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BOUNDEDNESS OF WEAK SOLUTIONS TO SOME LINEAR ELLIPTIC EQUATIONS WITH MEASURE DATA

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Abstract. We prove boundedness of weak solutions u to some linear elliptic equations in divergence form

$$-div(a(x)Du(x)) = \mu \,.$$

On the right-hand side we have a Radon measure μ with a suitable decay on balls $0 \leq \mu(B(x,\rho)) \leq c\rho^s$.

1. Introduction

Let us consider linear equations in divergence form

$$-\sum_{i=1}^{n} D_i\Big(\sum_{j=1}^{n} a_{ij}(x) D_j u(x)\Big) = -\sum_{i=1}^{n} D_i f_i(x) + f_0(x),\ x \in \Omega \subset \mathbb{R}^n, \quad (1.1)$$

where $D_i = \partial/\partial x_i$, the coefficients $a_{ij} : \Omega \to \mathbb{R}$ are measurable, bounded, and elliptic, and $u : \Omega \to \mathbb{R}$ belongs to the Sobolev space $W^{1,r}(\Omega), r \geq 1$.

When the gradient Du is in $L^2(\Omega)$ and $f_0 = f_1 = ... = f_n = 0$, the solution u is Hölder continuous, [8]. On the contrary, if the gradient Du is no longer assumed in $L^2(\Omega)$ but in some $L^r(\Omega)$ with $r < 2$, then u may be unbounded, even if $f_0 = f_1 = ... = f_n = 0$, [14]. This happens when the coefficients a_{ij} are only measurable, bounded, and elliptic.

If we restrict our study to the case of $\theta-$Hölder continuous coefficients a_{ij}, then higher integrability on the right-hand side, $f_0, f_1, ...f_n \in L^p(\Omega)$, improves the integrability of the gradient Du, [6], [11], [15],[9]; then the Sobolev imbedding theorem gives the continuity of solutions u, if p is large enough.

In [10] it is proved that, for suitable θ's, boundedness of u occurs without higher integrability on the right-hand side, but only assuming that f_i belongs

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to a suitable Morrey space $L^{1,\lambda}(\Omega)$. This means that we require a suitable decay on balls

$$\int_{B(x,\rho)} |f_i(y)|dy \leq c\rho^\lambda. \tag{1.2}$$

In the present paper we deal with weak solutions to linear elliptic equations with a measure on the right-hand side; that is

$$-\sum_{i=1}^{n} D_i\Big(\sum_{j=1}^{n} a_{ij}(x)D_j u(x)\Big) = \mu, \ \ x \in \Omega \subset \mathbb{R}^n, \tag{1.3}$$

where the coefficients a_{ij} are $\theta-$Hölder continuous and elliptic.

We assume that there exist two positive Radon measures μ_1, μ_2 such that $\mu = \mu_1 - \mu_2$. We require a decay similar to (1.2):

$$\mu_i(B(x,\rho)) \leq c_i\rho^\lambda\,, \ \forall\, i = 1, 2, \tag{1.4}$$

for a suitable λ.

We prove boundedness for weak solutions $u \in W^{1,r}(\Omega)$ of (1.3) under the assumption

$$\frac{n}{1+\theta} < r\,, \ \ 0 \leq n-2 < \lambda. \tag{1.5}$$

This result improves on [10] since any function $f_0 \in L^{1,\lambda}(\Omega)$ generates the measure $\mu \equiv f_0 \cdot \mathcal{H}^n\lfloor\,\Omega$ with the prescribed decay (1.4); here and in the following $\mathcal{H}^s$ is the $s-$dimensional Hausdorff measure. Our result can be also applied to a low-dimensional Hausdorff measure $\mathcal{H}^s$ restricted to "thin" sets or to "small" sets: the first example is $\mu = \mathcal{H}^1\lfloor\, L$ where L is a line segment in $\mathbb{R}^2$; we can also consider $\mu = \mathcal{H}^2\lfloor\, S$ where S is a disk in $\mathbb{R}^3$; the third example is $\mu = \mathcal{H}^{s_k}\lfloor\, C_k$ where C_k is the Cantor set in $\mathbb{R}^2$ with parameter $k \in (0, 1/2)$ and $s_k = log4/log(1/k)$.

Let us remark that the condition $n-2 < \lambda$ in (1.5) is sharp, as we show in section 4. Let us mention that in the last decade a great effort has been devoted to the understanding of elliptic equations with measures on the right-hand side: starting from [2], several authors have been studying existence of solutions; we address the reader to the references in [1]. As far as uniqueness is concerned, we quote [4], [7], [3], [1] and their references. The present paper deals with regularity of solutions. In section 2 we give the precise statement, whose proof is contained in section 3; the final section 4 is devoted to a counterexample showing that our result is sharp, as far as the condition $n-2 < \lambda$ is concerned.

2. Notation and results

Let Ω be a bounded open subset of $\mathbb{R}^n$, $n \geq 2$. For $i, j = 1, ..., n$ we consider functions $a_{ij} : \Omega \to \mathbb{R}$ and we assume that the matrix $\{a_{ij}\}$ is elliptic: there exist two constants $0 < l \leq L$ such that

$$l|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(x)\xi_j\xi_i \leq L|\xi|^2 \quad \forall \xi \in \mathbb{R}^n\,, \forall x \in \Omega\,. \tag{2.1}$$

We also assume $\theta-$Hölder continuity: for $\theta \in (0, 1]$ we have

$$a_{ij} \in C^{0,\theta}\left(\overline{\Omega}\right), \quad \forall\, i, j = 1, ..., n. \tag{2.2}$$

Let μ be a signed measure such that

$$\mu = \mu_1 - \mu_2 \tag{2.3}$$

where μ_1, μ_2 are positive Radon measures satisfying

$$\sup_{x\in\Omega,\, \rho>0} \rho^{-s}\mu_1\left(B(x, \rho) \cap \Omega\right) < +\infty\,, \tag{2.4}$$

and

$$\sup_{x\in\Omega,\, \rho>0} \rho^{-s}\mu_2\left(B(x, \rho) \cap \Omega\right) < +\infty \tag{2.5}$$

for $0 \leq (n-2) < s$, where $B(x, \rho)$ is the open ball around x with radius ρ.

Let $u : \Omega \to \mathbb{R}$ belong to the Sobolev space $W^{1,r}(\Omega)$ and satisfy

$$\int_\Omega \sum_{i,j=1}^{n} a_{ij}(x)D_j u(x)D_i\phi(x)dx = \int_\Omega \phi(x)d\mu(x) \ \ \forall\, \phi \in C_0^\infty(\Omega). \tag{2.6}$$

Theorem 2.1. *Assume that* (2.1), (2.2), (2.3), (2.4), (2.5) *hold. If* $u \in W^{1,r}(\Omega)$ *satisfies* (2.6) *with*

$$\frac{n}{1+\theta} < r, \tag{2.7}$$

then

$$u \in L^\infty_{loc}(\Omega)\,. \tag{2.8}$$

Now we give some examples of Radon measures satisfying the decay (2.4) with $n - 2 < s$.

Example 1. Let us consider the $n-$dimensional Hausdorff measure in $\mathbb{R}^n$; then

$$\mathcal{H}^n(B(x, \rho)) = \omega_n\rho^n, \tag{2.9}$$

that is, (2.4) holds with $s = n$.

Example 2. Let f be a nonnegative function such that $f \in L^\infty(\Omega)$. We consider the measure $\mu \equiv f \cdot \mathcal{H}^n \lfloor \Omega$ defined as follows

$$\mu(A) = (f \cdot \mathcal{H}^n \lfloor \Omega)(A) = \int_{A\cap\Omega} f(y)dy \tag{2.10}$$

for all $A \subset \mathbb{R}^n$ $\mathcal{H}^n-$measurable. Then

$$\mu(B(x,\rho)) \le \|f\|_{L^\infty(\Omega)}\omega_n\rho^n;$$

that is, (2.4) holds with $s = n$.

Example 3. Let $A = [a,b] \times \{y_0\} \subset \mathbb{R}^2$ with $a, b \in \mathbb{R}$, $a < b$. We consider in $\mathbb{R}^2$ the measure $\mu \equiv \mathcal{H}^1 \lfloor A$ defined as follows

$$\mu(B) = \mathcal{H}^1(A \cap B)\,, \tag{2.11}$$

for all $B \subset \mathbb{R}^2$. Then

$$\mu(B(x,\rho)) = \mathcal{H}^1(A \cap B(x,\rho)) \le 2\rho\,,$$

that is, (2.4) holds with $s = 1 > 0 = n - 2$.

Example 4. Let us consider in $\mathbb{R}^3$ the set

$$A = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \le R^2\} \times \{z_0\}\,.$$

We set $\mu \equiv \mathcal{H}^2 \lfloor A$; that is,

$$\mu(B) = \mathcal{H}^2(A \cap B)\,, \tag{2.12}$$

for all $B \subset \mathbb{R}^3$. Then

$$\mu(B(x,\rho)) = \mathcal{H}^2(A \cap B(x,\rho)) \le \pi\rho^2\,,$$

that is, (2.4) holds with $s = 2 > 1 = n - 2$.

Example 5. Let $0 < k < 1/2$ and $C_k \subset \mathbb{R}^2$ be the Cantor set. We consider the s_k-dimensional Hausdorff measure restricted to C_k, $\mu \equiv \mathcal{H}^{s_k} \lfloor C_k$, thus

$$\mu(B) = \mathcal{H}^{s_k}(C_k \cap B)\,, \tag{2.13}$$

for all $B \subset \mathbb{R}^2$. For $s_k = \dfrac{\log 4}{\log(1/k)}$, it follows that $0 < \mathcal{H}^{s_k}(C_k) < +\infty$, and

$$\mu\left(B(x,\rho)\right) \le \alpha_k \rho^{s_k}\,, \tag{2.14}$$

for a suitable constant α_k, thus we have (2.4) with $s = s_k \in (0,2)$ and $s_k > 0 = n - 2$.

Remark 2.2. In the last three examples there is no function f in L^1 such that (2.10) holds true.

Remark 2.3. Let us mention that existence of a solution u in the Sobolev class $W^{1,r}$ has been proved in [2] for $r < n/(n-1)$; in order to get our regularity result we need restriction (2.7): $n/(1+\theta) < r$; thus regularity merges into existence provided $n = 2$.

Remark 2.4. When the measure μ is absolutely continuous with respect to Lebesgue measure, with density f in the Morrey space, one could ask if better regularity on the exponent r can be obtained. This could be justified by [12], where the rate of decay on balls, guaranteed by the Morrey space, merges into difference quotient technique, provided the radii of balls suitably follow the size of the increment of the difference quotient. This allows us to get some fractional differentiability, which improves on the integrability. However, the application of this idea to our framework still requires some more effort; thus the question "whether f being in Morrey space improves on the integrability exponent r or not" is still open.

We end this section by thanking the referee for valuable remarks.

3. Proof of the Theorem

We follow [10]. Let $x^0 \in \Omega$ be an arbitrary fixed point. We split the matrix $a(x^0)$ into its symmetric part, $a^+(x^0)$, and skew symmetric one, $a^-(x^0)$

$$a_{ij}{}^+(x^0) = \frac{a_{ij}(x^0) + a_{ji}(x^0)}{2} \qquad a_{ij}{}^-(x^0) = \frac{a_{ij}(x^0) - a_{ji}(x^0)}{2}$$

for all $i, j = 1, ..., n$. Since

$$\sum_{i,j=1}^{n} a_{ij}{}^-(x^0)\xi_j\xi_i = 0\,,$$

then (2.1) yields

$$l|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}^+(x^0)\xi_j\xi_i \le L|\xi|^2 \quad \forall\xi \in \mathbb{R}^n\ \forall x^0 \in \Omega\,. \tag{3.1}$$

Because of (3.1) the $n \times n$ real matrix $a^+(x^0) = \{a_{ij}(x^0)\}$ is symmetric and positive, thus it has $\lambda^1, ..., \lambda^n$ positive real eigenvalues with $0 < l \le \lambda^1, ..., \lambda^n \le L$.

Let us select $w^1, ..., w^n$ eigenvectors such that $\mathcal{B} \equiv \{w^1, ..., w^n\}$ is an orthonormal basis in $\mathbb{R}^n$. By means of $\lambda^1, ..., \lambda^n$ and $\mathcal{B}$ we define the linear mapping $G : \mathbb{R}^n \to \mathbb{R}^n$ as follows

$$G_i(x) = \sum_{j=1}^{n} \left(\lambda^i\right)^{-1/2} w_j{}^i x_j, \quad \forall\, i = 1, ..., n, \tag{3.2}$$

where $w^i = \left(w_1{}^i, ..., w_n{}^i\right)$ and $x = (x_1, ..., x_n)$. Since the eigenvalues and eigenvectors of $a^+(x^0)$ depend on x^0, G depends on x^0 too: $G = G_{x^0}$.

Because of the ellipticity, we can give the following estimates independent of x^0:

$$L^{-1}|x-y|^2 \le |G(x)-G(y)|^2 \le l^{-1}|x-y|^2, \quad \forall\, x, y \in \mathbb{R}^n, \tag{3.3}$$

$$L^{-n/2} \le |\det JG| \le l^{-n/2}, \tag{3.4}$$

$$\sum_{i,j=1}^{n} (JG)_{ij}{}^2 \le nl^{-1}, \tag{3.5}$$

$$\sum_{i,j=1}^{n} (JG)_{\alpha i}\, a_{ij}{}^+(x^0)\, (JG)_{\beta j} = \delta_{\alpha\beta}, \quad \forall\, \alpha, \beta = 1, ..., n\,., \tag{3.6}$$

where $JG = \left\{\, (JG)_{ij} \,\right\}_{i,j} = \left\{\dfrac{\partial G_i}{\partial x_j}\right\}_{i,j}$ is the Jacobian matrix of G and $\delta_{\alpha\beta}$ is the Kronecker symbol. We set $y^0 = G(x^0)$. Using (3.3) we get

$$dist\left(x^0, \partial\Omega\right) \le \sqrt{L} dist\left(y^0, \partial[G\,(\Omega)]\right) \tag{3.7}$$

and

$$B(x^0, \sqrt{l}R) \subset G^{-1}\left(B\left(y^0, R\right)\right) \subset B(x^0, \sqrt{L}R) \quad \text{for all} \quad R > 0\,, \tag{3.8}$$

where ∂A is the boundary of the set A.

For every σ with $\sqrt{L}\sigma < dist\left(x^0, \partial\Omega\right)$, we have $\overline{G^{-1}\,(B(y^0,\sigma))} \subset \Omega$. Thus, (2.6) holds true for every $\phi \in C_0^\infty(G^{-1}\left(B(y^0,\sigma)\right)) \subset C_0^\infty(\Omega)$ and we can write

$$\begin{aligned}
&\int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} a_{ij}(x^0) D_j u(x) D_i \phi(x) dx \\
&= \int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} [a_{ij}(x^0) - a_{ij}(x)] D_j u(x) D_i \phi(x) dx \\
&\quad + \int_{G^{-1}(B(y^0,\sigma))} \phi(x) d\mu(x)\,,
\end{aligned} \tag{3.9}$$

for all $\phi \in C_0^\infty(G^{-1}\left(B(y^0,\sigma)\right))$ and $0 < \sqrt{L}\sigma < dist\left(x^0, \partial\Omega\right)$. Now

$$\int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} a_{ij}(x^0) D_j u(x) D_i \phi(x) dx$$

$$= \int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} a_{ij}^{+}(x^0) D_j u(x) D_i \phi(x) dx$$
$$+ \int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} a_{ij}^{-}(x^0) D_j u(x) D_i \phi(x) dx$$

and

$$\int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} a_{ij}^{-}(x^0) D_j u(x) D_i \phi(x) dx$$
$$= - \int_{G^{-1}(B(y^0,\sigma))} u(x) \sum_{i,j=1}^{n} a_{ij}^{-}(x^0) D_j D_i \phi(x) dx = 0\,,$$

because

$$\sum_{i,j=1}^{n} a_{ij}^{-}(x^0) D_j D_i \phi(x) = 0\,.$$

Then (3.9) becomes

$$\int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} {a_{ij}}^{+}(x^0) D_j u(x) D_i \phi(x) dx \tag{3.10}$$
$$= \int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} [a_{ij}(x^0) - a_{ij}(x)] D_j u(x) D_i \phi(x) dx + \int_{G^{-1}(B(y^0,\sigma))} \phi(x) d\mu(x),$$

for all $\phi \in C_0^\infty(G^{-1}\left(B(y^0,\sigma)\right))$ and $0 < \sqrt{L}\sigma < dist\left(x^0, \partial\Omega\right)$.

At this point, we insert into (3.10) a particular test function $\tilde{\phi}$; we choose $\tilde{\phi} \equiv \phi \vee 0$, where

$$\phi = \sigma^2 - |G(x) - y^0|^2\,; \tag{3.11}$$

see [5]. We remark that $\tilde{\phi} \notin C_0^\infty(G^{-1}\left(B(y^0,\sigma)\right))$: we only have $\tilde{\phi} \in C^{0,1}(\mathbb{R}^n)$ and $\tilde{\phi} = 0$ on $\Omega \setminus [G^{-1}\left(B(y^0,\sigma)\right)]$. We approximate $\tilde{\phi}$ by means of $\tilde{\phi}_k \in C_0^\infty(G^{-1}\left(B(y^0,\sigma)\right))$ in such a way that

(1) $\tilde{\phi}_k \xrightarrow[k\to+\infty]{} \tilde{\phi}$ uniformly in $\overline{G^{-1}\left(B(y^0,\sigma)\right)}$,

(2) $D\tilde{\phi}_k \xrightarrow[k\to+\infty]{} D\tilde{\phi}$ a.e. in $G^{-1}\left(B(y^0,\sigma)\right)$,

(3) $0 \le \tilde{\phi}_k \le \sup_{x\in\mathbb{R}^n} \tilde{\phi}(x)$ in $\mathbb{R}^n$,

(4) $|D\tilde{\phi}_k| \le Lip(\tilde{\phi})$ a.e. in $\mathbb{R}^n$.

Thus, we are allowed to use (3.10) with $\tilde{\phi}_k$ and to pass to the limit as k goes to infinity: in this way (3.10) is valid for $\tilde{\phi}$ too. Note that $\tilde{\phi}(x) = \sigma^2 - |G(x) - y^0|^2$ for every $x \in G^{-1}\left(B(y^0, \sigma)\right)$.

From now on, ϕ will be the function $\phi(x) = \sigma^2 - |G(x) - y^0|^2$. Let us deal with the left-hand side of (3.10): changing variables, setting $x = G^{-1}(y)$, $v(y) = u\left(G^{-1}(y)\right)$, $\psi(y) = \phi\left(G^{-1}(y)\right)$, and using (3.6) yield

$$I_1 \equiv \int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} a_{ij}^{+}(x^0) D_j u(x) D_i \phi(x) dx$$

$$= |detJG|^{-1} \int_{B(y^0,\sigma)} \sum_{i,j=1}^{n} a_{ij}^{+}(x^0) \sum_{\beta=1}^{n} \frac{\partial v}{\partial y_\beta}(y) \left(JG\right)_{\beta j} \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial y_\alpha}(y) \left(JG\right)_{\alpha i} dy$$
$$= |detJG|^{-1} \int_{B(y^0,\sigma)} \sum_{\alpha=1}^{n} \frac{\partial v}{\partial y_\alpha}(y) \frac{\partial \psi}{\partial y_\alpha}(y) dy\,.$$

Integrating by parts and keeping in mind the choice of our test function ϕ, the previous equation becomes

$$\begin{aligned} I_1 = |\det JG|^{-1} \Big\{ \int_{\partial B(y^0,\sigma)} v(y) \sum_{\alpha=1}^{n} \frac{\partial \psi}{\partial y_\alpha}(y) N_\alpha(y) dH^{n-1}(y) \\ - \int_{B(y^0,\sigma)} \sum_{\alpha=1}^{n} v(y) \frac{\partial^2 \psi}{\partial y_\alpha^2}(y) dy \Big\} \qquad (3.12) \\ = |\det JG|^{-1} \Big\{ -2\sigma \int_{\partial B(y^0,\sigma)} v(y) dH^{n-1}(y) + 2n \int_{B(y^0,\sigma)} v(y) dy \Big\}. \end{aligned}$$

Now, let us deal with the first term on the right-hand side of (3.10):

$$I_2 \equiv \Big| \int_{G^{-1}(B(y^0,\sigma))} \sum_{i,j=1}^{n} \left[a_{ij}(x^0) - a_{ij}(x)\right] D_j u(x) D_i \phi(x) dx \Big|$$
$$\leq \int_{G^{-1}(B(y^0,\sigma))} \Big[\sum_{i,j=1}^{n} \left|a_{ij}(x^0) - a_{ij}(x)\right|^2 \Big]^{1/2} |Du(x)||D\phi(x)| dx\,.$$

Because of (3.11), (3.5), and (3.8), it turns out that

$$|\phi(x)| \leq \sigma^2\,, \quad |D\phi(x)| \leq 2\sqrt{\frac{n}{l}}\sigma$$

for all $x \in G^{-1}\left(B(y^0,\sigma)\right)$; then

$$I_2 \leq 2\sqrt{\frac{n}{l}}\sigma \int_{B\left(x^0,\sqrt{L}\sigma\right)} \Big[\sum_{i,j=1}^{n} |a_{ij}(x^0)-a_{ij}(x)|^2\Big]^{1/2} |Du(x)|\,dx\,.$$

Moreover, because of (2.2),

$$\Big[\sum_{i,j=1}^{n} |a_{ij}(x^0)-a_{ij}(x)|^2\Big]^{1/2} \leq \left(\sqrt{L}\,\sigma\right)^{\theta}[a]\ ,$$

with

$$[a] \equiv \Big(\sum_{i,j=1}^{n} [a_{ij}]^2\Big)^{1/2};$$

then

$$I_2 \leq 2\sqrt{\frac{n}{l}}\,\sigma(\sigma\sqrt{L})^{\theta}\,[a]\int_{B(x^0,\sqrt{L}\sigma)} |Du(x)|\,dx\,.$$

Using the Hölder inequality, we write

$$\begin{aligned} I_2 &\leq 2\sqrt{\frac{n}{l}}\,\sigma^{\theta+1}\,[a]\,(\sqrt{L})^{\theta}[\mathcal{H}^n(B(x^0,\sqrt{L}\sigma))]^{1-1/r}\Big[\int_{B\left(x^0,\sqrt{L}\sigma\right)} |Du(x)|^r dx\Big]^{1/r} \\ &\leq 2\sqrt{\frac{n}{l}}\;[a]\,(\sqrt{L})^{\theta}\,\sigma^{1+\theta}\|Du\|_{L^r(\Omega)}\,((\sigma\sqrt{L})^n\omega_n)^{1-1/r}\,, \end{aligned} \tag{3.13}$$

where $\omega_n = \mathcal{H}^n(B(0,1))$.

Let us deal with the second term on the right-hand side of (3.10). Because of (3.11), (3.8), (2.4), and (2.5) it turns out that

$$\begin{aligned} I_3 &\equiv \Big|\int_{G^{-1}(B(y^0,\sigma))} \phi(x)\,d\mu(x)\Big| \leq \sigma^2\left[\mu_1(B(x^0,\sqrt{L}\sigma)) + \mu_2(B(x^0,\sqrt{L}\sigma))\right] \\ &\leq \sigma^2[C_1(\sqrt{L}\sigma)^s + C_2(\sqrt{L}\sigma)^s] = (\sqrt{L})^s(C_1+C_2)\sigma^{s+2}\,. \end{aligned} \tag{3.14}$$

Equality (3.12) and the estimates for I_2, I_3 merge into

$$\begin{aligned} &2\,|\det JG|^{-1}\Big\{-\sigma\int_{\partial B(y^0,\sigma)} v(y)dH^{n-1}(y) + n\int_{B(y^0,\sigma)} v(y)dy\Big\} \\ &\leq 2\sqrt{\frac{n}{l}}\;[a]\,(\sqrt{L})^{\theta+n(1-1/r)}\|Du\|_{L^r(\Omega)}\omega_n^{1-1/r}\sigma^{\theta+1+n(1-1/r)} \\ &\qquad + (\sqrt{L})^s(C_1+C_2)\sigma^{1+(s+1)}. \end{aligned}$$

Because of (3.4) we can write

$$-\sigma\int_{\partial B(y^0,\sigma)} v(y)dH^{n-1}(y) + n\int_{B(y^0,\sigma)} v(y)dy \leq d_1\,\sigma^{\gamma+1}\,, \tag{3.15}$$

where

$$\gamma = \min\left\{\theta + n\Big(1 - \frac{1}{r}\Big), s+1\right\},$$

and

$$d_1 = \frac{l^{-n/2}}{2}\left\{2\sqrt{n/l}[a](\sqrt{L})^{\theta+n(1-1/r)}\omega_n^{1-1/r}\|Du\|_{L^r(\Omega)} + (C_1+C_2)(\sqrt{L})^s\right\}.$$

The inequality (3.15) holds true for almost every $\sigma \in (0,1]$, with $\sqrt{L}\sigma < dist(x^0, \partial\Omega)$. If we set

$$h(\sigma) = \int_{B(y^0,\sigma)} v(y)dy\,,$$

then inequality (3.15) can be written as

$$\sigma\frac{d}{d\sigma}h(\sigma) \geq nh(\sigma) - d_1\sigma^{1+\gamma}\,. \tag{3.16}$$

Set $\tilde{h}(\sigma) = h(\sigma) + K\sigma^{1+\gamma}$, and let us find $K \geq 0$ such that

$$\sigma\frac{d}{d\sigma}\tilde{h}(\sigma) \geq n\tilde{h}(\sigma); \tag{3.17}$$

that is, taking into account (3.16), $K(1+\gamma-n) \geq d_1$. Since $s > n-2$ and $r > n/(1+\theta)$, it follows that $\gamma > n-1$ and $K = \frac{d_1}{1+\gamma-n} \geq 0$ is an admissible value, thus (3.17) is true. Then the function $\sigma \longrightarrow \sigma^{-n}\tilde{h}(\sigma)$ is increasing and this yields

$$\rho^{-n}\int_{B(y^0,\rho)} v(y)dy \leq \sigma^{-n}\Big(\int_{B(y^0,\sigma)} v(y)dy + K\sigma^{1+\gamma}\Big), \tag{3.18}$$

for all ρ and σ with $0 < \rho \leq \sigma \leq \min\{1; \frac{1}{2\sqrt{L}}\, dist(x^0, \partial\Omega)\}$.

Changing variables, coming back to $u(x) = v(G(x))$ and using (3.4) and (3.8) we have

$$⨍_{G^{-1}(B(y^0,\rho))} u(x)dx \leq \sigma^{-n}\,(L/l)^{n/2}\,\omega_n^{-1}\left(\|u\|_{L^1(\Omega)}l^{-n/2} + K\sigma^{1+\gamma}\right), \tag{3.19}$$

where $⨍_A u(x)dx \equiv \frac{1}{|A|}\int_A u(x)dx$ and $|A| = \mathcal{H}^n(A)$.

Now, let us consider the function $-u$. Since $-u$ solves equation (2.6) with $-\mu$ instead of μ, inequality (3.19) holds true for $-u$ too. Thus,

$$\left|⨍_{G^{-1}(B(y^0,\rho))} u(x)dx\right| \leq \sigma^{-n}\,(L/l)^{n/2}\,\omega_n^{-1}\left(\|u\|_{L^1(\Omega)}l^{-n/2} + K\sigma^{1+\gamma}\right), \tag{3.20}$$

for every $x^0 = G^{-1}(y^0)$, $0 < \rho \leq \sigma \leq \min\{1; \frac{1}{\sqrt{4L}}\, dist(x^0, \partial\Omega)\}$.

Now, we may use Theorem 8.8 in [13] that gives

$$\lim_{\rho\to 0} \fint_{G^{-1}(B(y^0,\rho))} u(x)dx = u(x^0)\,, \tag{3.21}$$

for almost every $x^0 \in \Omega$.

When x^o approaches the boundary of Ω, σ^{-n} goes to infinity and (3.20) becomes useless. Thus, in order to stay at some distance from $\partial\Omega$, for all $\varepsilon \in (0, 2\sqrt{L}]$ let us define the set $\Omega_\varepsilon \equiv \{x \in \Omega : dist(x, \partial\Omega) > \varepsilon\}$.

We can use inequality (3.20) for all $x^o \in \Omega_\varepsilon$ and $0 < \rho \le \sigma = \frac{\varepsilon}{\sqrt{4L}}$. We let ρ go to zero and we get

$$|u(x^0)| \le \left(\frac{2L}{\varepsilon\sqrt{l}}\right)^n \omega_n^{-1} \left(\|u\|_{L^1(\Omega)} l^{-n/2} + K\right)\,,$$

for almost every $x^0 \in \Omega_\varepsilon$, for every $\varepsilon \in (0, 2\sqrt{L}]$. Then

$$\|u\|_{L^\infty(\Omega_\varepsilon)} \le \left(\frac{2L}{\varepsilon\sqrt{l}}\right)^n \omega_n^{-1} \left(\|u\|_{L^1(\Omega)} l^{-n/2} + K\right) \tag{3.22}$$

for all $\varepsilon \in (0, 2\sqrt{L}]$. □

4. A Counterexample

Let us consider $u(x) = \log|x|$ for $x \in \mathbb{R}^2 \setminus \{0\}$. Then

$$u \in W^{1,p}(B(0,R)) \tag{4.1}$$

for all $R > 0$ and for all $p \in [1, 2)$.

Moreover, for all $R > 0$, u satisfies the following equation

$$\int_{B(0,R)} \sum_{i=1}^{2} D_i u(x) D_i \phi(x) dx = -2\pi\phi(0), \tag{4.2}$$

for all $\phi \in C_0^\infty(B(0,R))$. Recall that

$$2\pi\phi(0) = \int_{B(0,R)} \phi(x)\, d\mu(x), \tag{4.3}$$

where $\mu = -2\pi\delta_0$ and δ_0 is the Dirac measure related to the origin $0 \in \mathbb{R}^2$.

Then, the function u satisfies (2.6) with $\Omega = B(0,1) \subset \mathbb{R}^2$ and $a_{ij}(x) = \delta_{ij}$. So, (2.1) holds true with $l = L = 1$, (2.2) is verified with $\theta = 1$, and (2.3) holds true with $\mu_1 = 0$ and $\mu_2 = 2\pi\delta_0$.

Thus, (2.4) is satisfied for all values of $s \ge 0$, while (2.5) is satisfied only with $s = 0 = n - 2$. Since $u \in W^{1,p}(B(0,1))$ for all $p \in [1, 2)$, then $\dfrac{n}{1+\theta} = 1 < p$ and condition (2.7) holds true. Furthermore, $u \notin L^\infty_{loc}(B(0,1))$. This example satisfies all the hypotheses of our theorem except the one $n-2<s$. This example shows that the theorem is sharp with respect to the condition

$$n-2<s \tag{4.4}$$

on the decay exponent s of the measure μ.

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