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ALMOST PERIODIC SOLUTIONS FOR FIRST-ORDER DIFFERENTIAL EQUATIONS

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Abstract. We study here the almost periodic solutions of first-order differential equations. We give sufficient conditions for the existence and uniqueness. The method relies on penalization and a priori estimates. One of the main difficulties consists of verifying that the limit of the sequence of perturbed solutions remains almost periodic. We introduce the notions of minimal/maximal solutions.

1. Introduction

The theory of almost periodic functions has been developed in connection with problems of differential equations, stability theory, dynamical systems, and so on. The applications include not only ordinary differential equations, but also partial differential equations or equations in Banach spaces. There are several results concerning the existence and uniqueness of almost periodic solutions for first-order differential equations. Demidovitch [5] proved that if $f : \mathbb{R} \to \mathbb{R}$ is an almost periodic function with bounded primitive $F(t) = \int_0^t f(s)\, ds$, $t \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is monotone C^1 function, then all bounded solutions of

$$x'(t) + g(x(t)) = f(t), \quad t \in \mathbb{R}, \tag{1.1}$$

are almost periodic. This result was generalized by Gheorghiu [7] for first-order differential equations

$$x'(t) + g(t, x(t)) = 0, \quad t \in \mathbb{R}. \tag{1.2}$$

He proved that if g is C^1 almost periodic in t, uniformly with respect to $x \in [-R, R]$, $\forall R > 0$ and

$$\forall R > 0,\ \exists \gamma_R \in \mathbb{R} : \frac{\partial g}{\partial x} \geq \gamma_R > 0 \text{ (respectively } \tfrac{\partial g}{\partial x} \leq \gamma_R < 0),$$

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$\forall(t,x) \in \mathbb{R} \times [-R,R]$, then all bounded solutions of (1.2) are almost periodic. The previous result was extended by Opial [8] for functions $g = g(t,x)$ almost periodic in t, uniformly with respect to x on bounded sets and monotone with respect to x. Other results have been obtained by Amerio [1], Corduneanu [4], Favard [6]. In all these works the authors suppose the existence of bounded solutions and prove that these solutions are almost periodic. The aim of this paper is to give sufficient conditions in terms of the functions f, g in (1.1), (1.2) which ensure the existence of almost periodic solution for first-order differential equations. We indicate also a necessary condition. In order to present the ideas let us analyze the periodic solutions of (1.1) where f is a T periodic continuous function and g is a nondecreasing continuous function. Notice that if there is at least one T periodic solution for (1.1) then

$$\int_0^T g(x(t))\,dt = \int_0^T f(t)\,dt,$$

and therefore, by the mean value theorem, we deduce that there is $t_0 \in [0,T]$ such that

$$g(x(t_0)) = \langle f \rangle := \frac{1}{T}\int_0^T f(t)\,dt.$$

We obtain the following necessary condition

$$\langle f \rangle \in g(\mathbb{R}). \tag{1.3}$$

Conversely, we can prove that the above condition is sufficient for the existence of time periodic solutions for (1.1). For this it is convenient to use the penalization method. For all $\alpha > 0$ it is easy to construct the unique T periodic solution for the perturbed equation

$$\alpha x_\alpha(t) + x'_\alpha(t) + g(x_\alpha(t)) = f(t), \quad t \in \mathbb{R}. \tag{1.4}$$

As usual we obtain a solution for (1.1) by passing $\alpha \searrow 0$ in the sequence $(x_\alpha)_{\alpha>0}$. This can be done by using the Arzela-Ascoli theorem if we find uniform bounds with respect to $\alpha > 0$ for $\|x_\alpha\|_{L^\infty(\mathbb{R})}$. Suppose that the condition (1.3) is satisfied and let us derive a priori estimates for $(x_\alpha)_{\alpha>0}$. As before, by using the mean value theorem we obtain

$$\exists x_\alpha = x_\alpha(t_\alpha), \quad \alpha x_\alpha + g(x_\alpha) = \langle f \rangle,$$

which can be written

$$\alpha x_\alpha + g(x_\alpha) - g(x_0) = 0, \tag{1.5}$$

where $g(x_0) = \langle f \rangle$. After multiplication of (1.5) by $x_\alpha - x_0$ and by using the monotonicity of g we deduce that the sequence $(x_\alpha(t_\alpha))_\alpha$ is bounded

$$|x_\alpha(t_\alpha)| \le |x_0|, \ \ \forall \alpha > 0,$$

and after standard computations we obtain

$$\|x_\alpha\|_{L^\infty(\mathbb{R})} \le |x_0| + \|f - \langle f \rangle\|_{L^1(]0,T[)}, \ \ \forall \alpha > 0.$$

The same method applies for evolution equations with periodic source term

$$x'(t) + Ax(t) = f(t), \ \ t \in \mathbb{R},$$

where $A : D(A) \subset H \to H$ is a linear maximal monotone symmetric operator on a Hilbert space H and f is T periodic. We can prove that there is a periodic solution if and only if $\langle f \rangle \in \mathrm{Range}(A)$, *i.e.,* there exists $x_0 \in D(A)$ such that $\langle f \rangle = Ax_0$. For details the reader can refer to [2], [3].

The main results of this paper are the following sufficient conditions which guarantee the existence of an almost periodic solution for first-order differential equations

$$x'(t) + g(t, x(t)) = 0, \ \ t \in \mathbb{R}. \tag{1.6}$$

Theorem 3.3. *Assume that $g = g(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is nondecreasing with respect to x and almost periodic in t, uniformly with respect to x on bounded sets. If there is $M > 0$ such that*

$$g(t, -M) \le 0 \le g(t, M), \ \ \forall t \in \mathbb{R},$$

then there is at least one almost periodic solution $x(\cdot)$ for (1.6) satisfying

$$-M \le x(t) \le M, \ \ \forall t \in \mathbb{R}.$$

Theorem 4.1. *Assume that $g = g(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is nondecreasing with respect to x and almost periodic in t, uniformly with respect to x on bounded sets. If there is $X \in \mathbb{R}$ such that*

$$\langle g(\cdot, X) \rangle := \lim_{T \to +\infty} \frac{1}{T} \int_0^T g(t, X)\, dt = 0$$

and

$$\sup_{s,t \in \mathbb{R}} \Big(- \int_s^t g(\sigma, X)\, d\sigma \Big) < +\infty,$$

then there is at least one almost periodic solution $x(\cdot)$ for (1.6) satisfying

$$\|x(\cdot) - X\|_{L^\infty(\mathbb{R})} \le \sup_{s,t \in \mathbb{R}} \Big(- \int_s^t g(\sigma, X)\, d\sigma \Big).$$

We give also a uniqueness result for the almost periodic solution of

$$x'(t) + g(x(t)) = f(t), \quad t \in \mathbb{R}. \tag{1.7}$$

Theorem 5.2. *Assume that* $g : \mathbb{R} \to \mathbb{R}$ *is continuous, nondecreasing and* $f : \mathbb{R} \to \mathbb{R}$ *is almost periodic such that* $\langle f \rangle \in g(\mathbb{R})$ *and*

$$\sup_{s,t \in \mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right) < +\infty.$$

Then there is at least one almost periodic solution for (1.7) *and the solution is unique if and only if*

$$diam(g^{-1}\langle f \rangle) \leq \sup_{s,t \in \mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right).$$

The content of this paper is organized as follows. We start our analysis by studying the existence and uniqueness of bounded solution for first-order differential equations. We recall the notions of sub/supersolutions and we introduce the concept of minimal/maximal solutions. In Section 3 we prove our first existence result about almost periodic solutions (see Theorem 3.3). Actually we prove that the minimal solution is almost periodic. Moreover we deduce that all bounded solutions are almost periodic. In the next section we prove our second existence result about almost periodic solutions (see Theorem 4.1). In Section 5 we study the asymptotic behavior of almost periodic solutions for large frequencies. We end with some uniqueness and stability results for almost periodic solutions.

2. Bounded solutions for first-order differential equations

As we will see later, under appropriate hypotheses, the classes of bounded solutions and almost periodic solutions of first-order differential equations coincide. In this section we analyze the existence and uniqueness of bounded solutions for the equation

$$x'(t) + g(t, x(t)) = 0, \quad t \in \mathbb{R}, \tag{2.1}$$

where $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, nondecreasing with respect to x

$$g(t, x) \leq g(t, y), \quad \forall t \in \mathbb{R}, \quad \forall \, x \leq y. \tag{2.2}$$

Note that it is also possible to study the equation (2.1) when g is nonincreasing with respect to x. For this observe that $x(\cdot)$ is a solution of (2.1) if and only if $y(t) = x(-t)$ is a solution of

$$y'(t) - g(-t, y(t)) = 0, \quad t \in \mathbb{R}.$$

In the following we always suppose that g is nondecreasing with respect to x.

2.1. **Sub/supersolutions for first-order differential equations.** In this paragraph we study the properties of sub/supersolutions of (2.1).

Definition 2.1. *Let $x, y : \mathbb{R} \to \mathbb{R}$ be C^1 functions.*

1) *We say that $x(\cdot)$ is a subsolution of* (2.1) *if and only if*

$$x'(t) + g(t, x(t)) \leq 0, \quad \forall t \in \mathbb{R}.$$

2) *We say that $y(\cdot)$ is a supersolution of* (2.1) *if and only if*

$$y'(t) + g(t, y(t)) \geq 0, \quad \forall t \in \mathbb{R}.$$

The main tool is the following classical comparison result for bounded sub/supersolutions. We assume that g satisfies the hypothesis

$$\exists \gamma > 0 \ : \ g(t, x) - \gamma x \leq g(t, y) - \gamma y, \quad \forall t \in \mathbb{R}, \quad \forall x \leq y. \tag{2.3}$$

Proposition 2.1. *Assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and satisfies* (2.3)*. Let $x(\cdot)$ be a bounded (from above) subsolution of* (2.1) *and $y(\cdot)$ be a bounded (from below) supersolution of* (2.1)*. Then we have the inequality*

$$x(t) \leq y(t), \quad \forall t \in \mathbb{R}.$$

Proof. The arguments are standard. Consider $z_\delta(t) = x(t) - y(t) - \delta(t^2+1)^{\frac{1}{2}}$, $t \in \mathbb{R}, \delta > 0$. Since $x(\cdot)$ is bounded from above and $y(\cdot)$ is bounded from below we deduce that $\lim_{|t| \to +\infty} z_\delta(t) = -\infty$ and therefore there is $t_\delta \in \mathbb{R}$ such that

$$x(t) - y(t) - \delta(t^2+1)^{\frac{1}{2}} = z_\delta(t) \leq z_\delta(t_\delta) = x(t_\delta) - y(t_\delta) - \delta(t_\delta^2+1)^{\frac{1}{2}}, \quad \forall t \in \mathbb{R}. \tag{2.4}$$

In particular we obtain

$$x'(t_\delta) - y'(t_\delta) - \delta \frac{t_\delta}{(t_\delta^2 + 1)^{\frac{1}{2}}} = 0. \tag{2.5}$$

Since $x(\cdot)$ is a subsolution and $y(\cdot)$ is a supersolution we have

$$x'(t_\delta) - y'(t_\delta) + g(t_\delta, x(t_\delta)) - g(t_\delta, y(t_\delta)) \leq 0. \tag{2.6}$$

The hypothesis (2.3) implies that

$$g(t_\delta, x(t_\delta)) - g(t_\delta, y(t_\delta)) = (\gamma + r_\delta) \cdot (x(t_\delta) - y(t_\delta)), \tag{2.7}$$

where $r_\delta \geq 0$. Combining (2.5), (2.6), (2.7) yields

$$\delta \frac{t_\delta}{(t_\delta^2 + 1)^{\frac{1}{2}}} + (\gamma + r_\delta) \cdot (x(t_\delta) - y(t_\delta)) \leq 0, \tag{2.8}$$

which implies

$$x(t_\delta) - y(t_\delta) \le \frac{\delta}{\gamma + r_\delta} \le \frac{\delta}{\gamma}. \tag{2.9}$$

By using (2.4), (2.9) we deduce that

$$x(t) - y(t) \le \delta(t^2+1)^{\frac{1}{2}} + \frac{\delta}{\gamma}, \ \ \forall t \in \mathbb{R}, \forall \delta > 0.$$

The conclusion follows by keeping t fixed and by passing $\delta \searrow 0$. □

As a direct consequence of the above comparison result we obtain the uniqueness of bounded solutions for (2.1).

Corollary 2.1. *Assume that g is continuous and satisfies* (2.3). *Then the equation* (2.1) *has at most one bounded solution.*

When g satisfies only (2.2) it is possible to show that the difference between two solutions of (2.1) keeps a constant sign on $\mathbb{R}$.

Proposition 2.2. *Assume that g is continuous and satisfies* (2.2). *Consider x, y two solutions of* (2.1). *Then $x(t) \le y(t)$, for all $t \in \mathbb{R}$ or $x(t) \ge y(t)$, for all $t \in \mathbb{R}$.*

Proof. Suppose that there is t_1, t_2 such that $(x(t_1)-y(t_1))\cdot(x(t_2)-y(t_2)) < 0$. For example consider the case $x(t_1) < y(t_1)$ and $x(t_2) > y(t_2)$. We have

$$\frac{1}{2}\frac{d}{dt}|x-y|^2 + (g(t,x(t)) - g(t,y(t)))\cdot(x(t)-y(t)) = 0, \ \ t \in \mathbb{R},$$

and thus $t \to |x(t) - y(t)|$ is nonincreasing. There is $t_3 = \theta t_1 + (1-\theta)t_2$, $\theta \in]0,1[$ such that $x(t_3) = y(t_3)$. We deduce that $x(t) = y(t)$, for all $t \ge t_3$ which is impossible since we have $x(t_4) \ne y(t_4)$ with $t_4 = \max(t_1, t_2) > t_3$. Hence the difference $x(t) - y(t)$ keeps a constant sign for $t \in \mathbb{R}$. □

2.2. **Existence of bounded solutions.** We prove now the existence of bounded solutions for (2.1) when g satisfies hypothesis (2.3). In this case the proof is standard.

Proposition 2.3. *Assume that g is continuous satisfying* (2.3) *and*

$$\sup_{t\in\mathbb{R}} |g(t,0)| = C < +\infty.$$

Then there is a unique bounded solution for (2.1).

Proof. The uniqueness of bounded solutions was already proved. Let us construct a bounded solution. For all $n \ge 0$ we consider x_n the unique classical solution for

$$x_n'(t) + g(t, x_n(t)) = 0, \ \ t > -n, \ \ x_n(-n) = 0. \tag{2.10}$$

Note that the existence and uniqueness for the solution of (2.10) holds for every continuous, nondecreasing with respect to x function g. Indeed, the uniqueness follows easily by using the monotonicity of g and the initial condition. In order to prove the existence, construct $\tilde{x}_n : [-n, \tau_n[$ the maximal solution of (2.10). We show that $\tau_n = +\infty$. Suppose that $\tau_n < +\infty$ and observe that

$$\frac{1}{2}\frac{d}{dt}|\tilde{x}_n(t)|^2 + (g(t,\tilde{x}_n(t)) - g(t,0)) \cdot \tilde{x}_n(t) = -g(t,0) \cdot \tilde{x}_n(t), \;\; -n \le t < \tau_n,$$

which implies

$$\frac{1}{2}|\tilde{x}_n(t)|^2 \le \int_{-n}^{t} |g(s,0)| \cdot |\tilde{x}_n(s)| \, ds, \;\; -n \le t < \tau_n.$$

By using Bellman's lemma we obtain

$$|\tilde{x}_n(t)| \le \int_{-n}^{t} |g(s,0)| \, ds \le C(\tau_n + n), \;\; \forall -n \le t < \tau_n,$$

and therefore the maximal solution remains bounded on $[-n, \tau_n[$, which is not possible. Thus $\tau_n = +\infty$. We prove now that $(\|x_n\|_{L^\infty(]-n,+\infty[)})_n$ is bounded. We can write

$$\frac{1}{2}\frac{d}{dt}|x_n(t)|^2 + (g(t,x_n(t)) - g(t,0)) \cdot x_n(t) = -g(t,0) \cdot x_n(t), \;\; t \ge -n. \tag{2.11}$$

By hypothesis (2.3) we deduce that

$$(g(t,x_n(t)) - g(t,0)) \cdot x_n(t) \ge \gamma |x_n(t)|^2,$$

and therefore we obtain

$$\frac{1}{2}\frac{d}{dt}|x_n(t)|^2 + \gamma|x_n(t)|^2 \le |g(t,0)| \cdot |x_n(t)|, \;\; t \ge -n.$$

By using Bellman's lemma we have

$$|x_n(t)|e^{\gamma t} \le \int_{-n}^{t} |g(s,0)| e^{\gamma s} \, ds \le \frac{C}{\gamma}(e^{\gamma t} - e^{-\gamma n}),$$

and finally one gets

$$|x_n(t)| \le \frac{C}{\gamma}, \;\; \forall t \ge -n, \forall n \ge 0.$$

We can prove also that $(x_n)_n$ converges uniformly on every interval $[\tau, +\infty[$, $\tau \in \mathbb{R}$. Indeed, as before we obtain

$$\frac{1}{2}\frac{d}{dt}|x_n(t) - x_m(t)|^2 + \gamma|x_n(t) - x_m(t)|^2 \le 0, \;\; t \ge \max(-n,-m),$$

which implies

$$|x_n(t) - x_m(t)| \le e^{-\gamma(t-t_0)}|x_n(t_0) - x_m(t_0)| \le \frac{2C}{\gamma} e^{-\gamma(t-t_0)},$$

for all $t \ge t_0 \ge \max(-n, -m)$. Assume that $m \ge n$ and take $t_0 = -n$. We have for all $t \ge \tau \ge -n$

$$|x_n(t) - x_m(t)| \le \frac{2C}{\gamma} e^{-\gamma(t+n)} \le \frac{2C}{\gamma} e^{-\gamma(\tau+n)}.$$

We deduce that $(x_n)_n$ is a Cauchy sequence in $C^0([\tau, +\infty[)$ and therefore converges uniformly on $[\tau, +\infty[$. We denote by x the limit function. Since $|x_n(t)| \le \frac{C}{\gamma}$, for all $t \ge -n$, for all $n \ge 0$ we deduce that $|x(t)| \le \frac{C}{\gamma}$, for all $t \in \mathbb{R}$. Let us check that x is a solution for (2.1). For n large enough we have

$$x_n(t) - x_n(\tau) + \int_\tau^t g(s, x_n(s))\, ds = 0, \ \ \forall t \ge \tau \ge -n.$$

By passing to the limit for $n \to +\infty$ we obtain

$$x(t) - x(\tau) + \int_\tau^t g(s, x(s))\, ds = 0, \ \ \forall t \ge \tau,$$

and therefore $x \in C^1(\mathbb{R})$ and $x'(t) + g(t, x(t)) = 0$, for all $t \in \mathbb{R}$. □

2.3. **Minimal/maximal solutions.** In this section we suppose that the function g is only nondecreasing with respect to x. Generally, in this case, we can not prove the uniqueness of solutions for (2.1). We need to distinguish some particular solutions. We suppose that g satisfies the hypothesis

$$\forall R > 0, \exists C_R > 0 \ : \ |g(t,x)| \le C_R, \ \forall (t,x) \in \mathbb{R} \times [-R, R]. \tag{2.12}$$

Proposition 2.4. *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (2.2)*,* (2.12)*. Consider x_0 a bounded subsolution of* (2.1) *and y_0 a bounded supersolution of* (2.1) *such that*

$$x_0(t) \le y_0(t), \ \ \forall t \in \mathbb{R}.$$

For $\alpha > 0$ we denote by x_α the unique bounded solution of the equation

$$\alpha(x_\alpha(t) - x_0(t)) + x'_\alpha(t) + g(t, x_\alpha(t)) = 0, \ \ t \in \mathbb{R}. \tag{2.13}$$

Then the family $(x_\alpha)_\alpha$ converges uniformly on compact sets towards a solution x of (2.1) *verifying*

$$x_0(t) \le x(t) \le y_0(t), \ \ \forall t \in \mathbb{R}.$$

In particular there is at least one bounded solution for (2.1)*.*

Proof. Note that the hypotheses of Proposition 2.3 hold for the function

$$g_\alpha(t,x) = \alpha x + g(t,x) - \alpha x_0(t), \quad (t,x) \in \mathbb{R} \times \mathbb{R},$$

and therefore x_α is well defined for all $\alpha > 0$. Moreover x_0 is a bounded subsolution of (2.13) and by Proposition 2.1 we obtain $x_0 \le x_\alpha$ (*i.e.*, $x_0(t) \le x_\alpha(t)$, for all $t \in \mathbb{R}$). Since $x_0 \le y_0$ we check easily that y_0 is a bounded supersolution for (2.13) and thus by Proposition 2.1 we have $x_\alpha \le y_0$. In fact it is possible to prove that if $0 < \alpha \le \beta$ then

$$x_0 \le x_\beta \le x_\alpha \le y_0.$$

Indeed, it is sufficient to prove that x_β is a subsolution for (2.13). We can write

$$\begin{aligned}
&\alpha(x_\beta(t) - x_0(t)) + x'_\beta(t) + g(t, x_\beta(t)) \\
&= (\alpha - \beta) \cdot (x_\beta(t) - x_0(t)) + \beta(x_\beta(t) - x_0(t)) + x'_\beta(t) + g(t, x_\beta(t)) \\
&\le \beta(x_\beta(t) - x_0(t)) + x'_\beta(t) + g(t, x_\beta(t)) = 0.
\end{aligned}$$

We denote $x(t) = \sup_{\alpha>0} x_\alpha(t) = \lim_{\alpha \searrow 0} x_\alpha(t)$. Obviously we have

$$x_0 \le x_\alpha \le x \le y_0.$$

Take $R_0 = \max(\|x_0\|_{L^\infty(\mathbb{R})}, \|y_0\|_{L^\infty(\mathbb{R})})$ and note that we have the inequalities

$$|g(t, x_\alpha(t))| \le \max(|g(t, x_0(t))|, |g(t, y_0(t))|) \le C_{R_0}, \quad \forall t \in \mathbb{R}, \alpha > 0.$$

We deduce that the solutions $(x_\alpha)_\alpha$ are uniformly Lipschitz

$$|x'_\alpha(t)| \le y_0(t) - x_0(t) + C_{R_0} \le 2R_0 + C_{R_0}, \quad \forall t \in \mathbb{R}, 0 < \alpha \le 1,$$

and therefore $(x_\alpha)_\alpha$ converges to x uniformly on compact sets. It follows easily that x is a classical solution of (2.1) and $|x'(t)| \le C_{R_0}$, for all $t \in \mathbb{R}$. □

Definition 2.2. *Under the hypotheses of Proposition* 2.4 *we say that* $x = \sup_{\alpha>0} x_\alpha$ *is the minimal solution of* (2.1)*.*

Proposition 2.5. *Under the hypotheses of Proposition* 2.4*, the minimal solution verifies the following minimal property : if* z *is a supersolution of* (2.1) *such that* $x_0 \le z \le x$ *then* $z = x$*.*

Proof. We need to prove the inequality $z \ge x$ which is equivalent to $z \ge x_\alpha$, $\forall \alpha > 0$. For this it is sufficient to show that z is a supersolution for (2.13). We have

$$\alpha(z(t) - x_0(t)) + z'(t) + g(t, z(t)) \ge z'(t) + g(t, z(t)) \ge 0, \quad \forall t \in \mathbb{R},$$

and therefore, by Proposition 2.1 we obtain $z \ge x_\alpha$, for all $\alpha > 0$. □

Similarly we define the notion of maximal solution.

Proposition 2.6. *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (2.2), (2.12). *Consider x_0 a bounded subsolution of* (2.1) *and y_0 a bounded supersolution of* (2.1) *such that*

$$x_0(t) \leq y_0(t), \quad \forall t \in \mathbb{R}.$$

For $\alpha > 0$ we denote by y_α the unique bounded solution of the equation

$$\alpha(y_\alpha(t) - y_0(t)) + y'_\alpha(t) + g(t, y_\alpha(t)) = 0, \quad t \in \mathbb{R}. \tag{2.14}$$

Then the family $(y_\alpha)_\alpha$ converges uniformly on compact sets towards a solution y of (2.1) *satisfying*

$$x_0(t) \leq y(t) \leq y_0(t), \quad \forall t \in \mathbb{R}.$$

Proof. Similar to those of Proposition 2.4. In this case we prove that if $0 < \alpha \leq \beta$ then

$$x_0 \leq y_\alpha \leq y_\beta \leq y_0,$$

and therefore $y = \lim_{\alpha \searrow 0} y_\alpha = \inf_{\alpha > 0} y_\alpha$. □

Definition 2.3. *Under the hypotheses of Proposition* 2.6 *we say that $y = \inf_{\alpha>0} y_\alpha$ is the maximal solution of* (2.1).

Proposition 2.7. *Under the hypotheses of Proposition* 2.6, *the maximal solution verifies the following maximal property : if z is a subsolution of* (2.1) *such that $y \leq z \leq y_0$ then $z = y$.*

For all bounded subsolutions x of (2.1) and bounded supersolutions y of (2.1) such that $x \leq y$ we denote by $x_{min}(x, y)$, $y_{max}(x, y)$ the minimal, respectively maximal solutions constructed before. Let us give some easy properties of the minimal/maximal solutions. The proofs are immediate and are left to the reader.

Proposition 2.8. *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying* (2.2), (2.12).

1) *If x is a bounded subsolution, y is a bounded supersolution such that $x \leq y$, then $x_{min}(x, y) \leq y_{max}(x, y)$;*

2) *if x is a bounded subsolution and y_1, y_2 are bounded supersolutions such that $x \leq \min(y_1, y_2)$, then $x_{min}(x, y_1) = x_{min}(x, y_2)$;*

3) *if x_1, x_2 are bounded subsolutions and y is a bounded supersolution such that $x_1 \leq x_2 \leq y$, then $x_{min}(x_1, y) \leq x_{min}(x_2, y)$;*

4) *if x_1, x_2 are bounded subsolutions and y is a bounded supersolution such that $y \geq \max(x_1, x_2)$, then $y_{max}(x_1, y) = y_{max}(x_2, y)$;*

5) *if x is a bounded subsolution and y_1, y_2 are bounded supersolutions such that $x \leq y_1 \leq y_2$, then $y_{max}(x, y_1) \leq y_{max}(x, y_2)$.*

3. Almost periodic solutions for first-order differential equations

In the previous section we studied the existence and uniqueness of bounded solutions for (2.1) when g is a continuous function satisfying (2.3) or only (2.2). Now we are interested in almost periodic solutions for (2.1). We recall here the notions of almost periodic function and normal function.

Definition 3.1. *Consider $f : \mathbb{R} \to \mathbb{R}$ a continuous function. We say that f is almost periodic if and only if for all $\varepsilon > 0$, there exists $l(\varepsilon) > 0$ such that all intervals of length $l(\varepsilon)$ contain a number τ satisfying*

$$|f(t+\tau) - f(t)| \leq \varepsilon, \quad \forall t \in \mathbb{R}.$$

The number τ is called ε-almost period of the function f. We deduce immediately the following properties of almost periodic functions.

Proposition 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be an almost periodic function. Then f is bounded and uniformly continuous on $\mathbb{R}$.*

Another important property of almost periodic functions is the existence of the average (see [4]).

Proposition 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be an almost periodic function. Then*

$$\frac{1}{T}\int_a^{a+T} f(t)\, dt$$

converges as $T \to +\infty$ uniformly with respect to $a \in \mathbb{R}$. Moreover the limit does not depend on a and is called the average of f

$$\exists\, \langle f \rangle := \lim_{T\to+\infty} \frac{1}{T}\int_a^{a+T} f(t)\, dt, \text{ uniformly with respect to } a \in \mathbb{R}.$$

Definition 3.2. *Consider $f : \mathbb{R} \to \mathbb{R}$ a continuous function. We say that f is a normal function if and only if for all real sequences $(h_n)_n$ there is a subsequence $(h_{n_k})_k$ such that $(f(\cdot + h_{n_k}))_k$ converges uniformly on $\mathbb{R}$.*

We have the following result (see [4]).

Theorem 3.1. (*Bochner*) *Let $f : \mathbb{R} \to \mathbb{R}$ be a real function. Then f is almost periodic function if and only if f is a normal function.*

We introduce also the notion of almost periodic function of t, uniformly with respect to x on compact sets.

Definition 3.3. *Consider $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function. We say that g is almost periodic in t, uniformly with respect to x on compact sets if* *and only if for all $R > 0$, for all $\varepsilon > 0$, there exists $l(\varepsilon, R) > 0$ such that all intervals of length $l(\varepsilon, R)$ contain a number τ satisfying*

$$|g(t+\tau, x) - g(t, x)| \leq \varepsilon, \quad \forall (t, x) \in \mathbb{R} \times [-R, R]. \tag{3.1}$$

We have the analogous result.

Proposition 3.3. *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be almost periodic in t uniformly with respect to x on compact sets. Then g is bounded and uniformly continuous on $\mathbb{R} \times [-R, R]$, for all $R > 0$.*

Definition 3.4. *Consider $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ a continuous function. We say that g is a normal function of t uniformly with respect to x on compact sets if and only if for all real sequence $(h_n)_n$ there is a subsequence $(h_{n_k})_k$ such that $(g(\cdot + h_{n_k}, \cdot))_k$ converges uniformly on $\mathbb{R} \times [-R, R]$, for all $R > 0$.*

By adapting the proof of Bochner's theorem we obtain

Theorem 3.2. *Let $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be a real function. Then g is almost periodic in t uniformly with respect to x on compact sets if and only if g is normal in t uniformly with respect to x on compact sets.*

3.1. **Existence and uniqueness of almost periodic solutions when $\gamma > 0$.** In this paragraph we establish the existence and uniqueness of the almost periodic solution when g satisfies (2.3).

Proposition 3.4. *Assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies* (2.3), (3.1). *Then there is a unique almost periodic solution for* (2.1).

Proof. The uniqueness follows from Corollary 2.1 since all almost periodic functions are bounded functions. For the existence use Proposition 2.3. Indeed, by Proposition 3.3 we deduce that $\sup_{t\in\mathbb{R}} |g(t, 0)| < +\infty$ and therefore there is a unique bounded solution x for (2.1). It remains to prove that x is almost periodic. Take $R_0 = \|x\|_{L^\infty(\mathbb{R})}$, $\varepsilon > 0$ and $l = l(\varepsilon\gamma, R_0) > 0$. All intervals of length l contains a number τ such that

$$|g(t+\tau, x) - g(t, x)| \leq \varepsilon\gamma, \quad \forall (t, x) \in \mathbb{R} \times [-R_0, R_0].$$

We have for all $t \in \mathbb{R}$

$$\begin{aligned} &x'(t+\tau) - x'(t) + g(t+\tau, x(t+\tau)) - g(t+\tau, x(t)) \\ &= -(g(t+\tau, x(t)) - g(t, x(t))), \end{aligned}$$

and after multiplication by $x(t+\tau) - x(t)$ one gets

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}|x(t+\tau) - x(t)|^2 + \gamma|x(t+\tau) - x(t)|^2 \\ &\leq |g(t+\tau, x(t)) - g(t, x(t))| \cdot |x(t+\tau) - x(t)|, \qquad t \in \mathbb{R}. \end{aligned}$$

We obtain by using Bellman's lemma

$$e^{\gamma t}|x(t+\tau)-x(t)| \leq e^{\gamma t_0}|x(t_0+\tau)-x(t_0)| + \Big(\int_{t_0}^{t} e^{\gamma s} ds\Big) \sup_{(r,y)\in\mathbb{R}\times[-R_0,R_0]} |g(r+\tau,y)-g(r,y)|, \;\; \forall t \geq t_0.$$

Finally, we deduce

$$|x(t+\tau)-x(t)| \leq e^{-\gamma(t-t_0)}|x(t_0+\tau)-x(t_0)| + \varepsilon, \;\; \forall t \geq t_0.$$

By keeping t fixed and passing $t_0 \to -\infty$ we obtain

$$|x(t+\tau)-x(t)| \leq \varepsilon, \;\; \forall t \in \mathbb{R},$$

and thus x is an almost periodic function. □

We mention here the following result which will be used in the next paragraph.

Proposition 3.5. *Assume that $g : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ satisfies* (2.3),

$$\sup_{t\in\mathbb{R}} |g(t,0)| = C < +\infty$$

and consider $(h_n)_n$ a real sequence such that $(g(t+h_n,x))_n$ converges uniformly on $\mathbb{R}\times[-R,R]$, for all $R>0$

$$\lim_{n\to+\infty} g(t+h_n,x) =: \tilde{g}(t,x), \;\; \textit{uniformly on } \mathbb{R}\times[-R,R], \; \forall R>0.$$

Denote by x the unique bounded solution of (2.1)*. Then $(x(\cdot+h_n))_n$ converges uniformly on $\mathbb{R}$ towards the unique bounded solution of* (2.1) *associated to the function $\tilde{g}$.*

Proof. Note that the function $\tilde{g}$ satisfies (2.3) and $\sup_{t\in\mathbb{R}} |\tilde{g}(t,0)| = C < +\infty$ and therefore, by Proposition 2.3, we deduce that there is a unique bounded solution $\tilde{x}$ for

$$\tilde{x}' + \tilde{g}(t,\tilde{x}(t)) = 0, \;\; \forall t \in \mathbb{R}.$$

We introduce the notation $x_n(t) = x(t+h_n), g_n(t,x) = g(t+h_n,x)$. By the computations in the proof of Proposition 2.3 we know that

$$\max(\|x\|_{L^\infty(\mathbb{R})}, \|x_n\|_{L^\infty(\mathbb{R})}, \|\tilde{x}\|_{L^\infty(\mathbb{R})}) \leq \frac{C}{\gamma} =: R_0.$$

We obtain as before that for all $t \in \mathbb{R}$, for all n,

$$\frac{1}{2}\frac{d}{dt}|x_n-\tilde{x}|^2 + \gamma|x_n(t)-\tilde{x}(t)|^2 \leq \|g_n-\tilde{g}\|_{L^\infty(\mathbb{R}\times[-R_0,R_0])} \cdot |x_n(t)-\tilde{x}(t)|,$$

which implies that

$$\|x_n - \tilde{x}\|_{L^\infty(\mathbb{R})} \leq \frac{1}{\gamma}\|g_n - \tilde{g}\|_{L^\infty(\mathbb{R}\times[-R_0,R_0])}, \quad \forall n.$$

3.2. **Existence of an almost periodic solution when** $\gamma = 0$**.** We want to establish now the existence of an almost periodic solution for (2.1) when g satisfies (2.2), (3.1). We assume also that there are constant sub/super-solutions for (2.1)

$$\exists M > 0 \ : \ g(t, -M) \leq 0 \leq g(t, M), \quad \forall t \in \mathbb{R}. \tag{3.2}$$

We need several easy lemmas concerning almost periodic functions.

Lemma 3.1. *Let* $f : \mathbb{R} \to \mathbb{R}^+$ *be a nonnegative almost periodic function such that* $\langle f \rangle = 0$*. Then* $f = 0$*.*

Proof. Suppose that there is $t_0 \in \mathbb{R}$ such that $f(t_0) > 0$. We deduce that there is $\delta_0 > 0$ such that $f(t) \geq 2\varepsilon_0$, for all $t \in [t_0 - \delta_0, t_0 + \delta_0]$. Consider $l_0 = l(\varepsilon) > 0$ such that all intervals of length l_0 contain an ε_0-almost period of f. Without loss of generality we can assume that $\delta_0 < l_0$. For all $k \geq 0$ the interval $[2kl_0 + \delta_0, 2(k+1)l_0[$ contains an ε_0-almost period of f denoted τ_k. Observe that we have the inclusion

$$[t_0 - \delta_0 + \tau_k, t_0 + \tau_k[\subset [t_0 + 2kl_0, t_0 + (2k+2)l_0[,$$

and the inequality

$$f(t + \tau_k) = f(t) + f(t + \tau_k) - f(t) \geq 2\varepsilon_0 - \varepsilon_0 = \varepsilon_0, \quad \forall t \in [t_0 - \delta_0, t_0 + \delta_0].$$

We have

$$\begin{aligned}
&\frac{1}{2Nl_0}\int_{t_0}^{t_0+2Nl_0} f(t)\,dt = \frac{1}{2Nl_0}\sum_{k=0}^{N-1}\int_{t_0+2kl_0}^{t_0+(2k+2)l_0} f(t)\,dt \\
&\geq \frac{1}{2Nl_0}\sum_{k=0}^{N-1}\int_{t_0-\delta_0+\tau_k}^{t_0+\tau_k} f(t)\,dt = \frac{1}{2Nl_0}\sum_{k=0}^{N-1}\int_{t_0-\delta_0}^{t_0} f(t+\tau_k)\,dt \\
&\geq \frac{1}{2Nl_0}\sum_{k=0}^{N-1}\delta_0\varepsilon_0 = \frac{\delta_0\varepsilon_0}{2l_0}, \quad \forall N \geq 1.
\end{aligned}$$

By passing to the limit for $N \to +\infty$ we deduce that $\langle f \rangle \geq \frac{\delta_0\varepsilon_0}{2l_0}$ which contradicts our hypothesis. Therefore, $f = 0$. $\square$

Generally, when g satisfies (2.2) we have no uniqueness for the almost periodic solution of (2.1). Nevertheless it is possible to show that the difference between two almost periodic solutions is a constant.

Proposition 3.6. *Assume that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (2.2), (3.1) *and consider* x, y *two almost periodic solutions of* (2.1). *Then there is a constant* $C \in \mathbb{R}$ *such that* $x(t) - y(t) = C$, *for all* $t \in \mathbb{R}$.

Proof. We have

$$\frac{1}{2}\frac{d}{dt}|x(t)-y(t)|^2 + (g(t,x(t)) - g(t,y(t))) \cdot (x(t)-y(t)) = 0, \quad \forall t \in \mathbb{R}. \tag{3.3}$$

Since g satisfies (3.1) and x, y are almost periodic, we check easily that the functions $t \to g(t,x(t))$, $t \to g(t,y(t))$ are almost periodic and thus $t \to H(t) := (g(t,x(t)) - g(t,y(t))) \cdot (x(t)-y(t))$ is almost periodic and nonnegative. After integration of (3.3) we deduce

$$\frac{1}{2T}|x(T)-y(T)|^2 - \frac{1}{2T}|x(0)-y(0)|^2 + \frac{1}{T}\int_0^T H(t)\,dt = 0.$$

Since x, y are bounded we deduce that $\langle H \rangle = 0$ and by Lemma 3.1 one gets $H = 0$ which implies that

$$g(t,x(t)) = g(t,y(t)), \quad \forall t \in \mathbb{R}.$$

Therefore we obtain

$$x'(t) = -g(t,x(t)) = -g(t,y(t)) = y'(t), \quad \forall t \in \mathbb{R},$$

and thus there is a constant $C \in \mathbb{R}$ such that $x(t) - y(t) = C$, $\forall t \in \mathbb{R}$. □

In the following propositions we establish two important properties of the minimal/maximal solutions.

Proposition 3.7. *Assume that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (2.2), (3.1), (3.2). *We denote by* $x = x_{min}(-M, M)$, $y = y_{max}(-M, M)$ *the minimal and maximal solutions of* (2.1) *respectively. Then there is a constant* $C \in \mathbb{R}$ *such that*

$$y(t) - x(t) = C, \quad \forall t \in \mathbb{R}.$$

Proof. We have $(x(t), y(t)) = (\sup_{\alpha>0} x_\alpha(t), \inf_{\alpha>0} y_\alpha(t))$ for all $t \in \mathbb{R}$, where x_α, y_α solve

$$\alpha(x_\alpha(t) + M) + x'_\alpha(t) + g(t, x_\alpha(t)) = 0, \quad t \in \mathbb{R},$$

$$\alpha(y_\alpha(t) - M) + y'_\alpha(t) + g(t, y_\alpha(t)) = 0, \quad t \in \mathbb{R}.$$

By Proposition 3.4 we know that x_α, y_α are almost periodic functions. We introduce the notation $z_\alpha(t) = y_\alpha(t) - x_\alpha(t)$, $z(t) = y(t) - x(t)$, $t \in \mathbb{R}$. Obviously we have $\lim_{\alpha \searrow 0} z_\alpha = z$. We have the inequalities

$$-M \le x_\alpha \le x \le y \le y_\alpha \le M, \quad \forall \alpha > 0,$$

which implies $z_\alpha \geq z$, for all $\alpha > 0$. By the hypothesis (2.2) we deduce easily that z is nonincreasing. Assume that there is $t_1 < t_2$ such that $z(t_1) > z(t_2)$ and denote $\eta = z(t_1) - z(t_2) > 0$. For α small enough we have

$$z_\alpha(t_2) \leq z(t_2) + \frac{\eta}{2}.$$

Observe that for $t \leq t_1$ we have

$$z_\alpha(t) \geq z(t) \geq z(t_1) = \eta + z(t_2) \geq z_\alpha(t_2) + \frac{\eta}{2}. \tag{3.4}$$

Take now τ a $\frac{\eta}{4}$-almost period of z_α such that $\tau \geq t_2 - t_1$. We have

$$z_\alpha(t_2 - \tau) \leq z_\alpha(t_2) + \frac{\eta}{4}. \tag{3.5}$$

Combining (3.4) and (3.5) one gets

$$z_\alpha(t_2) + \frac{\eta}{2} \leq z_\alpha(t_2 - \tau) \leq z_\alpha(t_2) + \frac{\eta}{4},$$

which gives a contradiction. Therefore z is a constant function. □

Proposition 3.8. *Assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies* (2.2), (3.1), (3.2) *and consider $(h_n)_n$ a real sequence such that $(g(\cdot + h_n, \cdot))_n$ converges uniformly on $\mathbb{R} \times [-R, R]$, for all $R > 0$. We denote by $x = x_{min}(-M, M)$, $y = y_{max}(-M, M)$ the minimal and maximal solutions of* (2.1) *respectively. Then we have the convergence*

$$\lim_{n\to+\infty} (x(t + h_n), y(t + h_n)) = (\tilde{x}(t), \tilde{y}(t)), \quad \textit{uniformly on } \mathbb{R},$$

where $\tilde{x} = \tilde{x}_{min}(-M, M)$ and $\tilde{y} = \tilde{y}_{max}(-M, M)$ are respectively the minimal and maximal solutions of (2.1) *associated to the function $\tilde{g}(t, x) := \lim_{n\to+\infty} g(t + h_n, x)$, $(t, x) \in \mathbb{R} \times \mathbb{R}$.*

Proof. Notice that the function $\tilde{g}$ satisfies the same hypotheses as g. In particular the solutions $\tilde{x}, \tilde{y}$ are well defined (cf. Propositions 2.4, 2.6). Recall that

$$x = \sup_{\alpha>0} x_\alpha, \quad y = \inf_{\alpha>0} y_\alpha, \quad \tilde{x} = \sup_{\alpha>0} \tilde{x}_\alpha, \quad \tilde{y} = \inf_{\alpha>0} \tilde{y}_\alpha,$$

where for $t \in \mathbb{R}$,,

$$\alpha(x_\alpha(t)+M)+x'_\alpha(t)+g(t, x_\alpha(t)) = 0, \quad \alpha(\tilde{x}_\alpha(t)+M)+\tilde{x}'_\alpha(t)+\tilde{g}(t, \tilde{x}_\alpha(t)) = 0,$$

$$\alpha(y_\alpha(t)-M)+y'_\alpha(t)+g(t, y_\alpha(t)) = 0, \quad \alpha(\tilde{y}_\alpha(t)-M)+\tilde{y}'_\alpha(t)+\tilde{g}(t, \tilde{y}_\alpha(t)) = 0.$$

By Proposition 3.5 we have the convergence

$$\lim_{n\to+\infty} (x_\alpha(t + h_n), y_\alpha(t + h_n)) = (\tilde{x}_\alpha(t), \tilde{y}_\alpha(t)), \quad \text{uniformly on } \mathbb{R}.$$

By Proposition 3.7 there are the constants $C, \tilde{C} \in \mathbb{R}$ such that

$$y(t) - x(t) = C, \quad \tilde{y}(t) - \tilde{x}(t) = \tilde{C}, \quad \forall t \in \mathbb{R}.$$

We prove that $C = \tilde{C}$. Indeed, we can write

$$C = y(t+h_n) - x(t+h_n) \le y_\alpha(t+h_n) - x_\alpha(t+h_n),$$

and by passing to the limit for $n \to +\infty$ one gets

$$C \le \tilde{y}_\alpha(t) - \tilde{x}_\alpha(t), \forall t \in \mathbb{R}, \alpha > 0.$$

We deduce that

$$C \le \lim_{\alpha \searrow 0} (\tilde{y}_\alpha(t) - \tilde{x}_\alpha(t)) = \tilde{y}(t) - \tilde{x}(t) = \tilde{C}.$$

For the reverse inequality observe that since $(g(\cdot + h_n, \cdot))_n$ converges towards $\tilde{g}$ uniformly on $\mathbb{R} \times [-R, R]$, for all $R > 0$, then $(\tilde{g}(\cdot - h_n, \cdot))_n$ converges towards g uniformly on $\mathbb{R} \times [-R, R]$, for all $R > 0$ and we deduce as before that $\tilde{C} \le C$. We prove now that $(x(t+h_n), y(t+h_n))$ converges towards $(\tilde{x}(t), \tilde{y}(t))$, for all $t \in \mathbb{R}$. Indeed, we have for all $\alpha > 0$

$$\begin{aligned}\limsup_{n\to+\infty} y(t+h_n) - \liminf_{n\to+\infty} x(t+h_n) &\le \lim_{n\to+\infty} y_\alpha(t+h_n) - \lim_{n\to+\infty} x_\alpha(t+h_n)\\ &= \tilde{y}_\alpha(t) - \tilde{x}_\alpha(t).\end{aligned}$$

By taking the limit as $\alpha \searrow 0$ we obtain the inequality

$$\limsup_{n\to+\infty} y(t+h_n) - \liminf_{n\to+\infty} x(t+h_n) \le C. \tag{3.6}$$

By using the equality $y(t+h_n) - x(t+h_n) = C$, for all $t \in \mathbb{R}$, for all n, we obtain also

$$\limsup_{n\to+\infty} y(t+h_n) = \limsup_{n\to+\infty} x(t+h_n) + C. \tag{3.7}$$

Combining (3.6), (3.7) yields

$$\limsup_{n\to+\infty} x(t+h_n) \le \liminf_{n\to+\infty} x(t+h_n), \quad \forall t \in \mathbb{R}, \tag{3.8}$$

and therefore $(x(t+h_n))_n, (y(t+h_n))_n$ converge for all $t \in \mathbb{R}$. Now we can write

$$\lim_{n\to+\infty} y(t+h_n) \le \inf_{\alpha>0} \lim_{n\to+\infty} y_\alpha(t+h_n) = \inf_{\alpha>0} \tilde{y}_\alpha(t) = \tilde{y}(t), \tag{3.9}$$

and

$$\lim_{n\to+\infty} x(t+h_n) \ge \sup_{\alpha>0} \lim_{n\to+\infty} x_\alpha(t+h_n) = \sup_{\alpha>0} \tilde{x}_\alpha(t) = \tilde{x}(t). \tag{3.10}$$

Note also that

$$\lim_{n\to+\infty} y(t+h_n) - \lim_{n\to+\infty} x(t+h_n) = \lim_{n\to+\infty} (y(t+h_n) - x(t+h_n)) \tag{3.11}$$

$$= C = \tilde{y}(t) - \tilde{x}(t).$$

We deduce from (3.9), (3.10), (3.11) that $\lim_{n\to+\infty}(x(t+h_n), y(t+h_n)) = (\tilde{x}(t), \tilde{y}(t))$. It remains to prove that the above convergence is uniform on $\mathbb{R}$. We use the same method as in [8]. Since $y - x = C$ it is sufficient to treat only the convergence $\lim_{n\to+\infty} x(t+h_n) = \tilde{x}(t)$. Assume that the convergence is not uniform

$$\exists \varepsilon_0 > 0 \ : \ |x(t_k + h_{n_k}) - \tilde{x}(t_k)| \geq \varepsilon_0, \ \ \forall k, \tag{3.12}$$

for a sequence $(t_k)_k$ and a subsequence $(h_{n_k})_k$. Then Theorem 3.2 implies (after extraction if necessary) that

$$\lim_{k\to+\infty} g(t + t_k + h_{n_k}, x) = \tilde{\tilde{g}}(t, x), \text{uniformly for } (t, x) \in \mathbb{R} \times [-R, R], \forall R > 0.$$

Therefore for all $R > 0, \varepsilon > 0$, there exists $k_1 = k_1(R, \varepsilon)$ such that

$$|g(t + t_k + h_{n_k}, x) - \tilde{\tilde{g}}(t, x)| < \varepsilon, \forall (t, x) \in \mathbb{R} \times [-R, R], \forall k \geq k_1. \tag{3.13}$$

Recall that for all $R > 0, \varepsilon > 0$, there exists $k_2 = k_2(R, \varepsilon)$ such that

$$|g(t + h_{n_k}, x) - \tilde{g}(t, x)| < \varepsilon, \forall (t, x) \in \mathbb{R} \times [-R, R], \forall k \geq k_2.$$

In particular we obtain

$$|g(t + t_k + h_{n_k}, x) - \tilde{g}(t + t_k, x)| < \varepsilon, \forall (t, x) \in \mathbb{R} \times [-R, R], \forall k \geq k_2. \tag{3.14}$$

From (3.13), (3.14) we obtain the convergence

$$\lim_{k\to+\infty} \tilde{g}(t + t_k, x) = \tilde{\tilde{g}}(t, x), \text{uniformly on } \mathbb{R} \times [-R, R], \forall R > 0.$$

We denote by $\tilde{x}, \tilde{\tilde{x}}$ the minimal solutions associated to the functions $\tilde{g}, \tilde{\tilde{g}}$. The convergence $\lim_{k\to+\infty} g(t + t_k + h_{n_k}, x) = \tilde{\tilde{g}}(t, x)$ uniformly on $\mathbb{R} \times [-R, R]$, for all $R > 0$ implies the convergence $\lim_{k\to+\infty} x(t + t_k + h_{n_k}) = \tilde{\tilde{x}}(t)$, for all $t \in \mathbb{R}$. In particular we have

$$\lim_{k\to+\infty} x(t_k + h_{n_k}) = \tilde{\tilde{x}}(0). \tag{3.15}$$

The convergence $\lim_{k\to+\infty} \tilde{g}(t + t_k, x) = \tilde{\tilde{g}}(t, x)$ uniformly on $\mathbb{R} \times [-R, R]$, for all $R > 0$ implies the convergence $\lim_{k\to+\infty} \tilde{x}(t + t_k) = \tilde{\tilde{x}}(t)$, for all $t \in \mathbb{R}$. In particular we have

$$\lim_{k\to+\infty} \tilde{x}(t_k) = \tilde{\tilde{x}}(0). \tag{3.16}$$

From (3.15), (3.16) we deduce that

$$\lim_{k\to+\infty} (x(t_k + h_{n_k}) - \tilde{x}(t_k)) = 0,$$

which contradicts the assumption (3.12). □

Now we can prove the following existence result for almost periodic solutions.

Theorem 3.3. *Assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies* (2.2), (3.1), (3.2). *Then there is at least one almost periodic solution x for* (2.1) *satisfying*

$$-M \leq x(t) \leq M, \quad \forall t \in \mathbb{R}.$$

Proof. Consider $x = x_{min}(-M, M)$ the minimal solution of (2.1), cf. Proposition 2.4. By construction we have $-M \leq x(t) \leq M$, for all $t \in \mathbb{R}$. We need to check that x is almost periodic. We use Bochner's theorem. Consider $(h_n)_n$ a real sequence. After extraction of a subsequence we have

$$\lim_{k \to +\infty} g(t + h_{n_k}, x) = \tilde{g}(t, x), \quad \text{uniformly on } \mathbb{R} \times [-R, R], \forall R > 0.$$

By Proposition 3.8 we have

$$\lim_{k \to +\infty} x(t + h_{n_k}) = \tilde{x}(t), \quad \text{uniformly on } \mathbb{R},$$

where $\tilde{x} = \tilde{x}_{min}(-M, M)$ is the minimal solution associated to $\tilde{g}$. Therefore $(x(\cdot + h_{n_k}))_k$ converges uniformly and hence the minimal solution is almost periodic. □

Remark 3.1. In the above theorem we can replace (2.2) by $g|_{\mathbb{R} \times [-M,M]}$ nondecreasing with respect to x (apply the previous theorem with the function $g_1(t, x) = \mathbf{1}_{]-\infty,-M[}(x) g(t, -M) + \mathbf{1}_{[-M,M]}(x) g(t, x) + \mathbf{1}_{]M,+\infty[}(x) g(t, M)$, for all $(t, x) \in \mathbb{R} \times \mathbb{R}$).

In fact it is possible to show that all bounded solutions of (2.1) are almost periodic. For the completeness of the presentation we recall here the following result (cf. [8]).

Proposition 3.9. *Assume that $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ satisfies* (2.2), (3.1) *and consider x an almost periodic solution and y a bounded solution for* (2.1). *Then there is a constant $C \in \mathbb{R}$ such that $x(t) - y(t) = C$, for all $t \in \mathbb{R}$. In particular y is an almost periodic solution.*

Proof. From Proposition 2.2 we deduce that $x(t) - y(t) \geq 0$, for all $t \in \mathbb{R}$ or $x(t) - y(t) \leq 0$, for all $t \in \mathbb{R}$. We analyze the case $x(t) - y(t) \leq 0$, for all $t \in \mathbb{R}$; the other case follows in a similar way. As usual we obtain

$$\frac{1}{2} \frac{d}{dt} |x(t) - y(t)|^2 + (g(t, x(t)) - g(t, y(t))) \cdot (x(t) - y(t)) = 0, \quad \forall t \in \mathbb{R},$$

and therefore $y(t) - x(t)$ is nonincreasing on $\mathbb{R}$ which implies that $y'(t) - x'(t) \leq 0$, for all $t \in \mathbb{R}$. Assume that there is $t_0 \in \mathbb{R}$ such that $y'(t_0) - x'(t_0) <$

0. We deduce that there is $k > 0$ and $[a, b] \subset \mathbb{R}$ such that $x'(t) - y'(t) = g(t, y(t)) - g(t, x(t)) \geq 2k$, for all $t \in [a, b]$. For $t \leq a$ we have

$$y(t) - x(t) \geq y(a) - x(a) = \eta \geq 0, \tag{3.17}$$

and thus

$$x'(t) - y'(t) = g(t, y(t)) - g(t, x(t)) \geq g(t, x(t) + \eta) - g(t, x(t)), \quad \forall t \leq a. \tag{3.18}$$

It is easy to check that $h(t) = g(t, x(t) + \eta) - g(t, x(t)) \geq 0$ is an almost periodic function. Moreover, for all $t \in [a, b]$ we have

$$y(t) - x(t) \leq y(a) - x(a) = \eta,$$

and therefore

$$h(t) \geq g(t, y(t)) - g(t, x(t)) = x'(t) - y'(t) \geq 2k. \tag{3.19}$$

After integration of (3.18) one gets

$$0 \geq x(a) - y(a) \geq x(t) - y(t) + \int_t^a h(s)\, ds, \quad \forall t \leq a,$$

which implies

$$\int_t^a h(s)\, ds \leq \|x\|_{L^\infty(\mathbb{R})} + \|y\|_{L^\infty(\mathbb{R})}, \quad \forall t \leq a. \tag{3.20}$$

Combining (3.19), (3.20) and the almost periodicity of $h(t)$ provides a contradiction, since $\lim_{t \to -\infty} \int_t^a h(s)\, ds = +\infty$. Therefore $x'(t) - y'(t) = 0$, for all $t \in \mathbb{R}$ and there is a constant $C \in \mathbb{R}$ such that $x(t) - y(t) = C$, for all $t \in \mathbb{R}$. □

4. Necessary and sufficient conditions for the existence of almost periodic solutions

In the previous section we indicated sufficient conditions for the existence of an almost periodic solution for (2.1) when g is nondecreasing with respect to x (see Theorem 3.3). In fact the hypothesis (3.2) is not necessary for the existence of an almost periodic solution. For this we can take the easy example

$$x'(t) + g(t, x(t)) = 0, \quad \forall t \in \mathbb{R}, \tag{4.1}$$

where $g(t, x) = \sin t$, for all $t \in \mathbb{R}$. Note that $x(t) = \cos t$, for all $t \in \mathbb{R}$, is a periodic solution for (4.1) but that the hypothesis (3.2) is not satisfied. In this section we analyze other sufficient conditions and we give also a necessary condition for the existence of an almost periodic solution for (2.1). We start with the following necessary condition.

Proposition 4.1. *Assume that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (2.2), (3.1). *If there is a bounded solution for* (2.1), *then there is* $X \in \mathbb{R}$ *such that*

$$\langle g(\cdot, X)\rangle := \lim_{T\to+\infty} \frac{1}{T}\int_0^T g(t,X)\,dt = 0. \tag{4.2}$$

Proof. Let us denote by $G : \mathbb{R} \to \mathbb{R}$ the average function

$$G(x) = \lim_{T\to+\infty} \frac{1}{T}\int_0^T g(t,x)\,dt, \quad \forall x \in \mathbb{R}.$$

We check easily that G is a continuous nondecreasing function. Consider x a bounded solution for (2.1) and $m, M \in \mathbb{R}$ such that

$$m \le x(t) \le M, \quad \forall t \in \mathbb{R}.$$

We have

$$g(t,m) \le g(t,x(t)) \le g(t,M), \quad \forall t \in \mathbb{R},$$

and thus

$$\frac{1}{T}\int_0^T g(t,m)\,dt \le \frac{1}{T}(x(0)-x(T)) \le \frac{1}{T}\int_0^T g(t,M)\,dt, \quad \forall T > 0.$$

After passing to the limit for $T \to +\infty$ we find $G(m) \le 0 \le G(M)$, and hence there exists $X \in [m, M]$ such that $G(X) = 0$. □

We denote by C the convex closed set $C = \{X \in \mathbb{R} : G(X) = 0\}$. We intend to construct an almost periodic solution for (2.1) by taking the limit $x = \lim_{\alpha\searrow 0} x_\alpha^X$ where $x_\alpha^X = x_\alpha$ is the unique almost periodic solution for

$$\alpha(x_\alpha(t) - X) + x_\alpha'(t) + g(t, x_\alpha(t)) = 0, \quad t \in \mathbb{R}, \tag{4.3}$$

for all $\alpha > 0$, for all $X \in C$ (cf. Proposition 3.4). We need to find uniform estimates with respect to $\alpha > 0$. We use the following hypothesis

$$\exists X \in \mathbb{R} \;:\; \sup_{s,t\in\mathbb{R}} \Big(-\int_s^t g(\sigma, X)\,d\sigma\Big) < +\infty. \tag{4.4}$$

Note that if $X \in \mathbb{R}$ satisfies (4.4), then $X \in C$. Moreover, if $Y \in C$ then

$$\langle g(\cdot, X)\rangle - \langle g(\cdot, Y)\rangle = 0.$$

Assume that $X \ge Y$ which implies $g(t,X) \ge g(t,Y)$, for all $t \in \mathbb{R}$. By Lemma 3.1 we obtain that $g(t,X) = g(t,Y)$, for all $t \in \mathbb{R}$ and therefore the hypothesis (4.4) holds for all $Y \in C$. We establish now a priori estimates for $(x_\alpha)_\alpha$.

Proposition 4.2. *Assume that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (2.2), (3.1) *and* (4.4). *Then*

$$\|x_\alpha^X - X\|_{L^\infty(\mathbb{R})} \le \sup_{s,t\in\mathbb{R}} \Big(- \int_s^t g(\sigma, X)\ d\sigma\Big), \ \ \forall \alpha > 0, \forall X \in C.$$

Proof. We write

$$\alpha(x_\alpha(t)-X)+\frac{d}{dt}(x_\alpha(t)-X)+g(t,x_\alpha(t))-g(t,X) = -g(t,X), \forall t \in \mathbb{R}, \forall \alpha > 0.$$

Since $g(t, \cdot)$ is nondecreasing we have for some $r_\alpha(t) \ge 0$

$$g(t, x_\alpha(t)) - g(t, X) = r_\alpha(t)(x_\alpha(t) - X), \ \ \forall t \in \mathbb{R}, \alpha > 0,$$

and thus we obtain

$$(\alpha + r_\alpha(t))(x_\alpha(t) - X) + \frac{d}{dt}(x_\alpha(t) - X) = -g(t, X).$$

After integration we find for all $t_0 \le t$

$$\begin{aligned} x_\alpha(t) - X &= e^{-(A_\alpha(t)-A_\alpha(t_0))}(x_\alpha(t_0) - X) - e^{-A_\alpha(t)} \int_{t_0}^t g(s, X)e^{A_\alpha(s)}\ ds \\ &= I_1 + I_2, \end{aligned} \tag{4.5}$$

where $A_\alpha(t) = \int_0^t \{\alpha + r_\alpha(s)\}\ ds$, for all $t \in \mathbb{R}$. By Proposition 2.3 applied with the function $\tilde{g}(t, x) = \alpha(x - X) + g(t, x)$ we have

$$\|x_\alpha\|_{L^\infty(\mathbb{R})} \le \frac{\sup_{t\in\mathbb{R}} |g(t, 0)|}{\alpha} + |X|,$$

and thus we obtain

$$|I_1| \le e^{-\alpha(t-t_0)}(\alpha^{-1} \sup_{t\in\mathbb{R}} |g(t, 0)| + 2|X|). \tag{4.6}$$

In order to estimate the second term I_2 we introduce the function

$$F(s) = -\int_s^t g(\sigma, X)\ d\sigma, \ \ \forall s \in \mathbb{R}.$$

We have

$$\begin{aligned} I_2 &= -e^{-A_\alpha(t)} \int_{t_0}^t F'(s)e^{A_\alpha(s)}\ ds \\ &= e^{-(A_\alpha(t)-A_\alpha(t_0))}F(t_0) + e^{-A_\alpha(t)} \int_{t_0}^t F(s)e^{A_\alpha(s)}A'_\alpha(s)\ ds \\ &\le e^{-(A_\alpha(t)-A_\alpha(t_0))}F(t_0) + (1 - e^{-(A_\alpha(t)-A_\alpha(t_0))}) \sup_{t_0\le s\le t} F(s) \\ &\le \sup_{t_0\le s\le t} F(s) \le \sup_{s\le t} F(s). \end{aligned} \tag{4.7}$$

Similarly, we can prove that

$$I_2 \geq \inf_{t_0 \leq s \leq t} F(s) \geq \inf_{s \leq t} F(s). \tag{4.8}$$

Combining (4.5), (4.6), (4.7), (4.8) we deduce that for all $t_0 \leq t$, we have

$$\begin{aligned} &- e^{-\alpha(t-t_0)}(\|x_\alpha\|_{L^\infty} + |X|) + \inf_{s \leq t} F(s) \leq x_\alpha(t) - X \\ &\leq e^{-\alpha(t-t_0)}(\|x_\alpha\|_{L^\infty} + |X|) + \sup_{s \leq t} F(s). \end{aligned}$$

After passing to the limit for $t_0 \to -\infty$ we obtain for all $t \in \mathbb{R}$

$$\begin{aligned} - \sup_{t_1, s \in \mathbb{R}} \Big(- \int_{t_1}^{s} g(\sigma, X) d\sigma \Big) \leq \inf_{s \leq t} F(s) \leq x_\alpha(t) - X \\ \leq \sup_{s \leq t} F(s) \leq \sup_{s, t_1 \in \mathbb{R}} \Big(- \int_{s}^{t_1} g(\sigma, X) d\sigma \Big), \end{aligned}$$

and therefore the sequence $(x_\alpha)_\alpha$ is bounded

$$\|x_\alpha - X\|_{L^\infty(\mathbb{R})} \leq \sup_{s,t \in \mathbb{R}} \Big(- \int_{s}^{t} g(\sigma, X) \, d\sigma \Big), \ \ \forall \alpha > 0.$$

□

Remark 4.1. In the above proof we have used the following important inequalities

$$\inf_{t_0 \leq s \leq t} \int_{s}^{t} h(\sigma) \, d\sigma \leq e^{-A_\alpha(t)} \int_{t_0}^{t} e^{A_\alpha(s)} h(s) \, ds \leq \sup_{t_0 \leq s \leq t} \int_{s}^{t} h(\sigma) \, d\sigma.$$

We state now the following existence result for almost periodic solutions.

Theorem 4.1. *Assume that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (2.2), (3.1), (4.4). *Then there is at least one almost periodic solution for* (2.1).

Proof. Take $X \in C$ and consider the unique almost periodic solution x_α for (4.3), for all $\alpha > 0$. By Proposition 4.2 we have the inequalities

$$-D \leq x_\alpha(t) - X \leq D, \forall \alpha > 0, \forall t \in \mathbb{R},$$

where $D = \sup_{s,t \in \mathbb{R}} \Big(- \int_s^t g(\sigma, X) \, d\sigma \Big)$. We check easily that $(x'_\alpha)_{0 < \alpha \leq 1}$ is bounded since

$$\begin{aligned} |x'_\alpha(t)| &= |\alpha(x_\alpha(t) - X) + g(t, x_\alpha(t))| \\ &\leq D + \sup_{(t,x) \in \mathbb{R} \times [X-D, X+D]} |g(t, x)| < +\infty. \end{aligned}$$

By using the theorem of Arzela-Ascoli we can extract a sequence $(x_{\alpha_n})_n$ which converges uniformly on compact sets to a bounded solution x of (2.1), satisfying

$$-D \le x(t) - X \le D, \quad \forall t \in \mathbb{R}.$$

It remains to prove that x is almost periodic. We consider also the function

$$g_1(t,x) = g(t,x) + (x + D - X)\mathbf{1}_{\{x \le -D+X\}} + (x - D - X)\mathbf{1}_{\{x \ge D+X\}},$$

for all $(t,x) \in \mathbb{R} \times \mathbb{R}$. Note that the function g_1 satisfies the hypotheses (2.2), (3.1), (3.2) with $M = |X| + D + \sup_{t\in\mathbb{R}} |g(t,0)|$. By Theorem 3.3 we know that there is at least one almost periodic solution x_1 for the equation

$$x_1'(t) + g_1(t, x_1(t)) = 0, \quad t \in \mathbb{R}. \tag{4.9}$$

Note that since $-D \le x(t) - X \le D$, for all $t \in \mathbb{R}$ we have

$$g(t, x(t)) = g_1(t, x(t)), \quad t \in \mathbb{R},$$

and therefore x is also a bounded solution for (4.9). By applying Proposition 3.9 we deduce that there is a constant $K \in \mathbb{R}$ such that $x(t) = x_1(t) + K$, for all $t \in \mathbb{R}$. Hence x is an almost periodic function. □

Remark 4.2. In the previous theorem we can replace hypothesis (2.2) by $g|_{\mathbb{R}\times[X-D,X+D]}$ nondecreasing with respect to x.

Remark 4.3. The above theorem contains the well-known result concerning the almost periodicity of the primitive of almost periodic functions. Indeed, take $g(t,x) = -f(t)$ where f is almost periodic such that $F(t) = \int_0^t f(s)\, ds$ is bounded. Therefore we have for all $X \in \mathbb{R}$

$$\Big| \int_s^t g(\sigma, X)\, d\sigma \Big| = \Big| \int_s^t f(\sigma)\, d\sigma \Big| \le 2\|F\|_{L^\infty(\mathbb{R})}$$

and thus there is at least one almost periodic solution x for $x'(t) = f(t), t \in \mathbb{R}$. Since $F(t)$ satisfies also $F'(t) = f(t)$, for all $t \in \mathbb{R}$ we deduce that $F(t) = x(t) + K$, for all $t \in \mathbb{R}$ for some constant $K \in \mathbb{R}$. Thus F is almost periodic.

We end this section with several examples. Consider the linear first-order differential equation

$$x'(t) + a(t)\, x(t) = f(t), \quad t \in \mathbb{R}, \tag{4.10}$$

where $a, f : \mathbb{R} \to \mathbb{R}$ are almost periodic functions, $a \ge 0$. By Theorem 4.1 the hypothesis

$$\exists\, x_0 \in \mathbb{R} \;:\; \sup_{s,t\in\mathbb{R}} \Big(\int_s^t \{f(\sigma) - x_0\, a(\sigma)\}\, d\sigma \Big) < +\infty, \tag{4.11}$$

guarantees the existence of almost periodic solutions for (4.10). In particular existence holds if

$$\sup_{s,t\in\mathbb{R}}\Big(\int_s^t f(\sigma)\,d\sigma\Big) < +\infty.$$

Observe also that if a, f are trigonometric polynomials and $\langle a\rangle > 0$ then condition (4.11) is verified with $x_0 = \langle f\rangle\ \langle a\rangle^{-1}$. When $\inf_{\mathbb{R}} a > 0$ one can also deduce the existence of almost periodic solutions for (4.10) by using Theorem 3.3. In this case we obtain the existence of a solution $x(\cdot)$ satisfying

$$-\frac{\|f\|_{L^\infty}}{\inf_{\mathbb{R}} a} \le x(t) \le \frac{\|f\|_{L^\infty}}{\inf_{\mathbb{R}} a}.$$

More generally let us consider the nonlinear equation

$$x'(t) + a(t)\ g(x(t)) = f(t),\ \ t\in\mathbb{R}, \tag{4.12}$$

where $g:\mathbb{R}\to\mathbb{R}$ is continuous, nondecreasing and $a, f:\mathbb{R}\to\mathbb{R}$ are almost periodic, $a \ge 0$. In this case one gets existence under the hypothesis

$$\exists\, x_0\in\mathbb{R}\ :\ \sup_{s,t\in\mathbb{R}}\Big(\int_s^t \{f(\sigma) - g(x_0)\ a(\sigma)\}\,d\sigma\Big) < +\infty. \tag{4.13}$$

When a, f are trigonometric polynomials and $\langle a\rangle > 0$, then existence holds if and only if $\langle f\rangle\ \langle a\rangle^{-1} \in Range(g)$.

5. Other properties of almost periodic solutions

In this section we analyze the asymptotic behavior of almost periodic solutions for large frequencies. We present also uniqueness and stability results.

5.1. **Homogenization.** Assume that $g:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ satisfies (2.2), (3.1), (4.4). By Theorem 4.1 we know that there is at least one almost periodic solution, for example $x = \lim_{\alpha\searrow 0} x_\alpha^X$ where $x_\alpha^X = x_\alpha$ are the unique almost periodic solution of (4.3). Consider the function $g^\varepsilon(t,x) = g\left(\frac{t}{\varepsilon}, x\right)$, $(t,x)\in\mathbb{R}\times\mathbb{R}, \varepsilon > 0$. Note that the functions g^ε satisfy the same hypotheses as g and $C^\varepsilon = \{X\in\mathbb{R} : \langle g^\varepsilon(\cdot,X)\rangle = 0\} = \{X\in\mathbb{R} : \langle g(\cdot,X)\rangle = 0\} = C$. Therefore the equation

$$x'(t) + g^\varepsilon(t,x(t)) = 0,\ \ t\in\mathbb{R}, \tag{5.1}$$

has at least one almost periodic solution, for example $x^\varepsilon = \lim_{\alpha\searrow 0} x_\alpha^{\varepsilon,X}$, where $x_\alpha^{\varepsilon,X} = x_\alpha^\varepsilon$ is the unique almost periodic solution of

$$\alpha(x(t) - X) + x'(t) + g^\varepsilon(t,x(t)) = 0,\ \ t\in\mathbb{R}. \tag{5.2}$$

We want to establish the convergence of $(x^\varepsilon)_\varepsilon$ as $\varepsilon\searrow 0$.

Theorem 5.1. *Assume that* $g : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *satisfies* (2.2), (3.1), (4.4). *With the previous notation we have*

$$\lim_{\varepsilon \searrow 0} x^\varepsilon(t) = X, \quad \textit{uniformly for } t \in \mathbb{R},$$

and

$$\|x^\varepsilon - X\|_{L^\infty(\mathbb{R})} \le \varepsilon \sup_{s,t\in\mathbb{R}} \Big(- \int_s^t g(\sigma, X)\, d\sigma \Big).$$

Proof. Consider the function $y^\varepsilon_\alpha(t) = x^\varepsilon_\alpha(\varepsilon\, t)$, for all $t \in \mathbb{R}$. We deduce that y^ε_α is solution for

$$\alpha\varepsilon(y^\varepsilon_\alpha(t) - X) + \frac{d}{dt} y^\varepsilon_\alpha(t) + \varepsilon g(t, y^\varepsilon_\alpha(t)) = 0, \quad t \in \mathbb{R}.$$

From Proposition 4.2 we deduce that

$$\|y^\varepsilon_\alpha - X\|_{L^\infty(\mathbb{R})} \le \varepsilon \sup_{s,t\in\mathbb{R}} \Big(- \int_s^t g(\sigma, X)\, d\sigma \Big),$$

which implies that

$$|x^\varepsilon_\alpha(\tau) - X| \le \varepsilon \sup_{s,t\in\mathbb{R}} \Big(- \int_s^t g(\sigma, X)\, d\sigma \Big), \quad \forall \tau \in \mathbb{R}, \alpha > 0, \varepsilon > 0.$$

After passing to the limit for $\alpha \searrow 0$ we deduce

$$|x^\varepsilon(\tau) - X| \le \varepsilon \sup_{s,t\in\mathbb{R}} \Big(- \int_s^t g(\sigma, X)\, d\sigma \Big), \quad \forall \tau \in \mathbb{R}, \varepsilon > 0.$$

5.2. **Uniqueness.** In this paragraph we consider the following equation

$$x'(t) + g(x(t)) = f(t), \quad t \in \mathbb{R}, \tag{5.3}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous, nondecreasing and $f : \mathbb{R} \to \mathbb{R}$ is almost periodic. Note that the function $(t, x) \to g(x) - f(t)$, for all $(t, x) \in \mathbb{R} \times \mathbb{R}$ satisfies the hypotheses (2.2), (3.1). In this case the set C is given by

$$C = \{X \in \mathbb{R} : g(X) = \langle f \rangle\} = g^{-1}\langle f \rangle.$$

We suppose that the function f satisfies the hypothesis

$$\sup_{s,t\in\mathbb{R}} \Big(\int_s^t \{f(\sigma) - \langle f \rangle\}\, d\sigma \Big) < +\infty. \tag{5.4}$$

From Theorem 4.1 we deduce that under the hypotheses (5.4) there is at least one almost periodic solution for (5.3). In this case it is possible to give also necessary and sufficient conditions for the uniqueness of the almost periodic solution. We use the following easy lemma.

Lemma 5.1. *Assume that $h : \mathbb{R} \to \mathbb{R}$ is almost periodic. Then we have the equalities*

$$\sup_{s\le t}\{h(t)-h(s)\} = \sup_{s,t\in\mathbb{R}}\{h(t)-h(s)\} = \sup_{s\ge t}\{h(t)-h(s)\} = D,$$

$$\inf_{s\le t}\{h(t)-h(s)\} = \inf_{s,t\in\mathbb{R}}\{h(t)-h(s)\} = \inf_{s\ge t}\{h(t)-h(s)\} = d,$$

$$d + D = 0.$$

Proof. We show only the first and last equality. Obviously we have

$$\sup_{s\le t}\{h(t)-h(s)\} \le \sup_{s,t\in\mathbb{R}}\{h(t)-h(s)\}.$$

For all $\varepsilon > 0$ take $s_\varepsilon, t_\varepsilon \in \mathbb{R}$ such that

$$\sup_{s,t\in\mathbb{R}}\{h(t)-h(s)\} \le \frac{\varepsilon}{2} + h(t_\varepsilon) - h(s_\varepsilon).$$

Take τ large enough (such that $t_\varepsilon + \tau \ge s_\varepsilon$) an $\frac{\varepsilon}{2}$-almost period for h. We have

$$\sup_{s,t\in\mathbb{R}}\{h(t)-h(s)\} \le \frac{\varepsilon}{2} + h(t_\varepsilon+\tau) + \frac{\varepsilon}{2} - h(s_\varepsilon) \le \varepsilon + \sup_{s\le t}\{h(t)-h(s)\}.$$

Finally, one gets that $\sup_{s,t\in\mathbb{R}}\{h(t)-h(s)\} \le \sup_{s\le t}\{h(t)-h(s)\}$ and the first equality follows. For the last equality we write

$$\begin{aligned} D &= \sup_{s,t\in\mathbb{R}}\{h(t)-h(s)\} = \sup_{s\le t}\{h(t)-h(s)\} \\ &= -\inf_{s\le t}\{h(s)-h(t)\} = -\inf_{s,t\in\mathbb{R}}\{h(t)-h(s)\} = -d. \end{aligned}$$

Theorem 5.2. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is continuous nondecreasing and $f : \mathbb{R} \to \mathbb{R}$ is almost periodic satisfying* (5.4) *and such that $\langle f\rangle \in g(\mathbb{R})$. Then, for all $X \in g^{-1}\langle f\rangle$ there is at least one almost periodic solution x for* (5.3) *satisfying*

$$\|x - X\|_{L^\infty(\mathbb{R})} \le \sup_{s,t\in\mathbb{R}} \Big(\int_s^t \{f(\sigma) - \langle f\rangle\}\, d\sigma\Big). \tag{5.5}$$

Moreover, the almost periodic solution is unique if and only if

$$diam(g^{-1}\langle f\rangle) \le \sup_{s,t\in\mathbb{R}} \Big(\int_s^t \{f(\sigma) - \langle f\rangle\}\, d\sigma\Big).$$

Proof. The existence of almost periodic solutions follows from Theorem 4.1. Suppose now that

$$diam(C) > \sup_{s,t\in\mathbb{R}} \Big(\int_s^t \{f(\sigma) - \langle f\rangle\}\, d\sigma \Big)$$

and let us construct two different almost periodic solutions. Denote $F(t) = \int_0^t \{f(\sigma) - \langle f\rangle\}\, d\sigma$, for all $t \in \mathbb{R}$, which is also an almost periodic function. We take ε small enough such that $diam(C) > \sup_{s,t\in\mathbb{R}}\{F(t) - F(s)\} + \varepsilon = \sup F - \inf F + \varepsilon$. Consider $t_\varepsilon \in \mathbb{R}$ such that

$$F(t_\varepsilon) \le \inf F + \frac{\varepsilon}{2},$$

and $X, Y \in C$ such that $Y - X > diam(C) - \frac{\varepsilon}{4}$. Let

$$x_1(t) = X + \int_{t_\varepsilon}^t \{f(\sigma) - \langle f\rangle\}\, d\sigma + \frac{\varepsilon}{2} = X + F(t) - F(t_\varepsilon) + \frac{\varepsilon}{2},\ \ \forall t \in \mathbb{R}.$$

Observe that

$$x_1(t) \ge X + \inf F - F(t_\varepsilon) + \frac{\varepsilon}{2} \ge X, \forall t \in \mathbb{R},$$

and that

$$x_1(t) \le X + \sup F - \inf F + \frac{\varepsilon}{2} < X - \frac{\varepsilon}{2} + diam(C) < Y - \frac{\varepsilon}{4},$$

which implies that $x_1(t) \in C$, for all $t \in C$. Therefore we have $g(x_1(t)) = \langle f\rangle$, for all $t \in \mathbb{R}$ and thus x_1 is an almost periodic solution of (5.3)

$$x_1'(t) + g(x_1(t)) = f(t) - \langle f\rangle + \langle f\rangle = f(t),\ \ \forall t \in \mathbb{R}.$$

Consider now the function $x_2(t) = x_1(t) + \frac{\varepsilon}{4}$. As before we have

$$X + \frac{\varepsilon}{4} \le x_2(t) < Y,\ \ \forall t \in \mathbb{R},$$

and therefore x_2 is another almost periodic solution for (5.3). Suppose now that

$$diam(C) \le \sup_{s,t\in\mathbb{R}} \Big(\int_s^t \{f(\sigma) - \langle f\rangle\}\, d\sigma \Big), \tag{5.6}$$

and let us prove that we have uniqueness of the almost periodic solution. Take x, y two almost periodic solutions. There is a constant $K_1 \in \mathbb{R}$ such that $x(t) - y(t) = K_1$, for all $t \in \mathbb{R}$ and $g(x(t)) = g(y(t))$, for all $t \in \mathbb{R}$. If $K_1 \ne 0$ we deduce easily by using the monotonicity of g that $g(x(t)) = g(y(t)) = K_2$, for all $t \in \mathbb{R}$, for some constant $K_2 \in \mathbb{R}$. In fact K_2 must be the average of f and thus $x(t), y(t) \in C$, for all $t \in \mathbb{R}$ and

$$x(t) - x(s) = \int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma = y(t) - y(s), \;\; \forall s, t \in \mathbb{R}.$$

We deduce that

$$\sup_{s,t \in \mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right) \leq diam(C),$$

and by (5.6) we obtain

$$\begin{aligned} \sup_{s,t \in \mathbb{R}} \{x(t) - x(s)\} &= \sup_{s,t \in \mathbb{R}} \{y(t) - y(s)\} \\ &= \sup_{s,t \in \mathbb{R}} \left(\int_s^t \{f(\sigma) - \langle f \rangle\} \, d\sigma \right) = diam(C). \end{aligned}$$

Take $0 < \varepsilon < \frac{|K_1|}{2}$ and $s_\varepsilon, t_\varepsilon \in \mathbb{R}$ such that

$$\inf(C) + \varepsilon > x(s_\varepsilon), \;\; \sup(C) - \varepsilon < x(t_\varepsilon).$$

If $K_1 > 0$ we obtain that $y(s_\varepsilon) = x(s_\varepsilon) - K_1 < \inf(C) - \frac{K_1}{2} < \inf(C)$ and if $K_1 < 0$ we obtain $y(t_\varepsilon) = x(t_\varepsilon) - K_1 > \sup(C) - \frac{K_1}{2} > \sup(C)$. In both cases we obtained a contradiction since we have already proved that $y(t) \in C$, for all $t \in \mathbb{R}$. Therefore $K_1 = 0$ and the almost periodic solution is unique. □

5.3. **Stability.** We consider the equations

$$x'(t) + g(x(t)) = f_1(t), \;\; t \in \mathbb{R}, \tag{5.7}$$

and

$$x'(t) + g(x(t)) = f_2(t), \;\; t \in \mathbb{R}, \tag{5.8}$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous nondecreasing and $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are almost periodic satisfying $\langle f_1 \rangle = \langle f_2 \rangle \in g(\mathbb{R})$ and

$$\sup_{s,t \in \mathbb{R}} \left(\int_s^t \{f_1(\sigma) - f_2(\sigma)\} \, d\sigma \right) < +\infty. \tag{5.9}$$

By Theorem 4.1 we know that for all $X \in g^{-1}\langle f_1 \rangle = g^{-1}\langle f_2 \rangle$ we can construct the solutions $x_1 = \lim_{\alpha \searrow 0} x_{1\alpha}^X$ of (5.7) and $x_2 = \lim_{\alpha \searrow 0} x_{2\alpha}^X$ of (5.8) where $x_{k\alpha} = x_{k\alpha}^X, k \in \{1, 2\}$ are the unique almost periodic solutions of

$$\alpha(x_{k\alpha}(t) - X) + x'_{k\alpha}(t) + g(x_{k\alpha}(t)) = f_k(t), \;\; t \in \mathbb{R}, \; k \in \{1, 2\}.$$

We can prove the following stability result.

Proposition 5.1. *Assume that $g : \mathbb{R} \to \mathbb{R}$ is continuous nondecreasing and $f_1, f_2 : \mathbb{R} \to \mathbb{R}$ are almost periodic satisfying $\langle f_1 \rangle = \langle f_2 \rangle \in g(\mathbb{R})$ and (5.9). Then, with the previous notation, we have the inequality*

$$\|x_1 - x_2\|_{L^\infty(\mathbb{R})} \le \sup_{s,t\in\mathbb{R}} \Big(\int_s^t \{f_1(\sigma) - f_2(\sigma)\}\, d\sigma \Big). \tag{5.10}$$

Proof. We have the equality

$$\alpha(x_{1\alpha}(t) - x_{2\alpha}(t)) + x'_{1\alpha}(t) - x'_{2\alpha}(t) + g(x_{1\alpha}(t)) - g(x_{2\alpha}(t)) = f_1(t) - f_2(t),\ t \in \mathbb{R}.$$

Since g is nondecreasing we can write

$$g(x_{1\alpha}(t)) - g(x_{2\alpha}(t)) = r_\alpha(t)(x_{1\alpha}(t) - x_{2\alpha}(t)), \quad t \in \mathbb{R},$$

where $r_\alpha(t) \ge 0$, for all $t \in \mathbb{R}$. With the notation $z_\alpha = x_{1\alpha} - x_{2\alpha}$, $h = f_1 - f_2$ one gets

$$(\alpha + r_\alpha(t))z_\alpha(t) + z'_\alpha(t) = h(t), \quad t \in \mathbb{R}.$$

As in the proof of Proposition 4.2 we obtain

$$\|z_\alpha\|_{L^\infty(\mathbb{R})} \le \sup_{s,t\in\mathbb{R}} \Big(\int_s^t h(\sigma)\, d\sigma \Big).$$

The conclusion follows easily by passing to the limit for $\alpha \searrow 0$ in the previous inequality. □

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