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# ON THE EXTERIOR NEUMANN PROBLEM WITH CRITICAL GROWTH

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**Abstract.** In this paper we investigate the solvability of the nonlinear Neumann problem (1.1) involving a critical Sobolev nonlinearity in an exterior domain. We examine the common effect of the shape of the graph of the weight function, the mean curvature of the boundary and a nonlinear perturbation of lower order on the existence of solutions of problem (1.1).

## 1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a smooth boundary $\partial\Omega$ and put $\Omega^c = \mathbb{R}^N - \bar{\Omega}$. In this paper we apply variational methods to investigate the existence of a solution of the exterior Neumann problem

$$\begin{cases} -\Delta u &= Q(x)|u|^{2^*-2}u + \lambda f(x,u) \ \text{ in } \Omega^c, \\ \frac{\partial u}{\partial \nu} &= 0 \ \text{ on } \partial\Omega,\ u > 0 \text{ on } \Omega^c, \end{cases} \tag{1.1}$$

where $\lambda > 0$ is a parameter, $N \geq 3$, $2^* = \frac{2N}{N-2}$ is a critical Sobolev exponent and $Q$ is a positive continuous function on $\bar{\Omega}^c$.

We impose the following assumption on the nonlinearity $f$:

($f_1$) $f \in C(\bar{\Omega}^c \times \mathbb{R}, \mathbb{R})$ and $f(x,0) = 0$ on $\bar{\Omega}^c$. Since we are looking for nonnegative solutions, we may assume that $f(x,s) = 0$ for $s \leq 0$ and $x \in \Omega^c$.

($f_2$) We choose $R_\circ > 0$ so that $\Omega \subset B(0,R_\circ)$ and put $\Omega_R = \Omega^c \cap B(0,R)$ for $R \geq R_\circ$. We assume that for every $R \geq R_\circ$ there exist a constant $\theta_R \in [2, 2^*)$ and constants $a_R$, $b_R > 0$ such that

$$|f(x,s)| \leq a_R s^{\theta_R - 1} + b_R$$

for every $x \in \Omega_R$ and $s \geq 0$.

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($f_3$) There exist constants $r_1, r_2 \in (1, 2^*)$ such that

$$f(x,s) \leq c_1(x)s^{r_1-1} + c_2(x)s^{r_2-1}$$

for every $s \geq 0$ and $x \in \Omega^c$, where $c_i \in L^{\frac{2^*}{2^*-r_i}}(\Omega^c)$, $i = 1, 2$.

($f_4$) There exist constants $\mu \in (1, 2^*)$, $2 < \tau < 2^*$ such that

$$\frac{1}{\tau} f(x,s)s - F(x,s) \geq c_3(x)s^{\mu}$$

for every $x \in \Omega^c$ and $s \geq 0$, where $F(x,t) = \int_0^t f(x,s)\,ds$ and $c_3 \in L^{\frac{2^*}{2^*-\mu}}(\Omega^c)$.

Assumption ($f_4$) replaces the usual Ambrosetti-Rabinowitz condition when we apply the mountain-pass principle. Throughout this paper we assume that the limit

$$Q(\infty) = \lim_{|x|\to\infty} Q(x)$$

exists and $Q(\infty) > 0$. Since $f(x,0) = 0$, $u \equiv 0$ is a solution of problem (1.1). Therefore, our objective is to find a nontrivial solution.

**Example.** We now give an example of a function satisfying $(f_1), \dots, (f_4)$. Let $A$ and $B$ be continuous functions on $\bar{\Omega}^c$ such that $A \in L^{\frac{2^*}{2^*-r_1}}(\Omega^c)$, $A(x) > 0$ on $\bar{\Omega}^c$, $B(x) < 0$ for $|x| \geq R$ for some $R > 0$ with $\Omega \subset B(0,R)$ and $2 < p < r_1 < 2^*$. We define

$$f(x,s) = \begin{cases} A(x)s^{r_1-1} + B(x)s^{p-1} & \text{for } s \geq 0, \\ 0 & \text{for } s < 0. \end{cases}$$

Since $p < r_1$, condition ($f_2$) holds with $\theta_R = r_1$ for every $R > 0$ such that $\Omega \subset B(0,R)$. Condition ($f_3$) holds with $c_1(x) = A(x)$, $c_2(x) = B^+(x)$ and $r_2 = p$, where $a^+ = \max(0,a)$. If $p < \tau < r_1$, then

$$\begin{aligned} \frac{1}{\tau} f(x,s)s \quad - \quad F(x,s) &= \Big(\frac{1}{\tau} - \frac{1}{r_1}\Big)A(x)s^{r_1} + \Big(\frac{1}{\tau} - \frac{1}{p}\Big)B(x)s^p \\ &\geq \Big(\frac{1}{\tau} - \frac{1}{p}\Big)B^+(x)s^p. \end{aligned}$$

Hence, ($f_4$) is satisfied with $c_3(x) = \big(\frac{1}{\tau} - \frac{1}{p}\big)B^+(x)$. Finally, we point out that

$$F(x,s) \geq a s^{p-1} \quad \text{on} \quad \Omega_R$$

with $a = \inf_{x\in\bar{\Omega}_R} B(x)$. This condition will be used in Section 3. Since we do not assume any growth condition on $B(x)$ for large $|x|$ (or an integrability condition), the variational functional

$$I(u) = \frac{1}{2}\int_{\Omega^c} |\nabla u|^2\,dx - \frac{1}{2^*}\int_{\Omega^c} Q(x)|u|^{2^*}\,dx - \lambda\int_{\Omega^c} F(x,u)\,dx$$

is not well defined on $D^{1,2}(\Omega^c)$.

The main results of this paper are Theorems 4.1 and 4.2 in which we establish the existence of a solution of problem (1.1) under two different assumptions on the shape of the graph of the coefficient $Q$ with the nonlinearity satisfying conditions $(f_1),\dots,(f_4)$. The main point in the proof of these theorems is the use of the mountain-pass principle and the Palais-Smale condition. These ideas are presented in Proposition 3.3 which also gives a general existence result for problem (1.1). In fact, Theorems 4.1 and 4.2 are direct consequences of Proposition 3.3.

If $Q \equiv 1$ on $\Omega^c$ and $f(x,s) = -\lambda s$, $\lambda > 0$, some existence results can be found in [11] and [8]. The extension of these results to the case when $Q$ is not constant is given in [5]. Solutions in these papers are sought as minimizers of the variational problem for $\lambda \geq 0$

$$S(\Omega^c, Q, \lambda) = \inf\Big\{ \int_{\Omega^c} \big(|\nabla u|^2 + \lambda u^2\big)\,dx;\ u \in H^1(\Omega^c) \text{ and } \int_{\Omega^c} Q(x)|u|^{2^*}\,dx = 1\Big\}.$$

If $\lambda = 0$ the space $H^1(\Omega^c)$ in the above definition is replaced by the Sobolev space

$$D^{1,2}(\Omega^c) = \Big\{u;\ \nabla u \in L^2(\Omega^c) \text{ and } u \in L^{2^*}(\Omega^c)\Big\}.$$

The norm in $D^{1,2}(\Omega^c)$ is given by $\|u\|_{D^{1,2}} = \|\nabla u\|_2$. We recall that $H^1(\Omega^c)$ is a Sobolev space equipped with the norm

$$\|u\|_{H^1} = \Big(\int_{\Omega^c} \big(|\nabla u|^2 + u^2\big)\,dx\Big)^{\frac{1}{2}}.$$

It follows from [8] that

$$0 < S(\Omega^c, 1, \lambda) \leq \frac{S}{2^{\frac{2}{N}}}, \tag{1.2}$$

where $S$ is the best Sobolev constant defined by

$$S = \inf\Big\{ \int_{\mathbb{R}^N} |\nabla u|^2\,dx;\ u \in D^{1,2}(\mathbb{R}^N),\ \int_{\mathbb{R}^N} |u|^{2^*}\,dx = 1\Big\}.$$

Moreover, if $S(\Omega^c, 1, \lambda) < \frac{S}{2^{\frac{2}{N}}}$ for some $\lambda \ge 0$, then $S(\Omega^c, 1, \lambda)$ is achieved. If $u$ is a minimizer then $S(\Omega^c, 1, \lambda)^{\frac{1}{2^*-2}} u$ is a solution of the problem

$$\begin{cases} -\Delta u + \lambda u & = |u|^{2^*-2} u \ \text{ in } \ \Omega^c, \\ \frac{\partial u}{\partial \nu} & = 0 \ \text{ on } \ \partial\Omega. \end{cases}$$

These are the least energy solutions of this problem. If $Q$ is not constant, then $S(\Omega^c, Q, \lambda)$ with $\lambda \ge 0$ is achieved if

$$S(\Omega^c, Q, \lambda) < \min\left\{ \frac{S}{2^{\frac{2}{N}} Q_m^{\frac{N-2}{N}}}, \frac{S}{Q_M^{\frac{N-2}{N}}}, \frac{S}{Q(\infty)^{\frac{N-2}{N}}} \right\} \tag{1.3}$$

(see [5]), where

$$Q_M = \max_{x \in \bar{\Omega}^c} Q(x) \ \text{ and } \ Q_m = \max_{x \in \partial\Omega} Q(x).$$

Conditions guaranteeing (1.3) can be found in the paper [5]. In general these conditions depend on the shape of the graph of $Q$ and the mean curvature at some points of the boundary $\partial\Omega$. Motivated by the paper [9] we extend the above results to problem (1.1). Since the equation in (1.1) contains a lower-order term, a constrained minimization is not applicable. As we mentioned earlier, under assumptions $(f_1), \dots, (f_4)$ the variational functional corresponding to problem (1.1) is not well defined on the space $D^{1,2}(\Omega^c)$. Therefore, we look for a solution in $D^{1,2}(\Omega^c)$ in the sense of distributions. We recall that $u \in D^{1,2}(\Omega^c)$ is a solution of (1.1) in the sense of distributions if

$$\int_{\Omega^c} \nabla u \nabla \phi \, dx = \int_{\Omega^c} Q(x) |u|^{2^*-2} u \phi \, dx + \lambda \int_{\Omega^c} f(x, u) \phi \, dx \tag{1.4}$$

for every $\phi \in C^1_\circ(\bar{\Omega}^c)$. (Here we denote by $C^1_\circ(\bar{\Omega}^c)$ the space of continuously differentiable functions with compact supports in $\bar{\Omega}^c$.) To obtain a solution of (1.1) we consider problem (1.1) with truncated nonlinearity $f$. Following the paper [9], a solution will be obtained as a suitable limit of a sequence of "almost critical points" of the mountain-pass type of approximating functionals.

The paper is organized as follows. In Section 2 we consider an approximating problem. This technical result will be applied in Sections 3 and 4 to obtain, under various assumptions on $Q$, the existence of solutions to problem (1.1).

Let $\phi : X \to \mathbb{R}$ be a $C^1$ functional on a Banach space $X$. We recall that a sequence $\{x_n\} \subset X$ is a Palais-Smale sequence for $\phi$ at level $c$ (a $(PS)_c$ sequence for short) if $\phi(x_n) \to c$ and $\phi'(x_n) \to 0$ in $X^*$. Finally, we say that the functional $\phi$ satisfies the Palais-Smale condition at level $c$ ($(PS)_c$ condition for short) if each $(PS)_c$ sequence is relatively compact in $X$.

Throughout this paper we denote a strong convergence by " $\to$ " and a weak convergence by " $\rightharpoonup$ ". The norms in the Lebesgue spaces $L^p(\mathbb{R}^N)$ are denoted by $\|\cdot\|_p$.

## 2. Approximating problem

In this section we consider the following semi-linear problem

$$\begin{cases} -\Delta u &= g(x,u) \text{ in } \Omega^c \\ \frac{\partial u}{\partial \nu} &= 0 \text{ on } \partial\Omega, \end{cases} \tag{2.1}$$

where $g \in C(\bar{\Omega}^c \times \mathbb{R}, \mathbb{R})$ and satisfies the following two conditions:

($g_1$) For every $R > R_\circ$ there exist constants $a_R, b_R > 0$ such that

$$|g(x,s)| \le a_R |s|^{2^*-1} + b_R$$

for all $x \in \Omega_R$ and $s \in \mathbb{R}$, where $\Omega_R$ is a set defined in Section 1.

We put

$$I(u) = \frac{1}{2}\int_{\Omega^c} |\nabla u|^2\,dx - \int_{\Omega^c} G(x,u)\,dx,$$

where $G(x,t) = \int_0^t g(x,s)\,ds$. Under assumption ($g_1$) $I(u)$ is not well defined on $D^{1,2}(\Omega^c)$. We now introduce a stronger condition

($g_2$) There exist a constant $a > 0$ and $b \in C_\circ(\bar{\Omega}^c)$ such that

$$|g(x,s)| \le a|s|^{2^*-1} + b(x)$$

for every $x \in \bar{\Omega}^c$ and $s \in \mathbb{R}$. Here we denote by $C_\circ(\bar{\Omega}^c)$ the space of continuous functions with compact supports in $\bar{\Omega}^c$.

It is clear that under assumption ($g_2$) the functional $I$ is of class $C^1$ on $D^{1,2}(\Omega^c)$. Finally, we impose the following condition on $g$:

($g_3$) Let $n_\circ \in \mathbb{N}$ be such that $\Omega \subset B(0,n_\circ)$. For every $n \ge n_\circ$ there exists a function $g_n \in C(\bar{\Omega}^c \times \mathbb{R}, \mathbb{R})$ satisfying ($g_2$) such that $g(x,s) = g_n(x,s)$ for every $x \in \Omega_n := \Omega^c \cap B(0,n)$ and $s \in \mathbb{R}$.

We define variational functionals

$$I_n(u) = \frac{1}{2}\int_{\Omega^c} |\nabla u|^2\,dx - \int_{\Omega^c} G_n(x,u)\,dx,$$

where $G_n(x,t) = \int_0^t g_n(x,s)\,ds$.

**Proposition 2.1.** *Suppose that* $g \in C(\bar{\Omega}^c \times \mathbb{R}, \mathbb{R})$ *and satisfies* $(g_1)$ *and* $(g_3)$*. Then every bounded sequence* $\{u_n\} \subset D^{1,2}(\Omega^c)$ *such that* $I_n'(u_n) \to 0$ *has a weakly convergent subsequence to a solution of problem (2.1).*

**Proof.** Since $\{u_n\}$ is bounded in $D^{1,2}(\Omega^c)$ we may assume that $u_n \rightharpoonup u$ in $D^{1,2}(\Omega^c)$ and $u_n \to u$ almost everywhere on $\Omega^c$. It is easy to check that for every compact set $K \subset \bar{\Omega}^c$ there exist $n_\circ \in \mathbb{N}$ and a constant $M = M(K) > 0$ such that

$$\int_K |g_n(x, u_n)|^{\frac{2^*}{2^*-1}}\, dx \le M \tag{2.2}$$

for every $n \ge n_\circ$. Indeed, we select $n_\circ \in \mathbb{N}$ so that $K \subset B(0, n_\circ)$. Then $g_n(x, u_n(x)) = g(x, u_n(x))$ for $n \ge n_\circ$ and $x \in K$. Since $\{u_n\}$ is bounded in $D^{1,2}$, we easily derive (2.2) using $(g_2)$. Let $\phi \in C_\circ^1(\bar{\Omega}^c)$. We choose $n_\circ \in \mathbb{N}$ so that $\operatorname{supp} \phi \subset \Omega_{n_\circ}$. Then $g_n(x, s) = g(x, s)$ for every $x \in \operatorname{supp} \phi$, $s \in \mathbb{R}$ and $n \ge n_\circ$. By virtue of $(g_1)$ we have

$$|g_n(x, s)\phi| \le |\phi|\big(a_{n_\circ}|s|^{2^*-1} + b_{n_\circ}\big).$$

Estimate (2.2) applied to $\bar{\Omega}_{n_\circ}$ shows that $g_n(x, u_n(x))\phi(x)$ is uniformly integrable in $L^1(\Omega^c)$. Thus by Vitali's theorem we have

$$\lim_{n\to\infty} \int_{\Omega^c} g_n(x, u_n(x))\phi(x)\, dx = \int_{\Omega^c} g(x, u(x))\phi(x)\, dx.$$

Since $u_n \rightharpoonup u$ in $D^{1,2}(\Omega^c)$, we see that $u$ is a solution of (1.1) in the sense of distributions. □

## 3. Application of the mountain-pass principle

In order to apply the mountain-pass principle to problem (1.1) we consider the sequence of functionals $I_{\lambda,n}(u)$ on $D^{1,2}(\Omega^c)$ defined as follows. Let $\Phi$ be a function in $C_\circ^1(\mathbb{R}^N)$ such that $0 \le \Phi(x) \le 1$ on $\mathbb{R}^N$, $\Phi(x) = 1$ on $B(0,1)$ and $\Phi(x) = 0$ on $\mathbb{R}^N - B(0,2)$. We put $\Phi_n(x) = \Phi\big(\frac{x}{n}\big)$, $n \in \mathbb{N}$. We choose $n_\circ \in \mathbb{N}$ so that $\Omega \subset B(0, n_\circ)$ and put $f_n(x,s) = f(x,s)\Phi_n(x)$ for $n \ge n_\circ$. For every $n \ge n_\circ$ we consider a functional $I_{\lambda,n}$ on $D^{1,2}(\mathbb{R}^N)$ defined by

$$I_{\lambda,n}(u) = \frac{1}{2}\int_{\Omega^c} |\nabla u|^2\, dx - \frac{1}{2^*}\int_{\Omega^c} Q(x)|u^+|^{2^*}\, dx - \lambda \int_{\Omega^c} F_n(x,u)\, dx,$$

where $F_n(x,t) = \int_0^t f_n(x,s)\, ds$. It is clear that $I_{\lambda,n}$ is $C^1$ on $D^{1,2}(\mathbb{R}^N)$.

**Lemma 3.1.** *Suppose that* $f$ *satisfies* $(f_1)$, $(f_2)$ *and* $(f_3)$*. Then*

(a) *if* $1 < r_j \le 2$ *for* $j = 1, 2$, *then there is a* $\bar\lambda > 0$ *such that for every* $\lambda \in (0, \bar\lambda)$, *there exist constants* $\kappa > 0$ *and* $\rho_\circ > 0$ (*independent of* $n$) *such that*

$$I_{\lambda,n}(u) \ge \kappa \quad \textit{for} \quad \|u\|_{D^{1,2}} = \rho_\circ.$$

(b) *if* $2 < r_j < 2^*$ *for* $j = 1, 2$, *assertion (a) is true for every* $\lambda > 0$.

**Proof.** It follows from $(f_3)$ and the Sobolev inequality that

$$I_{\lambda,n}(u) \ge \frac{1}{2}\int_{\Omega^c} |\nabla u|^2\,dx - \frac{1}{2^*}\int_{\Omega^c} Q|u^+|^{2^*}\,dx$$
$$- \frac{\lambda}{r_1 S_\circ^{\frac{r_1}{2}}}\|c_1\|_{\frac{2^*}{2^*-r_1}}\Big(\int_{\Omega^c}|\nabla u|^2\,dx\Big)^{\frac{r_1}{2}} - \frac{\lambda}{r_2 S_\circ^{\frac{r_2}{2}}}\|c_2\|_{\frac{2^*}{2^*-r_2}}\Big(\int_{\Omega^c}|\nabla u|^2\,dx\Big)^{\frac{r_2}{2}},$$

where $S_\circ = S(\Omega^c, 1, 0)$ (see (1.2)). If $1 < r_j \le 2$ for $j = 1, 2$, then

$$\begin{aligned} I_{\lambda,n}(u) \quad &\ge \quad \|u\|^2_{D^{1,2}}\Big[\frac{1}{2} - \frac{Q_M}{2^* S_\circ^{\frac{2^*}{2}}}\|u\|^{2^*-2}_{D^{1,2}}\Big] \\ &- \quad \frac{\lambda}{r_1 S_\circ^{\frac{r_1}{2}}}\|c_1\|_{\frac{2^*}{2^*-r_1}}\|u\|^{r_1}_{D^{1,2}} - \frac{\lambda}{r_2 S_\circ^{\frac{r_2}{2}}}\|c_2\|_{\frac{2^*}{2^*-r_2}}\|u\|^{r_2}_{D^{1,2}}. \end{aligned}$$

First we choose $\rho_\circ > 0$ so that

$$\frac{1}{2} - \frac{Q_M \rho_\circ^{2^*-2}}{2^* S_\circ^{\frac{2^*}{2}}} > \frac{1}{4}.$$

We then select $\bar\lambda > 0$ so that

$$\lambda\Big(\frac{\|c_1\|_{\frac{2^*}{2^*-r_1}}\rho_\circ^{r_1}}{r_1 S_\circ^{\frac{r_1}{2}}} + \frac{\|c_2\|_{\frac{2^*}{2^*-r_2}}\rho_\circ^{r_2}}{r_2 S_\circ^{\frac{r_2}{2}}}\Big) \le \frac{\rho_\circ^2}{8}$$

for every $\lambda \in (0, \bar\lambda)$. The result follows with $\kappa = \frac{\rho_\circ^2}{8}$. In the case $2 < r_j < 2^*$ for $j = 1, 2$, we have

$$\begin{aligned} I_{\lambda,n} \quad &\ge \quad \|u\|^2_{D^{1,2}}\Big(\frac{1}{2} - \frac{Q_M}{2^* S_\circ^{\frac{2^*}{2}}}\|u\|^{2^*-2}_{D^{1,2}} - \frac{\lambda}{r_1 S_\circ^{\frac{r_1}{2}}}\|u\|_{\frac{2^*}{2^*-2}}\|u\|^{r_1-2}_{D^{1,2}} \\ &- \quad \frac{\lambda}{r_2 S_\circ^{\frac{r_1}{2}}}\|c_2\|_{\frac{2^*}{2^*-r_2}}\|u\|^{r_2-2}_{D^{1,2}}\Big). \end{aligned}$$

Choosing $\rho_\circ > 0$ small enough we obtain the assertion (b). □

To proceed further with the verification of the mountain-pass geometry of $I_{\lambda,n}$, we need the additional assumption:

($f_5$) There exist a bounded subset $\Omega_* \subset \bar{\Omega}^c$, $|\Omega_*| > 0$, constants $a \in \mathbb{R}$ and $1 < q < 2^*$ such that

$$F(x,s) \geq a|s|^q$$

for all $s \in \mathbb{R}$ and $x \in \Omega_*$.

**Lemma 3.2.** *Let $n_\circ \in \mathbb{N}$ be such that $\Omega \cup \Omega_* \subset B(0, n_\circ)$. Suppose that ($f_1$), ($f_2$), ($f_3$) and ($f_5$) hold. Then for every $\lambda > 0$ there exists $w \in D^{1,2}(\Omega^c)$ such that $I_{\lambda,n}(w) < 0$, $\|w\|_{D^{1,2}} > \rho_\circ$ for every $n \geq n_\circ$.*

**Proof.** We choose $w \in D^{1,2}(\Omega^c)$ with supp $w \subset \Omega_*$ and $\|w\|_{D^{1,2}} > 0$. Then for every $t > 0$ we have

$$I_{\lambda,n}(tw) \leq \frac{t^2}{2}\int_{\Omega_*} |\nabla w|^2\,dx - \frac{Q_* t^{2^*}}{2^*}\int_{\Omega_*} |w|^{2^*}\,dx - at^q\lambda\int_{\Omega_*} |w|^q\,dx,$$

where $0 < Q_* = \min_{x\in\bar{\Omega}_*} Q(x)$. Taking $t > 0$ sufficiently large the result follows. □

We shall use the following version of the concentration-compactness principle [3], [7]. Let $u_m \rightharpoonup u$ in $D^{1,2}(\Omega^c)$. We define two quantities which measure the loss of mass at infinity of the sequence $\{u_m\}$. We put

$$\nu_\infty = \lim_{R\to\infty}\limsup_{m\to\infty}\int_{\Omega^c\cap(|x|>R)} |u_m|^{2^*}\,dx \tag{3.1}$$

and

$$\mu_\infty = \lim_{R\to\infty}\limsup_{m\to\infty}\int_{\Omega^c\cap(|x|>R)} |\nabla u_m|^2\,dx. \tag{3.2}$$

It follows from the Sobolev inequality that

$$S\nu_\infty^{\frac{2}{2^*}} \leq \mu_\infty. \tag{3.3}$$

The concentration-compactness principle can be formulated in the following way. Let $u_m \rightharpoonup u$ in $D^{1,2}(\Omega^c)$ be such that $|u_m|^{2^*} \rightharpoonup \nu$ and $|\nabla u_m|^2 \rightharpoonup \mu$ weakly in the sense of measure. Then there exist numbers $\nu_j > 0$, $\mu_j > 0$ and points $x_j \in \Omega^c \cup \partial\Omega$, $j \in J$, where $J$ is an at most countable set, such that

$$\nu = |u|^{2^*} + \sum_{j\in J}\nu_j\delta_{x_j} + \nu_\infty\delta_\infty \tag{3.4}$$

and

$$\mu = |\nabla u|^2 + \sum_{j\in J}\mu_j\delta_{x_j} + \mu_\infty\delta_\infty. \tag{3.5}$$

Moreover, if $x_j \in \Omega^c$, then

$$S\nu_j^{\frac{2}{2^*}} \leq \mu_j \tag{3.6}$$

and if $x_j \in \partial\Omega$, then

$$\frac{S}{2^{\frac{2}{N}}}\nu_j^{\frac{2}{2^*}} \leq \mu_j, \tag{3.7}$$

and $\sum_{j\in J}\nu_j^{\frac{2}{2^*}} < \infty$. We point out here that the symbol $\delta_\infty$ plays the role of the delta Dirac at infinity; that is,

$$\int_{\Omega^c} d\nu = \int_{\Omega^c} |u|^{2^*}\,dx + \sum_{j\in J}\nu_j + \nu_\infty.$$

For every $n \geq n_\circ$ we define the mountain-pass level (see [2])

$$c_{\lambda,n} = \inf_{\gamma\in\Gamma}\max_{u\in\gamma} I_{\lambda,n}(u),$$

where $\Gamma = \{\gamma \in C([0,1], D^{1,2}(\Omega^c)) : \gamma(0) = 0,\ \gamma(1) = w\}$, and $w$ is a function from Lemma 3.2. If $f$ satisfies $(f_1),(f_2),(f_3)$ and $(f_5)$, then Lemmas 3.1, 3.2 hold. It is well known that each $c_{\lambda,n}$ generates a Palais-Smale sequence. Therefore for every $n \geq n_\circ$ there exists $\{u_j^n\} \subset D^{1,2}(\Omega^c)$ such that

$$I_{\lambda,n}(u_j^n) \to c_{\lambda,n} \quad \text{and} \quad I'_{\lambda,n}(u_j^n) \to 0 \ \text{ in } \ D^{1,2}(\Omega^c) \ \text{ as } \ j \to \infty.$$

We put

$$S_\infty = \min\left\{\frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \frac{S^{\frac{N}{2}}}{NQ(\infty)^{\frac{N-2}{2}}}\right\}.$$

**Proposition 3.3.** *Suppose that $(f_1),\dots,(f_5)$ hold and let $\bar\lambda$ be a constant determined by Lemma* 3.1.

(i) *If $1 < r_j \leq 2$ for $j = 1, 2$, $0 < \lambda < \bar\lambda$ and*

$$c_{\lambda,n} \leq d_\lambda < S_\infty \tag{3.8}$$

*for $n \geq n_\circ$ and some constant $d_\lambda$ independent of $n$, then problem* (1.1) *possesses a solution.*

(ii) *If $2 < r_j < 2^*$ forj $= 1, 2$, and if for every $\lambda > 0$ there exists a constant $d_\lambda > 0$ satisfying* (3.8) *for $n \geq n_\circ$, then problem* (1.1) *has a solution.*

**Proof.** Since $\kappa \leq c_{\lambda,n} \leq d_\lambda$ we may assume that $c_\lambda = \lim_{n\to\infty} c_{\lambda,n} \in [\kappa, d_\lambda]$. For a given $0 < \epsilon$ with $c_\lambda + \epsilon < S_\infty$ we can find $n_1 \geq n_\circ$ such that $c_{\lambda,n} \in$

$(c_\lambda-\epsilon, c_\lambda+\epsilon)$ for $n\geq n_1$. Now for every $n\geq n_1$, there exists $u_n=u^n_{j_n}$ such that

$$c_\lambda-\epsilon\leq I_{\lambda,n}(u_n)\leq c_\lambda+\epsilon \quad\text{and}\quad \|I'_{\lambda,n}(u_n)\|\leq\frac{1}{n}. \tag{3.9}$$

In the next step we show that $\{u_n\}$ is bounded in $D^{1,2}(\Omega^c)$. Using assumption $(f_4)$ we write

$$\begin{aligned} I_{\lambda,n}(u_n) &- \frac{1}{\tau}\langle I'_{\lambda,n}(u_n),u_n\rangle \qquad\qquad (3.10)\\ &= \big(\frac{1}{2}-\frac{1}{\tau}\big)\int_{\Omega^c}|\nabla u_n|^2\,dx+\big(\frac{1}{\tau}-\frac{1}{2^*}\big)\int_{\Omega^c}Q(x)|u_n^+|^{2^*}\,dx\\ &+ \lambda\int_{\Omega^c}\big(\frac{1}{\tau}f_n(x,u_n)u_n-F_n(x,u_n)\big)\,dx\\ &\geq \big(\frac{1}{2}-\frac{1}{\tau}\big)\int_{\Omega^c}|\nabla u_n|^2\,dx+\big(\frac{1}{\tau}-\frac{1}{2^*}\big)\int_{\Omega^c}Q(x)|u_n^+|^{2^*}\,dx\\ &+ \lambda\int_{\Omega^c}c_3(x)|u_n|^\mu\Phi_n\,dx\\ &\geq \big(\frac{1}{2}-\frac{1}{\tau}\big)\int_{\Omega^c}|\nabla u_n|^2\,dx+\big(\frac{1}{\tau}-\frac{1}{2^*}\big)\int_{\Omega^c}Q(x)|u_n^+|^{2^*}\,dx\\ &- \frac{\lambda}{S_\circ^{\frac{\mu}{2}}}\|c_3\|_{\frac{2^*}{2^*-\mu}}\|u_n\|_{2^*}^\mu. \end{aligned}$$

By (3.9) we also have

$$I_{\lambda,n}(u_n)-\frac{1}{\tau}\langle I'_{\lambda,n}(u_n),u_n\rangle\leq C\big(1+\|\nabla u_n\|_2\big)$$

for some constant $C>0$ independent of $n$. This combined with (3.10) shows that $\{u_n\}$ is bounded in $D^{1,2}(\Omega^c)$. It now follows from Proposition 2.1 that (up to a subsequence) $u_n\rightharpoonup u$ in $D^{1,2}(\Omega^c)$ and $u$ is a solution of (1.1). We may also assume that $u_n\to u$ in $L^p_{\text{loc}}(\Omega^c)$ for $2\leq p<2^*$. It remains to show that $u\not\equiv 0$. Testing $I_{\lambda,n}(u_n)$ with $u_n^-$ we see that $u_n^-\to 0$ in $D^{1,2}(\Omega^c)$. Arguing by contradiction, assume that $u\equiv 0$. Since $u_n\rightharpoonup 0$ in $D^{1,2}(\Omega^c)$, the sequence $\{u_n\}$ must concentrate. Applying the concentration-compactness principle we obtain points $x_j$, numbers $\nu_j>0$, $\mu_j>0$, $j\in J$ such that (3.4), (3.5), (3.6) and (3.7) hold. Let $\psi_\delta$, $\delta>0$ be a family of smooth functions concentrating at $x_i$, $x_i$ being fixed. It then follows from $(f_3)$ that

$$\int_{\Omega^c}\psi_\delta|\nabla u_n|^2\,dx \quad+\quad \int_{\Omega^c}\nabla u_n u_n\nabla\psi_\delta\,dx\leq\int_{\Omega^c}Q(x)\psi_\delta|u_n^+|^{2^*}\,dx$$

$$+ \quad \lambda \int_{\Omega^c} c_1(x)|u_n|^{r_1}\psi_\delta\, dx + \lambda \int_{\Omega^c} c_2(x)|u_n|^{r_2}\psi_\delta\, dx.$$

Letting $n \to \infty$ and then $\delta \to 0$ we derive

$$\mu_i \leq Q(x_i)\nu_i. \tag{3.11}$$

If $\nu_i > 0$ with $x_i \in \Omega^c$, then by (3.6) and (3.11) we obtain

$$\nu_i \geq \frac{S^{\frac{N}{2}}}{Q(x_i)^{\frac{N}{2}}} \quad \text{and} \quad \mu_i \geq \frac{S^{\frac{N}{2}}}{Q(x_i)^{\frac{N-2}{2}}}. \tag{3.12}$$

Thus,

$$\begin{aligned} S_\infty \quad &> \quad \lim_{n\to\infty} I_{\lambda,n}(u_n) \geq \lim_{n\to\infty} [I_{\lambda,n}(u_n) - \frac{1}{\tau}\langle I'_{\lambda,n}(u_n), u_n\rangle] \qquad (3.13)\\ &= \quad \lim_{n\to\infty} \Big[ \big(\frac{1}{2} - \frac{1}{\tau}\big) \int_{\Omega^c} |\nabla u_n|^2\, dx + \big(\frac{1}{\tau} - \frac{1}{2^*}\big) \int_{\Omega^c} Q(x)|u_n^+|^{2^*}\, dx \\ &+ \quad \int_{\Omega^c} c_3(x)|u_n|^\mu \Phi_n\, dx\Big]. \end{aligned}$$

Since $c_3 \in L^{\frac{2^*}{2^*-\mu}}(\Omega^c)$ and $\mu \in (1, 2^*)$, it is easy to check that

$$\lim_{n\to\infty} \int_{\Omega^c} c_3(x)|u_n|^\mu\, dx = 0.$$

Hence, letting $n \to \infty$ in (3.13) and using (3.12) we obtain

$$\begin{aligned} S_\infty \quad &> \quad \lim_{n\to\infty} I_{\lambda,n}(u_n) \geq \big(\frac{1}{2} - \frac{1}{\tau}\big)\mu_i + \big(\frac{1}{\tau} - \frac{1}{2^*}\big)Q(x_i)\nu_i \\ &\geq \quad \big(\frac{1}{2} - \frac{1}{\tau}\big)\frac{S^{\frac{N}{2}}}{Q(x_i)^{\frac{N-2}{2}}} + \big(\frac{1}{\tau} - \frac{1}{2^*}\big)\frac{S^{\frac{N}{2}}}{Q(x_i)^{\frac{N-2}{2}}} \geq \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}, \end{aligned}$$

which is impossible. Using (3.7) and (3.11) in a similar manner we rule out the concentration of $\{u_n\}$ at the boundary points of $\Omega$. Finally, we show that a concentration of $\{u_n\}$ cannot occur at infinity. For each $R > R_\circ$ we define a function $\Phi_R \in C^1_\circ(\mathbb{R}^N)$ such that $\Phi_R(x) = 0$ on $B(0, R)$, $\Phi(x) = 1$ on $\mathbb{R}^N - B(0, R+1)$ and $0 \leq \Phi(x) \leq 1$ on $\mathbb{R}^N$. It then follows from $(f_3)$ that

$$\begin{aligned} \langle I'_{\lambda,n}(u_n), u_n\Phi_R\rangle \quad &\geq \quad \int_{\Omega^c} |\nabla u_n|^2 \Phi_R\, dx + \int_{\Omega^c} \nabla u_n \nabla \Phi_R u_n\, dx \qquad (3.14)\\ &- \quad \int_{\Omega^c} \Phi_R Q(x)|u_n|^{2^*}\, dx \\ &- \quad \lambda \int_{\Omega^c} \Phi_R c_1(x)|u_n|^{r_1}\, dx - \lambda \int_{\Omega^c} \Phi_R c_2(x)|u_n|^{r_2}\, dx. \end{aligned}$$

It is easy to show that $\lim_{n\to\infty}\int_{\Omega^c} c_i(x)|u_n|^{r_i}\Phi_R\,dx = 0$, $i=1,2$. Using the Hölder inequality and the fact that $\{u_n\}$ is bounded in $D^{1,2}(\Omega^c)$ we have

$$\limsup_{n\to\infty}\int_{\Omega^c}|\nabla u_n||\nabla\Phi_R||u|\,dx \le \frac{C}{R}\Big(\int_{\Omega_{R+1}-\Omega_R} u^2\,dx\Big)^{\frac{1}{2}}$$
$$\le \frac{C}{R}|B(0,R+1)-B(0,R)|^{\frac{1}{N}}\Big(\int_{\Omega_{R+1}-\Omega_R}|u|^{2^*}\,dx\Big)^{\frac{1}{2^*}}$$

for some constant $C>0$ independent of $R$ and $n$. Hence

$$\lim_{R\to\infty}\limsup_{n\to\infty}\int_{\Omega^c}|\nabla u_n||\nabla\Phi_R||u_n|\,dx = 0.$$

Letting $n\to\infty$ in (3.14) and then $R\to\infty$ we deduce that $\mu_\infty \le Q(\infty)\nu_\infty$. This combined with (3.3) yields $\nu_\infty \ge \frac{S^{\frac{N}{2}}}{Q(\infty)^{\frac{N}{2}}}$ and $\mu_\infty \ge \frac{S^{\frac{N}{2}}}{Q(\infty)^{\frac{N-2}{2}}}$. By an argument similar to that used above we arrive at the inequality $S_\infty > \frac{S^{\frac{N}{2}}}{Q(\infty)^{\frac{N-2}{2}}}$. This contradiction completes the proof. □.

## 4. Verification of inequality (3.8)

To apply Proposition 3.3 we need conditions guaranteeing the validity of the inequality $c_{\lambda,n}\le d_\lambda < S_\infty$. To verify this inequality we use a family of functions

$$U_{\epsilon,y}(x) = \epsilon^{-\frac{N-2}{2}}U\Big(\frac{x-y}{\epsilon}\Big),\ \ \epsilon>0,\ y\in\mathbb{R}^N,$$

where $U(x) = \frac{c_N}{\left(N(N-2)+|x|^2\right)^{\frac{N-2}{2}}}$ and $c_N>0$ is a constant depending on $N$. The function $U$ (which is called an instanton) satisfies the equation

$$-\Delta U = U^{2^*-1} \quad \text{in} \quad \mathbb{R}^N.$$

We have $\int_{\mathbb{R}^N}|\nabla U|^2\,dx = \int_{\mathbb{R}^N}U^{2^*}\,dx = S^{\frac{N}{2}}$. If $y=0$ we write $U_{\epsilon,0}$.

We only consider the cases: $S_\infty = \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}$ and $S_\infty = \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}$. One can also obtain some existence results in the third case, however under additional assumptions on the behaviour of $Q(x)$ for large $|x|$ as in [5], and we omit the details.

We commence with the case $Q_m \le 2^{\frac{2}{N-2}}Q_M$. In this case we have

$$S_\infty = \frac{S^{\frac{N}{2}}}{2NQ_m^{\frac{N-2}{2}}}.$$

Let $Q_m = Q(y)$ for some $y \in \partial\Omega$. For simplicity we assume $y = 0$. We assume in condition $(f_5)$ that the set $\Omega_*$ has the form $\Omega_* = \Omega^c \cap B(0, r_\circ)$. Let $\psi$ be a function in $C^1_\circ(\mathbb{R}^N)$ such that $\psi(x) = 1$ for $x \in B(0, \frac{r_\circ}{2})$ and $\psi(x) = 0$ for $x \in \mathbb{R}^N - B(0, r_\circ)$ and $0 \le \psi(x) \le 1$ on $\mathbb{R}^N$. We need the following estimate ([1], [5])

$$\begin{aligned} &\frac{\left(\int_{\Omega^c} |\nabla\left(U_\epsilon\psi\right)|^2 \, dx\right)^{\frac{N}{2}}}{\left(\int_{\Omega^c} |\psi U_\epsilon|^{2^*} \, dx\right)^{\frac{N-2}{2}}} \\ &= \frac{S^{\frac{N}{2}}}{2} + \begin{cases} A_N H(0)\epsilon \log \frac{1}{\epsilon} + O(\epsilon) & N = 3, \\ A_N H(0)\epsilon + a_N \epsilon^2 \log \frac{1}{\epsilon} + O\left(\epsilon^2 \log \frac{1}{\epsilon}\right) & N = 4, \\ A_N H(0)\epsilon + O\left(\epsilon^2\right) & N \ge 5, \end{cases} \end{aligned} \tag{4.1}$$

where $A_N > 0$ and $a_N > 0$ are constants depending on $N$. Here $H(y)$ denotes the mean curvature of $\partial\Omega$ at $y \in \partial\Omega$, when viewed from inside $\Omega$.

**Theorem 4.1.** *Let $Q_M \le 2^{\frac{2}{N-2}} Q_m$, $N \ge 5$, $Q(y) = Q_m$ for some $y \in \partial\Omega$, with $H(y) < 0$ when viewed from inside $\Omega$, and*

$$|Q(x) - Q(y)| = o(|x - y|) \quad \text{for } x \text{ near } y.$$

*Suppose that $(f_1), \ldots, (f_5)$ hold and that, in assumption $(f_5)$, $\Omega_* = \Omega^c \cap B(0, r_\circ)$ and $\frac{N}{N-2} < q < \frac{2(N-1)}{N-2}$.*

(i) *If $1 < r_j \le 2$ for $j = 1, 2$, and $0 < \lambda < \bar{\lambda}$, then problem* (1.1) *possesses a solution.*

(ii) *If $2 < r_j < 2^*$ for $j = 1, 2$, then problem* (1.1) *has a solution for every $\lambda > 0$.*

**Proof.** We assume that $y = 0$ and choose $n_\circ \in \mathbb{N}$ so that $\Omega \cup \Omega_* \subset B(0, n_\circ)$. Then $F_n(x, s) = F(x, s)$ for $x \in \Omega_*$, $s \ge 0$ and $n \ge n_\circ$. For every $\epsilon > 0$ small enough there exists $t_\epsilon > 0$ such that

$$\begin{aligned} &I_{\lambda,n}(t_\epsilon \psi U_\epsilon) \\ &= \max_{t \ge 0} \Big[\frac{t^2}{2} \int_{\Omega^c} |\nabla\left(\psi U_\epsilon\right)|^2 \, dx - \frac{t^{2^*}}{2^*} \int_{\Omega^c} Q(x) |\psi U_\epsilon|^{2^*} \, dx - \int_{\Omega^c} F(x, t\psi U_\epsilon) \, dx\Big]. \end{aligned}$$

Since $1 < q < 2^*$, it is easy to see that $k \le t_\epsilon \le K$ for every $\epsilon > 0$ small, where $0 < k < K$ are constants independent of $\epsilon$. Moreover, we have

$$I_{\lambda,n}(t_\epsilon \psi U_\epsilon) \le \frac{1}{N} \frac{\left(\int_{\Omega^c} |\nabla\left(U_\epsilon\psi\right)|^2 \, dx\right)^{\frac{N}{2}}}{\left(\int_{\Omega^c} Q |\psi U_\epsilon|^{2^*} \, dx\right)^{\frac{N-2}{2}}} + \lambda |a| K^q \int_{\Omega^c} |\psi U_\epsilon|^q \, dx. \tag{4.2}$$

For the second term of the right-hand side of this inequality we have

$$\int_{\Omega^c} |\psi U_\epsilon|^q \, dx = o(\epsilon)$$

because $\frac{N}{N-2} < q < \frac{2(N-1)}{N-2}$. We now observe that

$$\int_{\Omega^c} Q(x)|\psi U_\epsilon|^{2^*} \, dx = Q(0) \int_{\Omega^c} |\psi U_\epsilon|^{2^*} \, dx + o(\epsilon).$$

Hence the result follows from (4.2) and (4.1) in both cases (i) and (ii). $\square$

We now consider the case $S_\infty = \frac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}$. This occurs if $Q_M > 2^{\frac{2}{N-2}} Q_m$. Moreover, we assume that $Q_M$ is attained at some point of $\Omega^c$.

**Theorem 4.2.** *Let $Q_M > 2^{\frac{2}{N-2}} Q_m$ and let $y \in \Omega^c$ be such that $Q(y) = Q_M$ and $N \geq 5$. Suppose that $(f_1), \ldots, (f_5)$ hold and that, in assumption $(f_5)$, $\Omega_* = B(0, r) \subset \Omega^c$ for some $r > 0$, $\frac{2(N-1)}{N-2} < q < 2^*$ and $a > 0$. Moreover, assume that $|Q(x) - Q(y)| = o(|x - y|)$ for $x$ near $y$.*

(i) *If $1 < r_j \leq 2$ for $j = 1, 2$, and $\lambda \in (0, \bar{\lambda})$, then problem* (1.1) *has a solution.*

(ii) *If $2 < r_j < 2^*$ for $j = 1, 2$, then problem* (1.1) *has a solution for every $\lambda > 0$.*

**Proof.** Let $\psi$ be a $C_\circ^1(\mathbb{R}^N)$ function such that $\psi(x) = 1$ for $x \in B(y, \frac{r}{2})$, $\psi(x) = 0$ for $x \in \mathbb{R}^N - B(0, r)$ and $0 \leq \psi(x) \leq 1$ for $x \in \mathbb{R}^N$. As in Theorem 1.1 we have

$$\begin{aligned} & I_{\lambda,n}(t_\epsilon U_{\epsilon,y}\psi) \\ &= \max_{t \geq 0} I_{\lambda,n}(t U_{\epsilon,y}\psi) \leq \frac{1}{N} \frac{\left(\int_{\Omega^c} |\nabla\left(\psi U_{\epsilon,y}\right)|^2 \, dx\right)^{\frac{N}{2}}}{\left(\int_{\Omega^c} Q|\psi U_{\epsilon,y}|^{2^*} \, dx\right)^{\frac{N-2}{2}}} - \lambda a K \int_{\Omega^c} |\psi U_\epsilon|^q \, dx \end{aligned} \tag{4.3}$$

with $0 < k < t_\epsilon < K$ for $\epsilon > 0$ small enough. We now observe that

$$\begin{aligned} \int_{\Omega^c} |\nabla\left(\psi U_{\epsilon,y}\right)|^2 dx &\leq \int_{\Omega^c} |\nabla U_{\epsilon,y}|^2 \, dx + \int_{\Omega^c} |\nabla\psi||\nabla U_{\epsilon,y}| U_{\epsilon,y} \, dx \\ &+ \int_{\Omega^c} U_{\epsilon,y}^2 |\nabla\psi|^2 \, dx \leq S^{\frac{N}{2}} + O\left(\epsilon^{N-2}\right) \end{aligned}$$

and

$$\int_{\Omega^c} Q(x)|\psi U_{\epsilon,y}|^{2^*} \, dx = Q(y) S^{\frac{N}{2}} + o(\epsilon).$$

Since

$$\int_{\Omega^c} |\psi U_{\epsilon,y}|^q \, dx \geq c\epsilon^{N-\frac{q(N-2)}{2}} \quad \text{and} \quad N - \frac{q(N-2)}{2} < 1$$

for some $c > 0$, the result follows from (4.3) by taking $\epsilon > 0$ sufficiently small. □

We now apply Theorems 4.1, 4.2 to the problem

$$\begin{cases} -\Delta u &= Q(x)u^{2^*-1} + \lambda a(x)u^{q-1} \quad \text{in} \quad \Omega^c, \\ \frac{\partial u}{\partial \nu} = & 0 \quad \text{on} \quad \partial\Omega, \; u > 0 \quad \text{on} \quad \Omega^c. \end{cases} \tag{4.4}$$

We assume that

(A) $a$ is continuous on $\bar{\Omega}^c$, $a(x_\circ) > 0$ for some $x_\circ \in \bar{\Omega}^c$ and $a^+ \in L^{\frac{2^*}{2^*-q}}$ for some $1 < q < 2^*$.

In this case $f(x,s) = a(x)s^{q-1}$ for $s \geq 0$. Assumption $(f_4)$ holds with $c_1(x) = a^+(x)$, $r = q$ and $c_2(x) \equiv 0$. If we choose $q < \tau < 2^*$, then $(f_4)$ is satisfied with $c_3(x) = \big(\frac{1}{\tau} - \frac{1}{q}\big)a^+(x)$.

**Theorem 4.3.** *Let $Q_M \leq 2^{\frac{2}{N-2}} Q_m$, $Q(\bar{x}) = Q_M$ for some $\bar{x} \in \partial\Omega$, $\frac{N}{N-2} < q < \frac{2(N-1)}{N-2}$, $N \geq 5$. Suppose that $|Q(x) - Q(\bar{x})| = o(|x - \bar{x}|)$ for $x$ near $\bar{x}$ and $H(\bar{x}) < 0$, when viewed from inside $\Omega$.*

(i) *If $\frac{N}{N-2} < q \leq 2$, then there exists a $\bar{\lambda} > 0$ such that problem (4.4) has a solution for $\lambda \in (0, \bar{\lambda})$.*

(ii) *If $2 < q < \frac{2(N-1)}{N-2}$, then problem (4.4) with $\lambda a(x)$ replaced by $a(x)$ has a solution.*

We now consider the case $S_\infty = \dfrac{S^{\frac{N}{2}}}{NQ_M^{\frac{N-2}{2}}}$.

**Theorem 4.4.** *Let $Q_M > 2^{\frac{2}{N-2}} Q_m$, $N \geq 5$ and $Q_M = Q(\bar{x})$ for some $\bar{x} \in \Omega^c$ with $a(\bar{x}) > 0$. Suppose that $\frac{2(N-1)}{N-2} < q < 2^*$ and that*

$$|Q(x) - Q(\bar{x})| = o(|x - \bar{x}|) \quad \text{for} \;\; x \;\; \text{close to} \;\; \bar{x}.$$

*Then problem (4.4) with $\lambda a(x)$ replaced by $a(x)$ has a solution.*

We now assume that $a(x) < 0$ on $\Omega^c$. In this case $f(x,s) = a(x)s^{q-1}$ for $s \geq 0$ satisfies $(f_2)$. Condition $(f_3)$ holds with $c_1 = c_2 \equiv 0$. If we choose $\max(2,q) < \tau$, then

$$\frac{1}{\tau} f(x,s)s - F(x,s) = \big(\frac{1}{\tau} - \frac{1}{q}\big)a(x) > 0,$$

so $(f_4)$ is satisfied with $c_3 \equiv 0$. Let $\Omega_* \subset \Omega^c$ be a bounded domain. Then

$$F(x,s) \geq \bar{a}s^q \quad \text{for } x \in \Omega_* \text{ and } s \geq 0, \tag{4.5}$$

where $\bar{a} = \min_{x\in\bar{\Omega}_*} a(x)$. Hence $(f_5)$ holds. Since $a(x) < 0$ on $\Omega^c$, the mountain-pass geometry for the corresponding truncated functional holds for every $\lambda > 0$. Therefore we consider problem (4.4) with $\lambda a(x)$ replaced by $a(x)$; that is,

$$\begin{cases} -\Delta u & = Q(x)u^{2^*-1} + a(x)u^{q-1} \text{ in } \Omega^c, \\ \frac{\partial u}{\partial \nu} & = 0 \text{ on } \partial\Omega \; u > 0 \text{ on } \Omega^c. \end{cases} \tag{4.6}$$

Theorem 4.1 implies the following existence result for problem (4.6). In this theorem we use condition (4.6) with $\Omega_* = B(y,r)$ for some $y \in \partial\Omega$ and $r > 0$.

**Theorem 4.5.** *Let $Q_M \leq 2^{\frac{2}{N-2}} Q_m$, $N \geq 5$, $Q(y) = Q_m$ for some $y \in \partial\Omega$ with $H(y) < 0$ when viewed from inside $\Omega$. Suppose that*

$$|Q(x) - Q(y)| = o(|x-y|) \quad \textit{for } x \textit{ close to } y.$$

*If $\frac{N}{N-2} < q < \frac{2(N-1)}{N-2}$, then problem (4.6) admits a solution.*

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