Differential and Integral Equations Volume 18, Number 12 (2005), Pages 1397–1426

BILINEAR ESTIMATES WITH APPLICATIONS TO THE GENERALIZED BENJAMIN-ONO-BURGERS EQUATIONS

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(Submitted by: Tohru Ozawa)

Abstract. In this paper, we deal with the well-posedness issues of the generalized Benjamin-Ono-Burgers (gBOB) equations which are interpolated between the ordinary BOB equation and the KdV-Burgers equation with respect to the dispersive terms. We solve the initial-value problem (IVP) with data below $H^{-1/2}$, where $s = -1/2$ is the threshold for the well posedness of the Burgers equation. Our proof is based on the method by L. Molinet and F. Ribaud, which is analogous to that developed by J. Bourgain and C.E. Kenig, G. Ponce, and L. Vega. Interestingly, it is known that such a method cannot be applied to the Benjamin-Ono equation with initial data in $H^s(\mathbb{R})$, $s \in \mathbb{R}$.

1. Introduction

We are concerned with the initial-value problem (IVP) for the generalized Benjamin-Ono-Burgers (gBOB) equations

$$\begin{cases} \partial_t u + u\partial_x u - \partial_x D_x^{1+a} u - \partial_x^2 u = 0, \quad x, t \in \mathbb{R}, \\ u(x,0) = u_0(x) \in H^s(\mathbb{R}), \end{cases} \tag{1.1}$$

where $0 \leq a < 1$ and D_x^{1+a} is the Fourier multiplier operator with symbol $|\xi|^{1+a}$. Note that when $a = 0$, (1.1) is the ordinary BOB equation [7, 10, 11, 14, 29], and that when $a = 1$, (1.1) is the KdV-Burgers equation (see [21, 23] and the references therein).

In [23], L. Molinet and F. Ribaud have proved that the KdV-Burgers equation is globally well posed for $s > -1$, provided u is real valued. In the present paper, we shall deal with the well-posedness issue of the gBOB equations (1.1) when $0 \leq a < 1$.

It is known that the Burgers' equation $\partial_t u + u\partial_x u - \partial_x^2 u = 0$, x, $t \in \mathbb{R}$ is locally well posed in $H^{-1/2}$ for small initial data [1]. The exponent $s = -1/2$

Accepted for publication: June 2005.
AMS Subject Classifications: 35A07, 35M10, 35Q53, 76B15.

is optimal since the uniqueness fails when $s < -1/2$ [9]. We shall show in this paper that the gBOB equations (1.1) are globally well posed for $s > -(1+a)/2$, provided u is real valued. Note that when $a > 0$, the value $s = -(1+a)/2$ is lower than the threshold $s = -1/2$ for the well posedness of the Burgers equation. This result is due to the effect of the dispersive term of (1.1).

We shall solve the gBOB equation in the following function space:

Definition 1.1. For $s, b \in \mathbb{R}$, $X^{s,b}$ denotes the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ with respect to the norm

$$\|F\|_{X^{s,b}} = \left(\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \langle\, i(\tau - \xi|\xi|^{1+a}) + \xi^2 \,\rangle^{2b} \langle \xi \rangle^{2s} |\widehat{F}(\xi,\tau)|^2 d\xi d\tau \right)^{1/2}, \tag{1.2}$$

where $\langle \cdot \rangle = (1 + |\cdot|^2)^{1/2}$. For $T > 0$, we define the localized spaces $X_T^{s,b}$ with the norm

$$\|f\|_{X_T^{s,b}} = \inf_{g \in X^{s,b}} \{\, \|g\|_{X^{s,b}} : g(t) = f(t) \text{ on } [0,T]\}. \tag{1.3}$$

Hereafter, $\widehat{\cdot}$ or $\mathcal{F}$ denotes the Fourier transform with respect to space-time variables. Note that $i(\tau - \xi|\xi|^{1+a}) + \xi^2$ is the symbol of the linear part of the gBOB equation.

Definition 1.2. Let $U(t) = \exp(itP(D_x))$ be the unitary operator associated with the linear generalized Benjamin-Ono (gBO) equation, where $P(D_x)$ is the Fourier multiplier with the symbol $P(\xi) = \xi|\xi|^{1+a}$. We denote by $\mathcal{W}(t)$ the semigroup associated with the linear gBOB equation;

$$\mathcal{F}_x\left(\mathcal{W}(t)\phi\right)(\xi) = \exp[-\xi^2 t + i\xi|\xi|^{1+a} t\,] \mathcal{F}_x(\phi)(\xi), \quad t \geq 0, \quad \phi \in \mathcal{S}. \tag{1.4}$$

And we extend $\mathcal{W}(t)$ to a linear operator defined on $\mathbb{R}$ by setting

$$\mathcal{F}_x\left(W(t)\phi\right)(\xi) = \exp[-\xi^2 |t| + i\xi|\xi|^{1+a} t\,] \mathcal{F}_x(\phi)(\xi), \quad t \in \mathbb{R}, \quad \phi \in \mathcal{S}. \tag{1.5}$$

Here $\mathcal{F}_x$ denotes the Fourier transform with respect to x.

Theorem 1.1. *Let $s > -(1+a)/2$ with $0 \leq a < 1$. Then for any $u_0 \in H^s(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^s}) > 0$, $b \in (1/2, 1)$, and a unique solution $u(t)$ of the IVP* (1.1) *satisfying*

$$u(t) \in C([0,T], H^s(\mathbb{R})), \tag{1.6}$$

$$u \in X^{s-(2b-1),b}, \tag{1.7}$$

$$u\partial_x u \in X^{s-(2b-1),b-1}, \quad \partial_t u,\, \partial_x^2 u \in X^{s-(2b+1),b-1}. \tag{1.8}$$

Moreover, the flow map $u_0 \mapsto u(t)$ *is locally Lipschitz from* $H^s(\mathbb{R})$ *to* $Z_T \equiv C([0,T], H^s(\mathbb{R})) \cap X^{s-(2b-1),b}$.

If the solution u *is real valued,* $u \in C((0,+\infty), H^\infty(\mathbb{R}))$.

Our proof of Theorem 1.1 is based on that of Molinet and Ribaud [23]. In a series of papers by Molinet and Ribaud [21, 22, 23], they treated the IVP for some dispersive dissipative equations with rough initial data (the dissipative KdV [21], the Kadomtsev-Petviashvili-Burgers [22] and the KdV-Burgers equations [23]). The method of these studies was based on the analogous arguments developed by J. Bourgain [5] and by C.E. Kenig, G. Ponce, and L. Vega [18].

We solve the IVP (1.1) via the following integral equation:

$$u(t) = \mathcal{W}(t)u_0(x) - \frac{1}{2}\int_0^t \mathcal{W}(t-t')\partial_x(u^2(t'))dt', \quad t \geq 0.$$

However, we actually apply a contraction mapping principle to the following version:

$$u(t) = \psi(t)\Big[W(t)u_0(x) - \frac{\chi_{\mathbb{R}_+(t)}}{2}\int_0^t W(t-t')\partial_x(\psi_T^2(t')u^2(t'))dt'\Big] \tag{1.9}$$

for $t \in \mathbb{R}$, where ψ is a cut-off function satisfying

$$\psi \in C_0^\infty(\mathbb{R}), \quad \text{supp}\,\psi \subset [-2,2], \quad \psi \equiv 1 \text{ on } [-1,1], \tag{1.10}$$

and $\psi_\delta(t) = \psi(t/\delta)$, and $\chi_{\mathbb{R}_+}(t)$ is the characteristic function of the interval $[0,\infty)$.

Interestingly, it is known that the following gBO equations with $0 \leq a < 1$

$$\partial_t u + u\partial_x u - \partial_x D_x^{1+a} u = 0, \quad x, t \in \mathbb{R} \tag{1.11}$$

cannot be solved by the Picard iteration scheme based on the Duhamel formula when initial data is in $H^s(\mathbb{R})$, $s \in \mathbb{R}$ (L. Molinet, J.-C. Saut, and N. Tzvetkov [24]). In particular, the method used in [5, 18] cannot be applied to the gBO equations with $0 \leq a < 1$. However, J. Colliander, C. Kenig, and G. Staffilani [6] showed by the iteration scheme that the gBO equations with $0 < a < 1$ are locally well posed in F^s with $s \geq 1/2 + a/2$, where F^s is defined as the completion of the Schwartz space with respect to the norm

$$\|f\|_{F^s} = \|f\|_{H^s} + \|xf\|_{H^{s-1-a}}. \tag{1.12}$$

Note that their results do not include the case of the ordinary BO equation (when $a = 0$).

Remark 1.1. We shall collect the known results of the IVP for the gBO equations with $0 \leq a < 1$. When $a = 0$, (1.11) is the ordinary BO equation, and many studies are known [15, 16, 19, 25, 26, 27]. The local well posedness of the gBO equations in H^s with $s > 9/8 - 3a/8$ was given by C. E. Kenig and K. D. Koenig [16] (see also [17], [26]). The global well posedness for $s \geq (1+a)/2 \geq 9/10$ was proved by Kenig, Ponce, and Vega [17]. In particular, T. Tao [27] has shown the global well posedness of the ordinary BO equation in $H^1(\mathbb{R})$. The gBO equations with $0 \leq a < 1$ are known to have the weak solution in $L^2(\mathbb{R})$ [13].

Remark 1.2. The Lipschitz continuity of the flow map of the gBOB equation can be naturally derived due to the Picard iteration argument. However, the flow map of the BO equation (when $a = 0$) is not uniformly continuous on bounded sets of $H^s(\mathbb{R})$ with $s > 0$ [20] and $s < -1/2$ [3]. In the cases of $0 < a < 1$, we may expect the appearance of similar phenomena, but it remains open.

Remark 1.3. Theorem 1.1 means the *conditional* well posedness with the auxiliary condition that the solution lies in $X^{s-(2b-1),b}$. However, the auxiliary condition may not be needed for the well posedness. See [4].

Remark 1.4. Shuji Yoshikawa [28] points out that if one takes the approach such as D. Bekiranov [1], then one can obtain the well-posedness result of the ordinary BOB equation for the critical case $s = -1/2$.

Notations. If there exists a harmless positive constant $c > 0$ such that $A \leq cB$ (respectively $A \geq cB$) for any positive A and B, we shall often write $A \lesssim B$ (respectively $A \gtrsim B$) for abbreviation. The notation $A \sim B$ means that $A \lesssim B \lesssim A$.

The rest of this paper is organized as follows: For the proof of Theorem 1.1, we need some linear and bilinear estimates. We shall collect the linear estimates in Section 2. Section 3 contains the preparatory lemmas for the construction of the bilinear estimate. Section 4 is devoted to the proof of the bilinear estimate. Theorem 1.1 will be proved in Section 5.

The author is grateful to Professor Keiichi Kato and to Professor Hikosaburo Komatsu for their kind advice on this subject. In particular, the author benefitted from the discussion on Proposition 3.2 with them. Thanks also to Professor Tosinobu Muramatu for his useful comments on Lemma 2.2 and his kind advice.

2. Linear estimates

In this section, we shall collect a few linear estimates for the proof of Theorem 1.1. The analogous estimates corresponding to the case of the KdV-Burgers equation (when $a = 1$) are given in [23]. Following Molinet and Ribaud [23], we can derive the estimates of other cases $0 \le a < 1$ with easy modifications. We also treat a linear estimate (Lemma 2.3) to construct Proposition 3.1.

Proposition 2.1. *Let $s \in \mathbb{R}$ and $b \in [0,1]$. There exists $C > 0$ such that*

$$\|\psi(t)W(t)\phi\|_{X^{s,b}} \le C\|\phi\|_{H^{s+2b-1}} \tag{2.1}$$

for any $\phi \in H^{s+2b-1}(\mathbb{R})$.

Proof. From the definition of the norm,

$$\begin{aligned}\|\psi(t)W(t)\phi\|_{X^{s,b}} &= \left\|\langle\xi\rangle^s \mathcal{F}_x(\phi)(\xi)\left\|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t\left(\psi(t)e^{-|t|\xi^2}\right)(\tau)\right\|_{L^2_\tau}\right\|_{L^2_\xi}\\ &\le C\left\|\langle\xi\rangle^s \mathcal{F}_x(\phi)(\xi)\left\|\langle\tau\rangle^b \mathcal{F}_t\left(\psi(t)e^{-|t|\xi^2}\right)(\tau)\right\|_{L^2_\tau}\right\|_{L^2_\xi}\\ &\quad + C\left\|\langle\xi\rangle^{s+2b} \mathcal{F}_x(\phi)(\xi)\left\|\mathcal{F}_t\left(\psi(t)e^{-|t|\xi^2}\right)(\tau)\right\|_{L^2_\tau}\right\|_{L^2_\xi}.\end{aligned} \tag{2.2}$$

Putting $g_\xi(\tau) = \mathcal{F}_t\left(\psi(t)e^{-|t|\xi^2}\right)(\tau)$, we obtain

$$\|\langle\tau\rangle^b g_\xi(\tau)\|_{L^2_\tau} \le C\langle\xi\rangle^{2b-1} \quad \text{for } 0 \le b \le 1. \tag{2.3}$$

For the proof of this estimate (2.3), see [23, Proposition 2.1].

Combining (2.2) with (2.3), we obtain the desired estimate. □

Proposition 2.2. *Let $s \in \mathbb{R}$ and let $b > 1/2$. For $\delta \in (0,1]$, we have*

$$\|\psi_\delta F\|_{X^{s,b}} \le C\delta^{(1-2b)/2}\|F\|_{X^{s,b}}. \tag{2.4}$$

Proof. The proof can be done by modifying that of Lemma 2.5 in [12] slightly. □

Lemmas 2.1 and 2.2 are needed for the proof of Proposition 2.3.

Lemma 2.1. *For $w \in \mathcal{S}(\mathbb{R}^2)$, we define k_ξ on $\mathbb{R}$ as follows:*

$$k_\xi(t) = \psi(t)\int_{\mathbb{R}} \frac{e^{it\tau} - e^{-\xi^2|t|}}{i\tau+\xi^2}\widehat{w}(\xi,\tau)d\tau. \tag{2.5}$$

Let $1/2 \le b < 1$. Then, it holds for any fixed $\xi \in \mathbb{R}$ that

$$\begin{aligned}&\left\|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t(k_\xi)\right\|^2_{L^2_\tau(\mathbb{R})}\\ &\le C\Big[\langle\xi\rangle^{2(2b-1)}\Big(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|}{\langle i\tau+\xi^2\rangle}d\tau\Big)^2 + \Big(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle^{2(1-b)}}d\tau\Big)\Big].\end{aligned} \tag{2.6}$$

Proof. We rewrite k_ξ in the following way:

$$\begin{aligned}
k_\xi(t) &= \psi(t)\int_{|\tau|\le 1}\frac{e^{it\tau}-1}{i\tau+\xi^2}\widehat{w}(\xi,\tau)d\tau + \psi(t)\int_{|\tau|\le 1}\frac{1-e^{-\xi^2|t|}}{i\tau+\xi^2}\widehat{w}(\xi,\tau)d\tau \\
&\quad + \psi(t)\int_{|\tau|\ge 1}\frac{e^{it\tau}}{i\tau+\xi^2}\widehat{w}(\xi,\tau)d\tau - \psi(t)\int_{|\tau|\ge 1}\frac{e^{-\xi^2|t|}}{i\tau+\xi^2}\widehat{w}(\xi,\tau)d\tau \\
&= I + II + III - IV. \qquad (2.7)
\end{aligned}$$

We have to estimate the contribution of these four terms to the left-hand side of (2.7).

Contribution of IV. Noting that $\langle i\tau+\xi^2\rangle \le C|i\tau+\xi^2|$ holds for $|\tau|\ge 1$,

$$\begin{aligned}
&\left\|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t(IV)\right\|^2_{L^2_\tau} \\
&\le C\int_{\mathbb{R}}\langle i\tau+\xi^2\rangle^{2b}\left|\mathcal{F}_t\left(\psi(t)e^{-\xi^2|t|}\right)(\tau)\right|^2 d\tau\Big(\int_{|\tau|\ge 1}\frac{|\widehat{w}(\xi,\tau)|}{\langle i\tau+\xi^2\rangle}d\tau\Big)^2. \qquad (2.8)
\end{aligned}$$

Set $g_\xi(\tau)=\mathcal{F}_t(\psi(t)e^{-\xi^2|t|})(\tau)$. By using (2.3), we have

$$\begin{aligned}
\int_{\mathbb{R}}\langle i\tau+\xi^2\rangle^{2b}|g_\xi(\tau)|^2 d\tau &\le C\int_{\mathbb{R}}\langle\tau\rangle^{2b}|g_\xi(\tau)|^2 d\tau + C|\xi|^{4b}\int_{\mathbb{R}}|g_\xi(\tau)|^2 d\tau \\
&\le C\langle\xi\rangle^{2(2b-1)}. \qquad (2.9)
\end{aligned}$$

Therefore we obtain

$$\left\|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t(IV)\right\|^2_{L^2_\tau} \le C\langle\xi\rangle^{2(2b-1)}\Big(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|}{\langle i\tau+\xi^2\rangle}d\tau\Big)^2. \qquad (2.10)$$

Contribution of III. Noting that $\langle i\tau+\xi^2\rangle^b \le C\langle\tau'\rangle^b + C|i(\tau-\tau')+\xi^2|^b$ and using Young's inequality,

$$\begin{aligned}
&\left\|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t(III)\right\|^2_{L^2_\tau} = \int_{\mathbb{R}}\langle i\tau+\xi^2\rangle^{2b}\Big|\int_{\mathbb{R}}\widehat{\psi}(\tau')\frac{\widehat{w}(\xi,\tau-\tau')\chi_{|\tau-\tau'|\ge 1}}{i(\tau-\tau')+\xi^2}d\tau'\Big|^2 d\tau \\
&\le \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}|\langle\tau'\rangle^b\widehat{\psi}(\tau')|\frac{|\widehat{w}(\xi,\tau-\tau')|}{|i(\tau-\tau')+\xi^2|}\chi_{|\tau-\tau'|\ge 1}d\tau'\Big)^2 d\tau \qquad (2.11) \\
&+ \int_{\mathbb{R}}\Big(\int_{\mathbb{R}}|\widehat{\psi}(\tau')|\frac{|\widehat{w}(\xi,\tau-\tau')|}{|i(\tau-\tau')+\xi^2|^{1-b}}\chi_{|\tau-\tau'|\ge 1}d\tau'\Big)^2 d\tau \le C\Big\|\frac{\widehat{w}(\xi,\tau)}{\langle i\tau+\xi^2\rangle^{1-b}}\Big\|^2_{L^2_\tau},
\end{aligned}$$

where $\|\langle\tau\rangle^b\widehat{\psi}\|_{L^1}\le C$ for $0\le b\le 1$.

Contribution of II. It follows from Schwarz inequality that

$$\left\|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t(II)\right\|^2_{L^2_\tau} \qquad (2.12)$$

$$\leq C \left\| \langle i\tau+\xi^2\rangle^b \mathcal{F}_t \left(\psi(t)[1-e^{-\xi^2|t|}]\right)(\tau)\right\|_{L^2_\tau}^2 \frac{\langle \xi^2\rangle}{|\xi|^4} \int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle} d\tau.$$

(i) Case of $|\xi| \geq 1$. It follows that

$$\begin{aligned} &\left\| \langle i\tau+\xi^2\rangle^b \mathcal{F}_t \left(\psi(t)[1-e^{-\xi^2|t|}]\right)(\tau)\right\|_{L^2_\tau}^2 \\ &\leq \left\| \langle i\tau+\xi^2\rangle^b \mathcal{F}_t(\psi)(\tau)\right\|_{L^2_\tau}^2 + \left\| \langle i\tau+\xi^2\rangle^b \mathcal{F}_t \left(\psi(t)e^{-\xi^2|t|}\right)(\tau)\right\|_{L^2_\tau}^2 \\ &\leq \left(\|\psi\|_{H^b}^2 + |\xi|^{4b}\|\psi\|_{L^2}^2\right) + C\langle\xi\rangle^{2(2b-1)} \leq C\langle\xi\rangle^{4b}, \end{aligned} \tag{2.13}$$

where we use (2.9) for the second term. Therefore, we have

$$\|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t(II)\|_{L^2_\tau}^2 \leq C\langle\xi\rangle^{2(2b-1)} \int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle} d\tau. \tag{2.14}$$

(ii) Case of $|\xi| \leq 1$. It follows that

$$\begin{aligned} &\|\langle i\tau +\xi^2\rangle^b \mathcal{F}_t(\psi(t)[1-e^{-\xi^2|t|}])(\tau)\Big\|_{L^2_\tau} \\ &\quad \leq C \left\| \langle \tau\rangle^b \mathcal{F}_t(\psi(t)[1-e^{-\xi^2|t|}])(\tau)\right\|_{L^2_\tau} = C\Big\| \sum_{n\geq 1} \frac{t^n\psi(t)|\xi|^{2n}}{n!}\Big\|_{H^b_\tau} \\ &\quad \leq C\sum_{n\geq 1} \frac{|\xi|^{2n}}{n!} \|t^n\psi(t)\|_{H^1_\tau} \leq C\sum_{n\geq 0}\frac{|\xi|^2}{n!} < C|\xi|^2. \end{aligned} \tag{2.15}$$

Hence,

$$\left\| \langle i\tau+\xi^2\rangle^b \mathcal{F}_t(II)\right\|_{L^2_\tau}^2 \leq C|\xi|^4 \frac{\langle\xi^2\rangle}{|\xi|^4} \int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle} d\tau \leq C\int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle} d\tau. \tag{2.16}$$

From (2.14) and (2.16), we obtain

$$\left\| \langle i\tau+\xi^2\rangle^b \mathcal{F}_t(II)\right\|_{L^2_\tau}^2 \leq C\langle\xi\rangle^{2(2b-1)} \int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle} d\tau \leq C\int_{\mathbb{R}} \frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle^{2(1-b)}} d\tau. \tag{2.17}$$

Contribution of I. We can rewrite I as

$$I = \psi(t)\int_{|\tau|\leq 1} \sum_{n\geq 1} \frac{(it\tau)^n}{n!} \frac{\widehat{w}(\xi,\tau)}{i\tau+\xi^2} d\tau. \tag{2.18}$$

It follows from the Schwarz inequality that

$$\|\langle i\tau+\xi^2\rangle^b \mathcal{F}_t(I)\|_{L^2_\tau} \leq C\|I\|_{H^b_\tau} + C|\xi|^{2b}\|I\|_{L^2_\tau}$$

$$\le C\sum_{n\ge 1}\Big[\Big\|\frac{t^n\psi(t)}{n!}\Big\|_{H^b_\tau}+|\xi|^{2b}\Big\|\frac{t^n\psi(t)}{n!}\Big\|_{L^2_\tau}\Big]\int_{|\tau|\le 1}\frac{|i\tau|^n}{|i\tau+\xi^2|}|\widehat{w}(\xi,\tau)|d\tau$$

$$\le C(1+|\xi|^{2b})\Big(\int_{|\tau|\le 1}\frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle}d\tau\Big)^{1/2}\Big(\int_{|\tau|\le 1}\frac{|\tau|^2\langle i\tau+\xi^2\rangle}{|i\tau+\xi^2|^2}d\tau\Big)^{1/2}. \quad (2.19)$$

If $|\xi|\le 1$, (2.19) is bounded by

$$C\Big(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle}d\tau\Big)^{1/2}. \quad (2.20)$$

If $|\xi|\ge 1$, (2.19) is bounded by

$$\frac{\langle\xi\rangle^{2b}}{\langle\xi\rangle}\Big(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle}d\tau\Big)^{1/2}\le C\Big(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle^{2(1-b)}}d\tau\Big)^{1/2}, \quad (2.21)$$

where we note that

$$\int_{|\tau|\le 1}\frac{|\tau|^2\langle i\tau+\xi^2\rangle}{|i\tau+\xi^2|^2}d\tau\le\frac{1}{\langle\xi\rangle^2}. \quad (2.22)$$

From (2.20) and (2.21), we get

$$\|\langle i\tau+\xi^2\rangle^b\mathcal{F}_t(I)\|_{L^2_\tau}\le C\Big(\int_{\mathbb{R}}\frac{|\widehat{w}(\xi,\tau)|^2}{\langle i\tau+\xi^2\rangle^{2(1-b)}}d\tau\Big)^{1/2}. \quad (2.23)$$

Summing up, from (2.10), (2.11), (2.17) and (2.23), we obtain the desired estimate (2.6). □

Lemma 2.2. *Let $0\le\sigma\le 1$, $\sigma\ne 1/2$. For $f\in H^\sigma(\mathbb{R})$ with $f(0)=0$,*

$$\big\|\chi_{\mathbb{R}_+}f\big\|_{H^\sigma}\le C_\sigma\|f\|_{H^\sigma}, \quad (2.24)$$

where $\chi_{\mathbb{R}_+}$ is the characteristic function of $[0,\infty)$.

Proposition 2.3. *Let $s\in\mathbb{R}$ and let $b>1/2$.*

(i) There exists $C>0$ such that, for any $v\in\mathcal{S}(\mathbb{R}^2)$,

$$\Big\|\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t W(t-t')v(t')dt'\Big\|_{X^{s,b}}\le \quad (2.25)$$

$$C\Big[\|v\|_{X^{s,b-1}}+\Big(\int_{\mathbb{R}}\langle\xi\rangle^{2s+2(2b-1)}\Big(\int_{\mathbb{R}}\frac{|\widehat{v}(\xi,\tau)|}{\langle i(\tau-\xi|\xi|^{1+a})+\xi^2\rangle}d\tau\Big)^2d\xi\Big)^{1/2}\Big].$$

(ii) For $0<\delta<1/2$, there exists $C_\delta>0$ such that, for any $v\in X^{s,b-1+\delta}$,

$$\Big\|\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t W(t-t')v(t')dt'\Big\|_{X^{s,b}}\le C_\delta\|v\|_{X^{s,b-1+\delta}}. \quad (2.26)$$

Proof. Assume that $v \in \mathcal{S}(\mathbb{R}^2)$. Setting $w(t') = U(-t')v(t')$, we get

$$\begin{aligned}
&\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t W(t-t')v(t')dt' \\
&\quad = U(t)\Big[\chi_{\mathbb{R}_+}(t)\psi(t)\int_{\mathbb{R}} e^{ix\xi}\int_0^t e^{-|t-t'|\xi^2}\mathcal{F}_x(U(-t')v(t'))(\xi)dt'd\xi\Big] \\
&\quad = U(t)\Big[\chi_{\mathbb{R}_+}(t)\psi(t)\int_{\mathbb{R}^2} e^{ix\xi}e^{-t\xi^2}\widehat{w}(\xi,\tau)\int_0^t e^{it'\tau}e^{t'\xi^2}dt'd\xi d\tau\Big] \\
&\quad = U(t)\Big[\chi_{\mathbb{R}_+}(t)\psi(t)\int_{\mathbb{R}^2} e^{ix\xi}\frac{e^{it\tau}-e^{-t\xi^2}}{i\tau+\xi^2}\widehat{w}(\xi,\tau)d\xi d\tau\Big].
\end{aligned} \tag{2.27}$$

Putting

$$k_\xi(t) = \psi(t)\int_{\mathbb{R}} \frac{e^{it\tau}-e^{-|t|\xi^2}}{i\tau+\xi^2}\widehat{w}(\xi,\tau)d\tau, \tag{2.28}$$

we can rewrite

$$\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t W(t-t')v(t')dt' = U(t)\mathcal{F}_\xi^{-1}\left(\chi_{\mathbb{R}_+}(t)k_\xi\right)(x,t). \tag{2.29}$$

Since $w(t) = U(-t)v(t) \in \mathcal{S}(\mathbb{R}^2)$, it is clear that for any fixed $\xi \in \mathbb{R}$, k_ξ is continuous on $\mathbb{R}$ and $k_\xi(0) = 0$. By virtue of Lemma 2.2, $\|\chi_{\mathbb{R}_+}k_\xi\|_{H_t^b} \le C_b\|k_\xi\|_{H_t^b}$ holds for $0 \le b \le 1$, $b \ne 1/2$. Thus, we find that

$$\begin{aligned}
&\Big\|\chi_{\mathbb{R}_+}(t)\ \psi(t)\int_0^t W(t-t')v(t')dt'\Big\|_{X^{s,b}} = \Big\|U(t)\mathcal{F}_\xi^{-1}\left(\chi_{\mathbb{R}_+}(t)k_\xi(t)\right)\Big\|_{X^{s,b}} \\
&\quad = \Big\|\langle i\tau+\xi^2\rangle^b\langle\xi\rangle^s\mathcal{F}_t\left(\chi_{\mathbb{R}_+}(t)k_\xi(t)\right)(\xi,\tau)\Big\|_{L^2_{\xi,\tau}} \\
&\quad \le \Big\|\langle\xi\rangle^s\left\|\chi_{\mathbb{R}_+}(t)k_\xi(t)\right\|_{H_t^b}\Big\|_{L^2_\xi} + \Big\|\langle\xi\rangle^{s+2b}\left\|\chi_{\mathbb{R}_+}(t)k_\xi(t)\right\|_{L^2_t}\Big\|_{L^2_\xi} \\
&\quad \le C\Big(\Big\|\langle\xi\rangle^s\|k_\xi(t)\|_{H_t^b}\Big\|_{L^2_\xi} + \Big\|\langle\xi\rangle^{s+2b}\|k_\xi(t)\|_{L^2_t}\Big\|_{L^2_\xi}\Big) \\
&\quad \le C\Big\|\langle\xi\rangle^s\Big\|\langle i\tau+\xi^2\rangle^b\mathcal{F}_t(k_\xi)(\tau)\Big\|_{L^2_\tau}\Big\|_{L^2_\xi}.
\end{aligned} \tag{2.30}$$

With the aid of Lemma 2.1, the statement (i) follows if we note that $\widehat{w}(\xi,\tau) = \widehat{v}(\xi,\tau+\xi|\xi|^{1+a})$. By using the Schwarz inequality and the density argument, we directly derive (ii) from (i). $\square$

Proposition 2.4. *Let $s \in \mathbb{R}$, $b \geq 1/2$ and $\delta > 0$. For all $f \in X^{s,b-1+\delta}$,*

$$t \mapsto \int_0^t W(t-t')f(t')dt' \in C(\mathbb{R}_+, H^{s+2\delta}(\mathbb{R})). \tag{2.31}$$

Moreover, we have

$$\Big\| \chi_{\mathbb{R}_+}(t)\psi(t) \int_0^t W(t-t')f(t')dt' \Big\|_{L^\infty(\mathbb{R}_+, H^{s+2\delta})} \leq C\|f\|_{X^{s,b-1+\delta}}. \tag{2.32}$$

Proof. We can set $s = 0$ without loss of generality. It suffices to prove that

$$t \mapsto U(-t) \int_0^t W(t-t')f(t')dt'$$

is continuous from $[0, \infty)$ to $H^{2\delta}(\mathbb{R})$ since U is a strongly continuous unitary group in $L^2(\mathbb{R})$.

Put $g(x,t) = (U(-t)f(t))(x)$. The statement follows if we show the continuity of

$$F : t \mapsto \langle \xi \rangle^{2\delta} \int_0^t e^{-\xi^2|t-t'|} \mathcal{F}_x(g(\cdot, t'))(\xi)dt' \tag{2.33}$$

for $\langle i\tau + \xi^2 \rangle^{b-1+\delta} \widehat{g} \in L^2_{\xi,\tau}(\mathbb{R}^2)$. We rewrite, for $t \geq 0$,

$$\begin{aligned} F(t) &= \langle \xi \rangle^{2\delta} e^{-\xi^2 t} \int_{\mathbb{R}} \widehat{g}(\xi, \tau) \int_0^t e^{(\xi^2 + i\tau)t'} dt' d\tau \\ &= \langle \xi \rangle^{2\delta} \int_{\mathbb{R}} \widehat{g}(\xi, \tau) \frac{e^{it\tau} - e^{-\xi^2|t|}}{\xi^2 + i\tau} d\tau. \end{aligned} \tag{2.34}$$

Hence,

$$F(t_1) - F(t_2) = \langle \xi \rangle^{2\delta} \int_{\mathbb{R}} \frac{\widehat{g}(\xi,\tau)}{i\tau + \xi^2} [(e^{i\tau t_1} - e^{i\tau t_2}) - (e^{-\xi^2|t_1|} - e^{-\xi^2|t_2|})] d\tau. \tag{2.35}$$

When $|\xi| \geq 1$, applying the Schwarz inequality, we obtain

$$\begin{aligned} &|F(t_1) - F(t_2)| \\ &\leq 4\langle \xi \rangle^{2\delta} \Big(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau + \xi^2 \rangle^{2(1-b)-2\delta}} d\tau \Big)^{1/2} \Big(\int_{\mathbb{R}} \frac{\langle i\tau + \xi^2 \rangle^{2(1-b)-2\delta}}{|i\tau + \xi^2|^2} d\tau \Big)^{1/2} \\ &\leq C\langle \xi \rangle^{2\delta} \Big(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau + \xi^2 \rangle^{2(1-b)-2\delta}} d\tau \Big)^{1/2} |\xi|^{1-2b-2\delta} \Big(\int_{\mathbb{R}} \frac{d\theta}{\langle \theta \rangle^{2b+2\delta}} \Big)^{1/2}, \end{aligned} \tag{2.36}$$

where we put $\tau = \xi^2\theta$. Hence it follows that for $|\xi| \geq 1$

$$|F(t_1) - F(t_2)| \leq C \Big(\int_{\mathbb{R}} \frac{|\widehat{g}(\xi,\tau)|^2}{\langle i\tau + \xi^2 \rangle^{2(1-b)-2\delta}} d\tau \Big)^{1/2}. \tag{2.37}$$

When $|\xi| \le 1$, we separate the two terms to estimate the right-hand side of (2.35). We may assume that $|t_1 - t_2| < 1$. It follows from the mean value theorem and the Schwarz inequality that

$$\begin{aligned}
&\Big|\int \frac{\widehat{g}(\xi,\tau)}{i\tau+\xi^2}(e^{i\tau t_1} - e^{i\tau t_2})d\tau\Big| \\
&\quad \le |t_1 - t_2| \int_{|\tau|\le 1} \frac{|\tau||\widehat{g}(\xi,\tau)|}{|i\tau+\xi^2|}d\tau + 2\int_{|\tau|\ge 1} \frac{|\widehat{g}(\xi,\tau)|}{|i\tau+\xi^2|}d\tau \\
&\quad \le C \left\| \langle i\tau+\xi^2\rangle^{b-1+\delta}\widehat{g}(\xi,\cdot)\right\|_{L^2_\tau} \\
&\qquad \times \Big[\Big(\int_{|\tau|\le 1} \langle\tau\rangle^{2(1-b)-2\delta}d\tau\Big)^{1/2} + \Big(\int_{|\tau|\ge 1}\langle\tau\rangle^{-2b-2\delta}d\tau\Big)^{1/2}\Big] \\
&\quad \le C \left\| \langle i\tau+\xi^2\rangle^{b-1+\delta}\widehat{g}(\xi,\cdot)\right\|_{L^2_\tau}.
\end{aligned} \tag{2.38}$$

Similarly it follows that

$$\begin{aligned}
&\Big|\int \frac{\widehat{g}(\xi,\tau)}{i\tau+\xi^2}(e^{-\xi^2|t_1|} - e^{-\xi^2|t_2|})d\tau\Big| \\
&\quad \le |t_1 - t_2||\xi|^2 \int_{|\tau|\le 1} \frac{|\widehat{g}(\xi,\tau)|}{|i\tau+\xi^2|}d\tau + 2\int_{|\tau|\ge 1} \frac{|\widehat{g}(\xi,\tau)|}{|i\tau+\xi^2|}d\tau \\
&\quad \le C \left\| \langle i\tau+\xi^2\rangle^{b-1+\delta}\widehat{g}(\xi,\cdot)\right\|_{L^2_\tau}.
\end{aligned} \tag{2.39}$$

Summing up, we obtain

$$|F(t_1) - F(t_2)| \le C \left\| \langle i\tau+\xi^2\rangle^{b-1+\delta}\widehat{g}(\xi,\cdot)\right\|_{L^2_\tau}. \tag{2.40}$$

Furthermore, we find that

$$\|F(t_1) - F(t_2)\|_{L^2(\mathbb{R})} \le C \left\| \langle i\tau+\xi^2\rangle^{b-1+\delta}\widehat{g}(\xi,\tau)\right\|_{L^2_{\xi,\tau}}. \tag{2.41}$$

It is clear that the integrand in (2.35) tends to 0 as $|t_1 - t_2| \to 0$, and is bounded uniformly in $|t_1 - t_2|$ by the integrand of the right-hand side of (2.40). Hence, $|F(t_1) - F(t_2)| \to 0$ as $|t_1 - t_2| \to 0$ for almost every $\xi \in \mathbb{R}$. Moreover, from (2.41) and the Lebesgue dominated convergence theorem, we infer that

$$\|F(t_1) - F(t_2)\|_{L^2(\mathbb{R})} \to 0 \qquad \text{as } |t_1 - t_2| \to 0. \tag{2.42}$$

To show (2.32), we first note that $F(0) = 0$. By repeating the above argument, we get

$$\|F(t)\|_{L^2(\mathbb{R})} = \|F(t) - F(0)\|_{L^2(\mathbb{R})} \le C\langle t\rangle \left\| \langle i\tau + \xi^2\rangle^{b-1+\delta}\widehat{g}(\xi,\tau)\right\|_{L^2_{\xi,\tau}}. \tag{2.43}$$

Hence, it follows that

$$\sup_{t\in\mathbb{R}_+} \chi_{\mathbb{R}_+}(t)\psi(t)\left\|U(-t)\int_0^t W(t-t')f(t')dt'\right\|_{H^{2\delta}} \le C\|f\|_{X^{0,b-1+\delta}}, \tag{2.44}$$

which implies (2.32). □

Finally, we introduce the following estimate to finish this section. Lemma 2.3 will be used in the proof of Proposition 3.1.

Lemma 2.3. *Let v be with compact support in time in $[-T,T]$. For any $\theta > 0$, there exists $\mu = \mu(\theta) > 0$ such that*

$$\left\|\mathcal{F}^{-1}_{x,t}\Big(\frac{\widehat{v}(\xi,\tau)}{\langle \tau - \xi|\xi|^{1+a}\rangle^{\theta}}\Big)\right\|_{L^2_{x,t}(\mathbb{R}^2)} \le CT^{\mu}\|v\|_{L^2_{x,t}(\mathbb{R}^2)}. \tag{2.45}$$

Proof. A similar estimate was verified by J. Ginibre, Y. Tsutsumi, and G. Velo [12, Lemma 3.1]. It suffices to modify the proof slightly. Therefore, we omit the proof of Lemma 2.3. □

3. BILINEAR ESTIMATES

Proposition 3.1. *For $s > -(1+a)/2$, there exists $b > 1/2$, C, μ, $\delta > 0$ such that for any u, $v \in X^{s,b}$ with compact support in $[-T,T]$, we have*

$$\|\partial_x(uv)\|_{X^{s,b-1+\delta}} \le CT^{\mu}\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}. \tag{3.1}$$

By a duality argument, this is equivalent to showing that for any $w \in X^{-s,1-b-\delta}$ with $\|w\|_{X^{-s,1-b-\delta}} \le 1$,

$$|I| = |\langle \partial_x(uv), w\rangle| \le CT^{\mu}\|u\|_{X^{s,b}}\|v\|_{X^{s,b}}\|w\|_{X^{-s,1-b-\delta}}. \tag{3.2}$$

Putting

$$\begin{aligned}
\widehat{f}(\xi,\tau) &= \langle i(\tau - \xi|\xi|^{1+a}) + \xi^2\rangle^{b}\langle\xi\rangle^{s}\widehat{u}(\xi,\tau),\\
\widehat{g}(\xi,\tau) &= \langle i(\tau - \xi|\xi|^{1+a}) + \xi^2\rangle^{b}\langle\xi\rangle^{s}\widehat{v}(\xi,\tau)\\
\text{and}\quad \widehat{h}(\xi,\tau) &= \langle i(\tau - \xi|\xi|^{1+a}) + \xi^2\rangle^{1-b-\delta}\langle\xi\rangle^{-s}\widehat{w}(\xi,\tau),
\end{aligned}$$

we see that (3.2) is equivalent to

$$|I| \le CT^{\mu}\|f\|_{L^2_xL^2_t}\|g\|_{L^2_xL^2_t}\|h\|_{L^2_xL^2_t}. \tag{3.3}$$

And we can rewrite

$$\begin{aligned} I &= \int_{\mathbb{R}^4} \frac{\xi\overline{\widehat{h}(\xi,\tau)}\langle\xi\rangle^s}{\langle i(\tau-\xi|\xi|^{1+a})+\xi^2\rangle^{1-b-\delta}} \frac{\widehat{g}(\xi_1,\tau_1)\langle\xi_1\rangle^{-s}}{\langle i(\tau_1-\xi_1|\xi_1|^{1+a})+\xi_1^2\rangle^b} \\ &\quad\times \frac{\widehat{f}(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{-s}}{\langle\, i(\tau-\tau_1-(\xi-\xi_1)|\xi-\xi_1|^{1+a}\,)+(\xi-\xi_1)^2\rangle^b} d\xi d\tau d\xi_1 d\tau_1 \qquad (3.4) \\ &= \int_{\mathbb{R}^4} \frac{\xi\overline{\widehat{h}(\xi,\tau)}\langle\xi\rangle^s}{\langle i\sigma+\xi^2\rangle^{1-b-\delta}} \frac{\widehat{g}(\xi_1,\tau_1)\langle\xi_1\rangle^{-s}}{\langle i\sigma_1+\xi_1^2\rangle^b} \frac{\widehat{f}(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{-s}}{\langle i\sigma_2+(\xi-\xi_1)^2\rangle^b} d\xi d\tau d\xi_1 d\tau_1, \end{aligned}$$

where

$$\sigma=\tau-\xi|\xi|^{1+a}, \quad \sigma_1=\tau_1-\xi_1|\xi_1|^{1+a}, \quad \sigma_2=\tau-\tau_1-(\xi-\xi_1)|\xi-\xi_1|^{1+a}. \qquad (3.5)$$

3.1 **Algebraic smoothing relation.** The following algebraic relation will be effectively used for the proof of Proposition 3.1:

Proposition 3.2. *Let $|\xi_1|\geq 1$ and $|\xi-\xi_1|\geq 1$, and let $0\leq a<1$. Then the following relation holds among σ, σ_1 and σ_2 defined above:*
(i) If $\xi_1(\xi-\xi_1)>0$ and $|\xi_1|\geq|\xi-\xi_1|,c$

$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\geq\frac{1+a}{3}|\xi_1|^{1+a}|\xi-\xi_1|. \qquad (3.6)$$

(ii) If $\xi_1(\xi-\xi_1)<0$ and $|\xi_1|\geq|\xi-\xi_1|$,

$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\geq\frac{1+a}{3}|\xi||\xi_1|^{a}|\xi-\xi_1|. \qquad (3.7)$$

(iii) If $\xi_1(\xi-\xi_1)>0$ and $|\xi_1|\leq|\xi-\xi_1|$,

$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\geq\frac{1+a}{3}|\xi_1||\xi-\xi_1|^{1+a}. \qquad (3.8)$$

(iv) If $\xi_1(\xi-\xi_1)<0$ and $|\xi_1|\leq|\xi-\xi_1|$,

$$\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\geq\frac{1+a}{3}|\xi||\xi_1||\xi-\xi_1|^{a}. \qquad (3.9)$$

Remark 3.1. When $a=1$, it follows from $\sigma_1+\sigma_2-\sigma=3\xi\xi_1(\xi-\xi_1)$ that $\max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\geq|\xi\xi_1(\xi-\xi_1)|$. See [5], [18].

Proof. We first set $\xi_2=\xi-\xi_1$. Then $\sigma_1+\sigma_2-\sigma=-\xi_1|\xi_1|^{1+a}-\xi_2|\xi_2|^{1+a}+(\xi_1+\xi_2)|\xi_1+\xi_2|^{1+a}$.

(i) We consider the case of $\xi_1>0$ and $\xi_2>0$. Putting $\xi_2=\beta\xi_1$, we find that $0<\beta\leq 1$ and that

$$\begin{aligned} \sigma_1+\sigma_2-\sigma &= -|\xi_1|^{2+a}-|\xi_2|^{2+a}+|\xi_1+\xi_2|^{2+a} \\ &= ((1+\beta)^{2+a}-1-\beta^{2+a})|\xi_1|^{2+a}. \qquad (3.10) \end{aligned}$$

Set $f(\beta) = (1+\beta)^{2+a} - 1 - \beta^{2+a}$. Since $f'(\beta) = (2+a)((1+\beta)^{1+a} - \beta^{1+a}) > 0$ and $f(0) = 0$, it follows that $f(\beta) \geq 0$ for any $\beta \geq 0$. From the mean value theorem, there exists $1 < c < 1+\beta$ such that $(1+\beta)^{2+a} - 1 = (2+a)c^{1+a}\beta$. Hence, for $0 < \beta \leq 1$

$$f(\beta) = \{(2+a)c^{1+a} - \beta^{1+a}\}\beta > (1+a)\beta, \tag{3.11}$$

from which it follows that $\sigma_1 + \sigma_2 - \sigma > (1+a)\beta|\xi_1|^{2+a} = (1+a)|\xi_1|^{1+a}|\xi_2|$. Therefore

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{1+a}{3}|\xi_1|^{1+a}|\xi_2|. \tag{3.12}$$

We next consider the case of $\xi_1 < 0$ and $\xi_2 < 0$. Putting $\xi_2 = \beta\xi_1$, then we find that $0 < \beta \leq 1$ and that

$$\sigma_1 + \sigma_2 - \sigma = |\xi_1|^{2+a} + |\xi_2|^{2+a} - |\xi_1 + \xi_2|^{2+a} = -f(\beta)|\xi_1|^{2+a}. \tag{3.13}$$

Therefore, repeating the same argument above, we obtain

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{1+a}{3}|\xi_1|^{1+a}|\xi_2|. \tag{3.14}$$

(ii) We first consider the case of $\xi_1 > 0$ and $\xi_2 < 0$. Putting $\xi_2 = -\beta\xi_1$, then we find that $0 < \beta \leq 1$ and that $\xi_1 + \xi_2 \geq 0$. Moreover,

$$\begin{aligned}\sigma_1 + \sigma_2 - \sigma &= -|\xi_1|^{2+a} + |\xi_2|^{2+a} + |\xi_1 + \xi_2|^{2+a} \\ &= ((1-\beta)^{2+a} - 1 + \beta^{2+a})|\xi_1|^{2+a} = -g(\beta)|\xi_1|^{2+a},\end{aligned} \tag{3.15}$$

where we set $g(\beta) = -(1-\beta)^{2+a} + 1 - \beta^{2+a}$. We get $g'(\beta) = -(2+a)(\beta^{1+a} - (1-\beta)^{1+a})$. Then $g(\beta) \geq 0$ for $0 \leq \beta \leq 1$. Indeed it suffices to note the following three points: (a) $g'(\beta) \geq 0$ for $0 \leq \beta \leq 1/2$, (b) $g'(\beta) \leq 0$ for $1/2 < \beta \leq 1$, (c) $g(0) = g(1) = 0$.

We next rewrite

$$g(\beta) = (1-\beta)\{1 - (1-\beta)^{1+a}\} + \beta(1 - \beta^{1+a}) \tag{3.16}$$

to apply the mean value theorem. It follows from the mean value theorem that there exist $(1-\beta) < c_1 < 1$ and $\beta < c_2 < 1$ satisfying

$$1 - (1-\beta)^{1+a} = (1+a)c_1^a\beta, \tag{3.17}$$

$$1 - \beta^{1+a} = (1+a)c_2^a(1-\beta). \tag{3.18}$$

Hence we have

$$\begin{aligned}g(\beta) &= (1+a)\beta(1-\beta)(c_1^a + c_2^a) \\ &> (1+a)\beta(1-\beta)\{(1-\beta)^a + \beta^a\} \geq (1+a)\beta(1-\beta),\end{aligned} \tag{3.19}$$

where we use the following fact in the last term: Let $0 < p \leq 1$. Then for a, $b \geq 0$ it follows that $(a+b)^p \leq a^p + b^p$.

Since $\sigma - \sigma_1 - \sigma_2 > (1+a)\beta(1-\beta)|\xi_1|^{2+a} = (1+a)|\xi_1|^a|\xi_2||\xi_1+\xi_2|$, we obtain

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{1+a}{3}|\xi_1|^a|\xi_2||\xi_1+\xi_2|. \tag{3.20}$$

We next consider the case of $\xi_1 < 0$ and $\xi_2 > 0$. Putting $\xi_2 = -\beta\xi_1$, we find that $0 < \beta \leq 1$ and that $\xi_1 + \xi_2 < 0$. Moreover,

$$\sigma_1 + \sigma_2 - \sigma = |\xi_1|^{2+a} - |\xi_2|^{2+a} - |\xi_1+\xi_2|^{2+a} = g(\beta)|\xi_1|^{2+a}. \tag{3.21}$$

Hence repeating the same argument above, we obtain

$$\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \geq \frac{1+a}{3}|\xi_1|^a|\xi_2||\xi_1+\xi_2|. \tag{3.22}$$

(iii)(iv) By symmetry between ξ_1 and $\xi_2 = \xi - \xi_1$, (iii) and (iv) are direct consequences of (i) and (ii) respectively.

Summing up, we establish our statement. □

Remark 3.2. Let $\rho = \frac{1+a}{2} - \frac{7+5a}{2}\epsilon$ and let $0 \leq a < 1$. The following exponents often appear throughout the proofs of Lemmas 3.1 to 3.4.

$$m_1 = 2\rho - (2+a)(1-5\epsilon) - 4\epsilon = -1-\epsilon, \tag{3.23}$$

$$m_2 = 2\rho - (2+a)(1+2\epsilon) + 10\epsilon = -1-\epsilon-7a\epsilon, \tag{3.24}$$

$$n_1 = 4\rho - (1+a)(1-5\epsilon) - 2(1+2\epsilon) = -1+a-(13+5a)\epsilon < 0. \tag{3.25}$$

3.2. **Preliminaries I.** For any fixed (ξ_1, τ_1) with $|\xi_1| \geq 1$, we introduce the following integral region: $A(\xi_1,\tau_1) = \{\, (\xi,\tau) \in \mathbb{R}^2 \colon |\xi| \leq 2|\xi_1|,\ |\xi - \xi_1| \geq 1 \,\}$.

Lemma 3.1. *Let $\rho = (1+a)/2 - (7+5a)\epsilon/2$. If $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi_1|^{1+a}|\xi - \xi_1|$ with $|\xi_1| \geq 1$ holds, then for any $\epsilon > 0$ there exists $C > 0$, depending only on ϵ, such that*

$$I = \frac{\langle \xi_1 \rangle^{2\rho}}{\langle i\sigma_1 + \xi_1^2 \rangle^{1+2\epsilon}} \iint_{A(\xi_1,\tau_1)} \frac{|\xi|^2 \langle \xi \rangle^{-2\rho} \langle \xi - \xi_1 \rangle^{2\rho}}{\langle i\sigma + \xi^2 \rangle^{1-5\epsilon} \langle i\sigma_2 + (\xi-\xi_1)^2 \rangle^{1+2\epsilon}} d\xi d\tau \leq C. \tag{3.26}$$

By symmetry between ξ_1 and $\xi - \xi_1$, we can easily derive the following corollary:

Corollary 3.1. *Let $\rho = (1+a)/2 - (7+5a)\epsilon/2$. If $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi_1||\xi - \xi_1|^{1+a}$ with $|\xi_1| \geq 1$, then (3.26) holds.*

Proof of Lemma 3.1. It follows that $|\xi - \xi_1| \leq 3|\xi_1|$ in $A(\xi_1,\tau_1)$.

We split $A(\xi_1,\tau_1)$ into three regions;

$A_1(\xi_1,\tau_1) = \{\, (\xi,\tau) \in A(\xi_1,\tau_1) : |\sigma| = \max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \,\}$,

$A_2(\xi_1,\tau_1) = \{\,(\xi,\tau) \in A(\xi_1,\tau_1) : |\sigma_1| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\,\}$,
$A_3(\xi_1,\tau_1) = \{\,(\xi,\tau) \in A(\xi_1,\tau_1) : |\sigma_2| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\,\}$.

Estimate in A_1. It follows from the assumption of the lemma that $\langle\sigma\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$ in A_1.

With the aid of $\langle\sigma\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned} I &\lesssim \iint_{A_1(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\xi_1^2\rangle^{1+2\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} d\xi d\tau \qquad (3.27)\\ &\lesssim \iint_{A_1(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)-2(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma_2\rangle^{1+2\epsilon}} d\xi d\tau. \end{aligned}$$

Since $|\xi| \le 2|\xi_1|$ and $|\xi-\xi_1| \le 3|\xi_1|$ hold, we have

$$\begin{aligned} &\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)-2(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)} \\ &\lesssim \langle\xi_1\rangle^{-(1+a)(1-5\epsilon)-4\epsilon}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)} \lesssim \langle\xi-\xi_1\rangle^{2\rho-(2+a)(1-5\epsilon)-4\epsilon}. \qquad (3.28) \end{aligned}$$

Hence it follows from Remark 3.2 that

$$I \le C \iint_{A_1(\xi_1,\tau_1)} \frac{d\xi d\tau}{\langle\xi-\xi_1\rangle^{1+\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} \le C. \qquad (3.29)$$

Estimate in A_2. It follows from the assumption of the lemma that $\langle\sigma_1\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$ in A_2.

(i) Case of $|\xi-\xi_1| \ge |\xi|$. In this case, it follows that $|\xi-\xi_1| \sim |\xi_1|$. We first note that $\langle\sigma\rangle^{1-5\epsilon}\langle\sigma_1\rangle^{1+2\epsilon} = \langle\sigma\rangle^{1-5\epsilon}\langle\sigma_1\rangle^{1-5\epsilon}\langle\sigma_1\rangle^{7\epsilon} \ge \langle\sigma\rangle^{1+2\epsilon}\langle\sigma_1\rangle^{1-5\epsilon}$ holds since $|\sigma_1| \ge |\sigma|$. With this inequality and $\langle\sigma_1\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned} I &\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho}\langle\xi-\xi_1\rangle^{2\rho}}{\langle\sigma\rangle^{1+2\epsilon}\langle\sigma_1\rangle^{1-5\epsilon}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{1+2\epsilon}} d\xi d\tau \\ &\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}\langle(\xi-\xi_1)^2\rangle^{1+2\epsilon}} d\xi d\tau \qquad (3.30)\\ &\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2(1+2\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}} d\xi d\tau. \end{aligned}$$

Since $|\xi| \le 2|\xi_1|$ and $|\xi-\xi_1| \sim |\xi_1|$ hold, we have

$$\begin{aligned} &\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2(1+2\epsilon)} \qquad (3.31)\\ &\lesssim \langle\xi_1\rangle^{2-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2-4\epsilon} \sim \langle\xi-\xi_1\rangle^{2\rho-(2+a)(1-5\epsilon)-4\epsilon}. \end{aligned}$$

Hence it follows from Remark 3.2 that

$$I \le C \iint_{A_2(\xi_1,\tau_1)} \frac{d\xi d\tau}{\langle \xi - \xi_1 \rangle^{1+\epsilon} \langle \sigma \rangle^{1+2\epsilon}} \le C. \tag{3.32}$$

(ii) Case of $|\xi - \xi_1| \le |\xi|$. In this case, it follows that $|\xi| \sim |\xi_1|$ holds. With the aid of $\langle \sigma_1 \rangle \gtrsim \langle \xi_1 \rangle^{1+a} \langle \xi - \xi_1 \rangle$, we have

$$\begin{aligned} I &\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle \xi \rangle^{2-2\rho} \langle \xi_1 \rangle^{2\rho-(1+a)(1+2\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1+2\epsilon)}}{\langle \xi^2 \rangle^{1-5\epsilon} \langle \sigma_2 \rangle^{1+2\epsilon}} d\xi d\tau \\ &\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle \xi \rangle^{-2\rho+10\epsilon} \langle \xi_1 \rangle^{2\rho-(1+a)(1+2\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1+2\epsilon)}}{\langle \sigma_2 \rangle^{1+2\epsilon}} d\xi d\tau. \end{aligned} \tag{3.33}$$

Since $|\xi| \sim |\xi_1|$ and $|\xi - \xi_1| \le |\xi|$ hold, we have

$$\begin{aligned} &\langle \xi \rangle^{-2\rho+10\epsilon} \langle \xi_1 \rangle^{2\rho-(1+a)(1+2\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1+2\epsilon)} \\ &\sim \langle \xi \rangle^{-(1+a)(1+2\epsilon)+10\epsilon} \langle \xi - \xi_1 \rangle^{2\rho-(1+2\epsilon)} \lesssim \langle \xi - \xi_1 \rangle^{2\rho-(2+a)(1+2\epsilon)+10\epsilon}. \end{aligned} \tag{3.34}$$

Hence it follows from Remark 3.2 that

$$I \le C \iint_{A_2(\xi_1,\tau_1)} \frac{d\xi d\tau}{\langle \xi - \xi_1 \rangle^{1+\epsilon+7a\epsilon} \langle \sigma_2 \rangle^{1+2\epsilon}} \le C. \tag{3.35}$$

Estimate in A_3. It follows from the assumption of the lemma that $\langle \sigma_2 \rangle \gtrsim \langle \xi_1 \rangle^{1+a} \langle \xi - \xi_1 \rangle$ in A_3. We first note that

$$\langle \sigma \rangle^{1-5\epsilon} \langle \sigma_2 \rangle^{1+2\epsilon} = \langle \sigma \rangle^{1-5\epsilon} \langle \sigma_2 \rangle^{1-5\epsilon} \langle \sigma_2 \rangle^{7\epsilon} \ge \langle \sigma \rangle^{1+2\epsilon} \langle \sigma_2 \rangle^{1-5\epsilon}$$

holds since $|\sigma_2| \ge |\sigma|$. With this inequality and $\langle \sigma_2 \rangle \gtrsim \langle \xi_1 \rangle^{1+a} \langle \xi - \xi_1 \rangle$, we have

$$\begin{aligned} I &\lesssim \iint_{A_3(\xi_1,\tau_1)} \frac{\langle \xi \rangle^{2-2\rho} \langle \xi_1 \rangle^{2\rho-(1+a)(1-5\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1-5\epsilon)}}{\langle \sigma \rangle^{1+2\epsilon} \langle \xi_1^2 \rangle^{1+2\epsilon}} d\xi d\tau \\ &\lesssim \iint_{A_3(\xi_1,\tau_1)} \frac{\langle \xi \rangle^{2-2\rho} \langle \xi_1 \rangle^{2\rho-(1+a)(1-5\epsilon)-2(1+2\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1-5\epsilon)}}{\langle \sigma \rangle^{1+2\epsilon}} d\xi d\tau. \end{aligned} \tag{3.36}$$

Since $|\xi| \le 2|\xi_1|$ and $|\xi - \xi_1| \le 3|\xi_1|$ hold, we have

$$\begin{aligned} &\langle \xi \rangle^{2-2\rho} \langle \xi_1 \rangle^{2\rho-(1+a)(1-5\epsilon)-2(1+2\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1-5\epsilon)} \\ &\lesssim \langle \xi_1 \rangle^{-(1+a)(1-5\epsilon)-4\epsilon} \langle \xi - \xi_1 \rangle^{2\rho-(1-5\epsilon)} \lesssim \langle \xi - \xi_1 \rangle^{2\rho-(2+a)(1-5\epsilon)-4\epsilon}. \end{aligned} \tag{3.37}$$

Hence it follows from Remark 3.2 that

$$I \le C \iint_{A_3(\xi_1,\tau_1)} \frac{d\xi d\tau}{\langle \xi - \xi_1 \rangle^{1+\epsilon} \langle \sigma \rangle^{1+2\epsilon}} \le C. \tag{3.38}$$

Summing up, we have the desired result. □

Lemma 3.2. *Let $\rho = (1+a)/2 - (7+5a)\epsilon/2$. If $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi||\xi_1|^a|\xi-\xi_1|$ with $|\xi_1| \geq 1$ holds, then for any $\epsilon > 0$ there exists $C > 0$, depending only on ϵ, such that*

$$I = \frac{\langle \xi_1 \rangle^{2\rho}}{\langle i\sigma_1 + \xi_1^2 \rangle^{1+2\epsilon}} \iint_{A(\xi_1,\tau_1)} \frac{|\xi|^2 \langle \xi \rangle^{-2\rho} \langle \xi - \xi_1 \rangle^{2\rho}}{\langle i\sigma + \xi^2 \rangle^{1-5\epsilon} \langle i\sigma_2 + (\xi-\xi_1)^2 \rangle^{1+2\epsilon}} d\xi d\tau \leq C. \tag{3.39}$$

By symmetry between ξ_1 and $\xi - \xi_1$, we can easily derive the following corollary:

Corollary 3.2. *Let $\rho = (1+a)/2 - (7+5a)\epsilon/2$. If $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi||\xi_1||\xi-\xi_1|^a$ with $|\xi_1| \geq 1$, then* (3.39) *holds.*

Proof of Lemma 3.2. We find that $|\xi - \xi_1| \leq 3|\xi_1|$ holds in $A(\xi_1, \tau_1)$. As in the proof of Lemma 3.1, we split $A(\xi_1, \tau_1)$ into three regions $A_1(\xi_1, \tau_1)$, $A_2(\xi_1, \tau_1)$ and $A_3(\xi_1, \tau_1)$.

Estimate in A_1. It follows from the assumption of the lemma that $\langle \sigma \rangle \gtrsim |\xi| \langle \xi_1 \rangle^a \langle \xi - \xi_1 \rangle$ in A_1. With the aid of $\langle \sigma \rangle \gtrsim |\xi| \langle \xi_1 \rangle^a \langle \xi - \xi_1 \rangle$, we have

$$\begin{aligned} I &\lesssim \iint_{A_1(\xi_1,\tau_1)} \frac{|\xi|^{1+5\epsilon} \langle \xi \rangle^{-2\rho} \langle \xi_1 \rangle^{2\rho - a(1-5\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1-5\epsilon)}}{\langle \xi_1^2 \rangle^{1+2\epsilon} \langle \sigma_2 \rangle^{1+2\epsilon}} d\xi d\tau \qquad (3.40) \\ &\lesssim \iint_{A_1(\xi_1,\tau_1)} \frac{\langle \xi \rangle^{1+5\epsilon-2\rho} \langle \xi_1 \rangle^{2\rho - a(1-5\epsilon) - 2(1+2\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1-5\epsilon)}}{\langle \sigma_2 \rangle^{1+2\epsilon}} d\xi d\tau. \end{aligned}$$

Since $|\xi| \leq 2|\xi_1|$ and $|\xi - \xi_1| \leq 3|\xi_1|$, we have

$$\begin{aligned} &\langle \xi \rangle^{1+5\epsilon-2\rho} \langle \xi_1 \rangle^{2\rho - a(1-5\epsilon) - 2(1+2\epsilon)} \langle \xi - \xi_1 \rangle^{2\rho-(1-5\epsilon)} \qquad (3.41) \\ &\lesssim \langle \xi_1 \rangle^{-(1+a)(1-5\epsilon)-4\epsilon} \langle \xi - \xi_1 \rangle^{2\rho-(1-5\epsilon)} \lesssim \langle \xi - \xi_1 \rangle^{2\rho-(2+a)(1-5\epsilon)-4\epsilon}. \end{aligned}$$

Hence it follows from Remark 3.2 that

$$I \leq C \iint_{A_1(\xi_1,\tau_1)} \frac{d\xi d\tau}{\langle \xi - \xi_1 \rangle^{1+\epsilon} \langle \sigma_2 \rangle^{1+2\epsilon}} \leq C. \tag{3.42}$$

Estimate in A_2. It follows from the assumption of the lemma that $\langle \sigma_1 \rangle \gtrsim |\xi| \langle \xi_1 \rangle^a \langle \xi - \xi_1 \rangle$ in A_2.

(i) Case of $|\xi - \xi_1| \geq |\xi|$. In this case, it follows that $|\xi_1| \sim |\xi - \xi_1|$.

We first note that $\langle \sigma \rangle^{1-5\epsilon} \langle \sigma_1 \rangle^{1+2\epsilon} \geq \langle \sigma \rangle^{1+2\epsilon} \langle \sigma_1 \rangle^{1-5\epsilon}$ holds since $|\sigma_1| \geq |\sigma|$. Hence we get

$$I \lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{|\xi|^2 \langle \xi \rangle^{-2\rho} \langle \xi_1 \rangle^{2\rho} \langle \xi - \xi_1 \rangle^{2\rho}}{\langle \sigma \rangle^{1+2\epsilon} \langle \sigma_1 \rangle^{1-5\epsilon} \langle i\sigma_2 + (\xi - \xi_1)^2 \rangle^{1+2\epsilon}} d\xi d\tau. \tag{3.43}$$

With the aid of $\langle\sigma_1\rangle \gtrsim |\xi|\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$, (3.43) is bounded by

$$\begin{aligned} &\iint_{A_2(\xi_1,\tau_1)} \frac{|\xi|^{1+5\epsilon}\langle\xi\rangle^{-2\rho}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}\langle(\xi-\xi_1)^2\rangle^{1+2\epsilon}} d\xi d\tau \qquad (3.44) \\ &\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2(1+2\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}} d\xi d\tau. \end{aligned}$$

Since $|\xi_1| \sim |\xi-\xi_1|$ and $|\xi| \le 2|\xi_1|$ hold,

$$\begin{aligned} &\langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2(1+2\epsilon)} \qquad (3.45) \\ &\sim \langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{4\rho-(1+a)(1-5\epsilon)-2(1+2\epsilon)} \lesssim \langle\xi\rangle^{2\rho-(2+a)(1-5\epsilon)-4\epsilon}, \end{aligned}$$

where $4\rho-(1+a)(1-5\epsilon)-2(1+2\epsilon) < 0$ (Remark 3.2). Therefore it follows from Remark 3.2 that

$$I \le C \iint_{A_2(\xi_1,\tau_1)} \frac{d\xi d\tau}{\langle\xi\rangle^{1+\epsilon}\langle\sigma\rangle^{1+2\epsilon}} \le C. \qquad (3.46)$$

(ii) Case of $|\xi-\xi_1| \le |\xi|$**.** In this case, it follows that $|\xi| \sim |\xi_1|$. By virtue of $\langle\sigma_1\rangle \gtrsim |\xi|\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned} I &\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{|\xi|^{1-2\epsilon}\langle\xi\rangle^{-2\rho}\langle\xi_1\rangle^{2\rho-a(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle\xi^2\rangle^{1-5\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} d\xi d\tau \qquad (3.47) \\ &\lesssim \iint_{A_2(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{-2\rho-(1+2\epsilon)+10\epsilon}\langle\xi_1\rangle^{2\rho-a(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle\sigma_2\rangle^{1+2\epsilon}} d\xi d\tau. \end{aligned}$$

Since $|\xi| \sim |\xi_1|$ and $|\xi-\xi_1| \le |\xi|$ hold, we have

$$\begin{aligned} &\langle\xi\rangle^{-2\rho-(1+2\epsilon)+10\epsilon}\langle\xi_1\rangle^{2\rho-a(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)} \qquad (3.48) \\ &\sim \langle\xi\rangle^{-(1+a)(1+2\epsilon)+10\epsilon}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)} \lesssim \langle\xi-\xi_1\rangle^{2\rho-(2+a)(1+2\epsilon)+10\epsilon}. \end{aligned}$$

Hence it follows from Remark 3.2 that

$$I \le C \iint_{A_2(\xi_1,\tau_1)} \frac{d\xi d\tau}{\langle\xi-\xi_1\rangle^{1+\epsilon+7a\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} \le C. \qquad (3.49)$$

Estimate in A_3**.** It follows from the assumption of the lemma that $\langle\sigma_2\rangle \gtrsim |\xi|\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$ in A_3.

(i) Case of $|\xi-\xi_1| \ge |\xi|$**.** In this case, it follows that $|\xi_1| \sim |\xi-\xi_1|$. Since $|\xi_1| \sim |\xi-\xi_1|$, this case is proved as in the region A_2 (i) above by using the symmetry between σ_1 and σ_2.

(ii) Case of $|\xi-\xi_1| \le |\xi|$. In this case, it follows that $|\xi| \sim |\xi_1|$. We first note that $\langle\sigma\rangle^{1-5\epsilon}\langle\sigma_2\rangle^{1+2\epsilon} \ge \langle\sigma\rangle^{1+2\epsilon}\langle\sigma_2\rangle^{1-5\epsilon}$ holds since $|\sigma_2| \ge |\sigma|$. Hence we get

$$I \lesssim \iint_{A_3(\xi_1,\tau_1)} \frac{|\xi|^2\langle\xi\rangle^{-2\rho}\langle\xi_1\rangle^{2\rho}\langle\xi-\xi_1\rangle^{2\rho}}{\langle\sigma\rangle^{1+2\epsilon}\langle i\sigma_1+\xi_1^2\rangle^{1+2\epsilon}\langle\sigma_2\rangle^{1-5\epsilon}} d\xi d\tau. \tag{3.50}$$

By virtue of $\langle\sigma_2\rangle \gtrsim |\xi|\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned} I &\lesssim \iint_{A_3(\xi_1,\tau_1)} \frac{|\xi|^{1+5\epsilon}\langle\xi\rangle^{-2\rho}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}\langle\xi_1^2\rangle^{1+2\epsilon}} d\xi d\tau \qquad (3.51) \\ &\lesssim \iint_{A_3(\xi_1,\tau_1)} \frac{\langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)-2(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma\rangle^{1+2\epsilon}} d\xi d\tau. \end{aligned}$$

Since $|\xi| \sim |\xi_1|$ and $|\xi-\xi_1| \le |\xi|$ hold, we have

$$\begin{aligned} &\langle\xi\rangle^{1-2\rho+5\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)-2(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)} \\ &\sim \langle\xi\rangle^{-(1+a)(1-5\epsilon)-4\epsilon}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)} \lesssim \langle\xi-\xi_1\rangle^{2\rho-(2+a)(1-5\epsilon)-4\epsilon}. \qquad (3.52) \end{aligned}$$

Hence it follows from Remark 3.2 that

$$I \le C \iint_{A_3(\xi_1,\tau_1)} \frac{d\xi d\tau}{\langle\xi-\xi_1\rangle^{1+\epsilon}\langle\sigma\rangle^{1+2\epsilon}} \le C. \tag{3.53}$$

Thus, we finish the proof. □

3.3. **Preliminaries II.** For any fixed (ξ,τ), we introduce the following integral region: $B(\xi,\tau) = \{(\xi_1,\tau_1) \in \mathbb{R}^2 : 2|\xi_1| \le |\xi|,\ \ |\xi_1| \ge 1,\ \ |\xi-\xi_1| \ge 1\}$.

Lemma 3.3. *Let $\rho = (1+a)/2 - (7+5a)\epsilon/2$. If $\max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \gtrsim |\xi_1|^{1+a}|\xi-\xi_1|$ holds, then for any $\epsilon > 0$ there exists $C > 0$, depending only on ϵ, such that*

$$I = \frac{|\xi|^2\langle\xi\rangle^{-2\rho}}{\langle i\sigma+\xi^2\rangle^{1-5\epsilon}} \iint_{B(\xi,\tau)} \frac{\langle\xi_1\rangle^{2\rho}\langle\xi-\xi_1\rangle^{2\rho}}{\langle i\sigma_1+\xi_1^2\rangle^{1+2\epsilon}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{1+2\epsilon}} d\xi_1 d\tau_1 \le C. \tag{3.54}$$

By symmetry between ξ_1 and $\xi-\xi_1$, we can easily derive the following corollary:

Corollary 3.3. *Let $\rho = (1+a)/2 - (7+5a)\epsilon/2$. If $\max\{|\sigma|,|\sigma_1|,|\sigma_2|\} \gtrsim |\xi_1||\xi-\xi_1|^{1+a}$, then (3.54) holds.*

Proof of Lemma 3.3. It follows that $|\xi| \sim |\xi-\xi_1|$ in $B(\xi,\tau)$.

We split $B(\xi,\tau)$ into three regions;
$B_1(\xi,\tau) = \{\,(\xi_1,\tau_1) \in B(\xi,\tau) : |\sigma| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\,\}$,
$B_2(\xi,\tau) = \{\,(\xi_1,\tau_1) \in B(\xi,\tau) : |\sigma_1| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\,\}$,
$B_3(\xi,\tau) = \{\,(\xi_1,\tau_1) \in B(\xi,\tau) : |\sigma_2| = \max\{|\sigma|,|\sigma_1|,|\sigma_2|\}\,\}$.

Estimate in B_1. It follows from the assumption of the lemma that $\langle\sigma\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$ in B_1.

With the aid of $\langle\sigma\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned} I &\lesssim \iint_{B_1(\xi,\tau)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma_1\rangle^{1+2\epsilon}\langle(\xi-\xi_1)^2\rangle^{1+2\epsilon}} d\xi_1 d\tau_1 \\ &\lesssim \iint_{B_1(\xi,\tau)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2(1+2\epsilon)}}{\langle\sigma_1\rangle^{1+2\epsilon}} d\xi_1 d\tau_1. \end{aligned} \tag{3.55}$$

Since $|\xi| \sim |\xi-\xi_1|$ and $2|\xi_1| \le |\xi|$ hold, we have

$$\begin{aligned} &\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2-4\epsilon} \\ &\sim \langle\xi\rangle^{-(1-5\epsilon)-4\epsilon}\langle\xi_1\rangle^{2\rho-(1+a)(1-5\epsilon)} \lesssim \langle\xi_1\rangle^{2\rho-(2+a)(1-5\epsilon)-4\epsilon}. \end{aligned} \tag{3.56}$$

Hence it follows from Remark 3.2 that

$$I \le C \iint_{B_1(\xi,\tau)} \frac{d\xi_1 d\tau_1}{\langle\xi_1\rangle^{1+\epsilon}\langle\sigma_1\rangle^{1+2\epsilon}} \le C. \tag{3.57}$$

Estimate in B_2. It follows from the assumption of the lemma that $\langle\sigma_1\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$ in B_2.

With the aid of $\langle\sigma_1\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned} I &\lesssim \iint_{B_2(\xi,\tau)} \frac{\langle\xi\rangle^{2-2\rho}\langle\xi_1\rangle^{2\rho-(1+a)(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle\xi^2\rangle^{1-5\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} d\xi_1 d\tau_1 \\ &\lesssim \iint_{B_2(\xi,\tau)} \frac{\langle\xi\rangle^{-2\rho+10\epsilon}\langle\xi_1\rangle^{2\rho-(1+a)(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle\sigma_2\rangle^{1+2\epsilon}} d\xi_1 d\tau_1. \end{aligned} \tag{3.58}$$

Since $|\xi| \sim |\xi-\xi_1|$ and $2|\xi_1| \le |\xi|$, we have

$$\begin{aligned} &\langle\xi\rangle^{-2\rho+10\epsilon}\langle\xi_1\rangle^{2\rho-(1+a)(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)} \\ &\sim \langle\xi\rangle^{-(1+2\epsilon)+10\epsilon}\langle\xi_1\rangle^{2\rho-(1+a)(1+2\epsilon)} \lesssim \langle\xi_1\rangle^{2\rho-(2+a)(1+2\epsilon)+10\epsilon}. \end{aligned} \tag{3.59}$$

Hence it follows from Remark 3.2 that

$$I \le C \iint_{B_2(\xi,\tau)} \frac{d\xi_1 d\tau_1}{\langle\xi_1\rangle^{1+\epsilon+7a\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} \le C. \tag{3.60}$$

Estimate in B_3. It follows from the assumption of the lemma that $\langle\sigma_2\rangle \gtrsim \langle\xi_1\rangle^{1+a}\langle\xi-\xi_1\rangle$ in B_3. By symmetry between $i\sigma_1+\xi_1^2$ and $i\sigma_2+(\xi-\xi_1)^2$, we can prove this case by following the analogous argument in B_2.

Summing up, our statement is established. □

Lemma 3.4. *Let* $\rho = (1+a)/2 - (7+5a)\epsilon/2$. *If* $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi||\xi_1|^a|\xi - \xi_1|$ *holds, then for any* $\epsilon > 0$ *there exists* $C > 0$, *depending only on* ϵ, *such that*

$$I = \frac{|\xi|^2\langle\xi\rangle^{-2\rho}}{\langle i\sigma + \xi^2\rangle^{1-5\epsilon}} \iint_{B(\xi,\tau)} \frac{\langle\xi_1\rangle^{2\rho}\langle\xi-\xi_1\rangle^{2\rho}}{\langle i\sigma_1 + \xi_1^2\rangle^{1+2\epsilon}\langle i\sigma_2 + (\xi-\xi_1)^2\rangle^{1+2\epsilon}} d\xi_1 d\tau_1 \le C. \tag{3.61}$$

By symmetry between ξ_1 and $\xi - \xi_1$, we can easily derive the following corollary:

Corollary 3.4. *Let* $\rho = (1+a)/2 - (7+5a)\epsilon/2$. *If* $\max\{|\sigma|, |\sigma_1|, |\sigma_2|\} \gtrsim |\xi||\xi_1||\xi-\xi_1|^a$, *then (3.61) holds.*

Proof of Lemma 3.4. It follows that $|\xi| \sim |\xi - \xi_1|$ in $B(\xi, \tau)$.

As in the proof of Lemma 3.3, we split $B(\xi,\tau)$ into three regions $B_1(\xi,\tau)$, $B_2(\xi,\tau)$ and $B_3(\xi,\tau)$.

Estimate in B_1. It follows from the assumption of the lemma that $\langle\sigma\rangle \gtrsim \langle\xi\rangle\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$ in B_1.

With the aid of $\langle\sigma\rangle \gtrsim \langle\xi\rangle\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned} I &\lesssim \iint_{B_1(\xi,\tau)} \frac{\langle\xi\rangle^{1+5\epsilon-2\rho}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)}}{\langle\sigma_1\rangle^{1+2\epsilon}\langle(\xi-\xi_1)^2\rangle^{1+2\epsilon}} d\xi_1 d\tau_1 \qquad (3.62)\\ &\lesssim \iint_{B_1(\xi,\tau)} \frac{\langle\xi\rangle^{1+5\epsilon-2\rho}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2(1+2\epsilon)}}{\langle\sigma_1\rangle^{1+2\epsilon}} d\xi_1 d\tau_1. \end{aligned}$$

Since $|\xi| \sim |\xi-\xi_1|$ and $2|\xi_1| \le |\xi|$ hold, we have

$$\begin{aligned} &\langle\xi\rangle^{-2\rho+2-(1-5\epsilon)}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1-5\epsilon)-2-4\epsilon}\\ &\sim \langle\xi\rangle^{-2(1-5\epsilon)-4\epsilon}\langle\xi_1\rangle^{2\rho-a(1-5\epsilon)} \lesssim \langle\xi_1\rangle^{2\rho-(2+a)(1-5\epsilon)-4\epsilon}. \end{aligned} \tag{3.63}$$

Hence it follows from Remark 3.2 that

$$I \le C \iint_{B_1(\xi,\tau)} \frac{d\xi_1 d\tau_1}{\langle\xi_1\rangle^{1+\epsilon}\langle\sigma_1\rangle^{1+2\epsilon}} \le C. \tag{3.64}$$

Estimate in B_2. It follows from the assumption of the lemma that $\langle\sigma_1\rangle \gtrsim \langle\xi\rangle\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$ in B_2.

By virtue of $\langle\sigma_1\rangle \gtrsim \langle\xi\rangle\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$, we have

$$\begin{aligned} I &\lesssim \iint_{B_2(\xi,\tau)} \frac{\langle\xi\rangle^{1-2\rho-2\epsilon}\langle\xi_1\rangle^{2\rho-a(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle\xi^2\rangle^{1-5\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} d\xi_1 d\tau_1\\ &\lesssim \iint_{B_2(\xi,\tau)} \frac{\langle\xi\rangle^{-2\rho-(1+2\epsilon)+10\epsilon}\langle\xi_1\rangle^{2\rho-a(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}}{\langle\sigma_2\rangle^{1+2\epsilon}} d\xi_1 d\tau_1. \qquad (3.65) \end{aligned}$$

Since $|\xi| \sim |\xi-\xi_1|$ and $2|\xi_1| \le |\xi|$ hold, we obtain

$$\begin{aligned}&\langle\xi\rangle^{-2\rho-(1+2\epsilon)+10\epsilon}\langle\xi_1\rangle^{2\rho-a(1+2\epsilon)}\langle\xi-\xi_1\rangle^{2\rho-(1+2\epsilon)}\\&\sim\langle\xi\rangle^{-2(1+2\epsilon)+10\epsilon}\langle\xi_1\rangle^{2\rho-a(1+2\epsilon)}\lesssim\langle\xi_1\rangle^{2\rho-(2+a)(1+2\epsilon)+10\epsilon}.\end{aligned}\tag{3.66}$$

Hence it follows from Remark 3.2 that

$$I \le C\iint_{B_2(\xi,\tau)}\frac{d\xi_1 d\tau_1}{\langle\xi_1\rangle^{1+\epsilon+7a\epsilon}\langle\sigma_2\rangle^{1+2\epsilon}} \le C.\tag{3.67}$$

Estimate in B_3. It follows from the assumption of the lemma that $\langle\sigma_2\rangle \gtrsim \langle\xi\rangle\langle\xi_1\rangle^a\langle\xi-\xi_1\rangle$ in B_3. By symmetry between $i\sigma_1+\xi_1^2$ and $i\sigma_2+(\xi-\xi_1)^2$, we can prove this case by following the analogous argument in B_2.

Summing up, we finish the proof. □

3.4. Preliminaries III.

Lemma 3.5. *Let $0<\rho\le 1$. For any fixed (ξ,τ), we introduce the following integral region:*

$D(\xi,\tau) = \{\,(\xi_1,\tau_1)\in\mathbb{R}^2 : |\xi_1|\le 1\,\}$.

Then for any $\epsilon>0$ there exists $C>0$, depending only on ϵ, such that

$$I=\frac{|\xi|^2\langle\xi\rangle^{-2\rho}}{\langle i\sigma+\xi^2\rangle^{1-5\epsilon}}\iint_{D(\xi,\tau)}\frac{\langle\xi_1\rangle^{2\rho}\langle\xi-\xi_1\rangle^{2\rho}}{\langle i\sigma_1+\xi_1^2\rangle^{1+2\epsilon}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{1+2\epsilon}}d\xi_1 d\tau_1 \le C.\tag{3.68}$$

Proof. Direct calculations show

$$\begin{aligned}I &\le C\frac{1}{\langle i\sigma+\xi^2\rangle^{\rho-5\epsilon}}\int_{\tau_1}\int_{|\xi_1|\le1}\frac{d\xi_1 d\tau_1}{\langle i\sigma_1+\xi_1^2\rangle^{1+2\epsilon}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{1-\rho+2\epsilon}}\\&\le C\int_{\tau_1}\int_{|\xi_1|\le1}\frac{d\xi_1 d\tau_1}{\langle\sigma_1\rangle^{1+2\epsilon}\langle\sigma_2\rangle^{1-\rho+2\epsilon}}\\&\le C\int_{|\xi_1|\le1}\int_{\tau_1}\frac{d\tau_1}{\langle\min\{|\sigma_1|,|\sigma_2|\}\rangle^{2-\rho+4\epsilon}}d\xi_1 = C\int_{|\xi_1|\le1}\int_{\tau_1}\frac{d\tau_1}{\langle\tau_1\rangle^{2-\rho+4\epsilon}}d\xi_1 \le C.\end{aligned}\tag{3.69}$$

Hence we establish our statement. □

4. Proof of Proposition 3.1

Let $s>-(1+a)/2$. In this section, we shall prove

$$|I| \le CT^{\mu}\|f\|_{L_x^2L_t^2}\|g\|_{L_x^2L_t^2}\|h\|_{L_x^2L_t^2},\tag{4.1}$$

where

$$I=\int_{\mathbb{R}^4}\frac{\xi\overline{\widehat{h}(\xi,\tau)}\langle\xi\rangle^s}{\langle i\sigma+\xi^2\rangle^{1-b-\delta}}\frac{\widehat{g}(\xi_1,\tau_1)\langle\xi_1\rangle^{-s}}{\langle i\sigma_1+\xi_1^2\rangle^b}\frac{\widehat{f}(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{-s}}{\langle i\sigma_2+(\xi-\xi_1)^2\rangle^b}d\xi d\tau d\xi_1 d\tau_1. \tag{4.2}$$

It suffices to show (4.1) only in the case $s=-\rho_0=-(1+a)/2+(7+5a)\epsilon/2$ since $\langle\xi\rangle^{s+\rho_0}\le\langle\xi-\xi_1\rangle^{s+\rho_0}\langle\xi\rangle^{s+\rho_0}$. By Fubini's theorem, we can assume that $\widehat{f},\widehat{g},\widehat{h}\ge 0$.

We divide $\mathbb{R}^4$ into five regions D_1, D_2, D_3, D_4 and D_5;
$D_1=\{\,(\xi,\xi_1,\tau,\tau_1)\in\mathbb{R}^4:|\xi_1|\ge 1,\ |\xi-\xi_1|\ge 1,\ (3.6)\text{ holds.}\,\}$,
$D_2=\{\,(\xi,\xi_1,\tau,\tau_1)\in\mathbb{R}^4:|\xi_1|\ge 1,\ |\xi-\xi_1|\ge 1,\ (3.7)\text{ holds.}\,\}$,
$D_3=\{\,(\xi,\xi_1,\tau,\tau_1)\in\mathbb{R}^4:|\xi_1|\ge 1,\ |\xi-\xi_1|\ge 1,\ (3.8)\text{ holds.}\,\}$,
$D_4=\{\,(\xi,\xi_1,\tau,\tau_1)\in\mathbb{R}^4:|\xi_1|\ge 1,\ |\xi-\xi_1|\ge 1,\ (3.9)\text{ holds.}\,\}$,
$D_5=\{\,(\xi,\xi_1,\tau,\tau_1)\in\mathbb{R}^4:\ |\xi_1|\le 1\text{ or }|\xi-\xi_1|\le 1\,\}$.

Furthermore, we split these regions into two parts respectively;
$D_j=D_{j,A}\cup D_{j,B}$ $(j=1,2,3,4)$, where
$D_{j,A}=\{\,(\xi,\xi_1,\tau,\tau_1)\in D_j:\ |\xi|\le 2|\xi_1|\,\}$, $D_{j,B}=\{\,(\xi,\xi_1,\tau,\tau_1)\in D_j:\ |\xi|\ge 2|\xi_1|\,\}$.

And we need not divide D_5.

According to these integral regions, we divide the integral I;

$$I=\sum_{j=1}^{4}I_{D_{j,A}}+\sum_{j=1}^{4}I_{D_{j,B}}+I_{D_5},$$

where

$$I_{\widetilde{D}}=\int_{\widetilde{D}}\frac{\xi\overline{\widehat{h}(\xi,\tau)}\langle\xi\rangle^s}{\langle i\sigma+\xi^2\rangle^{1-b-\delta}}\frac{\widehat{g}(\xi_1,\tau_1)\langle\xi_1\rangle^{-s}}{\langle i\sigma_1+\xi_1^2\rangle^b}\frac{\widehat{f}(\xi-\xi_1,\tau-\tau_1)\langle\xi-\xi_1\rangle^{-s}}{\langle i\sigma_2+(\xi-\xi_1)^2\rangle^b}\times d\xi d\tau d\xi_1 d\tau_1.$$

Each integral $I_{\widetilde{D}}$ is estimated according to the following two cases:

Case I. This case applies to the integral regions $\widetilde{D}=D_{1,A}\cup D_{2,A}\cup D_{3,A}\cup D_{4,A}$. Using the Schwarz inequality, we have

$$I_{\widetilde{D}}\le\int\frac{\langle\xi_1\rangle^{-s}\widehat{g}(\xi_1,\tau_1)}{\langle i\sigma_1+\xi_1^2\rangle^b}\times\Big(\int_{\widetilde{D}(\xi_1,\tau_1)}\frac{|\xi|^2\langle\xi-\xi_1\rangle^{-2s}\langle\xi\rangle^{2s}}{\langle i\sigma+\xi^2\rangle^{2(1-b)-3\delta}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{2b}}d\xi d\tau\Big)^{1/2}$$

$$\times \Big(\int_{\widetilde{D}(\xi_1,\tau_1)} \frac{|\widehat{h}(\xi,\tau)|^2}{\langle\sigma\rangle^{\delta}} |\widehat{f}(\xi-\xi_1,\tau-\tau_1)|^2 d\xi d\tau\Big)^{1/2} d\xi_1 d\tau_1,$$

where $\widetilde{D}(\xi_1,\tau_1) = \{(\xi,\tau)\in\mathbb{R}^2 : (\xi,\xi_1,\tau,\tau_1)\in\widetilde{D}\}$. Setting $s=-\rho_0=-(1+a)/2+(7+5a)\epsilon/2$, $\delta=\epsilon$ and $b=1/2+\epsilon$ to apply four lemmas in Section 3.2, we obtain

$$\begin{aligned} I_{\widetilde{D}} \le & \\ \sup_{\xi_1,\tau_1} & \Big[\frac{\langle\xi_1\rangle^{\rho_0}}{\langle i\sigma_1+\xi_1^2\rangle^{1/2+\epsilon}} \Big(\int_{\widetilde{D}(\xi_1,\tau_1)} \frac{|\xi|^2\langle\xi-\xi_1\rangle^{2\rho_0}\langle\xi\rangle^{-2\rho_0}}{\langle i\sigma+\xi^2\rangle^{1-5\epsilon}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{1+2\epsilon}} d\xi d\tau\Big)^{1/2}\Big] \\ & \times \int_{\mathbb{R}^2} \widehat{g}(\xi_1,\tau_1)\Big(\int_{\mathbb{R}^2}\frac{|\widehat{h}(\xi,\tau)|^2}{\langle\sigma\rangle^{\epsilon}}|\widehat{f}(\xi-\xi_1,\tau-\tau_1)|^2 d\xi d\tau\Big)^{1/2} d\xi_1 d\tau_1. \end{aligned} \tag{4.3}$$

Moreover, from the Schwarz inequality and Fubini's theorem, (4.3) is bounded by

$$C\|g\|_{L^2(\mathbb{R}^2)} \Big\|\frac{\widehat{h}}{\langle\sigma\rangle^{\epsilon/2}}\Big\|_{L^2(\mathbb{R}^2)} \|f\|_{L^2(\mathbb{R}^2)}. \tag{4.4}$$

By virtue of Lemma 2.3, we obtain

$$I_{\widetilde{D}} \le CT^{\mu}\|f\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)}\|h\|_{L^2(\mathbb{R}^2)}. \tag{4.5}$$

Case II. This case applies to the integral regions $\widetilde{D} = D_{1,B}\cup D_{2,B}\cup D_{3,B}\cup D_{4,B}\cup D_5$.

In the same way, we can show that

$$\begin{aligned} I_{\widetilde{D}} \le \sup_{\xi,\tau} & \Big[\frac{|\xi|\langle\xi\rangle^{-\rho_0}}{\langle i\sigma+\xi^2\rangle^{1/2-5\epsilon/2}} \\ & \times\Big(\int_{\widetilde{D}(\xi,\tau)} \frac{\langle\xi_1\rangle^{2\rho_0}\langle\xi-\xi_1\rangle^{2\rho_0}}{\langle i\sigma_1+\xi_1^2\rangle^{1+2\epsilon}\langle i\sigma_2+(\xi-\xi_1)^2\rangle^{1+2\epsilon}} d\xi_1 d\tau_1\Big)^{1/2}\Big] \\ & \times \int_{\mathbb{R}^2} \frac{\overline{\widehat{h}(\xi,\tau)}}{\langle\sigma\rangle^{\epsilon/2}} \Big(\int_{\mathbb{R}^2}|\widehat{g}(\xi_1,\tau_1)|^2|\widehat{f}(\xi-\xi_1,\tau-\tau_1)|^2 d\xi_1 d\tau_1\Big)^{1/2} d\xi d\tau, \end{aligned} \tag{4.6}$$

where $\widetilde{D}(\xi,\tau) = \{(\xi_1,\tau_1)\in\mathbb{R}^2 : (\xi,\xi_1,\tau,\tau_1)\in\widetilde{D}\}$ and we set $s=-\rho_0=-(1+a)/2+(7+5a)\epsilon/2$, $\delta=\epsilon$ and $b=1/2+\epsilon$. Moreover, by virtue of five lemmas in Sections 3.3 and 3.4 and Lemma 2.3, we obtain

$$I_{\widetilde{D}} \le CT^{\mu}\|f\|_{L^2(\mathbb{R}^2)}\|g\|_{L^2(\mathbb{R}^2)}\|h\|_{L^2(\mathbb{R}^2)}. \tag{4.7}$$

Therefore, Proposition 3.1 follows from (4.5) and (4.7). □

5. Proof of Theorem 1.1

5.1. **Existence.** Let $u_0(x) \in H^s(\mathbb{R})$ with $s > -(1+a)/2$. We may assume $T < 1$. Let us choose $0 < 8\epsilon < s+(1+a)/2$ and take b such that $2b-1 = 2\epsilon$.

We define the map

$$F(\omega) = \psi(t)W(t)u_0 - \frac{1}{2}\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t W(t-t')\partial_x(\psi_T\omega(t'))^2 dt' \qquad (5.1)$$

and suppose ω is in the ball

$$\mathcal{B}_M = \{u \in X^{s-(2b-1),b} : \|u\|_{X^{s-(2b-1),b}} \le M\}, \qquad (5.2)$$

where $M = 2C_0\|u_0\|_{H^s}$. In what follows, we shall show that $F(\omega)$ is a contraction on the ball $\mathcal{B}_M$ for $[0,T]$.

By virtue of Propositions 2.1, 2.3, 3.1 and 2.2, we have

$$\begin{aligned}\|F(u)\|_{X^{s-(2b-1),b}} &\le \|\psi(t)W(t)u_0\|_{X^{s-(2b-1),b}} \\ &\quad + \frac{1}{2}\left\|\chi_{\mathbb{R}_+}(t)\psi(t)\int_0^t W(t-t')\partial_x(\psi_T u(t'))^2 dt'\right\|_{X^{s-(2b-1),b}} \\ &\le C_0\|u_0\|_{H^s} + C_\delta\left\|\partial_x(\psi_T u)^2\right\|_{X^{s-(2b-1),b-1+\delta}} \\ &\le C_0\|u_0\|_{H^s} + C_\delta T^\mu \left\|\psi_T u\right\|^2_{X^{s-(2b-1),b}} \\ &\le C_0\|u_0\|_{H^s} + C_1 T^{\mu-2\epsilon}\|u\|^2_{X^{s-(2b-1),b}},\end{aligned} \qquad (5.3)$$

where $s-(2b-1) > -(1+a)/2+6\epsilon$ and $1-2b = -2\epsilon$. Therefore, for $u \in \mathcal{B}_M$

$$\|F(u)\|_{X^{s-(2b-1),b}} \le M/2 + C_1T^{\mu-2\epsilon}M^2. \qquad (5.4)$$

Hence it follows that, for $T = (4MC_1)^{-1/(\mu-2\epsilon)}$, $F(u) \in \mathcal{B}_M$.

Similarly, we obtain

$$\|F(u)-F(v)\|_{X^{s-(2b-1),b}} \le C_1T^{\mu-2\epsilon}\|u-v\|_{X^{s-(2b-1),b}}\|u+v\|_{X^{s-(2b-1),b}}. \quad (5.5)$$

Therefore it follows that for $u, v \in \mathcal{B}_M$

$$\begin{aligned}\|F(u)-F(v)\|_{X^{s-(2b-1),b}} &\le 2MC_1T^{\mu-2\epsilon}\|u-v\|_{X^{s-(2b-1),b}} \\ &= \frac{1}{2}\|u-v\|_{X^{s-(2b-1),b}},\end{aligned} \qquad (5.6)$$

from which F is a contraction on $\mathcal{B}_M$. By virtue of the contraction mapping principle, $F(u)$ has a fixed point in the ball $\mathcal{B}_M$. Therefore there exists a unique solution $u(t)$ in $\mathcal{B}_M$ for $T < (4MC_1)^{-1/(\mu-2\epsilon)}$ satisfying

$$u(t) = \psi(t)\Big[W(t)u_0 - \frac{1}{2}\chi_{\mathbb{R}_+}(t)\int_0^t W(t-t')\partial_x(\psi_T u(t'))^2 dt'\Big]. \qquad (5.7)$$

Hence $u(t)$ solves the integral equation associated to the IVP (1.1) in the time interval $[0, T]$.

5.2. **Uniqueness.** The uniqueness is analogously proved as in [2]. Let u_1 be the solution obtained above. And let u_2 be a solution to the integral equation with the same initial data u_0. We assume for some $M > 0$

$$\|u_1\|_{X^{s-(2b-1),b}},\ \|\psi u_2\|_{X^{s-(2b-1),b}} \leq M. \tag{5.8}$$

We may assume that $M > 1$ and $T < 1$. For $T^* < T$, we get

$$\psi u_2(t) = \psi(t) W(t) u_0 - \frac{1}{2}\psi(t)\chi_{\mathbb{R}_+}(t)\int_0^t W(t-t')\partial_x(\psi_{T^*}\psi u_2)^2(t')dt' \tag{5.9}$$

for $t \in [0, T^*]$.

From the definition of the norm, it follows that for any $\epsilon > 0$, there exists $w \in X^{s-(2b-1),b}$ such that for $t \in [0, T^*]$

$$w(t) = u_1(t) - \psi(t)u_2(t) \tag{5.10}$$

and

$$\|w\|_{X^{s-(2b-1),b}} \leq \|u_1 - \psi u_2\|_{X^{s-(2b-1),b}_{T^*}} + \epsilon. \tag{5.11}$$

We define for $t \in \mathbb{R}$

$$\omega(t) = -\frac{1}{2}\psi(t)\chi_{\mathbb{R}_+}(t)\int_0^t W(t-t')\partial_x[\psi^2_{T^*}(t')w(t')(u_1(t') + \psi u_2(t'))]dt'. \tag{5.12}$$

Since it follows that for $t \in [0, T^*]$

$$\omega(t) = w(t) = u_1(t) - \psi(t)u_2(t), \tag{5.13}$$

we have

$$\|u_1 - \psi u_2\|_{X^{s-(2b-1),b}_{T^*}} \leq \|\omega\|_{X^{s-(2b-1),b}}. \tag{5.14}$$

With a similar calculation as in Section 5.1, it follows that

$$\begin{aligned}
&\|u_1 - \psi u_2\|_{X^{s-(2b-1),b}_{T^*}} \leq \|\omega\|_{X^{s-(2b-1),b}} \\
&\leq C\|\partial_x[\psi^2_{T^*}w(t')(u_1 + \psi u_2)]\|_{X^{s-(2b-1),b}} \\
&\leq C_1 (T^*)^{\mu-2\epsilon}\left(\|u_1\|_{X^{s-(2b-1),b}} + \|\psi u_2\|_{X^{s-(2b-1),b}}\right)\|w\|_{X^{s-(2b-1),b}} \\
&\leq 2C_1(T^*)^{\mu-2\epsilon}M\|w\|_{X^{s-(2b-1),b}}.
\end{aligned} \tag{5.15}$$

If $T^* \leq (4C_1 M)^{-1/(\mu-2\epsilon)}$, for any $\epsilon > 0$ we have

$$\|u_1 - \psi u_2\|_{X^{s-(2b-1),b}_{T^*}} \leq \frac{1}{2}\|w\|_{X^{s-(2b-1),b}} \leq \frac{1}{2}\Big(\|u_1 - \psi u_2\|_{X^{s-(2b-1),b}_{T^*}} + \epsilon\Big), \tag{5.16}$$

where we use (5.11). Therefore,

$$\|u_1 - \psi u_2\|_{X_{T^*}^{s-(2b-1),b}} \le \epsilon, \tag{5.17}$$

which implies $u_1 = u_2$ on $[0, T^*]$.

Repeating this procedure, we obtain the uniqueness result for any existence interval.

5.3. **Continuous dependence.** In this section, we shall show the continuous dependence upon the initial data. We choose $0 < 8\epsilon < s + (1+a)/2$ and take b such that $2b - 1 = 2\epsilon$. Let u and v be the solutions obtained in Section 5.1 with data u_0 and v_0 respectively.

As in Section 5.1, with the aid of Propositions 2.1, 2.3, 3.1 and 2.2, we obtain

$$\begin{aligned}\|u-v\|_{X^{s-(2b-1),b}} &\le C_0\|u_0 - v_0\|_{H^s} + 2MC_1T^{\mu-2\epsilon}\|u-v\|_{X^{s-(2b-1),b}} \\ &\le C_0\|u_0 - v_0\|_{H^s} + \frac{1}{2}\|u-v\|_{X^{s-(2b-1),b}} \end{aligned}\tag{5.18}$$

for u, $v \in \mathcal{B}_M$ and for $T < (4MC_1)^{-1/(\mu-2\epsilon)}$. Hence,

$$\|u-v\|_{X^{s-(2b-1),b}} \le 2C_0\|u_0 - v_0\|_{H^s}. \tag{5.19}$$

Moreover, by virtue of Propositions 2.4, 3.1, 2.2 and (5.19), we have

$$\begin{aligned}\|u(t) - v(t)\|_{H^s} &\le \|\psi(t)W(t)(u_0 - v_0)\|_{H^s} \\ &\quad + \frac{1}{2}\left\|\psi(t)\chi_{\mathbb{R}_+}(t)\int_0^t W(t-t')\partial_x(\psi_T^2(u-v)(u+v)(t'))dt'\right\|_{H^s} \\ &\le C\|W(t)(u_0 - v_0)\|_{H^s} + C\left\|\partial_x(\psi_T^2(u-v)(u+v))\right\|_{X^{s-2\epsilon,b-1+\epsilon}} \\ &\le C_0\|u_0 - v_0\|_{H^s} + C_1T^{\mu-2\epsilon}\|u-v\|_{X^{s-2\epsilon,b}}\|u+v\|_{X^{s-2\epsilon,b}} \\ &\le C_0\|u_0 - v_0\|_{H^s} + 2C_1T^{\mu-2\epsilon}M\|u-v\|_{X^{s-2\epsilon,b}} \\ &\le C_0\|u_0 - v_0\|_{H^s} + \frac{1}{2}\|u-v\|_{X^{s-2\epsilon,b}} \le 2C_0\|u_0 - v_0\|_{H^s}, \end{aligned}\tag{5.20}$$

which implies the continuous dependence on the initial data.

5.4. **Global existence.** In this section, we assume that u is real valued.

Lemma 5.1. *Let $\lambda \in [0, \infty)$. Then for any $t > 0$ and $s \in \mathbb{R}$, $\mathcal{W}(t)$ is in $\mathcal{L}(H^s(\mathbb{R}), H^{s+\lambda}(\mathbb{R}))$ and satisfies*

$$\|\mathcal{W}(t)u_0\|_{H^{s+\lambda}} \le C_\lambda[1 + (2t)^{-\lambda}]^{1/2}\|u_0\|_{H^s} \tag{5.21}$$

for all $u_0 \in H^s(\mathbb{R})$.

Moreover, the map $t \mapsto \mathcal{W}(t)u_0$ belongs to $C((0,\infty), H^{s+\lambda}(\mathbb{R}))$.

Proof. It suffices to modify the proof of [14, Theorem 2.1]. □

We first note that $\mathcal{W}u_0$ belongs to $C([0,\infty), H^s) \cap C((0,\infty), H^\infty)$, where $H^\infty(\mathbb{R}) = \bigcap_{k\geq s} H^k(\mathbb{R})$. From Proposition 3.1, $u\partial_x u \in X^{s,-1/2+\delta}$. Hence, Proposition 2.4 implies that

$$t \mapsto \int_0^t W(t-t')\partial_x(u^2(t'))dt' \in C([0,T], H^{s+2\delta}). \tag{5.22}$$

Therefore, we obtain

$$u \in C([0,T], H^s) \cap C((0,T], H^{s+2\delta}). \tag{5.23}$$

Moreover, we deduce by induction that $u \in C((0,T], H^\infty)$.

We next multiply the gBOB equation by u and integrating by parts with respect to x, we get

$$\frac{1}{2}\partial_t \|u(t)\|_{L^2}^2 = -\|\partial_x u(t)\|_{L^2}^2. \tag{5.24}$$

Hence

$$\frac{1}{2}\partial_t \|u(t)\|_{L^2}^2 \leq 0. \tag{5.25}$$

Furthermore, integrating over $[0,t]$, we have

$$\|u(t)\|_{L^2} \leq \|u_0\|_{L^2}. \tag{5.26}$$

This show that the map $t \mapsto \|u(t)\|_{L^2}$ is nonincreasing on $(0,T]$. Since the time of the local existence T depends only on $\|u_0\|_{H^s}$, the solution is extended to be global in time.

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