Differential and Integral Equations Volume 18, Number 12 (2005), Pages 1333–1339

# A BILINEAR AIRY-ESTIMATE WITH APPLICATION TO GKDV-3

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**Abstract.** The Fourier restriction norm method is used to show local wellposedness for the Cauchy-Problem

$$u_t + u_{xxx} + (u^4)_x = 0, \qquad u(0) = u_0 \in H^s_x(\mathbf{R}), \;\; s > -\tfrac{1}{6}$$

for the generalized Korteweg-deVries equation of order three, for short gKdV-3. For real-valued data $u_0 \in L^2_x(\mathbf{R})$ global wellposedness follows by the conservation of the $L^2$ norm. The main new tool is a bilinear estimate for solutions of the Airy-equation.

The purpose of this note is to establish local wellposedness of the Cauchy-Problem

$$u_t + u_{xxx} + (u^4)_x = 0, \qquad u(0) = u_0 \in H^s_x(\mathbf{R}), \;\; s > -\tfrac{1}{6}$$

for the generalized Korteweg-deVries equation of order three, for short gKdV-3. So far, local wellposedness of this problem is known for data $u_0 \in H^s_x(\mathbf{R})$, $s \geq \frac{1}{12}$. This was shown by Kenig, Ponce and Vega in 1993; see Theorem 2.6 in [4]. Here we extend this result to data $u_0 \in H^s_x(\mathbf{R})$, $s > -\frac{1}{6}$. A standard scaling argument suggests that this is optimal (up to the endpoint). For real-valued data $u_0 \in L^2_x(\mathbf{R})$ we obtain global wellposedness by the conservation of the $L^2$ norm.

By the Fourier restriction norm method introduced in [1] and further developed in [5] and [2] matters reduce to the proof of the estimate

$$\|\partial_x \textstyle\prod_{i=1}^4 u_i\|_{X_{s,b'}} \leq c \displaystyle\prod_{i=1}^4 \|u_i\|_{X_{s,b}} \tag{0.1}$$

Accepted for publication: May 2005.
AMS Subject Classifications: 35Q53.

for suitable values of $s$, $b$ and $b'$. Here the space $X_{s,b}$ is the completion of the Schwartz class $\mathcal{S}(\mathbf{R}^2)$ with respect to the norm

$$\|u\|_{X_{s,b}} = \Big( \int d\xi d\tau < \tau - \xi^3 >^{2b} < \xi >^{2s} |\mathcal{F}u(\xi,\tau)|^2 \Big)^{\frac{1}{2}},$$

where $\mathcal{F}$ denotes the Fourier transform in both variables. The main new tool for the proof of (0.1) is the following bilinear Airy-estimate:

**Lemma 1.** *Let $I^s$ denote the Riesz potential of order $-s$ and let $I^s_-(f,g)$ be defined by its Fourier transform (in the space variable):*

$$\mathcal{F}_x I^s_-(f,g)(\xi) := \int_{\xi_1+\xi_2=\xi} d\xi_1 |\xi_1 - \xi_2|^s \mathcal{F}_x f(\xi_1) \mathcal{F}_x g(\xi_2).$$

*Then we have*

$$\|I^{\frac{1}{2}} I^{\frac{1}{2}}_-(e^{-t\partial^3}u_1, e^{-t\partial^3}u_2)\|_{L^2_{xt}} \le c\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}.$$

**Proof.** We will write for short $\hat{u}$ instead of $\mathcal{F}_x u$ and $\int_* d\xi_1$ for $\int_{\xi_1+\xi_2=\xi} d\xi_1$. Then, using Fourier-Plancherel in the space variable we obtain:

$$\begin{aligned}
&\|I^{\frac{1}{2}} I^{\frac{1}{2}}_-(e^{-t\partial^3}u_1, e^{-t\partial^3}u_2)\|^2_{L^2_{xt}} \\
&= c \int d\xi |\xi| dt \left| \int_* d\xi_1 |\xi_1 - \xi_2|^{\frac{1}{2}} e^{it(\xi_1^3+\xi_2^3)} \hat{u}_1(\xi_1)\hat{u}_2(\xi_2) \right|^2 \\
&= c \int d\xi |\xi| dt \int_* d\xi_1 d\eta_1 e^{it(\xi_1^3+\xi_2^3-\eta_1^3-\eta_2^3)} (|\xi_1-\xi_2||\eta_1-\eta_2|)^{\frac{1}{2}} \prod_{i=1}^2 \hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)} \\
&= c \int d\xi |\xi| \int_* d\xi_1 d\eta_1 \delta(\eta_1^3+\eta_2^3-\xi_1^3-\xi_2^3)(|\xi_1-\xi_2||\eta_1-\eta_2|)^{\frac{1}{2}} \prod_{i=1}^2 \hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)} \\
&= c \int d\xi |\xi| \int_* d\xi_1 d\eta_1 \delta(3\xi(\eta_1^2 - \xi_1^2 + \xi(\xi_1 - \eta_1))) \\
&\qquad \times (|\xi_1-\xi_2||\eta_1-\eta_2|)^{\frac{1}{2}} \prod_{i=1}^2 \hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)}.
\end{aligned}$$

Now we use $\delta(g(x)) = \sum_n \frac{1}{|g'(x_n)|}\delta(x - x_n)$, where the sum is taken over all simple zeros of $g$, in our case:

$$g(x) = 3\xi(x^2 + \xi(\xi_1 - x) - \xi_1^2)$$

with the zeros $x_1 = \xi_1$ and $x_2 = \xi - \xi_1$, hence $g'(x_1) = 3\xi(2\xi_1 - \xi)$, respectively $g'(x_2) = 3\xi(\xi - 2\xi_1)$. So the last expression is equal to

$$
\begin{aligned}
& c \int d\xi |\xi| \int_* d\xi_1 d\eta_1 \frac{1}{|\xi||2\xi_1 - \xi|} \delta(\eta_1 - \xi_1)(|\xi_1 - \xi_2||\eta_1 - \eta_2|)^{\frac{1}{2}} \prod_{i=1}^{2} \hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)} \\
& + c \int d\xi |\xi| \int_* d\xi_1 d\eta_1 \frac{1}{|\xi||2\xi_1 - \xi|} \delta(\eta_1 - (\xi - \xi_1)) \\
& \qquad \times (|\xi_1 - \xi_2||\eta_1 - \eta_2|)^{\frac{1}{2}} \prod_{i=1}^{2} \hat{u}_i(\xi_i)\overline{\hat{u}_i(\eta_i)} \\
& = c \int d\xi \int_* d\xi_1 \prod_{i=1}^{2} |\hat{u}_i(\xi_i)|^2 + c \int d\xi \int_* d\xi_1 \hat{u}_1(\xi_1)\overline{\hat{u}_1}(\xi_2)\hat{u}_2(\xi_2)\overline{\hat{u}_2}(\xi_1) \\
& \leq c(\prod_{i=1}^{2} \|u_i\|^2_{L^2_x} + \|\hat{u}_1\hat{u}_2\|^2_{L^1_\xi}) \leq c \prod_{i=1}^{2} \|u_i\|^2_{L^2_x}. \qquad \square
\end{aligned}
$$

Arguing as in the proof of Lemma 2.3 in [2] we get the following

**Corollary 1.** *Let $b > \frac{1}{2}$. Then the following estimate holds true:*

$$\|I^{\frac{1}{2}} I_-^{\frac{1}{2}}(u, v)\|_{L^2_{xt}} \leq c\|u\|_{X_{0,b}}\|v\|_{X_{0,b}}.$$

In the next lemma, some well-known Strichartz-type estimates for the Airy equation are gathered in terms of $X_{s,b}$ norms:

**Lemma 2.** *For $b > \frac{1}{2}$ the following estimates are valid:*

i) $\|u\|_{L^p_t(H^{s,q}_x)} \leq c\|u\|_{X_{0,b}}$, *whenever* $0 \leq s = \frac{1}{p} \leq \frac{1}{4}$ *and* $\frac{1}{q} = \frac{1}{2} - \frac{2}{p}$,

ii) $\|u\|_{L^p_t(L^q_x)} \leq c\|u\|_{X_{0,b}}$, *whenever* $0 < \frac{1}{q} = \frac{1}{2} - \frac{3}{p} \leq \frac{1}{2}$.

Quotation/proof: Theorem 2.1 in [3] gives in the case of the Airy-equation

$$\|e^{-t\partial^3} u_0\|_{L^p_t(\dot{H}^{s,q}_x)} \leq c\|u_0\|_{L^2_x},$$

provided $0 \leq s = \frac{1}{p} \leq \frac{1}{4}$ and $\frac{1}{q} = \frac{1}{2} - \frac{2}{p}$. Now Lemma 2.3 in [2] is applied to obtain

$$\|u\|_{L^p_t(\dot{H}^{s,q}_x)} \leq c\|u\|_{X_{0,b}}, \qquad b > \frac{1}{2} \tag{0.2}$$

for the same values of $s$, $p$ and $q$. From this ii) follows by Sobolev's embedding theorem (in the space variable). Especially we have

$$\|u\|_{L^8_{xt}} \leq c\|u\|_{X_{0,b}}, \qquad b > \frac{1}{2},$$

which, interpolated with the trivial case, gives

$$\|u\|_{L^4_{xt}} \leq c\|u\|_{X_{0,b}}, \qquad b > \tfrac{1}{3}.$$

Now let us see how to replace $\dot{H}^{s,q}_x$ by $H^{s,q}_x$ in (0.2) in the endpoint case, i. e. $s = \frac{1}{p} = \frac{1}{4}$, $q = \infty$: Using the projections $p = \mathcal{F}_x^{-1}\chi_{\{|\xi|\leq 1\}}\mathcal{F}_x$ and $P = Id - p$ we have

$$\|u\|_{L^4_t(H_x^{\frac{1}{4},\infty})} \leq \|Pu\|_{L^4_t(H_x^{\frac{1}{4},\infty})} + \|pu\|_{L^4_t(H_x^{\frac{1}{4},\infty})} =: I + II.$$

For $I$ we use (0.2) to obtain

$$I \leq c\|I^{-\frac{1}{4}}J^{\frac{1}{4}}Pu\|_{X_{0,b}} \leq c\|u\|_{X_{0,b}},$$

while for $II$ by Sobolev's embedding theorem we get

$$II \leq c\|pu\|_{L^4_t(H_x^{\frac{1}{2}+,4})} \leq c\|pu\|_{X_{\frac{1}{2}+,b}} \leq c\|u\|_{X_{0,b}}.$$

This gives i) in the endpoint case, from which the general case follows by interpolation with Sobolev's embedding theorem (in the time variable). □

**Remark.** The endpoint case in ii) is also valid - see e.g. Lemma 3.29 in [4] - but we shall not make use of this here.

Now we are prepared to prove the crucial nonlinear estimate:

**Theorem 1.** *For* $0 \geq s > -\frac{1}{6}$, $-\frac{1}{2} < b' < s - \frac{1}{3}$ *and* $b > \frac{1}{2}$ *the estimate* (0.1) *is valid.*

**Proof.** Writing $f_i(\xi,\tau) = <\tau - \xi^3>^b <\xi>^s \mathcal{F}u_i(\xi,\tau)$, $1 \leq i \leq 4$, we have

$$\|\partial_x \textstyle\prod_{i=1}^4 u_i\|_{X_{s,b'}}$$

$$= c\Big\| <\tau - \xi^3>^{b'} <\xi>^s |\xi| \int d\nu \prod_{i=1}^4 <\tau_i - \xi_i^3>^{-b} <\xi_i>^{-s} f_i(\xi_i,\tau_i)\Big\|_{L^2_{\xi,\tau}},$$

where $d\nu = d\xi_1..d\xi_3 d\tau_1..d\tau_3$ and $\sum_{i=1}^4(\xi_i,\tau_i) = (\xi,\tau)$. Now the domain of integration is divided into the regions $A$, $B$ and $C = (A \cup B)^c$, where in $A$ we assume $|\xi_{max}| \leq c$. (Here $\xi_{max}$ is defined by $|\xi_{max}| = \max_{i=1}^4 |\xi_i|$, similarly $\xi_{min}$.) Then for the region $A$ we have the upper bound

$$c\Big\|\int d\nu \prod_{i=1}^4 <\tau_i - \xi_i^3>^{-b} f_i(\xi_i,\tau_i)\Big\|_{L^2_{\xi,\tau}}$$

$$= c\Big\|\prod_{i=1}^4 J^s u_i\Big\|_{L^2_{x,t}} \leq c\prod_{i=1}^4 \|J^s u_i\|_{L^8_{x,t}} \leq c\prod_{i=1}^4 \|u_i\|_{X_{s,b}},$$

where in the last step Lemma 2 , part ii), with $p=q=8$ was applied.

Besides $|\xi_{max}| \geq c$ ($\Rightarrow < \xi_{max} > \leq c|\xi_{max}|$) we shall assume for the region $B$ that

i) $|\xi_{\min}| \leq 0.99|\xi_{\max}|$ or
ii) $|\xi_{\min}| > 0.99|\xi_{\max}|$, and there are exactly two indices $i \in \{1,2,3,4\}$ with $\xi_i > 0$.

Then the region $B$ can be split again into a finite number of subregions, so that for any of these subregions there exists a permutation $\pi$ of $\{1,2,3,4\}$ with

$$\begin{aligned}
&|\xi| < \xi >^s \prod_{i=1}^{4} < \xi_i >^{-s} \\
&\leq c|\xi_{\pi(1)} + \xi_{\pi(2)}|^{\frac{1}{2}} |\xi_{\pi(1)} - \xi_{\pi(2)}|^{\frac{1}{2}} < \xi_{\pi(3)} >^{-\frac{3s}{2}} < \xi_{\pi(4)} >^{-\frac{3s}{2}} .
\end{aligned}$$

Assume $\pi = id$ for the sake of simplicity now. Then we get the upper bound

$$\begin{aligned}
&\Big\| < \tau - \xi^3 >^{b'} \int d\nu |\xi_1 + \xi_2|^{\frac{1}{2}} |\xi_1 - \xi_2|^{\frac{1}{2}} < \xi_3 >^{-\frac{3s}{2}} \\
&\qquad \times < \xi_4 >^{-\frac{3s}{2}} \prod_{i=1}^{4} < \tau_i - \xi_i^3 >^{-b} f_i(\xi_i, \tau_i) \Big\|_{L^2_{\xi,\tau}} \\
&= c\Big\| (I^{\frac{1}{2}} I_-^{\frac{1}{2}} (J^s u_1, J^s u_2))(J^{-\frac{s}{2}} u_3)(J^{-\frac{s}{2}} u_4) \Big\|_{X_{0,b'}} .
\end{aligned}$$

To estimate the latter expression, we fix some Sobolev and Hölder exponents:

i) $\frac{1}{q_0} = \frac{1}{2} - b'$ so that $L_t^{q_0}(L_x^2) \subset X_{0,b'}$,
ii) $\frac{2}{p} = \frac{1}{q_0} - \frac{1}{2} = -b'$,
iii) $\frac{1}{q} = \frac{1}{2} - \frac{2}{p} = \frac{1}{2} + b'$ so that by Lemma 2 $\|J^{\frac{1}{p}} u\|_{L_t^p(L_x^q)} \leq c\|u\|_{X_{0,b}}$,
iv) $\epsilon = \frac{1}{p} + \frac{3s}{2} > \frac{1}{q}$ (since $s > \frac{1}{3} + b'$) so that $H_x^{\epsilon,q} \subset L_x^\infty$.

Now we have

$$\begin{aligned}
&\Big\| (I^{\frac{1}{2}} I_-^{\frac{1}{2}} (J^s u_1, J^s u_2))(J^{-\frac{s}{2}} u_3)(J^{-\frac{s}{2}} u_4) \Big\|_{X_{0,b'}} \\
&\leq c\Big\| (I^{\frac{1}{2}} I_-^{\frac{1}{2}} (J^s u_1, J^s u_2))(J^{-\frac{s}{2}} u_3)(J^{-\frac{s}{2}} u_4) \Big\|_{L_t^{q_0}(L_x^2)} \\
&\leq c\Big\| I^{\frac{1}{2}} I_-^{\frac{1}{2}} (J^s u_1, J^s u_2) \Big\|_{L^2_{xt}} \Big\| J^{-\frac{s}{2}} u_3 \Big\|_{L_t^p(L_x^\infty)} \Big\| J^{-\frac{s}{2}} u_4 \Big\|_{L_t^p(L_x^\infty)} .
\end{aligned}$$

Now by Lemma 0.1 the first factor can be controlled by $c\|u_1\|_{X_{s,b}}\|u_2\|_{X_{s,b}}$, while for the second we have the upper bound

$$c\|J^{-\frac{3s}{2}+\epsilon}J^s u_3\|_{L^p_t(L^q_x)} = c\|J^{\frac{1}{p}}J^s u_3\|_{L^p_t(L^q_x)} \leq c\|u_3\|_{X_{s,b}}.$$

The third factor can be treated in precisely the same way. So for the contributions of the region $B$ we have obtained the desired bound.

Finally we consider the remaining region $C$: Here the $|\xi_i|$, $1 \leq i \leq 4$, are all very close together and $\geq c < \xi_i >$. Moreover, at least three of the variables $\xi_i$ have the same sign. Thus for the quantity $c.q.$ controlled by the expressions $< \tau - \xi^3 >$, $< \tau_i - \xi_i^3 >$, $1 \leq i \leq 4$, we have in this region:

$$c.q. := \left|\xi^3 - \sum_{i=1}^4 \xi_i^3\right| \geq c\sum_{i=1}^4 < \xi_i >^3 \geq c < \xi >^3$$

and hence, since $s > \frac{1}{3} + b'$ is assumed,

$$|\xi| < \xi >^s \prod_{i=1}^4 < \xi_i >^{-s} \leq c(< \tau - \xi^3 >^{-b'} + \sum_{i=1}^4 < \tau_i - \xi_i^3 >^{-b'} \chi_{C_i}),$$

where in the subregion $C_i$, $1 \leq i \leq 4$, the expression $< \tau_i - \xi_i^3 >$ is dominant. The first contribution can be estimated by

$$\begin{aligned} & c\left\|\int d\nu \prod_{i=1}^4 < \tau_i - \xi_i^3 >^{-b} f_i(\xi_i, \tau_i)\right\|_{L^2_{\xi,\tau}} \\ = \; & c\left\|\prod_{i=1}^4 J^s u_i\right\|_{L^2_{x,t}} \leq c\prod_{i=1}^4 \|J^s u_i\|_{L^8_{x,t}} \leq c\prod_{i=1}^4 \|u_i\|_{X_{s,b}}, \end{aligned}$$

where we have used Lemma 2, part ii). For the contribution of the subregion $C_1$ we take into account that

$$< \tau_1 - \xi_1^3 >= \max\{< \tau - \xi^3 >, < \tau_i - \xi_i^3 >, 1 \leq i \leq 4\},$$

which gives

$$< \tau - \xi^3 >^{b+b'} |\xi| < \xi >^s \prod_{i=1}^4 < \xi_i >^{-s} \leq c < \tau_1 - \xi_1^3 >^b .$$

So, for this contribution we get the upper bound

$$c\left\|< \tau - \xi^3 >^{-b} \int d\nu < \tau_1 - \xi_1^3 >^b \prod_{i=1}^4 < \tau_i - \xi_i^3 >^{-b} f_i(\xi_i, \tau_i)\right\|_{L^2_{\xi,\tau}}$$

$$\leq c \Big\| \mathcal{F}^{-1} f_1 \prod_{i=2}^{4} J^s u_i \Big\|_{X_{0,-b}} \leq c \|\mathcal{F}^{-1} f_1 \textstyle\prod_{i=2}^{4} J^s u_i \|_{L_{xt}^{\frac{8}{7}}}$$

$$\leq c \|\mathcal{F}^{-1} f_1\|_{L_{xt}^2} \prod_{i=2}^{4} \|J^s u_i\|_{L_{x,t}^8} \leq c \prod_{i=1}^{4} \|u_i\|_{X_{s,b}}.$$

Here we have used the dual version of the $L^8$-Strichartz estimate, Hölder and the estimate itself. For the remaining subregions $C_i$ the same argument applies. □

**Corollary 2.** *For* $s \geq 0$, $-\frac{1}{2} < b' < -\frac{1}{3}$ *and* $b > \frac{1}{2}$ *the estimate* (0.1) *holds true.*

**Proof.** For $s = 0$ this is contained in the above theorem, while for $s > 0$ one only has to use $< \xi > \leq c \prod_{i=1}^{4} < \xi_i >$. □

**Acknowledgement.** I want to thank Professor Hartmut Pecher for numerous helpful conversations.

## References


[1] J. Bourgain, *Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations,* GAFA, 3 (1993), 107–156 and 209–262.

[2] J. Ginibre, Y. Tsutsumi, and G. Velo, *On the Cauchy-Problem for the Zakharov-System,* J. of Functional Analysis, 151 (1997), 384–436.

[3] C.E. Kenig, G. Ponce, and L. Vega, *Oscillatory integrals and regularity of dispersive equations,* Indiana Univ. Math. J., 40 (1991), 33–69.

[4] C.E. Kenig, G. Ponce, and L. Vega, *Wellposedness and scattering results for the generalized Korteweg-de Vries equation via the contraction principle,* CPAM, 46 (1993), 527–620.

[5] C.E. Kenig, G. Ponce, and L. Vega, *A bilinear estimate with applications to the KdV equation,* J. of the Amer. Math. Soc., 9 (1996), 573–603.