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SUBCRITICAL PSEUDODIFFERENTIAL EQUATION ON A HALF-LINE WITH NONANALYTIC SYMBOL

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Abstract. We study nonlinear pseudodifferential equations on a half-line with a nonanalytic symbol

$$\begin{cases} \partial_t u + \mathbb{K}u = \lambda \left|u\right|^{\sigma} u, \; x \in \mathbf{R}^+, \; t > 0, \\ u\left(0,x\right) = u_0\left(x\right), \, x \in \mathbf{R}^+, \end{cases}$$

where $0 < \sigma < 1$, $\lambda \in \mathbf{R}$ and

$$\mathbb{K}u = \frac{1}{2\pi i}\theta(x) \int_{-i\infty}^{i\infty} e^{px} K(p) \widehat{u}(t,p) dp, \qquad K(p) = \frac{p^2}{p^2-1}.$$

The aim of this paper is to prove the global existence of solutions to the initial-boundary-value problem and to find the main term of the asymptotic representation of solutions in subcritical case, when the nonlinear term of equation has the time decay rate less than that of the linear terms.

1. Introduction

This paper is devoted to the study of the initial- boundary-value problem for the pseudodifferential equation on a half-line

$$\begin{cases} \partial_t u + \mathbb{K}u = \lambda \left|u\right|^{\sigma} u, \; x \in \mathbf{R}^+, \; t > 0, \\ u\left(0,x\right) = u_0\left(x\right), \, x \in \mathbf{R}^+, \end{cases} \tag{1.1}$$

where $0 < \sigma < 1$, $\lambda \in \mathbf{R}$,

$$\mathbb{K}u = \frac{1}{2\pi i}\theta_1(x) \int_{-i\infty}^{i\infty} e^{px} K(p) \widehat{u}(t,p) dp, \qquad K(p) = \frac{p^2}{p^2-1},$$

and

$$\theta_1(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

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The Cauchy problem for nonlinear pseudoparabolic-type equations was studied in many papers (for example, see [3], [4],[5], [15], [16]). The large time asymptotic of solutions to the Cauchy problem was obtained in papers [7], [10], [14].

In this paper we study the initial-boundary-value problem (1.1) in the subcritical case, when the nonlinear term of equation (1.1) has a time decay rate less than that of the linear terms. Recently much attention was drown to the study of the global existence and large time asymptotic behavior of solutions to the Cauchy problems for nonlinear equations in the subcritical cases (see papers [1] , [2], [9], [13] and literature cited therein.). Critical and supercritical nonlinear pseudoparabolic-type equations

$$\partial_t(u-u_{xx})-u_{xx}=\lambda\left|u\right|^{\sigma}u,\sigma\geq 1. \tag{1.2}$$

on a half-line were studied in the papers [11], [12], where it was shown that it is necessary to put one boundary condition in the initial-boundary-value problem for equation (1.2) for its correct solvability. A general theory of nonlinear nonlocal equations on a half-line was developed in book [6]. As far as we know the case of nonanalytic symbols $K(p)$ in the right-half complex plane was not studied previously. In the present paper we fill this gap, considering for example the pseudodifferential equation (1.1) with a symbol $K(p)=\frac{p^2}{1-p^2}$. Note, that we do not put any boundary condition in the initial-boundary-value problem (1.1), since the order of the operator is equal to zero (see book [6]). Therefore initial-boundary-value problems (1.1) and (1.2) are completely different compared with the case of the corresponding Cauchy problem, when equations (1.1) and (1.2) are equivalent.

In the present paper we propose a new method for constructing the Green operator for problem (1.1) based on the introduction of some necessary condition at the singularity point $p=1$ of the symbol $K(p)$. Another difficulty which we overcome here is in evaluating the contribution of the boundary into the large-time asymptotic behavior of solutions. We will prove that in the case of the initial-boundary-value problem the solution obtains an additional decay comparing with the corresponding Cauchy problem. As a result the main term of the asymptotic part does not depend on the mean value of the solution; instead it is determined by the first moment of the solution. Thus to be able to obtain an optimal time decay estimate of the solution we have to estimate the evolution of the first moments.

Below $\hat{\phi}$ is the Laplace transform of ϕ defined by

$$\hat{\phi}(\xi) = \int_{\mathbf{R}^+} e^{-x\xi} \phi(x)\, dx$$

and

$$\mathcal{L}^{-1} \hat{\phi}(\xi) = (2\pi i)^{-1} \int_{-\mathbf{i}\infty}^{i\infty} e^{x\xi} \hat{\phi}(\xi)\, d\xi$$

is the inverse Laplace transform of ϕ. By $\mathbf{C}(\mathbf{I}; \mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. The usual Lebesgue space is denote by $\mathbf{L}^p$, $1 \leq p \leq \infty$, the weighted Lebesgue space $\mathbf{L}^{1,a}$ is defined by

$$\mathbf{L}^{p,a} = \left\{ \phi \in \mathbf{L}^p(\mathbf{R}^+); \|\phi\|_{\mathbf{L}^{p,a}} = \|\langle x \rangle^a \phi\|_{\mathbf{L}^p} < \infty \right\},$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$, $a \geq 0$. Also we denote by $\mathbf{H}^{k,a} = \mathbf{W}_2^{k,a}$.

Define

$$|\lambda| t^{\frac{\sigma-1}{2}} \int_{\mathbf{R}^+} x \left(G_0(t,x) \right)^\sigma G_0(t,x)\, dx = \eta,$$

where the heat kernel $G_0(t,x) = (4\pi t)^{-\frac{1}{2}} \frac{x}{t} e^{-\frac{x^2}{4t}}$. Denote

$$\theta = \int_{\mathbf{R}^+} x u_0(x) dx, \qquad \widetilde{G}(x) = (4\pi)^{-\frac{1}{2}} x e^{-\frac{|x|^2}{4}},$$

and

$$G_1(s,q) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{p^2} (e^{p(s-q)} - e^{p(s+q)}) dp.$$

Now we state the main result of this paper.

Theorem 1. *Let $0 < \sigma < 1, \eta > 0$. We assume that the initial data $u_0 \in \mathbf{L}^\infty \cap \mathbf{L}^{1,1+a}$, $a \in (0,1)$ are sufficiently small, $\|u_0\|_{\mathbf{L}^\infty} + \|u_0\|_{\mathbf{L}^{1,1+a}} \leq \varepsilon$, and $\lambda\theta \leq -C_1\varepsilon < 0$, where $\theta = \int_0^{+\infty} x u_0(x) dx$, $C_1 \in (0,1)$. Also we suppose that the value σ is close to 1, so that $1 - \sigma \leq C\varepsilon^\sigma$. Then the initial-boundary-value problem (1.1) has a unique global solution $u(t,x) \in \mathbf{C}\left([0,\infty); \mathbf{L}^\infty \cap \mathbf{L}^{1,1+a}\right)$, satisfying the following time decay estimates*

$$\|u(t)\|_{\mathbf{L}^\infty} \leq C \langle t \rangle^{-2+\sigma} \tag{1.3}$$

for large $t > 0$. Furthermore there exist a number A and a function $V \in \mathbf{L}^{1,1+a} \cap \mathbf{L}^\infty$ such that the asymptotic formula

$$u(t,x) = At^{-2+\sigma} V(xt^{-\frac{1}{2}}) + O(t^{-2+\sigma-\gamma}) \tag{1.4}$$

is valid for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^{+}$, where $\gamma = \frac{1}{2}\min(a, 1-\sigma t)$, and $V(\xi t)$ is the solution of the integral equation

$$V(\xi) = \widetilde{G}(\xi) - \frac{1}{\beta}\int_0^1 \frac{dz}{z^{\frac{3}{2}}(1-z)^{\frac{1}{2}}} \times \int_0^{+\infty} G_1\Big(\frac{(\cdot)}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}}y}{(1-z)^{\frac{1}{2}}}\Big) F(y)\, dy, \tag{1.5}$$

with

$$\beta = \frac{\sigma}{1-\sigma}\int_{\mathbf{R}^{+}} yV^{1+\sigma}(y)\, dy,$$

and

$$F(y) = V^{1+\sigma}(y) - V(y)\int_{\mathbf{R}^{+}} \xi V^{1+\sigma}(\xi)\, d\xi.$$

Remark 1. The condition that the value σ should be close to 1, so that $1-\sigma \le C\varepsilon^{\sigma}$, is rather technical and is caused by the application of the contraction mapping principle for proving global existence of solutions.

Now we give the sketch of the proof. In Section 2 we construct the Green's function for the linearized initial- boundary-value problem corresponding to (1.1)

$$\begin{cases} \partial_t u + \mathbb{K}u = f(t,x),\ x \in \mathbf{R}^{+},\ t > 0. \\ u(0,x) = u_0(x),\ x \in \mathbf{R}^{+}. \end{cases}$$

Using the Laplace transformation we represent the solution of this problem in the form

$$u(t,x) = \mathcal{G}(t)\, u_0 + \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau,$$

where the Green operator $\mathcal{G}(t)$ is given by

$$\mathcal{G}(t)\, g = \theta_1(x)\mathcal{L}^{-1}\Big\{e^{-K(p)t}\Big(\widehat{g}(p) - \widehat{g}(-p)\frac{1+p}{1-p}\Big)\Big\}.$$

In Section 3 we prepare some preliminary estimates of the operator $\mathcal{G}(t)$ in the Lebesgue norms $\|\phi\|_{\mathbf{L}^q}$ and $\|\phi\|_{\mathbf{L}^{1,a}} = \|\langle\cdot\rangle^a \phi\|_{\mathbf{L}^1}$, where $a \ge 0$, $1 \le q \le \infty$. We estimate the Green operator in our basic norm

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0}\Big(\langle t\rangle \|\phi(t)\|_{\mathbf{L}^\infty} + \langle t\rangle^{-\frac{a}{2}} \|\phi(t)\|_{\mathbf{L}^{1,1+a}}\Big),$$

where $a \in (0,1)$. We prove that if the function $f(t,x)$ has a zero first moment $\int_0^{+\infty} xf(t,x)\,dx = 0$, then the following inequality

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{X}} \le C \left\| \langle t \rangle f \right\|_{\mathbf{X}}$$

is valid. Also we formulate a local existence theorem. The next section is devoted to the proof of Theorem 1. Let us give a short motivation of our method. The solution of the initial-boundary-value problem (1.1) has the following integral representation

$$u(t) = \mathcal{G}(t)\, u_0 + \int_0^t \mathcal{G}(t-\tau) f(u(\tau))\, d\tau, \tag{1.6}$$

where

$$f(u(\tau)) = |u|^{\sigma} u, \qquad \int_0^{+\infty} x\, |u|^{\sigma} u \neq 0.$$

Since the nonlinearity $f(u(\tau)) = |u|^{\sigma} u$ is subcritical we have for $t > 1$

$$\| f(u(\tau)) \|_X < C t^{-1+\gamma}.$$

Therefore we can not obtain time decay estimates of the solution in our basic norm, using the integral formula (1.6) directly. The zero condition of the first moment $\int_0^{+\infty} xf(u)\,dx = 0$ permits us to obtain a desirable estimate

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau)\, d\tau \right\|_{\mathbf{X}} \le C \left\| \langle t \rangle f \right\|_{\mathbf{X}}.$$

So we make a change of the dependent variable $u(t,x) = v(t,x)\, e^{-\varphi(t)}$. Then for the new function $v(t,x)$ we get the following equation

$$v_t + \mathbb{K} v = \lambda e^{-\sigma\varphi} |v|^{\sigma} v + \varphi' v.$$

We choose $\varphi(t)$ such that the zero condition of the first moment is valid:

$$\int_{\mathbf{0}}^{+\infty} x \left(\lambda e^{-\sigma\varphi} |v|\, v + \varphi' v \right) dx = 0.$$

Thus we consider the initial -boundary-value problem for the new dependent variables $(v(t,x), \varphi(t))$

$$\begin{cases} \partial_t v + \mathbb{K} v = \lambda e^{-\sigma\varphi} \Big(|v|^{\sigma} - \dfrac{1}{\theta} \displaystyle\int_{\mathbf{R}^+} x\, |v|^{\sigma}\, v dx \Big) v, \\ \partial_t \varphi(t) = -\dfrac{\lambda}{\theta} e^{-\sigma\varphi} \displaystyle\int_{\mathbf{R}^+} x\, |v|^{\sigma}\, v dx, \\ v(0,x) = u_0(x), \ \varphi(0) = 0. \end{cases} \tag{1.7}$$

By the construction of the new nonlinearity we have

$$\int_{\mathbf{R}^+} dx \Big(|v|^\sigma - \frac{1}{\theta} \int_{\mathbf{R}^+} x\, |v|^\sigma\, v dx \Big) v = 0.$$

Hence we can prove that problem (1.7) has a unique solution $v(t,x) \in \mathbf{X}$, $h(t) = e^{\sigma\varphi(t)} \in \mathbf{C}(0,\infty)$ satisfying equations

$$\begin{cases} v(t) = \mathcal{G}(t)\, u_0 + \int_0^t \mathcal{G}(t-\tau) f\left(v(\tau), h(\tau)\right) d\tau, \\ h(t) = 1 - \dfrac{\sigma\lambda}{\theta} \int_0^t d\tau \int_{\mathbf{R}^+} x\, |v|^\sigma\, v dx, \end{cases} \tag{1.8}$$

where

$$f\left(v(\tau), h(\tau)\right) = \lambda h^{-1} \Big(|v|^\sigma - \frac{1}{\theta} \int_{\mathbf{R}^+} x\, |v|^\sigma\, v dx \Big) v.$$

Note that the terms $\mathcal{G}(t)\, u_0$ and $\int_0^t \mathcal{G}(t-\tau) f\left(v(\tau), h(\tau)\right) d\tau$ have the same time decay rate. Therefore to obtain asymptotic representations of the solutions v, h we consider the following integral equation:

$$V(\xi) = V_0(\xi) - \frac{1}{\beta} \int_0^1 \frac{dz}{z^{\frac{3}{2}} (1-z)^{\frac{1}{2}}} \int_0^{+\infty} G_1\Big(\frac{\xi}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}} y}{(1-z)^{\frac{1}{2}}} \Big) F(y)\, dy, \tag{1.9}$$

where $V_0(\xi) = \widetilde{G}(\xi)$,

$$\begin{aligned} F(y) &= V^{1+\sigma}(y) - V(y) \int_{\mathbf{R}^+} \xi V^{\sigma+1}(\xi)\, d\xi, \\ \beta &= \frac{\sigma}{1-\sigma} \int_{\mathbf{R}^+} \xi V^{\sigma+1}(\xi)\, d\xi, \end{aligned}$$

and

$$G_1(s,q) = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} e^{p^2} (e^{p(s-q)} - e^{p(s+q)}) dp,$$

which gives the self-similar asymptotic form of the solutions of (1.8). Equation (1.9) is obtained from (1.8) by substituting the main term of the asymptotics for the Green function and changing to the self-similar variables.

2. Green operator

Consider the linear initial- boundary-value problem on the half-line

$$\begin{cases} \partial_t u + \mathbb{K} u = f(t,x), \ x \in \mathbf{R}^+, \ t > 0, \\ u(0,x) = u_0(x), \ x \in \mathbf{R}^+, \end{cases} \tag{2.1}$$

where

$$\mathbb{K}u = \frac{1}{2\pi i}\theta_1(x)\int_{-i\infty}^{i\infty} e^{px}K(p)\widehat{u}(t,p)dp, \qquad K(p) = \frac{p^2}{p^2-1}.$$

Taking the Laplace transform with respect to the space variable we get

$$\begin{cases} \frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{1}{q-p}\big(\partial_t\widehat{u}(t,q)+K(q)\widehat{u}(t,q)-\widehat{f}(t,q)\big)dq = 0, \ x\in\mathbf{R}^+, t>0, \\ \widehat{u}\,(0,p)=\widehat{u}_0\,(p)\,, \ x\in\mathbf{R}^+ \end{cases} \tag{2.2}$$

where

$$\widehat{\cdot} = \int_0^{+\infty}\cdot dx.$$

We rewrite the equation (2.2) in the form

$$\partial_t\widehat{u}(t,p)+K(p)\widehat{u}(t,p)-\widehat{f}\,(t,p)=\frac{v(t)}{p-1}, \tag{2.3}$$

where $v(\tau)$ is some function, which we define below. Integrating equation (2.3) we obtain

$$\widehat{u}(t,p)=e^{-K(p)t}\widehat{u}_0\,(p)+\int_0^t e^{-K(p)(t-\tau)}\Big(f(\tau,p)+\frac{v(\tau)}{p-1}\Big)d\tau, \tag{2.4}$$

where $K(p)=\frac{p^2}{p^2-1}$. Note that the symbol $K(p)$ is not analytic in the right-half complex plane. Therefore we have for solution $u(t,x)$

$$u(t,x)=\frac{1}{2\pi i}\int_{1+\varepsilon-i\infty}^{1+\varepsilon+i\infty} e^{px}\widehat{u}(t,p)dp.$$

To satisfy the zero condition for $x\to+\infty$ we need to have

$$res(\widehat{u}(t,p),1)=0. \tag{2.5}$$

We have by definition

$$\begin{aligned} res(\widehat{u}(t,p),1) &= \lim_{\rho\to 0}\int_{C_{\rho,1}}\widehat{u}(t,p)dp \\ &= \lim_{\rho\to 0}\int_{C_{\rho,1}\cap D^+}\widehat{u}(t,p)dp+\lim_{\rho\to 0}\int_{C_{\rho,1}\cap D^-}\widehat{u}(t,p)dp, \end{aligned}$$

where $C_{\rho,1}=\left\{p\in\mathbb{C}: p=1+\rho e^{i\phi},\rho>0,\phi\in[0,2\pi)\right\}$, and by D^+,D^- we denote domains where $\mathrm{Re}K(p)>0$ and $\mathrm{Re}K(p)<0$.

Since for $p \in C_{\rho,1}$

$$K(p) = -\frac{(1+\rho e^{i\phi})^2}{\rho e^{i\phi}(2+\rho e^{i\phi})}$$

it is easy to see that

$$\lim_{\rho\to 0}\int_{C_{\rho,1,p\in D^+}} \widehat{u}(t,p)dp = 0.$$

We rewrite formula (2.4) in the domain where $\mathrm{Re}K(p) < 0$ in the form

$$\begin{aligned}\widehat{u}(t,p) &= e^{-K(p)t}\Big(\widehat{u}_0\,(p) + \int_0^{+\infty} e^{-K(p)(t-\tau)}\Big(f(\tau,p) + \frac{v(\tau)}{p-1}\Big)d\tau\Big) \\ &\quad - \int_t^{+\infty} e^{-K(p)(t-\tau)}\Big(f(\tau,p) + \frac{v(\tau)}{p-1}\Big)d\tau.\end{aligned}$$

We have

$$\lim_{\rho\to 0}\int_{C_{\rho,1,p\in D^-}} dp\int_t^{+\infty} e^{-K(p)(t-\tau)}\Big(f(\tau,p) + \frac{v(\tau)}{p-1}\Big)d\tau = 0.$$

To satisfy (2.5) we have to impose the following necessary condition for all $p \in D^-$,Re $p > 0$:

$$\widehat{u}_0\,(p) + \int_0^{+\infty} e^{K(p)\tau}\Big(f(\tau,p) + \frac{v(\tau)}{p-1}\Big)d\tau = 0. \tag{2.6}$$

Note that the last integral includes an unknown function $v(\tau)$. Also we have two roots $\phi_j(\xi)$ of equation $\xi = -K(p)$. By direct calculation we obtain

$$\phi_1 = \sqrt{\frac{\xi}{\xi+1}} \quad \text{and} \quad \phi_2 = -\sqrt{\frac{\xi}{\xi+1}}.$$

Since we are interested only in $\mathrm{Re}\xi > 0$ and $\mathrm{Re}p > 0$, making the change of variable $\xi = -K(p)$ we rewrite condition (2.6) as one equation with one unknown function $v(\tau)$

$$\widehat{u}_0\,(\phi_1) + \int_0^{+\infty} e^{-\xi\tau}\Big(\widehat{f}\,(\tau,\phi_1) + \frac{v(\tau)}{\phi_1(\xi)-1}\Big)d\tau = 0. \tag{2.7}$$

Solving the equation (2.7) we get

$$\mathcal{L}(v(t)) = (\phi_1 - 1)\big(\widehat{u}_0\,(\phi_1) + \widehat{\widehat{f}}\,(\xi,\phi_1)\,\big)$$

so

$$v(t) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{\xi t}(\phi_1 - 1)\big(\widehat{u}_0\,(\phi_1) + \widehat{\widehat{f}}\,(\xi,\phi_1)\,\big)d\xi.$$

Substituting this representation into (2.4) we obtain

$$\widehat{u}(t,p) = I_1(t,p) + I_2(t,p), \tag{2.8}$$

where

$$\begin{aligned} & I_1(t,p) \\ &= e^{-K(p)t}\widehat{u}_0\left(p\right) + \frac{1}{2\pi i}\int_0^t d\tau e^{-K(p)(t-\tau)}\frac{1}{p-1}\int_{-i\infty}^{i\infty} e^{\xi\tau}(\phi_1 - 1)\widehat{u}_0\left(\phi_1\right)d\xi \\ &= J_1 + J_2 \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} I_2(t,p) &= \int_0^t e^{-K(p)(t-\tau)}f(\tau,p)d\tau \\ &+\frac{1}{2\pi i}\int_0^t d\tau e^{-K(p)(t-\tau)}\frac{1}{p-1}\int_{-i\infty}^{i\infty} e^{\xi\tau}(\phi_1 - 1)\widehat{\widehat{f}}\left(\xi,\phi_1\right)d\xi. \end{aligned} \tag{2.10}$$

Now we consider I_1 in the representation (2.8).

Changing the order of integration we get

$$\begin{aligned} J_2 &= \frac{1}{2\pi i}\int_0^t d\tau e^{-K(p)(t-\tau)}\frac{1}{p-1}\int_{-i\infty}^{i\infty} e^{\xi\tau}(\phi_1 - 1)\widehat{u}_0\left(\phi_1\right)d\xi \\ &= e^{-K(p)t}\frac{1}{p-1}\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}(\phi_1 - 1)\widehat{u}_0\left(\phi_1\right)\frac{e^{(\xi+K(p))t}-1}{K(p)+\xi}d\xi. \end{aligned}$$

Since ϕ_1 is analytic in the right -half complex plane and $\mathrm{Re}\phi_1 > 0$,$\mathrm{Re}K(p) > 0$ for all Re $p = 0$,Re $\xi > 0$, via Cauchy's theorem we obtain

$$\int_{-i\infty}^{i\infty}\widehat{u}_0\left(\phi_1\right)\frac{\phi_1(\xi)-1}{K(p)+\xi}d\xi = 0.$$

Therefore we get

$$J_2 = \frac{1}{2\pi i}\frac{1}{p-1}\int_{-i\infty}^{i\infty}\widehat{u}_0\left(\phi_1\right)\frac{e^{\xi t}(\phi_1(\xi)-1)}{K(p)+\xi}d\xi.$$

Taking the inverse Laplace transform with respect to the space variable we obtain

$$\begin{aligned} L_x^{-1}(J_2) &= \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} dpe^{px}\frac{1}{p-1}\int_{-i\infty}^{i\infty}\widehat{u}_0\left(\phi_1\right)\left(\phi_1(\xi)-1\right)\frac{e^{\xi t}}{K(p)+\xi}d\xi \\ &= \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\widehat{u}_0\left(\phi_1\right)\left(\phi_1(\xi)-1\right)\int_{-i\infty}^{i\infty} dpe^{px}\frac{1}{(K(p)+\xi)(p-1)} \end{aligned}$$

$$= \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty} d\xi e^{\xi t}\widehat{u}_0\left(\phi_1\right)\left(\phi_1(\xi)-1\right)\int_{-i\infty}^{i\infty} dpe^{px}\frac{1+p}{p^2+\xi(p^2-1)}.$$

Since for $x>0$ due to Cauchy's theorem

$$\begin{aligned}\int_{-i\infty}^{i\infty} dpe^{px}\frac{p+1}{(1+\xi)p^2-\xi} &= \frac{1}{1+\xi}\int_{-i\infty}^{i\infty} dpe^{px}\frac{p+1}{(p-\phi_1)(p-\phi_2)}\\ &= 2\pi i e^{\phi_2 x}\frac{1}{1+\xi}\frac{\phi_2+1}{\phi_2-\phi_1}\end{aligned}$$

we get using $\phi_1+\phi_2=0$

$$L_x^{-1}(J_2) = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t+\phi_2 x}\frac{\phi_2+1}{2(\xi+1)\phi_2}\widehat{u}_0\left(-\phi_2\right)\left(1+\phi_2(\xi)\right).$$

We have

$$\phi_2' = \frac{1}{2\phi_2}\left(\frac{\xi}{\xi+1}\right)' = \frac{1}{2\phi_2}\frac{1}{(\xi+1)^2}.$$

Making the change of variables $p=\phi_2$ we obtain

$$\begin{aligned}\mathcal{L}_x^{-1}(J_2) &= -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty} d\xi e^{\xi t+\phi_2 x}\phi_2'\widehat{u}_0\left(-\phi_2\right)\left(\phi_2+1\right)^2(\xi+1)\\ &= \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} dpe^{-K(p)t+px}\widehat{u}_0\left(-p\right)(p+1)^2(-K(p)+1)\\ &= -\mathcal{L}^{-1}\Big\{e^{-K(p)t}\widehat{u}_0\left(-p\right)\frac{1+p}{1-p}\Big\}.\end{aligned}$$

Substituting this representation into (2.9) we get

$$\mathcal{L}_x^{-1}(I_1) = \mathcal{G}(t)u_0 \tag{2.11}$$

where the Green operator $\mathcal{G}\left(t\right)$ is given by

$$\mathcal{G}\left(t\right)g = \theta_1(x)\mathcal{L}^{-1}\Big\{e^{-K(p)t}\Big(\widehat{g}(p)-\widehat{g}(-p)\frac{1+p}{1-p}\Big)\Big\}. \tag{2.12}$$

Now we consider I_2 in the formula (2.10). In the same way we get

$$\mathcal{L}_x^{-1}(I_2) = \int_0^t d\tau\mathcal{G}(t-\tau)f(\tau). \tag{2.13}$$

From (2.8), (2.11), and (2.13) we obtain an integral formula for the solution of (2.1):

$$u(t,x) = \mathcal{G}\left(t\right)u_0 + \int_0^t \mathcal{G}(t-\tau)f\left(\tau\right)d\tau.$$

We first collect some preliminary estimates of the Green operator $\mathcal{G}(t)$ in the norms $\|\phi\|_{\mathbf{L}^p}$ and $\|\phi\|_{\mathbf{L}^{1,1+w}}$, where $w \in (0,1)$, $1 \le r \le \infty$.

3. Preliminary lemmas

We introduce an operator $\mathcal{G}_0(t)$ given by

$$\mathcal{G}_0(t)\phi = \int_0^{+\infty} G_1(t,x,y)\phi(y)\,dy,$$

where the kernel

$$G_1(t,x,y) = (4\pi t)^{-\frac{1}{2}}\left(e^{-\frac{(x-y)^2}{4t}} - e^{-\frac{(x+y)^2}{4t}}\right).$$

Denote

$$G_0(t,x) = \partial_y G_1(t,x,y)\big|_{y=0} = (4\pi t)^{-\frac{1}{2}}\frac{x}{t}e^{-\frac{x^2}{4t}}.$$

Lemma 1. *Let $\phi \in \mathbf{L}^r$;*

$$\|\mathcal{G}_0(t)\phi\|_{\mathbf{L}^q} \le C\langle t\rangle^{\frac{1}{2}(\frac{1}{q}-\frac{1}{r})}\|\phi\|_{\mathbf{L}^r}$$

is true for all $t > 0$, $1 \le q \le \infty$, $1 \le r \le \infty$. Furthermore we assume that $\phi \in \mathbf{L}^{1,1+a}$; then the estimate

$$\left\|(\cdot)^b\left(\mathcal{G}_0(t)\phi - \vartheta G_0(t)\right)\right\|_{\mathbf{L}^q} \le Ct^{-1+\frac{1}{2q}+\frac{b-a}{2}}\|\phi\|_{\mathbf{L}^{1,1+a}}$$

is valid for all $t > 0$, where $1 \le q \le \infty$, $b \in [0,1+a]$ and

$$\vartheta = \int_0^{+\infty} x\phi(x)\,dx.$$

Proof. Since $|G_1(t,x,y)| \le Ct^{-\frac{1}{2}}e^{-\frac{C}{t}|x-y|^2}$ for all $x,y \in \mathbf{R}^+$, by the Young inequality we have for $p = \frac{qr+r-q}{qr}$

$$\begin{aligned}\|\mathcal{G}_0(t)\phi\|_{\mathbf{L}^q} &\le Ct^{-\frac{1}{2}}\left\|\int_0^{+\infty} e^{-\frac{C}{t}|x-y|^2}\phi(y)\,dy\right\|_{\mathbf{L}^q} \\ &\le Ct^{-\frac{1}{2}}\left\|e^{-\frac{C}{t}|x|^2}\right\|_{\mathbf{L}^p}\|\phi\|_{\mathbf{L}^r} \le C\langle t\rangle^{\frac{1}{2}\left(\frac{1}{q}-\frac{1}{r}\right)}\|\phi\|_{\mathbf{L}^r}\end{aligned}$$

for all $t > 0$, where $1 \le q \le \infty$. Hence the first estimate of the lemma follows. For the second estimate we write

$$x^b\left(\mathcal{G}_0(t)\phi - \vartheta G_0(t,x)\right) = \int_0^{+\infty} x^b\left(G_1(t,x,y) - G_0(t,x)\,y\right)\phi(y)\,dy$$

for any $b \in [0, 1+a]$. Applying a Taylor expansion, we obtain

$$|G_1(t,x,y) - G_0(t,x)y| \le Ct^{-1-\frac{a}{2}} y^{1+a} \left(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2} \right)$$

for all $x, y \in \mathbf{R}^+$. Hence in the domain $y \le \frac{x}{2}$

$$x^b |G_1(t,x,y) - G_0(t,x)y| \le Ct^{-1-\frac{a}{2}} y^{a+1} x^b e^{-\frac{C}{t}|x|^2} \le Ct^{-1+\frac{b-a}{2}} y^{a+1} e^{-\frac{C}{t}|x|^2}.$$

By the Lagrange finite differences theorem we have

$$|G_1(t,x,y)| \le Ct^{-\frac{1+\nu}{2}} y^{\nu} e^{-\frac{C}{t}|x-y|^2}$$

for all $x, y \in \mathbf{R}^+$, where $\nu \in [0,1]$. Taking $\nu = 1 + a - b$, in the case $b \in [1, a+1]$ we get for $y \ge \frac{x}{2}$

$$\begin{aligned} &x^b |G_1(t,x,y) - G_0(t,x)y| \le x^b (|G_1(t,x,y)| + |G_0(t,x)y|) \\ &\le Ct^{-1+\frac{b-a}{2}} x^b y^{a+1-b} e^{-\frac{C}{t}|x-y|^2} + Ct^{-\frac{3}{2}} x^{b+1} y e^{-\frac{C}{t}|x|^2} \\ &\le Ct^{-1+\frac{b-a}{2}} y^{a+1} \left(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2} \right), \end{aligned}$$

and in the case $b \in [0,1]$ we write

$$\begin{aligned} &x^b |G_1(t,x,y) - G_0(t,x)y| \\ &\le x^b (|G_1(t,x,y)| + |G_0(t,x)y|)^b |G_1(t,x,y) - G_0(t,x)y|^{1-b} \\ &\le Ct^{-b} |y|^{(1+a)b} t^{-\left(1+\frac{a}{2}\right)(1-b)} |y|^{(a+1)(1-b)} \times \left(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2} \right) \\ &\le Ct^{-1+\frac{b-a}{2}} y^{1+a} \left(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2} \right), \end{aligned}$$

for all $x, y \in \mathbf{R}^+$, $y \ge \frac{x}{2}$. Thus we obtain the estimate

$$x^b |G_1(t,x,y) - G_0(t,x)y| \le Ct^{-1+\frac{b-a}{2}} |y|^{a+1} \left(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2} \right)$$

for all $x, y \in \mathbf{R}^+$, and for any $b \in [0, 1+a]$. Applying the above estimate with Young's inequality we find

$$\begin{aligned} &\left\| (\cdot)^b (\mathcal{G}_0(t)\phi - \vartheta G_0(t)) \right\|_{\mathbf{L}^p} = \left\| \int_0^{+\infty} x^b (G_1(t,x,y) - G_0(t,x)y)\phi(y)dy \right\|_{\mathbf{L}^q_x} \\ &\le Ct^{-1+\frac{b-a}{2}} \left\| \int_0^{+\infty} \left(e^{-\frac{C}{t}|x-y|^2} + e^{-\frac{C}{t}|x|^2} \right) y^{1+a} |\phi(y)| \, dy \right\|_{\mathbf{L}^q_x} \\ &\le Ct^{-1+\frac{1}{2q}+\frac{b-a}{2}} \|\phi\|_{\mathbf{L}^{1,1+a}}. \end{aligned}$$

Thus the second estimate of the lemma follows. The lemma is proved. □

Denote by $\mathcal{G}(t)$

$$\mathcal{G}(t)\phi = e^{-t}\mathcal{L}_x^{-1}\Big\{e^{\frac{t}{1-p^2}}\Big(\hat{\phi}(p)-\hat{\phi}(-p)\frac{p+1}{p-1}\Big)\Big\}.$$

Lemma 2. *Suppose that the function $\phi \in \mathbf{L}^{\infty}(\mathbf{R}^{+}) \cap \mathbf{L}^{1,1+a}(\mathbf{R}^{+})$, where $a \in (0,1)$. Then the estimates*

$$\|\mathcal{G}(t)\phi\|_{\mathbf{L}^r} \leq e^{-t}\|\phi\|_{\mathbf{L}^r} + C\langle t\rangle^{-\frac{1}{2}\left(\frac{1}{r_1}-\frac{1}{r}\right)}\|\phi\|_{\mathbf{L}^{r_1}},$$

$$\|\mathcal{G}(t)\phi - \vartheta G_0(t)\|_{\mathbf{L}^{\infty}} \leq Ct^{-1-\frac{a}{2}}\|\phi\|_{\mathbf{L}^{1,1+a}} + e^{-t}\|\phi\|_{\mathbf{L}^{\infty}},$$

and

$$\left\|(\cdot)^b(\mathcal{G}(t)\phi - \vartheta G_0)\right\|_{\mathbf{L}^1} \leq Ct^{-\frac{1}{2}+\frac{b-a}{2}}\|\phi\|_{\mathbf{L}^{1,1+a}} + e^{-t}\left\|(\cdot)^b\phi\right\|_{\mathbf{L}^1}$$

are valid for all $t>0$, where $1 \leq r \leq \infty$,$0 < b \leq a$.

Proof. Note that the Green operator $\mathcal{G}(t)$ can be represented as

$$\mathcal{G}(t)\phi = \mathcal{G}_0(t)\phi + e^{-t}\phi(x) + \mathcal{R}(t)\phi + \mathcal{R}_1(t)\phi, \tag{3.1}$$

where the remainder

$$\mathcal{R}(t)\phi = \int_0^{+\infty}(R(t,x-y) - R(t,x+y))\phi(y)\,dy$$

with a kernel

$$R(t,x) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{px}\widehat{R}(t,p)\,dp,$$

where $\widehat{R}(t,p) = e^{\frac{tp^2}{1-p^2}} - e^{tp^2} - e^{-t}$, and

$$\mathcal{R}_1(t)\phi = e^{-t}\mathcal{L}_x^{-1}\Big\{e^{\frac{t}{1-p^2}}\Big(\hat{\phi}(-p)\frac{2p}{p-1}\Big)\Big\}.$$

From Lemma 1 the operator $\mathcal{G}_0(t)$ satisfies the estimates of the lemma .

Now we estimate the remainder $\mathcal{R}(t)$. We represent

$$\widehat{R}(t,p) = e^{\frac{tp^2}{1-p^2}}\left(1 - e^{-t\frac{p^4}{1-p^2}}\right) - e^{-t}$$

for all $|p| \leq 1$, and

$$\widehat{R}(t,p) = -e^{tp^2} + e^{-t}\left(e^{\frac{\alpha t}{1-p^2}} - 1\right)$$

for all $|p| \geq 1$, then we see that

$$\left|\partial_p^j\widehat{R}(t,p)\right| \leq C\langle t\rangle^{\frac{j}{2}-1}e^{\frac{1}{2}tp^2} + C\langle t\rangle^2 e^{-t}\left(1-p^2\right)^{-3}$$

for all Re $p=0$, $t>0$, $0\leq j\leq 4$. Therefore, we have

$$|R(t,x)|\leq C\langle x\langle t\rangle^{-\frac{1}{2}}\rangle^{-4}\langle t\rangle^{-\frac{1}{2}-1}+C\langle x\rangle^{-4}\langle t\rangle^{2}e^{-t}\leq C\langle x\langle t\rangle^{-\frac{1}{2}}\rangle^{-4}\langle t\rangle^{-\frac{1}{2}-1}$$

for all $x\in\mathbf{R}$, $t>0$. Applying this estimate by the Young inequality we find

$$\left\|\mathcal{R}\left(t\right)\phi\right\|_{\mathbf{L}^{r}}\leq C\left\langle t\right\rangle^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}\left\|\phi\right\|_{\mathbf{L}^{q}}$$

for all $1\leq q\leq r\leq\infty$ and

$$\left\|\mathcal{R}\left(t\right)\phi\right\|_{\mathbf{L}^{1,w}}\leq C\left\langle t\right\rangle^{-1}\left(\left\langle t\right\rangle^{\frac{w}{2}}\left\|\phi\right\|_{\mathbf{L}^{1}}+\left\|\phi\right\|_{\mathbf{L}^{1,w}}\right)$$

for all $t>0$. By analogy to the proof of Lemma 1 we can prove that

$$\left\|\mathcal{R}_{1}\left(t\right)\phi\right\|_{\mathbf{L}^{r}}\leq C\left\langle t\right\rangle^{-\frac{1}{2}(\frac{1}{q}-\frac{1}{r})-1}\left\|\phi\right\|_{\mathbf{L}^{q}}$$

for all $1\leq q\leq r\leq\infty$ and

$$\left\|\mathcal{R}_{1}\left(t\right)\phi\right\|_{\mathbf{L}^{1,w}}\leq C\left\langle t\right\rangle^{-1}\left(\left\langle t\right\rangle^{\frac{w}{2}}\left\|\phi\right\|_{\mathbf{L}^{1}}+\left\|\phi\right\|_{\mathbf{L}^{1,w}}\right).$$

Now by representation (3.1) the estimates of the lemma follow. Lemma 2 is proved. □

In the next lemma we estimate the Green operator in our basic norm

$$\left\|\phi\right\|_{\mathbf{X}}=\sup_{t>0}\left(\left\langle t\right\rangle\left\|\phi\left(t\right)\right\|_{\mathbf{L}^{\infty}}+\left\langle t\right\rangle^{-\frac{a}{2}}\left\|\phi\left(t\right)\right\|_{\mathbf{L}^{1,1+a}}\right),$$

where $a\in(0,1)$. Note that the $\mathbf{L}^{1}$ norm is estimated by the norm $\mathbf{X}$

$$\begin{aligned}\left\langle t\right\rangle^{\frac{1}{2}}\left\|\phi\left(t\right)\right\|_{\mathbf{L}^{1}} &= \int_{0}^{\langle t\rangle}\left|\phi\left(t,x\right)\right|dx+\int_{\langle t\rangle}^{+\infty}\left|1+x\right|^{-1-\alpha}\left|x\right|^{1+\alpha}\left|\phi\left(t,x\right)\right|dx\\ &\leq C\left\langle t\right\rangle\left\|\phi\left(t\right)\right\|_{\mathbf{L}^{\infty}}+C\left\langle t\right\rangle^{-\frac{a}{2}}\left\|\phi\left(t\right)\right\|_{\mathbf{L}^{1,1+a}}\leq C\left\|\phi\right\|_{\mathbf{X}}.\end{aligned}$$

Lemma 3. *Let the function $f\left(t,x\right)$ have a zero first moment*

$$\int_{0}^{+\infty}xf\left(t,x\right)dx=0.$$

Then the following inequality

$$\left\|\int_{0}^{t}\mathcal{G}\left(t-\tau\right)f\left(\tau\right)d\tau\right\|_{\mathbf{X}}\leq C\left\|\left\langle t\right\rangle f\right\|_{\mathbf{X}}$$

is valid, provided that the right-hand side is finite.

Proof. In view of Lemma 2 we get

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^\infty} + \left\| \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^{1,1+a}} \leq C \| \langle t \rangle f \|_{\mathbf{X}} \leq C \| \langle t \rangle f \|_{\mathbf{X}}$$

for all $0 \leq t \leq 1$. We now consider $t > 1$.

Therefore, by virtue of Lemma 2 we obtain

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq C \int_0^{\frac{t}{2}} (t-\tau)^{-1-\frac{a}{2}} \left(\| f(\tau) \|_{\mathbf{L}^\infty} + \| f(\tau) \|_{\mathbf{L}^{1,1+a}} \right) d\tau + C \int_{\frac{t}{2}}^t \| f(\tau) \|_{\mathbf{L}^\infty} d\tau,$$

hence using the definition of the norm $\mathbf{X}$ we get

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^\infty}$$
$$\leq C \| \langle t \rangle f \|_{\mathbf{X}} \int_0^{\frac{t}{2}} (t-\tau)^{-1-\frac{a}{2}} \langle \tau \rangle^{\frac{a}{2}-1} d\tau + C \| \langle t \rangle f \|_{\mathbf{X}} \int_{\frac{t}{2}}^t \langle \tau \rangle^{-2} d\tau$$
$$\leq C t^{-1} \| \langle t \rangle f \|_{\mathbf{X}} \leq C t^{-1} \| \langle t \rangle f \|_{\mathbf{X}}$$

and similarly

$$\left\| \int_0^t \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^{1,1+a}} \leq C \int_0^t \| f(\tau) \|_{\mathbf{L}^{1,1+a}} d\tau$$
$$\leq \quad C \| \langle t \rangle f \|_{\mathbf{X}} \int_0^t \tau^{\frac{a}{2}-1} d\tau \leq C t^{\frac{a}{2}} \| \langle t \rangle f \|_{\mathbf{X}}$$

for all $t > 4$. Hence the result of the lemma follows. Lemma 3 is proved. □

By standard methods we can prove the local existence of weak solutions to the initial- boundary-value problem (1.1) (see, for example, [8]).

Proposition 1. *Let* $u_0 \in \mathbf{L}^{1,1+a}(\mathbf{R}^+) \cap \mathbf{L}^\infty(\mathbf{R}^+)$, $a \geq 0$. *Then for some* $T > 0$ *there exists a unique solution* $u \in \mathbf{C}\left([0,T]; \mathbf{L}^{1,1+a}(\mathbf{R}^+) \cap \mathbf{L}^\infty(\mathbf{R}^+)\right)$ *to the problem* (1.1).

4. Proof of Theorem 1

As in paper [8] we make a change of the dependent variable $u(t,x) = v(t,x) e^{-\varphi(t)}$. Then for the new function $v(t,x)$ we get the following equation

$$v_t + \mathbb{K} v - \lambda e^{-\sigma\varphi} |v|^\sigma v - \varphi' v = 0.$$

We assume that $\varphi(t)$ is such that $\varphi(0) = 1$ and

$$\int_{\mathbf{0}}^{+\infty} x\left(\lambda e^{-\sigma\varphi}|v|\,v + \varphi' v\right)dx = 0.$$

Since by construction

$$\int_0^{+\infty} x\mathbb{K}v dx = \partial_p(K(p)\widehat{v}(t,p))\,|_{p=0} = 0$$

the first moment of the new function $v(t,x)$ satisfies a conservation law:

$$\frac{d}{dt}\int_{\mathbf{0}}^{+\infty} xv(t,x)\,dx = 0,$$

hence,

$$\int_{\mathbf{0}}^{+\infty} xv(t,x)\,dx = \int_{\mathbf{0}}^{+\infty} xu_0(x)\,dx \quad \text{for all } t > 0.$$

Thus, we consider the initial -boundary-value problem for the new dependent variables $(v(t,x), \varphi(t))$

$$\begin{cases} \partial_t v + \mathbb{K}v = \lambda e^{-\sigma\varphi}\Big(|v|^\sigma - \dfrac{1}{\theta}\displaystyle\int_{\mathbf{R}^+} x|v|^\sigma v dx\Big)v, \\ \partial_t\varphi(t) = -\dfrac{\lambda}{\theta}e^{-\sigma\varphi}\displaystyle\int_{\mathbf{R}^+} x|v|^\sigma v dx, \\ v(0,x) = u_0(x),\ \ \varphi(0) = 0. \end{cases} \tag{4.1}$$

We denote $h(t) = e^{\sigma\varphi(t)}$ and write (4.1) as

$$\begin{cases} \partial_t v + \mathbb{K}v = f(v,h),\ \ v(0,x) = u_0(x), \\ \partial_t h = -\dfrac{\sigma\lambda}{\theta}\displaystyle\int_{\mathbf{R}^+} |v|^\sigma v dx,\ \ h(0) = 1, \end{cases} \tag{4.2}$$

where

$$f(v,h) = \lambda h^{-1}\Big(|v|^\sigma - \frac{1}{\theta}\int_{\mathbf{R}^+} x|v|^\sigma v dx\Big)v.$$

We note that the first moment of the nonlinearity $\int_{\mathbf{R}^+} f(v,h)(t,x)\,dx = 0$ for all $t > 0$. We now prove the existence of the solution $(v(t,x), h(t))$ for the initial-boundary-value problem (4.2) by the successive approximations $(v_m(t,x), h_m(t))$, $m = 1, 2, ...$, defined as follows

$$\begin{cases} \partial_t v_m + \mathbb{K}v_m = f(v_{m-1}, h_{m-1}), \\ \partial_t h_m = -\dfrac{\sigma\lambda}{\theta}\displaystyle\int_{\mathbf{R}^+} x|v_{m-1}|^\sigma v_{m-1} dx, \\ v_m(0,x) = u_0(x),\ \ h_m(0) = 1, \end{cases} \tag{4.3}$$

for all $m \geq 2$, where $v_1 = \mathcal{G}(t) u_0$,

$$h_1(t) = 1 + \frac{|\theta|^{\sigma} \eta}{1-\sigma} t^{1-\sigma} \ ,$$

and $\eta = \sigma\lambda (4\pi)^{-\sigma} (1+\sigma)^{-1}$. We now prove by induction the following estimates

$$\begin{array}{rcl} \|v_m\|_{\mathbf{X}} & \leq & C\varepsilon, \ \ \|v_m(t) - \mathcal{G}(t) u_0\|_{\mathbf{L}^{1,1}} \leq C\varepsilon^{1+\sigma}, \\ & & |h_m(t) - h_1(t)| \leq C\varepsilon^{\sigma} h_1(t) \end{array} \tag{4.4}$$

for all $m \geq 1$, where the norm $\|\cdot\|_{\mathbf{X}}$ is defined as above by

$$\|\phi\|_{\mathbf{X}} = \sup_{t>0} \left(\langle t \rangle \|\phi(t)\|_{\mathbf{L}^{\infty}} + \langle t \rangle^{-\frac{a}{2}} \|\phi(t)\|_{\mathbf{L}^{1,1+a}} \right).$$

By virtue of Lemma 2 we have

$$\begin{array}{l} \|\mathcal{G}(t) u_0\|_{\mathbf{L}^{\infty}} \leq C\varepsilon \langle t \rangle^{-1}, \ \ \|\mathcal{G}(t) u_0\|_{\mathbf{L}^{1,1}} \leq C\varepsilon, \\ \left\| |\cdot|^a \left(\mathcal{G}(t) u_0 - \theta t^{-1} \widetilde{G} \left(t^{-\frac{1}{2}} (\cdot) \right) \right) \right\|_{\mathbf{L}^{1,1}} \leq C\varepsilon \end{array}$$

and

$$\left\| t^{-1} (\cdot)^a \widetilde{G} \left(t^{-\frac{1}{2}} (\cdot) \right) \right\|_{\mathbf{L}^{1,1}} \leq C t^{\frac{a}{2}}.$$

Therefore estimates (4.4) are valid for $m = 1$. We assume that estimates (4.4) are true with m replaced by $m-1$. The integral equation associated with (4.3) is written as

$$\begin{cases} v_m(t) = \mathcal{G}(t) u_0 + \int_0^t \mathcal{G}(t-\tau) f(v_{m-1}(\tau), h_{m-1}(\tau)) \, d\tau, \\ h_m(t) = 1 - \dfrac{\sigma\lambda}{\theta} \int_0^t d\tau \int_{\mathbf{R}^+} |v_{m-1}|^{\sigma} v_{m-1} dx. \end{cases}$$

Since $(v_{m-1}(t), h_{m-1}(t))$ satisfies (4.4), we obtain

$$\begin{aligned} &\|(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^{\infty}} \leq C h_{m-1}^{-1}(t) \|v_{m-1}(t)\|_{\mathbf{L}^{\infty}}^{1+\sigma} \big(1 + \frac{1}{|\theta|} \|v_{m-1}(t)\|_{\mathbf{L}^{1,1}}\big) \\ &\leq C\varepsilon^{1+\sigma} \langle t \rangle^{-1-\sigma} \left(1 + \frac{\sigma |\theta|^{\sigma} \eta}{1-\sigma} t^{1-\sigma}\right)^{-1} \leq C\varepsilon (1-\sigma) \langle t \rangle^{-2}, \end{aligned} \tag{4.5}$$

and

$$\begin{array}{rl} & \|f(v_{m-1}(t), h_{m-1}(t))\|_{\mathbf{L}^{1,1+a}} \\ \leq & C h_{m-1}^{-1}(t) \|v_{m-1}(t)\|_{\mathbf{L}^{\infty}}^{\sigma} \|v_{m-1}(t)\|_{\mathbf{L}^{1,1+a}} \left(1 + \dfrac{1}{|\theta|} \|v_{m-1}(t)\|_{\mathbf{L}^{1,1}}\right) \\ \leq & C\varepsilon (1-\sigma) \langle t \rangle^{-1+\frac{\alpha}{2}} \end{array}$$

for all $t > 0$. This yields the estimate

$$\left\| \langle t \rangle f\left(v_{m-1}(t), h_{m-1}(t)\right) \right\|_{\mathbf{X}} \leq C\varepsilon^{1+\sigma}$$

if we suppose that $1 - \sigma \leq C\varepsilon^{\sigma}$. Since $f\left(v_{m-1}(\tau), h_{m-1}(\tau)\right)$ have a zero first moment we get via Lemma 3

$$\left\| \int_0^t \mathcal{G}(t-\tau) f\left(v_{m-1}(\tau), h_{m-1}(\tau)\right) d\tau \right\|_{\mathbf{X}} \leq C\varepsilon^{1+\sigma}$$

hence it follows that

$$\|v_m\|_{\mathbf{X}} \leq C\varepsilon, \left\| v_m(t) - \mathcal{G}(t) u_0 \right\|_{\mathbf{L}^{1,1}} \leq C\varepsilon^{1+\sigma}. \tag{4.6}$$

To prove the third estimate in (4.4) we prepare the following lemma, where we evaluate the large-time behavior of the first moment of the nonlinearity in equation (1.1) in the subcritical case.

Lemma 4. *Assume that* $u_0 \in \mathbf{L}^{\infty}(\mathbf{R}^+) \cap \mathbf{L}^{1,1+a}(\mathbf{R}^+)$, *the norm* $\|u_0\|_{\mathbf{L}^{\infty}} + \|u_0\|_{\mathbf{L}^{1,1+a}} = \varepsilon$ *is sufficiently small and*

$$\theta\lambda \leq -C\varepsilon < 0,$$

where $\theta = \int_{\mathbf{R}^+} x u_0(x)\, dx$. *Let a function* $v(t,x)$ *satisfy the estimates*

$$\|v\|_{\mathbf{L}^{\infty}} \leq C\varepsilon \langle t \rangle^{-1}, \quad \|v\|_{\mathbf{L}^{1,1}} \leq C\varepsilon \text{ and}$$
$$\left\| v(t) - \mathcal{G}(t) u_0 \right\|_{\mathbf{L}^{1,1}} \leq C\varepsilon^{1+\sigma}$$

for all $t > 0$. *Then the inequality*

$$\left| 1 - \frac{\sigma\lambda}{\theta} \int_0^t d\tau \int_{\mathbf{R}^+} x|v|^{\sigma} v(\tau,x) dx - h_1(t) \right| \leq C\varepsilon^{\sigma} h_1(t) + C\varepsilon^{\sigma} t^{1-\sigma-\frac{a}{2}} \tag{4.7}$$

is valid for all $t > 0$.

Proof. By the last estimate of Lemma 2 we get

$$\left\| \mathcal{G}(t) u_0 - \theta G_0(t,x) \right\|_{\mathbf{L}^{1,1}} \leq C\varepsilon \langle t \rangle^{-\frac{a}{2}}$$

for all $t > 0$. Hence we find

$$\begin{aligned}
& \left\| |v|^{\sigma} v - |\theta|^{\sigma} \theta \left(G_0(t,x)\right)^{\sigma+1} \right\|_{\mathbf{L}^{1,1}} \\
\leq\ & C \left(\left\| v(t) - \mathcal{G}(t) u_0 \right\|_{\mathbf{L}^{1,1}} + \left\| \mathcal{G}(t) u_0 - \theta G_0(t,x) \right\|_{\mathbf{L}^{1,1}} \right) \\
& \times \left(\|v\|_{\mathbf{L}^{\infty}}^{\sigma} + \|\mathcal{G} u_0\|_{\mathbf{L}^{\infty}}^{\sigma} + |\theta|^{\sigma} \|G_0\|_{\mathbf{L}^{\infty}}^{\sigma} \right) \\
\leq\ & C\varepsilon^{1+\sigma} t^{-\sigma} \left(\varepsilon^{\sigma} + \langle t \rangle^{-\frac{a}{2}} \right)
\end{aligned}$$

for all $t > 0$. Since

$$t^{\sigma}\int_{\mathbf{R}^{+}} x\left(G_{0}\left(t,x\right)\right)^{\sigma+1}dx = \left(4\pi\right)^{-\sigma}\left(1+\sigma\right)^{-1} = \frac{\eta}{\sigma\lambda},$$

we get

$$\begin{aligned}&\Big|\int_{\mathbf{R}^{+}} x\left|v\right|^{\sigma}v\left(t,x\right)dx - \left|\theta\right|^{\sigma}\theta t^{-\sigma}\frac{\eta}{\sigma\lambda}\Big|\\ &\leq C\left\|\left|v\right|^{\sigma}v - \left|\theta\right|^{\sigma}\theta\left(G_{0}\left(t,x\right)\right)^{\sigma+1}\right\|_{\mathbf{L}^{1,1}} \leq C\varepsilon^{1+\sigma}t^{-\sigma}\left(\varepsilon^{\sigma}+\langle t\rangle^{-\frac{a}{2}}\right)\end{aligned}$$

for all $t > 0$, where $0 < \sigma < 1$. Therefore we have the inequality

$$\begin{aligned}&\Big|1-\frac{\sigma\lambda}{\theta}\int_{0}^{t}d\tau\int_{\mathbf{R}^{+}} x\left|v\right|^{\sigma}v\left(\tau,x\right)dx - h_{1}\left(t\right)\Big|\\ =\ &\Big|\frac{\sigma\lambda}{\theta}\int_{0}^{t}d\tau\int_{\mathbf{R}^{+}} x\left|v\right|^{\sigma}v\left(\tau,x\right)dx - \left|\theta\right|^{\sigma}\eta\left(1-\sigma\right)^{-1}t^{1-\sigma}\Big|\\ \leq\ &\frac{C\varepsilon^{2\sigma}}{1-\sigma}t^{1-\sigma}+C\varepsilon^{\sigma}t^{1-\sigma-\frac{a}{2}} \leq \varepsilon^{\sigma}h_{1}\left(t\right)+C\varepsilon^{\sigma}t^{1-\sigma-\frac{a}{2}}\end{aligned}$$

for all $t > 0$, which implies (4.7). Lemma 4 is proved. □

Using (4.6) we apply Lemma 4 to find that

$$\left|h_{m}\left(t\right)-h_{1}\left(t\right)\right| \leq C\varepsilon^{\sigma}h_{1}\left(t\right)$$

for all $t > 0$. Thus by induction we see that estimates (4.4) are valid for all $m \geq 1$. In the same way by induction we can prove that

$$\begin{aligned}\left\|v_{m}-v_{m-1}\right\|_{\mathbf{X}} &\leq \frac{1}{4}\left\|v_{m-1}-v_{m-2}\right\|_{\mathbf{X}},\\ \sup_{t>0}h_{1}^{-1}\left(t\right)\left|h_{m}\left(t\right)-h_{m-1}\left(t\right)\right| &\leq \frac{1}{4}\left\|v_{m-1}-v_{m-2}\right\|_{\mathbf{X}}\\ &\quad+\frac{1}{4}\sup_{t>0}h_{1}^{-1}\left(t\right)\left|h_{m-1}\left(t\right)-h_{m-2}\left(t\right)\right|\end{aligned}$$

for all $m > 2$. Therefore taking limits

$$\lim_{m\to\infty}v_{m}\left(t,x\right)=v\left(t,x\right)\ \text{ and }\ \lim_{m\to\infty}h_{m}\left(t\right)=h\left(t\right)$$

and using Proposition 1 we obtain a unique solution $v(t,x) \in \mathbf{X}$, $h(t) = e^{\sigma\varphi(t)} \in \mathbf{C}(0,\infty)$ satisfying equations

$$\begin{cases} v(t) = \mathcal{G}(t)\, u_0 + \int_0^t \mathcal{G}(t-\tau) f\left(v(\tau), h(\tau)\right) d\tau, \\ h(t) = 1 - \dfrac{\sigma\lambda}{\theta} \int_0^t d\tau \int_{\mathbf{R}^+} x\,|v|^\sigma\, v dx, \end{cases} \tag{4.8}$$

and estimates

$$\|v(t) - \mathcal{G}(t)\, u_0\|_{\mathbf{L}^{1,1}} \leq C\varepsilon^{1+\sigma}$$

and

$$|h(t) - h_1(t)| \leq C\varepsilon^\sigma h_1(t). \tag{4.9}$$

We also have by applying (4.5) to (4.8)

$$\|v(t) - \mathcal{G}(t)\, u_0\|_{\mathbf{L}^\infty} \leq C\varepsilon^{1+\sigma} \langle t\rangle^{-1}, \tag{4.10}$$

for large $t > 0$. Then via formulas $u(t,x) = e^{-\varphi(t)} v(t,x) = h^{-\frac{1}{\sigma}}(t)\, v(t,x)$, we find the estimates

$$\begin{aligned} &\left\|u(t) - \theta t^{-1}\widetilde{G}(t^{-\frac{1}{2}}(\cdot))e^{-\varphi(t)}\right\|_{\mathbf{L}^\infty} \leq \left\|u(t) - \left(\mathcal{G}(t)\, u_0\right) e^{-\varphi(t)}\right\|_{\mathbf{L}^\infty} \\ &+ \left\|(\mathcal{G}(t)u_0 - \theta t^{-1}\widetilde{G}(t^{-\frac{1}{2}}(\cdot)))e^{-\varphi(t)}\right\|_{\mathbf{L}^\infty} \leq C\varepsilon^{1+\sigma} \langle t\rangle^{-2+\sigma}, \end{aligned} \tag{4.11}$$

where we have used the estimate for $\widetilde{G}(x) = (4\pi)^{-\frac{1}{2}}\, x e^{-\frac{|x|^2}{4}}$

$$\left\|\left(\mathcal{G}(t)\, u_0 - \theta t^{-1}\widetilde{G}\left(t^{-\frac{1}{2}}(\cdot)\right)\right) e^{-\varphi(t)}\right\|_{\mathbf{L}^\infty} \leq C t^{-2+\sigma-\frac{a}{2}} \|u_0\|_{\mathbf{L}^{1,1+a}}$$

and (4.10). We also have by (4.9)

$$\begin{aligned} &\left\|\theta t^{-1}\widetilde{G}\left(t^{-\frac{1}{2}}(\cdot)\right) h^{-\frac{1}{\sigma}}(t) - \theta t^{-1}\widetilde{G}\left(t^{-\frac{1}{2}}(\cdot)\right) h_1^{-\frac{1}{\sigma}}(t)\right\|_{\mathbf{L}^\infty} \\ \leq\ & C\varepsilon t^{-1} h_1^{-1-\frac{1}{\sigma}}(t)\, |h(t) - h_1(t)| \end{aligned}$$

hence via (4.11) it follows that

$$\left\|u(t) - \theta t^{-1}\widetilde{G}\left(t^{-\frac{1}{2}}(\cdot)\right) h_1^{-\frac{1}{\sigma}}(t)\right\|_{\mathbf{L}^\infty} \leq C\varepsilon^{1+\sigma} \langle t\rangle^{-2+\sigma}.$$

Thus, estimate (1.3) is true.

We now compute the asymptotics of the solution. First we show the existence of solutions to the integral equation

$$V(\xi) = V_0(\xi) - \frac{1}{\beta}\int_0^1 \frac{dz}{z^{\frac{3}{2}}(1-z)^{\frac{1}{2}}}\int_0^{+\infty} G_1\Big(\frac{\xi}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}}y}{(1-z)^{\frac{1}{2}}}\Big)F(y)\,dy, \tag{4.12}$$

where $V_0(\xi) = \widetilde{G}(\xi)$,

$$\begin{aligned} F(y) &= V^{1+\sigma}(y) - V(y)\int_{\mathbf{R}^+}\xi V^{\sigma+1}(\xi)\,d\xi, \\ \beta &= \frac{\sigma}{1-\sigma}\int_{\mathbf{R}^+}\xi V^{\sigma+1}(\xi)\,d\xi, \end{aligned}$$

and

$$G_1(s,q) = \frac{1}{2\pi i}\int_{-i\infty}^{i\infty} e^{p^2}(e^{p(s-q)} - e^{p(s+q)})dp.$$

Note that the integral in (4.12) converges since $G_1(\frac{\xi}{(1-z)^{\frac{1}{2}}}, 0) = 0$. To prove the existence of the self-similar solutions for equation (4.12) we need the following lemma.

Lemma 5. *Let the function $F(x)$ have the zero first moment value*

$$\int_{\mathbf{R}^+} xF(x)\,dx = 0,$$

and the norm $\|F\|_{\mathbf{L}^{p,1+a}}$ be finite, where $0 < a \le 1$, $1 \le p \le \infty$. Then the following inequalities are valid

$$\Big\|\int_0^1 \frac{dz}{z^{\frac{3}{2}}(1-z)^{\frac{1}{2}}}\int_0^{+\infty} G_1\Big(\frac{\xi}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}}y}{(1-z)^{\frac{1}{2}}}\Big)F(y)dy\Big\|_{\mathbf{L}^{p,1+a}} \\ \le C\|F\|_{\mathbf{L}^{1,1+a}} + C\|F\|_{\mathbf{L}^{p,1+a}}$$

and

$$\Big\|\int_0^1 \frac{dz}{(1-z)^{\frac{1}{2}}z^{\frac{1}{2}}}\Big(\frac{1}{z} - \frac{t}{\langle tz\rangle}\Big)\int_0^{+\infty} G_1\Big(\frac{\xi}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}}y}{(1-z)^{\frac{1}{2}}}\Big)F(y)dy\Big\|_{\mathbf{L}^{p,a}} \\ \le C\langle t\rangle^{-\frac{a}{2}}(\|F\|_{\mathbf{L}^{1,1+a}} + \|F\|_{\mathbf{L}^{p,1+a}})$$

for all $t > 0$, where $1 \le p \le \infty$.

Proof. By definition we have

$$G_1\Big(\frac{x}{(1-z)^{\frac12}},\frac{z^{\frac12}y}{(1-z)^{\frac12}}\Big)=C\Big(e^{-\frac{(x-z^{\frac12}y)^2}{4(1-z)}}-e^{-\frac{(x+z^{\frac12}y)^2}{4(1-z)}}\Big)$$
$$=G_{11}\Big(\frac{x-z^{\frac12}y}{(1-z)^{\frac12}}\Big)+G_{12}\Big(\frac{x+z^{\frac12}y}{(1-z)^{\frac12}}\Big).$$

By the Young inequality for convolutions we get

$$\begin{aligned}&\Big\|\int_0^{+\infty}G_{11}\Big(\frac{(\cdot)-z^{\frac12}y}{(1-z)^{\frac12}}\Big)F(y)\,dy\Big\|_{\mathbf{L}^{p,a}}\\
&\le\Big\|\int_0^{+\infty}\Big\langle(\cdot)-yz^{\frac12}\Big\rangle^a G_{11}\Big(\frac{(\cdot)-z^{\frac12}y}{(1-z)^{\frac12}}\Big)|F(y)|\,dy\Big\|_{\mathbf{L}^p}\\
&+\Big\|\int_0^{+\infty}G_{11}\Big(\frac{(\cdot)-z^{\frac12}y}{(1-z)^{\frac12}}\Big)\langle y\rangle^a|F(y)|\,dy\Big\|_{\mathbf{L}^p}\\
&\le\Big\|G_1\Big((\cdot)(1-z)^{-\frac12}\Big)\Big\|_{\mathbf{L}^{1,a}}\|F\|_{\mathbf{L}^p}+\Big\|G_1\Big((\cdot)(1-z)^{-\frac12}\Big)\Big\|_{\mathbf{L}^1}\|F\|_{\mathbf{L}^{p,a}}\\
&\le(1-z)^{\frac12}\|F\|_{\mathbf{L}^{p,a}}\end{aligned}$$

for all $z\in\left[\frac12,1\right]$, since

$$\Big\|\int_{\mathbf{R}^+}\phi\Big(\cdot-yz^{\frac12}\Big)F(y)\,dy\Big\|_{\mathbf{L}^p}\le C\|\phi\|_{\mathbf{L}^1}\|F\|_{\mathbf{L}^p}.$$

In the same way we obtain for all $z\in\left[\frac12,1\right]$

$$\Big\|\int_0^{+\infty}G_{11}\Big(\frac{(\cdot)+z^{\frac12}y}{(1-z)^{\frac12}}\Big)F(y)\,dy\Big\|_{\mathbf{L}^{p,a}}\le(1-z)^{\frac12}\|F\|_{\mathbf{L}^{p,a}}.$$

Therefore,

$$\begin{aligned}&\Big\|\int_{\frac12}^1\frac{dz}{(1-z)^{\frac12}z^{\frac12}}\int_0^{+\infty}G_1\Big(\frac{\xi}{(1-z)^{\frac12}},\frac{z^{\frac12}y}{(1-z)^{\frac12}}\Big)F(y)\,dydz\Big\|_{\mathbf{L}^{p,a}}\\
&\le C\int_{\frac12}^1(1-z)^{-\frac12}\Big\|\int_0^{+\infty}G_1\Big(\frac{\xi}{(1-z)^{\frac12}},\frac{z^{\frac12}y}{(1-z)^{\frac12}}\Big)F(y)\,dy\Big\|_{\mathbf{L}^{p,a}}dz\\
&\le C\|F\|_{\mathbf{L}^{p,a}}\end{aligned}\tag{4.13}$$

and

$$\begin{aligned}&\Big\| \int_{\frac{1}{2}}^{1} \frac{dz}{(1-z)^{\frac{1}{2}} z^{\frac{1}{2}}} \Big(\frac{1}{z} - \frac{t}{\langle tz \rangle}\Big) \int_0^{+\infty} G_1\Big(\frac{\xi}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}} y}{(1-z)^{\frac{1}{2}}}\Big) F(y)\, dy \Big\|_{\mathbf{L}^{p,a}} \\ &\leq C \langle t \rangle^{-1} \int_{\frac{1}{2}}^{1} (1-z)^{-\frac{1}{2}} \Big\| \int_0^{+\infty} G_1\Big(\frac{\xi}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}} y}{(1-z)^{\frac{1}{2}}}\Big) F(y)\, dy \Big\|_{\mathbf{L}^{p,a}} dz \\ &\leq C \langle t \rangle^{-1} \|F\|_{\mathbf{L}^{p,a}}, \end{aligned} \tag{4.14}$$

where $1 \leq p \leq \infty$.

Via the condition $\int_0^{+\infty} yF(y)\, dy = 0$, we write

$$\begin{aligned}&\Big\| \int_0^{+\infty} G_1\Big(\frac{(\cdot)}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}} y}{(1-z)^{\frac{1}{2}}}\Big) F(y)\, dy \Big\|_{\mathbf{L}^{p,1+a}} \\ &\leq \int_\Gamma |dq|\, e^{-C|q|^2} \frac{1}{|q|^{1+a+\frac{1}{p}}} \Big| \int_0^{+\infty} \exp(q \frac{z^{\frac{1}{2}} y}{(1-z)^{\frac{1}{2}}}) - \exp(-q \frac{z^{\frac{1}{2}} y}{(1-z)^{\frac{1}{2}}}) F(y) dy \Big| \\ &\leq C z^{\frac{a+1}{2}} \int_0^{+\infty} \langle y \rangle^{1+a} |F(y)|\, dy \\ &+ \int_1^{+\infty} |F(y)|\, dy \int_{\Gamma, |q|>1} |dq|\, e^{-C|q|^2} \frac{\exp(|q| \frac{z^{\frac{1}{2}} y}{(1-z)^{\frac{1}{2}}})}{|q|^{1+a+\frac{1}{p}}} \leq C z^{\frac{a+1}{2}} \|F\|_{\mathbf{L}^{1,1+a}}\end{aligned}$$

for all $z \in \left(0, \frac{1}{2}\right)$. Thus

$$\begin{aligned}&\Big\| \int_0^{\frac{1}{2}} \frac{1}{z^{\frac{3}{2}} (1-z)^{\frac{1}{2}}} \int_0^{+\infty} G_1\Big(\frac{(\cdot)}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}} y}{(1-z)^{\frac{1}{2}}}\Big) F(y)\, dy dz \Big\|_{\mathbf{L}^{p,1+a}} \\ \leq \;& \int_0^{\frac{1}{2}} \frac{dz}{z^{\frac{3}{2}} (1-z)^{\frac{1}{2}}} \Big\| \int_0^{+\infty} G_1\Big(\frac{(\cdot)}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}} y}{(1-z)^{\frac{1}{2}}}\Big) F(y)\, dy \Big\|_{\mathbf{L}^{p,1+a}} \\ \leq \;& C \int_0^{\frac{1}{2}} \frac{dz}{z^{1-\frac{a}{2}} (1-z)^{\frac{1}{2}}} \|F\|_{\mathbf{L}^{1,1+a}} \leq C \|F\|_{\mathbf{L}^{1,1+a}}\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}&\Big\| \int_0^{\frac{1}{2}} \frac{dz}{(1-z)^{\frac{1}{2}} z^{\frac{1}{2}}} \Big(\frac{1}{z} - \frac{t}{\langle tz \rangle}\Big) \int_0^{+\infty} G_1\Big(\frac{(\cdot)}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}} y}{(1-z)^{\frac{1}{2}}}\Big) F(y) dy \Big\|_{\mathbf{L}^{p,1+a}} \\ &\leq \int_0^{\frac{1}{2}} \frac{dz}{(1-z)^{\frac{1}{2}} z^{\frac{1}{2}}} \Big(\frac{1}{z} - \frac{t}{\langle tz \rangle}\Big) \Big\| \int_0^{+\infty} G_1\Big(\frac{(\cdot)}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}} y}{(1-z)^{\frac{1}{2}}}\Big) F(y) dy \Big\|_{\mathbf{L}^{p,1+a}}\end{aligned}$$

$$\leq C\int_0^{\frac{1}{2}}\frac{z^{\frac{a}{2}}}{(1-z)^{\frac{1}{2}}}\Big(\frac{1}{z}-\frac{t}{\langle tz\rangle}\Big)dz\|F\|_{\mathbf{L}^{1,1+a}}\leq C\langle t\rangle^{-\frac{a}{2}}\|F\|_{\mathbf{L}^{1,1+a}}, \tag{4.16}$$

where $1\leq p\leq\infty$. Collecting estimates (4.13) - (4.16), we get the result of the lemma. Lemma 5 is proved. □

We define successive approximations $V_{k+1}=\mathcal{R}(V_k)$ for $k=0,1,2,...,$ where

$$\mathcal{R}(V_k)(\xi)=V_0(\xi)-\frac{1}{\beta_k}\int_0^1\frac{dz}{z^{\frac{3}{2}}(1-z)^{\frac{1}{2}}}\int_0^{+\infty}G_1\Big(\frac{\xi}{(1-z)^{\frac{1}{2}}},\frac{z^{\frac{1}{2}}y}{(1-z)^{\frac{1}{2}}}\Big)F_k(y)dy,$$

$$\begin{aligned} F_k(y) &= V_k^{\sigma+1}(y)-V_k(y)\int_0^{+\infty}\xi V_k^{\sigma+1}(\xi)\,d\xi \text{ and} \\ \beta_k &= \frac{\sigma}{1-\sigma}\int_0^{+\infty}\xi V_k^{\sigma+1}(\xi)\,d\xi\ . \end{aligned}$$

By induction via Lemma 5 we prove the estimates

$$\sup_{1\leq p\leq\infty}\|V_{k+1}-V_0\|_{\mathbf{L}^{p,1+a}}\leq C\varepsilon,\ \ \sup_{1\leq p\leq\infty}\|V_k\|_{\mathbf{L}^{p,1+a}}\leq C,\ \ \beta_k\geq C\varepsilon^{-1} \tag{4.17}$$

and

$$\sup_{1\leq p\leq\infty}\|V_{k+1}-V_k\|_{\mathbf{L}^p}\leq\frac{1}{2}\sup_{1\leq p\leq\infty}\|V_k-V_{k-1}\|_{\mathbf{L}^p} \tag{4.18}$$

for all $k\geq1$. To use Lemma 5 we have to show that

$$\int_0^{+\infty}yF_k(y)\,dy=0 \text{ and } \int_0^{+\infty}yV_k(y)\,dy=1. \tag{4.19}$$

Since $\int_0^{+\infty}yV_0(y)\,dy=1$ by the definition of $F_k(y)$, we see that (4.19) is true for $k=0$. We assume that (4.19) holds for some k. Denote

$$\mathcal{G}_0F=\int_0^{+\infty}G_1F\,dy.$$

Note that $\mathcal{G}_0F$ is a solution of the following initial-boundary-value problem

$$\begin{cases}(\mathcal{G}_0F)_t-(\mathcal{G}_0F)_{xx}=0, \quad t>0,x>0,\\ (\mathcal{G}_0F)(x,0)=F,\ x>0;\ (\mathcal{G}_0F)(0,t)=0,\ t>0.\end{cases}$$

Therefore, we get

$$\frac{d}{dt}\int_0^{+\infty}x\mathcal{G}_0F\,dx=\int_0^{+\infty}x(\mathcal{G}_0F)_{xx}dx.$$

Since

$$\int_0^{+\infty} x(\mathcal{G}_0F)_{xx}dx = 0,$$

we easily see that

$$\int_0^{+\infty} x\mathcal{G}_0F dx = \int_0^{+\infty} yF(y)\,dy.$$

Then by Lemma 5 we have

$$\begin{aligned}\int_0^{+\infty} \xi V_{k+1}(\xi)\,d\xi &= 1 - \frac{1}{\beta_k(4\pi)}\int_0^1 \frac{dz}{z^{\frac{3}{2}}(1-z)^{\frac{1}{2}}}\int_0^{+\infty}\xi d\xi \\ &\int_0^{+\infty} G_1\Big(\frac{\xi}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}}y}{(1-z)^{\frac{1}{2}}}\Big)F_k(y)\,dy = 1,\end{aligned}$$

hence it follows that $\int_0^{+\infty} yF_{k+1}(y)\,dy = 0$. Thus we get (4.19) for any k. We use Lemma 5 to obtain

$$\begin{aligned}&\sup_{1\le p\le\infty}\|V_{k+1}-V_0\|_{\mathbf{L}^{p,1+a}} \\ &= \frac{C}{\beta_k}\sup_{1\le p\le\infty}\Big\|\int_0^1 \frac{dz}{z^{\frac{3}{2}}(1-z)^{\frac{1}{2}}}\int_0^{+\infty} G_1\Big(\frac{(\cdot)}{(1-z)^{\frac{1}{2}}}, \frac{z^{\frac{1}{2}}y}{(1-z)^{\frac{1}{2}}}\Big)F_k(y)\Big\|_{\mathbf{L}^{p,a}} \\ &\le \frac{C}{\beta_k}\sup_{1\le p\le\infty}\Big\|\Big(V_k^{\sigma+1}(\cdot) - V_k(\cdot)\int_0^{+\infty}\xi V_k^{\sigma+1}(\xi)\,d\xi\Big)\Big\|_{\mathbf{L}^{p,1+a}} \\ &\le \frac{C}{\beta_k}\sup_{1\le p\le\infty}\|V_k\|_{\mathbf{L}^{p,1+a}}\Big(\Big(\sup_{1\le p\le\infty}\|V_k\|_{\mathbf{L}^{p,1}}\Big)^{\sigma} + \Big(\sup_{1\le p\le\infty}\|V_k\|_{\mathbf{L}^{p,1}}\Big)^{\sigma+1}\Big) \le C\varepsilon,\end{aligned}$$

since σ is close to 1, hence β_k is considered to be sufficiently large. Therefore, (4.17) is true for any k. In the same manner we have (4.18) for any $k \ge 1$. Hence $\mathcal{R}$ is a contraction mapping and there exists a unique solution $V(\xi)$ to the integral equation (4.12).

We are now in a position to prove asymptotics of solutions v. We prove the estimate

$$\Big\|v(t) - t^{-1}\theta V\Big((\cdot)\,t^{-\frac{1}{2}}\Big)\Big\|_{\mathbf{L}^{p,b}} < C\langle t\rangle^{\frac{b}{2}} t^{-(1-\frac{1}{p})-\gamma} \tag{4.20}$$

for all $t > 0$, $b \in [0,2]$, $1 \le p \le \infty$, where $\gamma = \frac{1}{2}\min(b,(1-\sigma))$. To the contrary we suppose that estimate (4.20) is violated for some time $t = T_1$. By the continuity in time we have

$$\langle t\rangle^{-\frac{b}{2}}\Big\|v(t) - t^{-1}\theta V\Big((\cdot)\,t^{-\frac{1}{2}}\Big)\Big\|_{\mathbf{L}^{p,b}} \le C\langle t\rangle^{\frac{b}{2}} t^{-\left(1-\frac{1}{p}\right)-\gamma} \tag{4.21}$$

for all $t \in (0, T_1]$. By Lemma 2 we get

$$\left\| \mathcal{G}(t) u_0 - t^{-1} \theta \widetilde{G}\left((\cdot) t^{-\frac{1}{2}} \right) \right\|_{\mathbf{L}^{p,b}} < C \langle t \rangle^{\frac{b}{2}} t^{-\left(1-\frac{1}{p}\right)} \langle t \rangle^{-\frac{a}{2}}, \tag{4.22}$$

$b < a$. Then from (4.21) it follows that

$$\begin{aligned} & \left| h(t) - |\theta|^{\sigma} \beta t^{1-\frac{n}{2}\sigma} \right| = \left| 1 - \frac{\sigma\lambda}{\theta} \int_0^t \int_0^{+\infty} x |v|^{\sigma} v(\tau, x) \, dx d\tau \right. \\ & \left. -t^{1-\sigma} \frac{\sigma \theta^{\sigma}}{1-\sigma} \int_0^{+\infty} \xi V^{1+\sigma}(\xi) \, d\xi \right| \\ \leq \; & 1 + \frac{C}{\theta} \int_0^t \int_0^{+\infty} \left| x |v|^{\sigma} v(\tau, x) - |\theta|^{\sigma} \theta \tau^{-\sigma-1} x V^{1+\sigma} \left(x \tau^{-\frac{1}{2}} \right) \right| dx d\tau \\ \leq \; & 1 + \frac{C}{\theta} \int_0^t \left(\|v\|_{\mathbf{L}^{\infty}} + \theta \tau^{-1} \|V\|_{\mathbf{L}^{\infty}} \right)^{\sigma} \\ & \times \left\| v(\tau, \cdot) - \theta \tau^{-1} V \left((\cdot) \tau^{-\frac{1}{2}} \right) \right\|_{\mathbf{L}^{1,1}} d\tau \\ \leq \; & 1 + \frac{C}{|\theta|} \int_0^t \tau^{-\sigma-\gamma} d\tau \leq 1 + C \beta t^{1-\sigma-\gamma} \end{aligned} \tag{4.23}$$

for all $t \in (0, T_1]$, $1 \leq p \leq \infty$. We denote

$$f(\tau) = \lambda |v|^{\sigma} v(\tau) - \frac{\lambda v(\tau)}{\theta} \int_0^{+\infty} x |v|^{\sigma} v(\tau) \, dx.$$

We have

$$\begin{aligned} & \langle t \rangle^{-\frac{b}{2}} \left\| \theta t^{-1} V \left((\cdot) t^{-\frac{1}{2}} \right) - v(t) \right\|_{\mathbf{L}^{p,b}} \qquad (4.24) \\ & = \langle t \rangle^{-\frac{b}{2}} \left\| \theta t^{-1} V \left((\cdot) t^{-\frac{1}{2}} \right) - \mathcal{G}(t) u_0 + \int_0^t h^{-1}(\tau) \mathcal{G}(t-\tau) f(\tau) \, d\tau \right\|_{\mathbf{L}^{p,b}} \\ & \leq C \langle t \rangle^{-\frac{b}{2}} \left\| \theta t^{-1} V_0 \left((\cdot) t^{-\frac{1}{2}} \right) - \mathcal{G}(t) u_0 \right\|_{\mathbf{L}^{p,b}} \\ & + C \langle t \rangle^{-\frac{b}{2}} \left\| \theta t^{-1} \left(V((\cdot) t^{-\frac{1}{2}}) - V_0((\cdot) t^{-\frac{1}{2}}) \right) + \int_0^t h^{-1}(\tau) \mathcal{G}(t-\tau) f(\tau) d\tau \right\|_{\mathbf{L}^{p,b}}. \end{aligned}$$

Changing the variables $\tau = zt$ and $\xi \tau^{-\frac{1}{2}} = y$ we obtain

$$\begin{aligned} & \frac{1}{\beta} \int_0^t \tau^{\sigma-1} \mathcal{G}_0(t-\tau) \tau^{-\sigma-1} F\left((\cdot) \tau^{-\frac{1}{2}} \right) d\tau \\ = \; & \frac{1}{\beta} \int_0^t d\tau \tau^{-1-1} (t-\tau)^{-\frac{1}{2}} \int_0^{+\infty} G_1 \Big(\frac{x}{(t-\tau)^{\frac{1}{2}}}, \frac{\xi}{(t-\tau)^{\frac{1}{2}}} \Big) F\left(\xi \tau^{-\frac{1}{2}} \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\beta}t^{-1}\int_0^1 \frac{dz}{z^{\frac{3}{2}}(1-z)^{\frac{1}{2}}}\int_0^{+\infty} G_1\Big(\frac{(\cdot)}{(1-z)^{\frac{1}{2}}},\frac{z^{\frac{1}{2}}y}{(1-z)^{\frac{1}{2}}}\Big)F(y)\,dy \\
&= t^{-1}\left(V_0\left(xt^{-\frac{1}{2}}\right)-V\left(xt^{-\frac{1}{2}}\right)\right).
\end{aligned}
$$

Substituting this formula into (4.24) we get

$$
\begin{aligned}
& \langle t\rangle^{-\frac{b}{2}}\left\|\theta t^{-1}V\left((\cdot)\,t^{-\frac{1}{2}}\right)-v(t)\right\|_{\mathbf{L}^{p,b}} \\
= {} & \langle t\rangle^{-\frac{b}{2}}\left\|\theta t^{-1}V\left((\cdot)\,t^{-\frac{1}{2}}\right)-\mathcal{G}(t)\,u_0+\int_0^t h^{-1}(\tau)\,\mathcal{G}(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{L}^{p,b}} \\
\le {} & C\langle t\rangle^{-\frac{b}{2}}\left\|\theta t^{-1}V_0\left((\cdot)\,t^{-\frac{1}{2}}\right)-\mathcal{G}(t)\,u_0\right\|_{\mathbf{L}^{p,b}} \\
& +C\langle t\rangle^{-\frac{b}{2}}\left\|\int_0^t\left(h^{-1}(\tau)-\frac{1}{\beta|\theta|^{\sigma}}\tau^{\sigma}\langle\tau\rangle^{-1}\right)\mathcal{G}(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{L}^{p,b}} \\
& +\frac{C\langle t\rangle^{-\frac{b}{2}}}{\beta|\theta|^{\sigma}}\left\|\int_0^t\left(\mathcal{G}(t-\tau)-\mathcal{G}_0(t-\tau)\right)f(\tau)\,\frac{\tau^{\sigma}d\tau}{\langle\tau\rangle}\right\|_{\mathbf{L}^{p,b}} \\
& +\frac{C\langle t\rangle^{-\frac{b}{2}}}{\beta|\theta|^{\sigma}}\left\|\int_0^t\left(\mathcal{G}_0(t-\tau)\Big(f(\tau)-\frac{|\theta|^{\sigma}\theta\lambda}{\tau^{(\sigma+1)}}F\big((\cdot)\tau^{-\frac{1}{2}}\big)\Big)\right)\frac{\tau^{\sigma}d\tau}{\langle\tau\rangle}\right\|_{\mathbf{L}^{p,b}} \\
& +\frac{C|\theta\lambda|\langle t\rangle^{-\frac{b}{2}}}{\beta}\left\|\int_0^t\mathcal{G}_0(t-\tau)\,F\left((\cdot)\,\tau^{-\frac{1}{2}}\right)\left(\frac{1}{\langle\tau\rangle}-\frac{1}{\tau}\right)\tau^{-\frac{n}{2}}d\tau\right\|_{\mathbf{L}^{p,b}} \\
\equiv {} & I_1+I_2+I_3+I_4+I_5.
\end{aligned}
$$

Since $\widetilde{G}\left((\cdot)\,t^{-\frac{1}{2}}\right)=V_0(\xi)$ from (4.22) we have

$$
I_1\le Ct^{-\left(1-\frac{1}{p}\right)}\langle t\rangle^{-\frac{a}{2}}.
$$

By (4.23) and Lemma 5 we obtain

$$
\begin{aligned}
& I_2\le C\langle t\rangle^{-\frac{b}{2}}\left\|\int_0^t\frac{\tau^{\sigma}}{\langle\tau\rangle}\left|\frac{\langle\tau\rangle}{\tau^{\sigma}}-\frac{h(\tau)}{\beta\theta^{\sigma}}\right|h^{-1}(\tau)\,\mathcal{G}(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{L}^{p,b}} \\
\le {} & C\langle t\rangle^{-\frac{b}{2}}\left\|\int_0^t\frac{\tau^{\sigma}}{\langle\tau\rangle}\Big(\frac{\langle\tau\rangle}{\tau^{\sigma}}-\tau^{1-\sigma}+\frac{1}{\beta|\theta|^{\sigma}}+C\tau^{1-\sigma-\gamma}\Big)\right. \\
& \left.\times\, h^{-1}(\tau)\,\mathcal{G}(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{L}^{p,b}} \\
\le {} & C\langle t\rangle^{-\frac{b}{2}}\left\|\int_0^t\langle\tau\rangle^{-\gamma}h^{-1}(\tau)\,\mathcal{G}(t-\tau)\,f(\tau)\,d\tau\right\|_{\mathbf{L}^{p,b}}\le Ct^{-(1-\frac{1}{p})-\gamma}
\end{aligned}
$$

for all $t\in(0,T_1]$, $1\le p\le\infty$. Via estimate (4.5) and Lemma 2 we find

$$\begin{aligned}
I_3 &\le C\langle t\rangle^{-\frac{b}{2}}\Big\|\int_0^t\left(\mathcal{G}(t-\tau)-\mathcal{G}_0(t-\tau)\right)f(\tau)\frac{\tau^\sigma d\tau}{\langle\tau\rangle}\Big\|_{\mathbf{L}^{p,b}}\\
&\le C\langle t\rangle^{-\frac{b}{2}}\int_0^{\frac{t}{2}}\tau^\sigma\langle\tau\rangle^{-1}\langle t-\tau\rangle^{-(1-\frac{1}{p})-\gamma}\left(\langle\tau\rangle^{\frac{b}{2}}+\langle t-\tau\rangle^{\frac{b}{2}}\right)d\tau\\
&+C\langle t\rangle^{-\frac{b}{2}}\int_{\frac{t}{2}}^{t}\tau^\sigma\langle\tau\rangle^{-1-(1-\frac{1}{p})}\langle t-\tau\rangle^{-\gamma}\left(\langle\tau\rangle^{\frac{b}{2}}+\langle t-\tau\rangle^{\frac{b}{2}}\right)d\tau\\
&\le Ct^{-(1-\frac{1}{p})-\gamma}
\end{aligned}$$

for all $t\in(0,T_1]$, $1\le p\le\infty$. In the same manner we get

$$\begin{aligned}
I_4 &= \frac{C\langle t\rangle^{-\frac{b}{2}}}{\beta|\theta|^\sigma}\Big\|\int_0^t\mathcal{G}_0(t-\tau)\left(f(\tau)-\tau^{-\sigma-1}|\theta|^\sigma\theta\lambda F\left((\cdot)\tau^{-\frac{1}{2}}\right)\right)\frac{\tau^\sigma d\tau}{\langle\tau\rangle}\Big\|_{\mathbf{L}^{p,b}}\\
&\le\frac{C}{\beta|\theta|^\sigma}t^{-(1-\frac{1}{p})-\gamma}\\
&\times\sup_{t>0}\sup_{1\le p\le\infty}t^{(1+\sigma-\frac{1}{p})+\gamma}\langle t\rangle^{-\frac{b}{2}}\left\|f(t)-t^{-\sigma-1}|\theta|^\sigma\theta\lambda F\left((\cdot)t^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^{p,b}}\\
&\le\frac{C}{\beta|\theta|^\sigma}t^{-(1-\frac{1}{p})-\gamma}\Big(\sup_{t>0}\left(t\|v(t)\|_{\mathbf{L}^\infty}+\theta\|V\|_{\mathbf{L}^\infty}\right)\Big)^\sigma\\
&\times\sup_{t>0}\sup_{1\le p\le\infty}t^{(1-\frac{1}{p})+\gamma}\langle t\rangle^{-\frac{b}{2}}\left\|v(t)-t^{-1}\theta V\left((\cdot)t^{-\frac{1}{2}}\right)\right\|_{\mathbf{L}^{p,b}}\le Ct^{-(1-\frac{1}{p})-\gamma}
\end{aligned}$$

for all $t\in(0,T_1]$, $1\le p\le\infty$. Finally, changing independent variables $\tau=zt$ and $\xi\tau^{-\frac{1}{2}}=y$ and applying Lemma 5 we obtain

$$\begin{aligned}
I_5 &= \frac{C|\theta\lambda|}{\beta\langle t\rangle^{\frac{b}{2}}}\Big\|\int_0^t\mathcal{G}_0(t-\tau)F\left((\cdot)\tau^{-\frac{1}{2}}\right)\Big(\frac{1}{\langle\tau\rangle}-\frac{1}{\tau}\Big)\tau^{-\frac{1}{2}}d\tau\Big\|_{\mathbf{L}^{p,b}}\\
&= \frac{C|\theta\lambda|\langle t\rangle^{-\frac{b}{2}}}{\beta}\Big\|\int_0^t d\tau\Big(\frac{1}{\langle\tau\rangle}-\frac{1}{\tau}\Big)\tau^{-1}(t-\tau)^{-\frac{1}{2}}\\
&\times\int_0^{+\infty}G_0\Big(\frac{x}{(t-\tau)^{\frac{1}{2}}},\frac{\xi}{(t-\tau)^{\frac{1}{2}}}\Big)F(\xi\tau^{-\frac{1}{2}})\Big\|_{\mathbf{L}^{p,b}}\\
&= Ct^{-(1-\frac{1}{p})}\Big\|\int_0^1\frac{dz}{(1-z)^{\frac{1}{2}}z^{\frac{1}{2}}}\Big(\frac{1}{z}-\frac{t}{\langle tz\rangle}\Big)\\
&\times\int_0^{+\infty}G_0\Big(\frac{(\cdot)}{(1-z)^{\frac{1}{2}}},\frac{z^{\frac{1}{2}}y}{(1-z)^{\frac{1}{2}}}\Big)F(y)\,dy\Big\|_{\mathbf{L}^{p,b}}\le Ct^{-(1-\frac{1}{p})-\gamma}
\end{aligned}$$

for all $t \in (0, T_1]$, $1 \leq p \leq \infty$. Hence (4.20) is true for all $t > 0$. Taking $b = 0$ in (4.20) we get

$$\left\| v(t) - t^{-1}\theta V\left((\cdot)t^{-\frac{1}{2}}\right) \right\|_{\mathbf{L}^p} \leq Ct^{-(1-\frac{1}{p})-\gamma}. \tag{4.25}$$

Hence by virtue of (4.23) and (4.25) we have the asymptotics

$$v(t) = t^{-1}\theta V\left((\cdot)\, t^{-\frac{1}{2}}\right) + O\left(t^{-1-\gamma}\right) \tag{4.26}$$

and

$$h(t) = |\theta|^{\sigma} \beta t^{1-\sigma}\left(1 + O\left(t^{-\gamma}\right)\right)$$

for $t \to \infty$ uniformly with respect to $x \in \mathbf{R}^+$. Therefore via the formula $u(t, x) = e^{-\varphi(t)} v(t, x)$ taking into account estimates (4.26) we obtain the asymptotics (1.4) of the solution $u(t, x)$ with a constant $A = \beta^{-\frac{1}{\sigma}}$. This completes the proof of Theorem 1.

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REFERENCES

[1] M. Escobedo and O. Kavian, *Asymptotic behavior of positive solutions of a non-linear heat equation*, Houston Journal of Mathematics, **13** (1987), 39–50..

[2] A. Gmira and L. Veron, *Large time behavior of the solutions of a semilinear parabolic equation in* $\mathbf{R}^n$, J. Diff. Eqs., **53** (1984), 258–276.

[3] G.V. Demidenko and S.V. Uspenskii, "Equations and systems that are not solved with respect to the highest derivative," Nauchnaya Kniga, Novosibirsk, 1998, 437 pp.

[4] I.E. Egorov, S.G. Pyatkov, and S.V. Popov, "Nonclassical operator-differential equations," Nauka, Novosibirsk, 2000. 336 pp.

[5] A. Favini and A. Yagi, "Degenerate differential equations in Banach spaces," **215**, Marcel Dekker, Inc., New York, 1999, 313 pp.

[6] N. Hayashi and E.I. Kaikina, "Nonlinear Theory of Pseudodifferential Equations on a Half-Line," North-Holland Mathematics Studies, **194** (2004), 340.

[7] E.I. Kaikina, P.I. Naumkin, and I.A. Shishmarev, *The Cauchy problem for Sobolev type equation with power like nonlinearity,* Izvestiya, Ser. Math., **69** (2005).

[8] N. Hayashi, E.I. Kaikina ,and P.I. Naumkin, *Large time behavior of solutions to the dissipative nonlinear Schrödinger equation,* Proceedings of the Royal Soc. Edingburgh, **130**-A (2000), 1029–1043.

[9] N. Hayashi, E.I. Kaikina, and P.I. Naumkin, *Landau-Ginzburg type equations in the sub critical case,* Commun. Contemp. Math., **5** (2003), 127–145.

[10] G. Karch, *Asymptotic behavior of solutions to some pseudoparabolic equations*, Mat. Meth. in the Applied Sciences, **20** (1997), 271–289.

[11] E.I. Kaikina, *Nonlinear pseudoparabolic type equations on half-line with large initial data,* Preprint, 2005.

[12] E.I. Kaikina, *Initial-boundary value problem for nonlinear pseudoparabolic type equations in critical case,* Preprint, 2005.

[13] O. Kavian, *Remarks on the large time behavior of a nonlinear diffusion equation*, Ann. Inst. Henri Poincaré, Analyse non linéaire, **4** (1987), 423–452.

[14] Prado R.,Zuazua E., *Asymptotic expansion for the generalized Benjamin-Bona-Majoni-Burgers equation,* J.Diff. and Int. Eq.,**15**, (2002),N.12, pp. 1409–1434.

[15] R.E. Showalter, "Monotone operators in Banach space and nonlinear partial differential equations," **49**, American Mathematical Society, Providence, RI, 1997, 278 pp.

[16] U. Stefanelli, *On a class of doubly nonlinear nonlocal evolution equations,* Differential Integral Equations, **15** (2002), 897–922.