

for every function $\varphi \in C^\infty(\mathbb{S})$ with compact support.

Theorem 4. *Let* $1 + \sigma < q \leq q^*$ *and* $\int_{\mathbb{R}^n} \omega(x)dx > 0$. *Then* (4) *has no nonnegative solutions on* $\mathbb{S}$.

Remark 1. Since the solutions of (2) as well as the nonnegative solutions of (3) and (4) are also solutions of (1), Theorems 2, 3, and 4 are simple consequences of Theorem 1.

Remark 2. If $q > q^*$, then from the existence of nontrivial nonnegative global solutions of the Cauchy problem for equation (4) on $\mathbb{S}$, with some nontrivial nonnegative globally integrable function ω (see [1, Theorem 2.1 (c)]), we arrive at the conclusion that there exist nontrivial nonnegative solutions of (1), (2) (3), and (4) on $\mathbb{S}$.

Remark 3. Since we impose no conditions on the behavior of solutions of (1), (2), (3) and (4) on the hyperplane $t = 0$, the corresponding blow-up results for global solutions of the Cauchy problem for (1) and (2), and, respectively, for global nonnegative solutions of the Cauchy problem for (3) and (4), with arbitrary initial data, are special cases of Theorems 1–4.

Remark 4. The results of Theorems 1–4 are also new in the case $\sigma = 0$.

Remark 5. Using the approach developed in this paper, one can improve the main result of [1] by showing that for $n > 2$ and $\sigma = 0$ the critical blow-up exponent $q^* = n/(n-2)$ is of the blow-up type for solutions of the Cauchy problem for equation (7), on the half-space $\mathbb{S}$ with nonnegative initial data $u_0(x)$ on $\mathbb{R}^n$, if $\int_{\mathbb{R}^n} \omega(x)dx > 0$. In this connection we cite the abstract to [1] and Theorem 2.1 (b) therein. A formal proof of this fact coincides almost word for word with the proof of Theorem 1. More precisely, it coincides word for word with the proof of Theorem 1 starting from inequality (9) with $\sigma = 0$ and $\eta \equiv 1$. To obtain (9) from (7), in terms of the proof of Theorem 1, it is sufficient to multiply both sides of equation (7) by the function $\varphi(t,x) = \zeta^s(t,x)\eta^2(t)$ with $\eta \equiv 1$ and integrate over $\mathbb{S}$. Then, by the assumption that the initial datum $u_0(x)$ is nonnegative, one obtains the inequality

$$-s\int_{\mathbb{S}} u\zeta_t\zeta^{s-1}dtdx - \int_{\mathbb{S}} u\Delta\zeta^s dtdx \geq \int_{\mathbb{S}} |u|^q\zeta^s dtdx + \int_{\mathbb{S}} \omega(x)\zeta^s dtdx, \qquad (8')$$

which is a direct analogue of inequality (8). Finally, inequality (9) with $\sigma = 0$ and $\eta \equiv 1$ follows immediately from (8′).

3. The Proofs

In what follows, a "smooth" function is a C^∞-function, $B(R) := \{x \in \mathbb{R}^n : |x| < R\}$, $\overline{B(R)}$ is the closure of $B(R)$, and $S(T,R) := (0,T) \times B(R)$.

Proof of Theorem 1. The proof is by contradiction. Let $1 + \sigma < q \leq q^*$, let $\omega : \mathbb{R}^n \to \mathbb{R}^1$ be a globally integrable function, let $\int_{\mathbb{R}^n} \omega(x)dx > 0$, and let there exist a solution $u(t,x)$ of (1) on $\mathbb{S}$. We show then that $\int_{\mathbb{R}^n} \omega(x)dx = 0$. To this end, let $0 < \tau < \infty$, $0 < r < R < \infty$, and let $\eta : [0,\infty) \to [0,1]$ be a smooth function which has nonnegative derivative η' and equals 0 on the interval $[0,\tau]$ and 1 on the interval $[2\tau,\infty)$. Let $\zeta(t,x):[0,\infty)\times\mathbb{R}^n\to[0,1]$ be a smooth function which equals 1 on $[0,T/2]\times\overline{B(r)}$ and 0 outside $[0,T)\times B(R)$. Substituting $\varphi(t,x)=\zeta^s(t,x)\eta^2(t)$ as a test function in $(1')$, where the positive constant $s\geq 2$ will be chosen below, we obtain

$$\begin{aligned}&-s\int_{\mathbb{S}}|u|\zeta_t\zeta^{s-1}\eta^2dtdx-2\int_{\mathbb{S}}|u|\zeta^s\eta'\eta dtdx-\int_{\mathbb{S}}u|u|^\sigma\eta^2\Delta\zeta^s dtdx\\&\geq\int_{\mathbb{S}}|u|^q\zeta^s\eta^2dtdx+\int_{\mathbb{S}}\omega(x)\zeta^s\eta^2dtdx.\end{aligned}\tag{8}$$

Since $\eta'\geq 0$ for all $t>0$, the second integral on the left-hand side of (8) is nonnegative. Therefore, (8) yields

$$\begin{aligned}&s\int_{\mathbb{S}}|u||\zeta_t|\zeta^{s-1}\eta^2dtdx+\int_{\mathbb{S}}|u|^{\sigma+1}\eta^2|\Delta\zeta^s|dtdx\\&\geq\int_{\mathbb{S}}|u|^q\zeta^s\eta^2dtdx+\int_{\mathbb{S}}\omega(x)\zeta^s\eta^2dtdx.\end{aligned}\tag{9}$$

Since

$$\Delta\zeta^s=s\zeta^{s-1}\Delta\zeta+s(s-1)\zeta^{s-2}|\nabla_x\zeta|^2,\tag{10}$$

it follows easily from (9) that

$$\begin{aligned}&\int_{\mathbb{S}}s|u||\zeta_t|\zeta^{s-1}\eta^2dtdx+\int_{\mathbb{S}}s|u|^{\sigma+1}\zeta^{s-1}|\Delta\zeta|\eta^2dtdx\\&+\int_{\mathbb{S}}s(s-1)|u|^{\sigma+1}\zeta^{s-2}|\nabla_x\zeta|^2\eta^2dtdx\\&\geq\int_{\mathbb{S}}|u|^q\zeta^s\eta^2dtdx+\int_{\mathbb{S}}\omega(x)\zeta^s\eta^2dtdx.\end{aligned}\tag{11}$$

Estimating all the integrands on the left-hand side of (11) by Young's inequality

$$AB\leq\alpha A^{\frac{\beta}{\beta-1}}+\alpha^{1-\beta}B^\beta\tag{12}$$

with $\alpha=\frac{1}{4}$ and suitable A,B, and β, we arrive at

$$\begin{aligned}&\frac{1}{4}\int_{\mathbb{S}}|u|^q\zeta^s\eta^2dtdx+c_1\int_{\mathbb{S}}|\zeta_t|^{\frac{q}{q-1}}\zeta^{s-\frac{q}{q-1}}\eta^2dtdx\\&+\frac{1}{4}\int_{\mathbb{S}}|u|^q\zeta^s\eta^2dtdx+c_1\int_{\mathbb{S}}|\Delta\zeta|^{\frac{q}{q-\sigma-1}}\zeta^{s-\frac{q}{q-\sigma-1}}\eta^2dtdx\\&+\frac{1}{4}\int_{\mathbb{S}}|u|^q\zeta^s\eta^2dtdx+c_1\int_{\mathbb{S}}|\nabla_x\zeta|^{\frac{2q}{q-\sigma-1}}\zeta^{s-\frac{2q}{q-\sigma-1}}\eta^2dtdx\end{aligned}$$

$$\geq \int_{\mathbb{S}} |u|^q \zeta^s \eta^2 dtdx + \int_{\mathbb{S}} \omega(x)\zeta^s \eta^2 dtdx. \tag{13}$$

We use the symbols c_i, $i = 1, 2, ...$, to denote constants depending, possibly, on n, q, s, σ, but not on τ, r, T, or R. Inequality (13), for any fixed $s \geq \frac{2q}{q-\sigma-1}$, yields

$$c_2 \int_{S(T,R)} |\zeta_t|^{\frac{q}{q-1}} dtdx + c_2 \int_{S(T,R)} |\Delta\zeta|^{\frac{q}{q-\sigma-1}} dtdx + c_2 \int_{S(T,R)} |\nabla_x \zeta|^{\frac{2q}{q-\sigma-1}} dtdx$$

$$\geq \int_{\mathbb{S}} |u|^q \zeta^s \eta^2 dtdx + \int_{\mathbb{S}} \omega(x)\zeta^s \eta^2 dtdx. \tag{14}$$

Now, we choose in (14) the function $\zeta(t,x)$ in the form $\zeta(t,x) := \psi(t/T)\xi(x)$, where $\psi : [0,\infty) \to [0,1]$ is a smooth function which equals 1 on $[0,1/2]$ and 0 on $[1,\infty)$, and $\xi : \mathbb{R}^n \to [0,1]$ is a smooth function which equals 1 on $\overline{B(r)}$ and 0 outside $B(R)$. Since for arbitrary $T > 0$ and $R > 0$ the volume of $S(T,R)$ is equal to $c_3 T R^n$ and since the function $\psi(t/T)$ can be chosen such that

$$|\zeta_t| \leq c_4 T^{-1}, \tag{15}$$

we obtain from (14) the inequality

$$c_5 T^{-\frac{1}{q-1}} R^n + c_5 T \int_{B(R)} |\nabla^2 \xi|^{\frac{q}{q-\sigma-1}} dx + c_5 T \int_{B(R)} |\nabla \xi|^{\frac{2q}{q-\sigma-1}} dx$$

$$\geq \int_{\mathbb{S}} \omega(x) \psi^s(t/T) \xi^s(x) \eta^2(t) dtdx. \tag{16}$$

Choosing in (16) $r = R/\sqrt{2}$ and the function $\xi(x)$ in the form $\xi(x) := \phi(|x|^2/R^2)$, where $\phi : [0,\infty) \to [0,1]$ is a smooth function which equals 1 on $[0,1/2]$ and 0 on $[1,\infty)$ and is such that, for arbitrary $R > 0$,

$$|\nabla \xi| \leq c_6 R^{-1} \quad \text{and} \quad |\nabla^2 \xi| \leq c_6 R^{-2}, \tag{17}$$

we obtain from (16) and (17) that the inequality

$$c_7 T^{-\frac{1}{q-1}} R^n + c_7 T R^{n - \frac{2q}{q-\sigma-1}} \geq \Psi(T) \int_{\mathbb{R}^n} \omega(x) \phi_R(x) dx \tag{18}$$

and, therefore, the inequality

$$c_8 T^{-\frac{q}{q-1}} R^n + c_8 R^{n - \frac{2q}{q-\sigma-1}} \geq \Psi(T) T^{-1} \int_{\mathbb{R}^n} \omega(x) \phi_R(x) dx, \tag{19}$$

where $\phi_R(x) := \phi^s(|x|^2/R^2)$ and $\Psi(T) := \int_0^T \psi^s(t/T)\eta^2(t)dt$, hold with arbitrary $T > 0$ and $R > 0$. It is easy to see that the function $\Psi(T)$ is continuous on $[0, +\infty)$ and is such that

$$\Psi(T)T^{-1} \geq \frac{1}{2} \tag{20}$$

for every $T > 0$. Let R_k be a sequence of positive numbers which converges monotonically to $+\infty$ as $k \to +\infty$. The sequence of functions $\omega(x)\phi_{R_k}(x)$, which are measurable on $\mathbb{R}^n$, converges almost everywhere to the measurable function $\omega(x)$. Moreover, $|\omega(x)\phi_{R_k}(x)| \leq |\omega(x)|$ and, by our assumptions, the function $|\omega(x)|$ is globally integrable on $\mathbb{R}^n$. Thus, Lebesgue's dominated convergence theorem (see, e.g., [7, page 303]) implies

$$\lim_{R_k \to +\infty} \int_{\mathbb{R}^n} \omega(x)\phi_{R_k}(x)dx = \int_{\mathbb{R}^n} \omega(x)dx. \tag{21}$$

Case 1. $q < q^*$. Letting in (19) $R = R_k$ and $T = (R_k)^\gamma$, where the positive constant γ will be chosen below, we easily obtain the inequality

$$\begin{aligned} &c_9 \limsup_{R_k \to +\infty} [(R_k)^{n - \frac{\gamma q}{q-1}} + (R_k)^{n - \frac{2q}{q-\sigma-1}}] \\ &\geq \liminf_{R_k \to +\infty} [\Psi((R_k)^\gamma)(R_k)^{-\gamma}] \liminf_{R_k \to +\infty} \Big[\int_{\mathbb{R}^n} \omega(x)\phi_{R_k}(x)dx \Big]. \end{aligned} \tag{22}$$

Further, since

$$n - \frac{2q}{q - \sigma - 1} < 0 \quad \text{for} \quad 1 + \sigma < q < q^* \tag{23}$$

and since we can choose $\gamma > 0$ sufficiently large such that $n - \frac{\gamma q}{q-1} < 0$, we have from (20)–(23) that $\int_{\mathbb{R}^n} \omega(x)dx = 0$.

Case 2. $q = q^*$. In this case q is a critical blow-up exponent. Let $n > 2$ and $q = \frac{n(1+\sigma)}{n-2}$. Then $\frac{q}{q-\sigma-1} = \frac{n}{2}$. Therefore, inequality (16) yields

$$\begin{aligned} &c_{10} T^{-\frac{q}{q-1}} R^n + c_{10} \int_{B(R)} |\nabla^2 \xi|^{\frac{n}{2}} dx + c_{10} \int_{B(R)} |\nabla \xi|^n dx \\ &\geq T^{-1} \int_{\mathbb{S}} \omega(x)\psi^s(t/T)\xi^s(x)\eta^2(t)dtdx \end{aligned} \tag{24}$$

for all $R > 0$ and $T > 0$. Choosing in (24) the function $\xi(x)$ in the form $\xi(x) := \phi\left(\frac{\ln(|x|/r)}{\ln(R/r)}\right)$ for arbitrary $R > r > 1$, where $\phi : [-\infty, +\infty) \to [0, 1]$ is a smooth function which equals 1 on $[-\infty, 0]$ and 0 on $[1, +\infty)$ and is such that the inequalities

$$|\nabla\xi| \leq \frac{c_{11}}{|x|\ln(R/r)} \quad \text{and} \quad |\nabla^2\xi| \leq \frac{c_{11}}{|x|^2\ln(R/r)} \tag{25}$$

hold on $\mathbb{R}^n$ with a certain positive constant c_{11} for arbitrary $R > r > 1$, we have from (24) and (25) the inequality

$$c_{12}T^{-\frac{q}{q-1}}R^n + c_{12}\int_{B(R)}(|x|^2\ln(R/r))^{-n/2}dx + c_{12}\int_{\mathbb{R}^n}(|x|\ln(R/r))^{-n}dx$$
$$\geq T^{-1}\int_{\mathbb{R}^n}\omega(x)\psi^s(t/T)\xi^s(x)\eta^2(t)dtdx. \tag{26}$$

Furthermore, letting $r = \sqrt{R}$ we obtain from (26) that the inequality

$$c_{13}T^{-\frac{q}{q-1}}R^n + c_{13}(\ln R)^{1-n} + c_{13}(\ln R)^{1-\frac{n}{2}} \geq \Psi(T)T^{-1}\int_{\mathbb{R}^n}\omega(x)\phi_R(x)dx \tag{27}$$

holds for arbitrary $T > 0$ and $R > 0$, where

$$\phi_R(x) := \phi^s\Big(\frac{\ln(|x|/\sqrt{R})}{\ln\sqrt{R}}\Big) \tag{28}$$

and, as above, $\Psi(T) := \int_0^T \psi^s(t/T)\eta^2(t)dt$. In addition, as above, let R_k be a sequence of positive numbers which converges monotonically to $+\infty$ as $k \to +\infty$. The sequence of functions $\omega(x)\phi_{R_k}(x)$, which are measurable on $\mathbb{R}^n$, converges almost everywhere to the measurable function $\omega(x)$. Moreover, $|\omega(x)\phi_{R_k}(x)| \leq |\omega(x)|$. Since, by our hypothesis, the function $|\omega(x)|$ is globally integrable on $\mathbb{R}^n$, Lebesgue's dominated convergence theorem (see, e.g., [7, page 303]) implies that equality (21) is satisfied with the function ϕ_R given by (28). Let $T_k = (R_k)^\gamma$, where the positive constant γ will be chosen below. Then it follows from (27) with $R = R_k$ and $T = T_k$ that the inequality

$$c_{14}\limsup_{R_k\to+\infty}[(R_k)^{n-\frac{\gamma q}{q-1}} + (\ln R_k)^{1-n} + (\ln R_k)^{1-\frac{n}{2}}]$$
$$\geq \liminf_{R_k\to+\infty}[\Psi((R_k)^\gamma)(R_k)^{-\gamma}]\liminf_{R_k\to+\infty}\Big[\int_{\mathbb{R}^n}\omega(x)\phi_{R_k}(x)dx\Big] \tag{29}$$

holds. Furthermore, since $n > 2$ and since we can choose $\gamma > 0$ sufficiently large such that $n - \frac{\gamma q}{q-1} < 0$, we obtain from (20), (21), and (29) that $\int_{\mathbb{R}^n}\omega(x)dx = 0$. Thus, for $1 + \sigma < q \leq q^*$, we have a contradiction to our assumption about the function $\omega(x)$. □

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