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HESSIAN ESTIMATES FOR VISCOUS HAMILTON-JACOBI EQUATIONS WITH THE ORNSTEIN-UHLENBECK OPERATOR

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Abstract. In this paper, we consider Hessian estimates of solutions of the Cauchy problem for parabolic PDEs with the Ornstein-Uhlenbeck operator. Our upper estimate on the Hessian matrix of solutions is a generalization of the result of Kružkov [4]. On the other hand, our lower estimate on the Hessian matrix of solutions is best possible in some sense.

1. Introduction

In this paper, we consider Hessian estimates of solutions of the Cauchy problem for the parabolic PDE

$$u_t(x,t) - \Delta u(x,t) + \alpha x \cdot Du(x,t) + H(Du(x,t)) = 0 \qquad \text{in } \mathrm{R}^N \times (0,\infty), \tag{1.1}$$

$$u(\cdot,0) = \varphi \qquad \text{in } \mathrm{R}^N. \tag{1.2}$$

Here, $u : \mathrm{R}^N \to \mathrm{R}$ is the unknown function, $\alpha > 0$ is a given constant, Δ denotes the Laplace operator in R^N, Du denotes the gradient of u, and $H \in \mathrm{Lip}_{loc}(\mathrm{R}^N)$ and $\varphi \in \mathrm{Lip}(\mathrm{R}^N)$ are given functions. We refer to the operator $\Delta - \alpha x \cdot D$ on $C^2(\mathrm{R}^N)$ as the *Ornstein-Uhlenbeck operator*.

The PDE (1.1) arises typically as the dynamic programming equation for stochastic optimal control or stochastic differential games of the systems described by controlled Ornstein-Uhlenbeck processes. In this viewpoint we often call (1.1) a viscous Hamilton-Jacobi equation, and H a Hamiltonian. Existence, uniqueness, gradient estimates and asymptotic behavior of solutions of Cauchy problem (1.1)-(1.2) have been considered in [3] under a more general setting. As for Hessian estimates, Kružkov [4] showed that even if φ is not semiconvex on R^N, the solution $u(\cdot,t)$ is semiconvex on R^N for $t \in (0,\infty)$ provided $\alpha = 0$ and H is uniformly convex on R^N. Here, a

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function $g \in C(\mathrm{R}^N)$ is said to be semiconcave on R^N, if there is a constant $b > 0$ such that $g(x) - b|x|^2/2$ is concave on R^N. This is equivalent to the condition such that $D^2 g \le bI_N$ in $\mathcal{D}'(\mathrm{R}^N)$, where I_N is the $N \times N$ identity matrix and $\mathcal{D}'(\mathrm{R}^N)$ is the set of all distribution functions on R^N. In the same way, a function $g \in C(\mathrm{R}^N)$ is said to be semiconvex on R^N, if there is a constant $b > 0$ such that $-bI_N \le D^2 g$ in $\mathcal{D}'(\mathrm{R}^N)$. The result of Kuržkov [4] said that if $\alpha = 0$ and there is a constant $\theta > 0$ such that $H(p) - \theta|p|^2/2$ is convex on R^N, then

$$D^2 u(x,t) \le \frac{1}{\theta t} I_N \quad \text{for } (x,t) \in \mathrm{R}^N \times (0,\infty). \tag{1.3}$$

Our goal of this paper is to give upper and lower estimates of the Hessian matrix of solutions of (1.1)-(1.2). Our upper estimate is a generalization of inequality (1.3). On the other hand, our lower estimate is best possible in some sense.

First, we explain our upper estimate. Let φ be semiconcave on R^N; we denote by

$$\lambda(\varphi) = \inf \left\{ b > 0 : D^2 \varphi \le bI_N \text{ in } \mathcal{D}'(\mathrm{R}^N) \right\}. \tag{1.4}$$

We show that if there is a constant $\theta > 0$ such that $H(p) - \theta|p|^2/2$ is convex on R^N, then

$$D^2 u(x,t) \le \frac{2\alpha\lambda(\varphi)e^{-2\alpha t}}{2\alpha + \theta\lambda(\varphi)(1 - e^{-2\alpha t})} I_N \quad \text{for } (x,t) \in \mathrm{R}^N \times (0,\infty). \tag{1.5}$$

Note that this result is a generalization of inequality (1.3). Indeed, letting $\lambda(\varphi)$ tend to ∞ formally in (1.5), we have

$$D^2 u(x,t) \le \frac{2\alpha e^{-2\alpha t}}{\theta(1 - e^{-2\alpha t})} I_N \quad \text{for } (x,t) \in \mathrm{R}^N \times (0,\infty). \tag{1.6}$$

By using the inf-convolution, we show that this procedure is justified even if φ is not semiconvex on R^N. Letting $\alpha \searrow 0$ in (1.6), we have (1.3). Another application of inequality (1.5) is that we can investigate asymptotic behavior of the Hessian matrix $D^2 u(x,t)$ as $t \to \infty$. Removing the decaying factor $e^{-2\alpha t}$ in (1.5), we obtain

$$\limsup_{t\to\infty} \left[e^{2\alpha t} D^2 u(x,t) \right] \le \frac{2\alpha\lambda(\varphi)}{2\alpha + \theta\lambda(\varphi)} I_N \quad \text{for } x \in \mathrm{R}^N. \tag{1.7}$$

Hence, we have

$$\limsup_{t\to\infty} \left[e^{2\alpha t} D^2 u(x,t) \right] \le \lambda(\varphi) I_N \quad \text{for } x \in \mathrm{R}^N. \tag{1.8}$$

Next, we consider lower estimates of solutions. Let φ be semiconvex on R^N, and denote by

$$\Lambda(\varphi) = \inf\left\{ b > 0 : -bI_N \le D^2\varphi \text{ in } \mathcal{D}'(\mathrm{R}^N) \right\}. \tag{1.9}$$

We show that if there is a constant $\Theta > 0$ such that $H(p) - \Theta|p|^2/2$ is concave on R^N and $\Lambda(\varphi)\Theta < 2\alpha$, then

$$-\frac{2\alpha\Lambda(\varphi)e^{-2\alpha t}}{2\alpha - \Theta\Lambda(\varphi)(1-e^{-2\alpha t})}\, I_N \le D^2u(x,t) \quad \text{for } (x,t) \in \mathrm{R}^N \times (0,\infty). \tag{1.10}$$

In particular, we have

$$-\frac{2\alpha\Lambda(\varphi)}{2\alpha - \Theta\Lambda(\varphi)}\, I_N \le \liminf_{t\to\infty} \left[e^{2\alpha t} D^2u(x,t)\right] \quad \text{for } x \in \mathrm{R}^N. \tag{1.11}$$

This inequality corresponds to (1.7). We give an example which shows that inequality (1.11) is best possible. Hence, contrary to (1.8), the inequality

$$-\Lambda(\varphi)\, I_N \le \liminf_{t\to\infty} \left[e^{2\alpha t} D^2u(x,t)\right] \quad \text{for } x \in \mathrm{R}^N \tag{1.12}$$

does not hold in general, because $-2\alpha\Lambda(\varphi)/(2\alpha - \Theta\Lambda(\varphi)) \le -\Lambda(\varphi)$.

The contents of this paper are as follows: In Section 2, we give some preliminaries. In Section 3, we consider the semiconcavity property of solutions, and show inequalities (1.5)-(1.8). In Section 4, we consider the semiconvexity property of solutions, and show inequalities (1.10)-(1.11). We also give an example which shows that inequality (1.11) is best possible, and show that inequality (1.12) does not hold in general.

2. PRELIMINARIES

In this section, we give some preliminaries. We assume the following:

$$\alpha > 0, \; H \in \mathrm{Lip}_{loc}(\mathrm{R}^N), \text{ and } \varphi \in \mathrm{Lip}(\mathrm{R}^N). \tag{2.1}$$

Let us denote by L_φ the Lipschitz constant of φ. We set $Q = \mathrm{R}^N \times (0,\infty)$, $B(a,R) = \left\{ x \in \mathrm{R}^N : |x-a| \le R \right\}$ for $a \in \mathrm{R}^N$, $R > 0$.

First of all, we give a comparison theorem and an existence theorem for solutions to Cauchy problem (1.1)-(1.2). The following Propositions 2.1 and 2.2 were proved in [3].

Proposition 2.1. *Assume* (2.1). *Let* $v \in \mathrm{USC}(Q)$ *and* $w \in \mathrm{LSC}(Q)$ *be a viscosity subsolution and a viscosity supersolution of* (1.1)*, respectively. Assume that for each* $T > 0$ *there is a constant* $C_T > 0$ *such that*

$$v(x,t) \vee (-w(x,t)) \le C_T(1+|x|) \quad \text{for } (x,t) \in R^N \times (0,T), \tag{2.2}$$

and that $v(x,0) \le \varphi(x) \le w(x,0)$ *for all* $x \in R^N$. *Then* $v \le w$ *in* $\overline{Q}$.

Next, we give an existence theorem for Cauchy problem (1.1)-(1.2). For $\gamma \in (0,1)$, we denote by $C^{\gamma,\gamma/2}(Q)$ the space of functions $u \in C(Q)$ such that

$$\sup\left\{ \frac{|u(x,t)-u(y,s)|}{|x-y|^{\gamma}+|t-s|^{\gamma/2}} : (x,t),\ (y,s) \in Q_1, (x,t) \neq (y,s) \right\} < \infty$$

for any compact set Q_1 of Q. For $\gamma \in (0,1)$, we denote by $C^{2+\gamma,1+(\gamma/2)}(Q)$ the space of functions $u \in C^{2,1}(Q)$ such that $\partial_x^{\beta} u$ ($\beta = 0,1,2$) and $\partial_t u$ belong to $C^{\gamma,\gamma/2}(Q)$

Proposition 2.2. *Assume* (2.1). *Then, for any $\gamma \in (0,1)$, there is a solution $u \in C^{2+\gamma,1+(\gamma/2)}(Q) \cap C(\overline{Q})$ of* (1.1)-(1.2) *satisfying the following:*

$$\sup_{(x,t)\in R^N \times [0,T]} \frac{|u(x,t)|}{1+|x|} < \infty \quad \textit{for } T > 0, \tag{2.3}$$

$$|Du(x,t)| \leq L_{\varphi} \quad \textit{for } (x,t) \in Q. \tag{2.4}$$

$$\textit{For any compact set } K \subset R^N,\ u \textit{ is uniformly continuous on } K \times [0,\infty) \tag{2.5}$$

Next, we consider an approximation of $H \in \mathrm{Lip}_{loc}(\mathrm{R}^N)$. For $0 < \epsilon < 1$, we define H_ϵ by

$$H_\epsilon(p) = \int_{\mathrm{R}^N} H(p-\epsilon y)\,\rho(y)\,dy \quad \text{for } p \in \mathrm{R}^N, \tag{2.6}$$

where ρ is a $C^\infty(\mathrm{R}^N)$ function such that

$$\rho \geq 0 \ \text{ on } \mathrm{R}^N, \qquad \mathrm{supp}(\rho) \subset B(0,1) \qquad \text{and} \qquad \int_{\mathrm{R}^N} \rho(y)\,dy = 1.$$

It is easy to see that $H_\epsilon \in C^\infty(\mathrm{R}^N)$.

Proposition 2.3. *Assume* (2.1). *Let u be the unique solution of* (1.1)-(1.2), *and u_ϵ the unique solution of* (1.1)-(1.2) *where H is replaced by H_ϵ. Then, we have the following:*

(*i*) *There is a constant $C > 0$, which is independent of ϵ, (x,t), and p, such that*

$$|H_\epsilon(p) - H(p)| \leq C\epsilon \quad \textit{for } p \in B(0, L_\varphi),\ 0 < \epsilon < 1, \tag{2.7}$$

$$|u_\epsilon(x,t) - u(x,t)| \leq C\epsilon t \quad \textit{for } (x,t) \in \overline{Q},\ 0 < \epsilon < 1. \tag{2.8}$$

(*ii*) $u_\epsilon \in C^{2,2}(Q)$.

(*iii*) *If there is a constant $b > 0$ such that $H(p) - b|p|^2/2$ is convex* (*respectively concave*) *on R^N, then $H_\epsilon(p) - b|p|^2/2$ is convex* (*respectively concave*) *on R^N.*

Proof. (i) Let $p \in B(0, L_\varphi)$. Then, we have

$$|H_\epsilon(p) - H(p)| \le \int_{\mathrm{R}^N} |H(p - \epsilon y) - H(p)|\rho(y)\,dy \le \int_{\mathrm{R}^N} \hat{L}\epsilon|y|\rho(y)\,dy =: C\epsilon,$$

where

$$\hat{L} = \sup\left\{ \frac{|H(p) - H(q)|}{|p-q|} : p, q \in B(0, L_\varphi + 1),\ p \ne q \right\}.$$

Hence, (2.7) follows. Next, we consider (2.8). By (2.4) and (2.7), we have

$$(u_\epsilon)_t - \Delta u_\epsilon + \alpha x \cdot Du_\epsilon + H(Du_\epsilon) = H(Du_\epsilon) - H_\epsilon(Du_\epsilon) \le C\epsilon \quad \text{on } Q.$$

Hence, the function $\eta(x,t) := u_\epsilon(x,t) - C\epsilon t$ satisfies

$$\begin{aligned} &\eta_t - \Delta\eta + \alpha x \cdot D\eta + H(D\eta) \le 0 && \text{in } Q \\ &\eta(\cdot, 0) = \varphi && \text{in } \mathrm{R}^N. \end{aligned}$$

By the comparison theorem in Proposition 2.1, we see that $\eta \le u$ on $\overline{Q}$, so that $u_\epsilon - u \le C\epsilon t$ on $\overline{Q}$. Similarly, we can show that $u_\epsilon - u \ge -C\epsilon t$ on $\overline{Q}$. Hence, we obtain (2.8).

(ii) By Proposition 2.1, we have $u_\epsilon \in C^{2+\gamma, 1+(\gamma/2)}(Q)$ for any $0 < \gamma < 1$. Then, $\partial_x^\beta H_\epsilon(Du_\epsilon) \in C^{\gamma,\gamma/2}(Q)$ for any $\beta = 0, 1$ and $0 < \gamma < 1$. By Schauder's interior estimate for parabolic equations, we have $\partial_x\partial_t u, \partial_x^3 u \in C^{\gamma,\gamma/2}(Q)$ for any $0 < \gamma < 1$. Using Schauder's interior estimate again, we have $\partial_t^2 u \in C^{\gamma,\gamma/2}(Q)$ for any $0 < \gamma < 1$.

(iii) This is clear from the definition of H_ϵ. □

Finally, we introduce an important function, which plays an essential role in our results. For given constants $C \ge 0, k \in \mathrm{R}$ with $Ck < 2\alpha$, we define the function $L_{C,k} \in C^\infty((0,\infty)) \cap C([0,\infty))$ by

$$L_{C,k}(t) = \frac{2\alpha C e^{-2\alpha t}}{2\alpha - Ck(1 - e^{-2\alpha t})} \quad \text{for } t \in [0,\infty). \tag{2.9}$$

Since the proof of the following proposition is easy, we omit it.

Proposition 2.4. *Let $C \ge 0$ and $k \in R$ be constants with $Ck < 2\alpha$. Then*

$$L_{C,k}(0) = C,\ L_{C,k} \text{ is nonincreasing on } [0,\infty) \text{ and } \lim_{t\to\infty} L_{C,k}(t) = 0, \tag{2.10}$$

$$L'_{C,k}(t) + 2\alpha L_{C,k}(t) - kL_{C,k}(t)^2 = 0 \quad \text{for } t \in (0,\infty). \tag{2.11}$$

3. SEMICONCAVITY OF SOLUTIONS

In this section, we consider the semiconcavity property of the unique solution of Cauchy problem (1.1)-(1.2). We assume the following:

$$\text{There is a constant } \theta > 0 \text{ such that } H(p) - \theta|p|^2/2 \text{ is convex on } \mathrm{R}^N. \tag{3.1}$$

Theorem 3.1. *Assume* (2.1) *and* (3.1). *Assume also that there is a constant* $b > 0$ *such that* $\varphi(x) - b|x|^2/2$ *is concave on* R^N. *Then, the unique solution* u *of Cauchy problem* (1.1)–(1.2) *satisfies*

$$D^2u(x,t) \leq \frac{2\alpha b e^{-2\alpha t}}{2\alpha + \theta b(1 - e^{-2\alpha t})} I_N \quad \text{for } (x,t) \in R^N \times (0,\infty). \tag{3.2}$$

In particular, we have (1.5), (1.7), *and* (1.8), *where* $\lambda(\varphi)$ *is the constant given by* (1.4).

To show Theorem 3.1, we prepare two lemmas. Since the proof of the first one is clear, we omit it.

Lemma 3.2. *Let* $\psi \in C(R^N)$ *and* $c \in R$. *Then, the function* $\psi(x) + c|x|^2/2$ *is convex on* R^N *if and only if*

$$\psi(x+hy) + \psi(x-hy) - 2\psi(x) \geq -ch^2|y|^2 \quad \text{for } x, y \in R^N,\ h \in R.$$

Next, let $f \in \mathrm{LSC}(\mathrm{R}^N)$ be coercive; i.e., $\lim_{|x|\to\infty} f(x)/|x| = \infty$. Define the convex envelope f_{**} of f by

$$f_{**}(x) = \inf\Big\{ \sum_{j=1}^{k} \beta_j f(x_j) : x = \sum_{j=1}^{k} \beta_j x_j,\ \sum_{j=1}^{k} \beta_j = 1, \\ \beta_j > 0,\ x_j \in \mathrm{R}^N,\ k \leq N+1 \Big\}.$$

By [5, Corollary 17.1.5, page 157], f_{**} is the largest convex function majorized by f on R^N.

Lemma 3.3. *Let* $g \in \mathrm{LSC}(R^N)$ *be a function such that there is a constant* $L > 0$ *satisfying*

$$-L(1+|x|) \leq g(x) \quad \text{for } x \in R^N.$$

Define the function f *by* $f(x) = g(x) + b|x|^2/2$ *for a constant* $b > 0$. *Then, there is a constant* $C > 0$ *such that*

$$-f_{**}(x) + \frac{b}{2}|x|^2 \leq C(1+|x|) \quad \text{for } x \in R^N.$$

Proof. Let us define the function V by

$$V(x) = \Big(|x| - \frac{L}{b}\Big)^2 \quad \text{for } x \in \mathrm{R}^N.$$

It is easy to see that

$$V_{**}(x) = \chi_{\{|x| \geq L/b\}}(x) \left(|x| - \frac{L}{b}\right)^2 \quad \text{for } x \in \mathrm{R}^N, \tag{3.3}$$

where $\chi_E(x)$ is the characteristic function of a set E. In the following of this proof, the infimum is always taken over all $\{(x_j, \beta_j)\}_{j=1}^k$ satisfying $x = \sum_{j=1}^k \beta_j x_j,\ \sum_{j=1}^k \beta_j = 1, \beta_j > 0,\ x_j \in \mathrm{R}^N$, and $k \leq N+1$. Then, we have

$$\begin{aligned} f_{**}(x) &= \inf\Big\{\sum_{j=1}^k \beta_j f(x_j)\Big\} \geq \inf\Big\{\sum_{j=1}^k \beta_j \Big[-L(1+|x_j|) + \frac{b}{2}|x_j|^2\Big]\Big\} \\ &= -L + \inf\Big\{\sum_{j=1}^k \beta_j \Big[\frac{b}{2}\Big(|x_j| - \frac{L}{b}\Big)^2 - \frac{L^2}{2b}\Big]\Big\} \\ &= -L - \frac{L^2}{2b} + \frac{b}{2}\ \inf\Big\{\sum_{j=1}^k \beta_j \Big(|x_j| - \frac{L}{b}\Big)^2\Big\} = -L - \frac{L^2}{2b} + \frac{b}{2}V_{**}(x). \end{aligned}$$

By (3.3), it is easy to see that there is a constant $C^* > 0$ such that

$$V_{**}(x) \geq |x|^2 - C^*(1+|x|) \quad \text{for } x \in \mathrm{R}^N.$$

The proof is complete. □

Proof of Theorem 3.1. 1. The function $L_{b,-\theta}$ of (2.9) is well defined, because $-b\theta < 0 < 2\alpha$. For simplicity, we write $L(t)$ for $L_{b,-\theta}(t)$. We will show that

$$\begin{cases} \text{the function } \frac{1}{2}|x|^2(L'(t) + 2\alpha L(t)) + H(L(t)x - p) \text{ is convex} \\ \text{on } \mathrm{R}^N \text{ for each } (p,t) \in \mathrm{R}^N \times (0,\infty). \end{cases} \tag{3.4}$$

Indeed, fix $(p,t) \in \mathrm{R}^N \times (0,\infty)$ arbitrarily. For all $x, y \in \mathrm{R}^N$ and $h \in \mathrm{R}$, we have

$$\begin{aligned} &H\Big(L(t)(x+hy) - p\Big) + H\Big(L(t)(x-hy) - p\Big) - 2H\Big(L(t)x - p\Big) \\ &= H\Big((L(t)x - p) + hL(t)y\Big) + H\Big((L(t)x-p) - hL(t)y\Big) - 2H\Big(L(t)x - p\Big) \\ &\geq \theta h^2 |L(t)y|^2 \qquad \text{(by (3.1) and Lemma 3.2)} \\ &= -h^2\left[L'(t) + 2\alpha L(t)\right]|y|^2 \qquad \text{(by (2.11) for } C = b \text{ and } k = -\theta). \end{aligned}$$

Using Lemma 3.2 again, we obtain (3.4).

2. Let u_ϵ be the unique solution of (1.1)-(1.2) where H is replaced by H_ϵ of (2.6). We set

$$w(x,t) := -u_\epsilon(x,t) + \frac{L(t)}{2}|x|^2 \quad \text{for } (x,t) \in \overline{Q}.$$

Then w is a classical solution of

$$\begin{aligned} &w_t - \Delta w + \alpha x \cdot Dw + NL(t) \\ &\qquad - \frac{1}{2}|x|^2(L'(t) + 2\alpha L(t)) - H_\epsilon(-Dw + L(t)x) = 0 \qquad \text{in } Q. \end{aligned} \tag{3.5}$$

We show that the convex envelope w_{**} of w is a viscosity supersolution of (3.5). For this purpose, we check that Proposition 7 of [1] is applicable. By Proposition 2.3-(ii), we have $w \in C^{2,2}(Q)$. By Proposition 2.3-(iii) and (3.4), the condition that

$$(x,q,X) \mapsto q - \mathrm{tr}X^{-1} + \alpha x \cdot p + NL(t) - \frac{1}{2}|x|^2(L'(t) + 2\alpha L(t)) - H_\epsilon(-p + L(t)x)$$

is concave on $\mathrm{R}^N \times \mathrm{R} \times \mathcal{S}^N_{++}$ for each $(p,t) \in \mathrm{R}^N \times (0,\infty)$ is satisfied, where

$$\mathcal{S}^N_{++} = \{X \in \mathcal{S}^N : X > 0\}.$$

The condition that

$$\lim_{|x|\to\infty} \frac{w(x,t)}{|x|} = \infty \quad \text{for } t \in [0,\infty)$$

is also satisfied, because $b > 0$ and (2.3) holds. Then, by Proposition 7 of [1], the convex envelope w_{**} of w is a supersolution of (3.5).

3. We will show that $w_{**} = w$ on $\overline{Q}$. By the definition of w_{**}, we have $w_{**} \le w$ on $\overline{Q}$. Hence, it suffices to show that $w_{**} \ge w$ on $\overline{Q}$. Let

$$v(x,t) := -w_{**}(x,t) + \frac{L(t)}{2}|x|^2 \quad \text{for } (x,t) \in \overline{Q}.$$

Since w_{**} is a supersolution of (3.5), v is a subsolution of (1.1) where H is replaced by H_ϵ. On the other hand, it follows from Proposition 2.2 that for each $T > 0$, there is a constant $C_{T,\epsilon} > 0$ such that

$$-C_{T,\epsilon}(1 + |x|) \le -u_\epsilon(x,t) \quad \text{for } (x,t) \in \mathrm{R}^N \times [0,T].$$

Then, by Lemma 3.3, for each $T > 0$, there is a constant $C'_{T,\epsilon} > 0$ such that

$$v(x,t) \le C'_{T,\epsilon}(1 + |x|) \quad \text{for } (x,t) \in \mathrm{R}^N \times [0,T].$$

Since $w(\cdot,0)$ is convex on R^N by our assumption, we have $w_{**}(\cdot,0) = w(\cdot,0)$ on R^N. Hence $v(\cdot,0) = \varphi$. Using the comparison theorem in Proposition

2.1, we conclude that $v \le u_\epsilon$ on $\mathrm{R}^N \times [0,T]$ for each $T > 0$. Hence, $w_{**} \ge w$ on $\overline{Q}$.

4. Since $w_{**}(\cdot,t)$ is convex on R^N, it follows from the result of 3. that $w(\cdot,t)$ is also convex on R^N. Letting $\epsilon \searrow 0$, we see that $-u(\cdot,t)+L(t)|x|^2/2$ is convex on R^N by (2.8). Then, we obtain (3.2), because $u(\cdot,t) \in C^2(\mathrm{R}^N)$ for $t \in (0,\infty)$. The proof is complete. □

Next, we show the following:

Theorem 3.4. *Assume* (2.1) *and* (3.1). *Then, the unique solution u of Cauchy problem* (1.1)-(1.2) *satisfies* (1.6), *even if φ is not semiconcave on R^N.*

Proof. 1. For $\delta > 0$, we define the inf-convolution function φ_δ of φ by

$$\varphi_\delta(x) = \inf\left\{ \varphi(y) + \frac{1}{2\delta}|x-y|^2 : y \in \mathrm{R}^N \right\} \quad \text{for } x \in \mathrm{R}^N.$$

We show that

$$\varphi_\delta \in \mathrm{Lip}(\mathrm{R}^N) \text{ and } \varphi_\delta(x) - |x|^2/(2\delta) \text{ is concave on } \mathrm{R}^N, \tag{3.6}$$

$$0 \le \varphi(x) - \varphi_\delta(x) \le 2L_\varphi^2\delta \text{ for all } x \in \mathrm{R}^N \text{ and } \delta > 0. \tag{3.7}$$

Since (3.6) is clear, we prove only (3.7). The inequality $0 \le \varphi - \varphi_\delta$ on R^N is trivial. Fix $x \in \mathrm{R}^N$ and $\delta > 0$. For any $\epsilon > 0$, there is a point $y_\epsilon \in \mathrm{R}^N$ such that

$$\varphi_\delta(x) + \epsilon > \varphi(y_\epsilon) + \frac{1}{2\delta}|x-y_\epsilon|^2.$$

Then, we have

$$0 \le \varphi(x)-\varphi_\delta(x) \le \varphi(x)-\varphi(y_\epsilon)-\frac{1}{2\delta}|x-y_\epsilon|^2+\epsilon \le L_\varphi|x-y_\epsilon|-\frac{1}{2\delta}|x-y_\epsilon|^2+\epsilon,$$

so that

$$0 \le L_\varphi|x-y_\epsilon| - \frac{1}{2\delta}|x-y_\epsilon|^2 + \epsilon,$$

where L_φ is the Lipschitz constant of φ. Hence, we obtain

$$0 \le |x-y_\epsilon| \le \delta\left(L_\varphi + \sqrt{L_\varphi^2 + (2\epsilon/\delta)}\right),$$

which leads to

$$0 \le \varphi(x) - \varphi_\delta(x) \le L_\varphi|x-y_\epsilon| + \epsilon \le L_\varphi\delta\left(L_\varphi + \sqrt{L_\varphi^2 + (2\epsilon/\delta)}\right) + \epsilon.$$

Letting ϵ tend to 0, we conclude (3.7).

2. Let u_δ be the unique solution of (1.1) with the initial condition $u_\delta(\cdot,0) = \varphi_\delta$. Then, by (3.7) and the comparison theorem in Proposition 2.1, we have

$$-2L_\varphi^2\delta \le u_\delta(x,t) - u(x,t) \le 0 \quad \text{for } (x,t) \in \overline{Q}. \tag{3.8}$$

3. By (3.6) and Theorem 3.1, we see that $u_\delta(x,t) - \frac{2\alpha e^{-2\alpha t}}{2\alpha\delta+\theta(1-e^{-2\alpha t})}\frac{|x|^2}{2}$ is concave on R^N for each $t \in (0,\infty)$. Letting $\delta \searrow 0$ and using (3.8), we conclude that $u(x,t) - \frac{2\alpha e^{-2\alpha t}}{\theta(1-e^{-2\alpha t})}\frac{|x|^2}{2}$ is concave on R^N for each $t \in (0,\infty)$. This completes the proof. □

4. Semiconvexity of solutions

In this section, we consider the semiconvexity property of the unique solution of Cauchy problem (1.1)-(1.2). We assume the following:

$$\text{There is a constant } \Theta > 0 \text{ such that } H(p) - \Theta|p|^2/2 \text{ is concave on } \mathrm{R}^N. \tag{4.1}$$

Theorem 4.1. *Assume* (2.1) *and* (4.1). *Let $b \in (0, 2\alpha/\Theta)$ be a constant such that $\varphi(x) + b|x|^2/2$ is convex on R^N. Then, the unique solution u of Cauchy problem* (1.1)-(1.2) *satisfies*

$$-\frac{2\alpha b e^{-2\alpha t}}{2\alpha - \Theta b(1-e^{-2\alpha t})} I_N \le D^2u(x,t) \quad \textit{for } (x,t) \in R^N \times (0,\infty). \tag{4.2}$$

In particular, we have (1.10) *and* (1.11), *where $\Lambda(\varphi)$ is the constant given by* (1.9).

Proof. 1. Since $b \in (0, 2\alpha/\Theta)$, the function $L_{b,\Theta}$ of (2.9) is well defined. For simplicity, we write $L(t)$ for $L_{b,\Theta}(t)$. By the same arguments as those of (3.4), we see that

$$\begin{cases} \text{the function } \frac{1}{2}|x|^2(L'(t) + 2\alpha L(t)) - H(p - L(t)x) \text{ is convex} \\ \text{on } \mathrm{R}^N \text{ for each } (p,t) \in \mathrm{R}^N \times (0,\infty). \end{cases} \tag{4.3}$$

The condition $b \in (0, 2\alpha/\Theta)$ is necessary for (4.3).

2. Let u_ϵ be the unique solution of (1.1)-(1.2) where H is replaced by H_ϵ of (2.6). We set

$$w(x,t) := u_\epsilon(x,t) + \frac{L(t)}{2}|x|^2 \quad \text{for } (x,t) \in \overline{Q}.$$

Then w is a classical solution of

$$\begin{aligned} w_t - \Delta w + \alpha x \cdot Dw + NL(t) \\ -\frac{1}{2}|x|^2(L'(t) + 2\alpha L(t)) + H_\epsilon(Dw - L(t)x) = 0 \qquad \text{in } Q. \end{aligned} \tag{4.4}$$

By the same arguments as those of the proof of Theorem 3.1, the convex envelope w_{**} of w is a supersolution of (4.4), and $w_{**} = w$ on $\overline{Q}$. Letting $\epsilon \searrow 0$ and using (2.8), we conclude (4.2). The proof is complete. □

We give an example which shows that inequality (1.11) is best possible. Hence, contrary to (1.8), inequality (1.12) does not hold in general, because $-2\alpha\Lambda(\varphi)/(2\alpha-\Theta\Lambda(\varphi)) \le -\Lambda(\varphi)$.

Proposition 4.2. *Let $H(p)=\Theta|p|^2/2$ for some constant $\Theta\in(0,2\alpha)$. For each $n\in N$, define the function $\varphi_n\in C^2(R^N)\cap \mathrm{Lip}(R^N)$ by*

$$\varphi_n(x)=\int_0^{|x|}\int_0^t \rho_n(s)\,ds\,dt \quad \text{for } x\in R^N,$$

where $\rho_n\in C(R)$ is given by

$$\rho_n(s)=\begin{cases} -1, & |s|\le n,\\ |s|-(n+1), & n<|s|<n+1,\\ 0, & n+1\le |s|.\end{cases}$$

Let $u_n\in C^{2,1}(Q)\cap C(\overline{Q})$ be the unique solution of (1.1) for $H(p)=\Theta|p|^2/2$ and $u_n(\cdot,0)=\varphi_n$. Then, we have the following:

(i) For each $n\in N$, $\Lambda(\varphi_n)=1$.

(ii) There is a sequence of $N\times N$ symmetric matrices $\{L_n\}$ such that

$$\lim_{n\to\infty} L_n = -\frac{2\alpha}{2\alpha-\Theta} I_N < -I_N, \tag{4.5}$$

$$\lim_{t\to\infty}\left[e^{2\alpha t}D^2u_n(x,t)\right]=L_n \quad \text{for } x\in R^N,\ n\in N. \tag{4.6}$$

Hence, inequality (1.11) is best possible, and inequality (1.12) does not hold.

Proof. 1. Note that $\varphi_n\in C^2(\mathrm{R}^N)$ and

$$D\varphi_n(0)=0,\qquad D^2\varphi_n(0)=-I_N,$$

$$D\varphi_n(x)=\frac{x}{|x|}\int_0^{|x|}\rho_n(s)\,ds \quad \text{for } x\in \mathrm{R}^N\setminus\{0\},$$

$$D^2\varphi_n(x)=\frac{I_N}{|x|}\int_0^{|x|}\rho_n(s)\,ds+\frac{x}{|x|}\otimes\frac{x}{|x|}\left[\rho_n(|x|)-\frac{1}{|x|}\int_0^{|x|}\rho_n(s)\,ds\right]$$

for $x\in\mathrm{R}^N\setminus\{0\}$.

2. Since

$$|D\varphi_n(x)|\le\int_0^\infty|\rho_n(s)|\,ds=\frac{n+2}{2} \quad \text{for } x\in\mathrm{R}^N,$$

we see that $\varphi_n\in\mathrm{Lip}(\mathrm{R}^N)$. Since $D^2\varphi_n(x)\ge D^2\varphi_n(0)=-I_N$ for $x\in\mathrm{R}^N$, we have $\Lambda(\varphi_n)=1$, so that we obtain (i).

3. Let us define the function $v_n\in C(\overline{Q})\times C^{2,1}(Q)$ by

$$v_n(x,t)=e^{-\Theta u_n(x,t)/2} \quad \text{for } (x,t)\in\mathrm{R}^N\times[0,\infty).$$

Then, v_n is the unique solution of the Cauchy problem

$$(v_n)_t(x,t) - \Delta v_n(x,t) + \alpha x \cdot Du_n(x,t) = 0 \qquad \text{in } \mathrm{R}^N \times (0,\infty), \tag{4.7}$$

$$v_n(x,0) = f_n(x) := e^{-\Theta\varphi_n(x)/2} \qquad \text{in } \mathrm{R}^N \times \{0\}. \tag{4.8}$$

Hence, v_n is given by $v_n(x,t) = R_t f_n(x)$, where $\{R_t\}$ is the Ornstein-Uhlenbeck semigroup defined by

$$R_t\psi(x) = \int_{\mathrm{R}^N} \psi(\, e^{-\alpha t}x + y\sqrt{1-e^{-2\alpha t}})\, d\nu(y), \quad (x,t) \in \mathrm{R}^N \times [0,\infty) \tag{4.9}$$

and $d\nu$ is the probability measure on R^N defined by

$$d\nu(y) = \left(\frac{\alpha}{2\pi}\right)^{N/2} e^{-\alpha|y|^2/2}\, dy.$$

Therefore, we have

$$u_n(x,t) = -\frac{2}{\Theta} \log\left(R_t f_n(x)\right) \quad \text{for } (x,t) \in \mathrm{R}^N \times [0,\infty).$$

Since

$$D(R_t f_n)(x) = e^{-\alpha t} R_t(Df_n)(x), \qquad D^2(R_t f_n)(x) = e^{-2\alpha t} R_t(D^2 f_n)(x),$$

we have

$$\begin{aligned} &e^{2\alpha t}\, D^2 u_n(x,t) \\ &= -\frac{2}{\Theta}\, \frac{1}{(R_t f_n)(x))^2} \left[\, R_t f_n(x) R_t(D^2 f_n)(x) - R_t(Df_n)(x) \otimes R_t(Df_n)(x) \,\right]. \end{aligned} \tag{4.10}$$

4. Since $0 < \Theta < 2\alpha$, we have

$$0 \le f_n(x) \le e^{\Theta|x|^2/4} \in L^1(\nu). \tag{4.11}$$

Since

$$|D\varphi_n(x)| \le \int_0^{|x|} |\rho_n(s)|\, ds = \int_0^{|x|} ds = |x| \quad \text{for } x \in \mathrm{R}^N,$$

we have

$$|Df_n(x)| \le \frac{\Theta}{2}|x|e^{\Theta|x|^2/4} \in L^1(\nu). \tag{4.12}$$

Since $-I_N \le D^2\varphi_n(x) \le 0$ for $x \in \mathrm{R}^N$, we have

$$\left|D^2 f_n(x)\right| \le \frac{\Theta}{2} e^{\Theta|x|^2/4} + \frac{\Theta^2}{4}|x|^2 e^{\Theta|x|^2/4} \in L^1(\nu). \tag{4.13}$$

5. Note that if $\psi \in C(\mathrm{R}^N) \bigcap L^1(\nu)$, we have

$$\lim_{t\to\infty} R_t\psi(x) = \int_{\mathrm{R}^N} \psi(y)\, d\nu(y) \quad \text{for } x \in \mathrm{R}^N.$$

By (4.10)-(4.13), we obtain

$$\lim_{t\to\infty} \left[e^{2\alpha t} D^2 u_n(x,t)\right] = -\frac{2}{\Theta} \frac{\displaystyle\int_{\mathrm{R}^N} D^2 f_n(y)\, d\nu(y)}{\displaystyle\int_{\mathrm{R}^N} f_n(y)\, d\nu(y)} \quad \text{for } x \in \mathrm{R}^N,\ n \in \mathrm{N}, \tag{4.14}$$

because

$$\int_{\mathrm{R}^N} Df_n(y)\, d\nu(y) = 0.$$

Now, we define the constant matrix L_n by the right-hand side of (4.14). Then L_n is an $N \times N$ symmetric matrix. Since

$$\lim_{n\to\infty} f_n(x) = e^{\Theta|x|^2/4} \quad \text{for } x \in \mathrm{R}^N,$$

it follows from (4.11) that

$$\lim_{n\to\infty} \int_{\mathrm{R}^N} f_n(y)\, d\nu(y) = \int_{\mathrm{R}^N} e^{\Theta|y|^2/4}\, d\nu(y) = \left(\frac{2\alpha}{2\alpha-\Theta}\right)^{N/2}.$$

On the other hand, since

$$\lim_{n\to\infty} D^2 f_n(x) = \frac{\Theta^2}{4} e^{\Theta|x|^2/4} x \otimes x + \frac{\Theta}{2} e^{\Theta|x|^2/4} I_N \quad \text{for } x \in \mathrm{R}^N,$$

it follows from (4.13) that

$$\lim_{n\to\infty} \int_{\mathrm{R}^N} D^2 f_n(y)\, d\nu(y) = \frac{\Theta}{2} \left(\frac{2\alpha}{2\alpha-\Theta}\right)^{(N+2)/2} I_N.$$

Hence, we obtain (ii) by (4.14). The proof is complete. □

Finally, we give a non-trivial example for (1.12).

Proposition 4.3. *Let $H(p) = \Theta|p|^2/2$ for some constant $\Theta > 0$. Define the function $\varphi \in C^2(R^N) \cap \mathrm{Lip}(R^N)$ by*

$$\varphi(x) = -\frac{2}{\Theta} \log(1 + |x|^2) \quad \text{for } x \in R^N.$$

Let $u \in C^{2,1}(Q) \cap C(\overline{Q})$ be the unique solution of Cauchy problem (1.1)–(1.2) for $H(p) = \Theta|p|^2/2$. Then, we have the following:

(i) $\Lambda(\varphi) = 4/\Theta$.

(ii)

$$\lim_{t\to\infty} \left[e^{2\alpha t} D^2 u(x,t)\right] = -\frac{4\alpha}{\Theta(\alpha+N)} \quad \text{for } x \in R^N. \tag{4.15}$$

Hence, we have

$$-\Lambda(\varphi)I_N < \lim_{t\to\infty}\left[e^{2\alpha t}D^2u(x,t)\right] \quad \text{for } x \in R^N. \tag{4.16}$$

Remark 4.4. The condition $\Lambda(\varphi)\Theta < 2\alpha$ in (1.10) can be read as $2 < \alpha$ in Proposition 4.3. However, note that this condition is not necessary in Proposition 4.3. Hence, the condition $\Lambda(\varphi)\Theta < 2\alpha$ is a sufficient condition for (1.10), but it is not a necessary condition for (1.10).

Proof. 1. It is easy to see that $\varphi \in C^2(\mathrm{R}^N) \cap \mathrm{Lip}(\mathrm{R}^N)$ and that

$$D^2\varphi(x) \geq D^2\varphi(0) = -\frac{4}{\Theta}I_N \quad \text{for } x \in \mathrm{R}^N,$$

so that $\Lambda(\varphi) = 4/\Theta$.

2. By the same arguments as those of the proof of Proposition 4.2, we have

$$u(x,t) = -\frac{2}{\Theta}\log\left(R_t f(x)\right) \quad \text{for } (x,t) \in \mathrm{R}^N \times [0,\infty),$$

where $\{R_t\}$ is the Ornstein-Uhlenbeck semigroup of (4.9) and $f(x) = 1+|x|^2$. Then, by the same arguments as those of (4.14), we obtain

$$\lim_{t\to\infty}\left[e^{2\alpha t}D^2u(x,t)\right] = -\frac{2}{\Theta}\frac{\displaystyle\int_{\mathrm{R}^N} D^2f(y)\,d\nu(y)}{\displaystyle\int_{\mathrm{R}^N} f(y)\,d\nu(y)} \quad \text{for } x \in \mathrm{R}^N.$$

Since

$$\int_{\mathrm{R}^N} f(y)\,d\nu(y) = 1 + \frac{N}{\alpha}, \qquad \int_{\mathrm{R}^N} D^2f(y)\,d\nu(y) = 2I_N,$$

we conclude (ii). The proof is complete. □

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