Differential and Integral Equations Volume 19, Number 1 (2006), 51–74

NON-HOMOGENEOUS LINEAR SYMMETRIC HYPERBOLIC SYSTEMS WITH CHARACTERISTIC BOUNDARY

E. Casella, P. Secchi, and P. Trebeschi
Dip. di Matematica, Università di Brescia
Facoltà di Ingegneria, Via Valotti 9, 25133 Brescia, Italy

(Submitted by: Reza Aftabizadeh)

Abstract. In this paper we consider the initial-boundary value problem for linear symmetric hyperbolic systems with non-homogeneous, maximally non-negative and characteristic boundary condition. We prove the existence of regular solutions in suitable functions spaces which take into account the loss of regularity in the normal direction to the characteristic boundary.

1. Introduction

Let Ω be the half-space $\mathbb{R}^n_+ = \{(x_1, x') \in \mathbb{R}^n : x_1 > 0, x' \in \mathbb{R}^{n-1}\}$. Consider the linear differential operator of first order

$$L = A_0\partial_t + \sum_{j=1}^{n} A_j\partial_j + B,$$

where $\partial_t = \frac{\partial}{\partial t}$, $\partial_j = \frac{\partial}{\partial x_j}$ and $A_j(x,t)$, $B(x,t)$ are $N \times N$ matrices. Let $\Gamma = \{0\} \times \mathbb{R}^{n-1}$ be the boundary of $\Omega, Q_T = \Omega \times (0,T)$ and, finally, $\Sigma_T = \Gamma \times (0,T)$, with $T > 0$. We study the following initial-boundary value problem

$$\begin{cases} Lu = F & \text{in} \quad Q_T, \\ Mu = G & \text{on} \quad \Sigma_T, \\ u(x,0) = f & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where M is a $d \times N$ matrix; u, F, f are N-vectors and G is a d-vector.

Denote by ν the unit out-normal to Γ. The boundary matrix is

$$A_\nu := \sum_{j=1}^{n} A_j\nu_j = -A_1.$$

Accepted for publication: August 2005.
AMS Subject Classifications: 35L50, 35L40, 35L45.

We study Problem (1.1) under the following assumptions.

(A) The matrices $A_j, j = 0, \dots, n$, are real symmetric;

(B) the matrix A_0 is positive definite on $\overline{Q}_T$, i.e. there exists a real positive constant a_0 such that $(A_0(x,t)u, u) \geq a_0|u|^2$, for any $(x,t) \in \overline{Q}_T$ and $u \in \mathbb{R}^N$; here $(\cdot,\cdot)$ and $|\cdot|$ denotes the inner product and the Euclidean norm in $\mathbb{R}^N$;

(C) the boundary matrix $A_\nu = -A_1$ is singular, with constant rank at Γ, i.e. $0 < \operatorname{rank} A_1(x,t) = \text{const} < N$ for $(x,t) \in \Sigma_T$. We write A_1 in the following block form

$$A_1 = \begin{pmatrix} A_1^{I,I} & A_1^{I,II} \\ & \\ A_1^{II,I} & A_1^{II,II} \end{pmatrix}$$

where $A_1^{I,I} A_1^{I,II}, A_1^{II,I} A_1^{II,II}$ are respectively $r \times r$, $r \times (N-r)$, $(N-r) \times r$, $(N-r) \times (N-r)$ sub-matrices. We also suppose that $A_1^{I,II}, A_1^{II,I} A_1^{II,II}$ vanish on $\Gamma \times [0,T]$ and that there exists a real positive constant μ such that $|\det A_1^{I,I}(x,t)| \geq \mu$, for any $(x,t) \in \Sigma_T$;

(D) the matrix M has the form $M = (I_d, 0)$, where I_d is the $d \times d$ unit matrix, with $d \leq r$;

(E) the boundary condition is strictly dissipative in the following sense:

$$\exists\, \delta > 0 : \ (A_\nu u, u) \geq \delta |Pu|^2 - \frac{1}{\delta}|Mu|^2, \quad \forall u \in \mathbb{R}^N, \forall (x,t) \in \Sigma_T,$$

where P denotes the orthogonal projection onto $(\ker A_\nu)^\perp$.

Observe that assumptions (C) and (D) yield $\ker A_\nu \subseteq \ker M$. Observe also that condition (E) implies the strict positivity

$$(A_\nu u, u) > 0, \quad \forall u \in \ker M,\ Pu \neq 0. \tag{1.2}$$

Actually, if A_ν is a continuous matrix function of (x,t) and $\partial\Omega$ is a compact set, we can show that (1.2) is equivalent to (E) (see Appendix Lemma 7.1).

The initial-boundary value problem (1.1) has been considered by many authors, in particular by Kreiss [3] under quite general boundary conditions in the case of non-characteristic boundaries, see also [7, 9]. The analysis has been extended by Majda & Osher [8] to characteristic boundaries in the case when the boundary matrix has constant rank in a neighborhood of the boundary. Unfortunately, this is not the case in many physical applications, such as the compressible Euler equations and the equations of ideal compressible magneto-hydrodynamics. The relevant case of characteristic boundary when the boundary matrix has constant rank *only* at the boundary has been studied in the linear case for maximally non-negative boundary conditions by Rauch [11], Ohno-Shizuta-Yanagisawa [10], Secchi [13, 14]; for the applications to quasilinear problems see Secchi [15, 16, 17, 20]. In the previous papers the boundary conditions are always linear and homogeneous. Motivated by the interest in relativity theory [2, 21], in the present paper we study the characteristic problem with non-homogeneous boundary conditions. We introduce condition (E) at the boundary which provides an optimal regularity for the trace at the boundary of the non-characteristic part of the solution; this seems to be a fundamental point in view of possible extensions to nonlinear boundary conditions. Condition (E) is a natural extension to characteristic boundaries of the strict positivity condition

$$\exists\, \delta > 0 : \quad (A_\nu u, u) \geq \delta |u|^2 - \frac{1}{\delta}|Mu|^2, \quad \forall u \in \mathbb{R}^N, \tag{1.3}$$

which in its turn is equivalent to (see [6])

$$(A_\nu u, u) > 0 \quad \forall u \in \ker M,\ u \neq 0.$$

Problem (1.1) was studied by Secchi [18] in the non-characteristic case under the previous assumption (1.3), when $G \neq 0$. He proved the existence of regular solutions in suitable Sobolev spaces. In the present paper we prove the existence of solutions in anisotropic weighted Sobolev spaces $H^m_*(\Omega)$, which take account of the loss of regularity in the normal direction to the characteristic boundary.

Recall that for characteristic mixed problems the full regularity can not be expected, in the sense that the regularity theory can not be stated in terms of the usual Sobolev spaces $H^m(\Omega)$. In this case the space $H^m_*(\Omega)$ seems to be more suitable. This function space was introduced by Chen Shuxing [4] and Yanagisawa & Matsumura [22]. The definition of $H^m_*(\Omega)$ is motivated by the observation that the normal differentiation of order one of the solutions results from the tangential differentiation of order two.

2. Function spaces

We denote by H^m the usual Sobolev space $H^m(\Omega)$ and by $||\cdot||_m$ its norm. For simplicity, we denote the norm of $L^2 = L^2(\Omega)$ by $||\cdot||$. Let $\sigma = \sigma(x_1)$ be a smooth and positive function such that $\sigma(x_1) = x_1$ in a neighborhood of the origin and $\sigma(x_1) = 1$ for x_1 large enough. The differential operator in the tangential direction is $\partial^\alpha_* = (\sigma\partial_1)^{\alpha_1}\partial_2^{\alpha_2}\dots\partial_n^{\alpha_n}$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index of length $|\alpha| = \alpha_1 + \dots + \alpha_n$.

Given an integer $m \geq 1$, we denote by $H^m_* = H^m_*(\Omega)$ the space of distributions $u \in L^2$ such that $\partial^\alpha_* \partial^k_1 u \in L^2$ for any multi-index α and non-negative integer k such that $|\alpha| + 2k \leq m$.

Observe that, by definition, in H^m_* tangential and normal derivatives have respectively order one and two. The norm in H^m_* is

$$||u||^2_{m,*} = \sum_{|\alpha|+2k \leq m} ||\partial^\alpha_* \partial^k_1 u||^2.$$

Note that $H^0_* = L^2$. In the sequel, we set

$$\partial^\alpha_\star = \partial^{\alpha_0}_t (\sigma \partial_1)^{\alpha_1} \partial^{\alpha_2}_2 \dots \partial^{\alpha_n}_n,$$

for any multi-index $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n)$, $|\alpha| = \alpha_0 + \alpha_1 + \dots + \alpha_n$.

As usual, for any given $T > 0$ and for any normed space X, the symbol $L^p(0,T;X)$, with $1 \leq p < +\infty$, denotes the set of all measurable functions $u(t)$ with values in X such that

$$\|u\|_{L^p(X)} = \Big(\int_0^T \|u(t)\|^p_X dt \Big)^{\frac{1}{p}} < +\infty,$$

where $\| \cdot \|_X$ is the norm in X.

The set of all essentially bounded (with respect to the norm of X) measurable functions of t with values in X is denoted by $L^\infty(0,T;X)$. We equip this space with the usual norm

$$\|u\|_{L^\infty(X)} = \sup_{t \in (0,T)} \|u(t)\|_X.$$

Let $C^m([0,T];X)$ denote the set of all X-valued m-times continuously differentiable functions of t, for $0 \leq t \leq T$. The norm in $C([0,T];L^2)$ is denoted by $|| \cdot ||_{0,T}$.

For $t \in [0,T]$, let $u(t)$ belong to H^m_*, and be such that $\partial^k_t u(t) \in H^{m-k}_*$ for $k = 1, \dots, m$. Then, we set

$$|||u(t)|||^2_{m,*} = \sum_{k=0}^m ||\partial^k_t u(t)||^2_{m-k,*}.$$

We define

$$\mathcal{L}^2_T(H^m_*) = \bigcap_{k=0}^m H^k(0,T;H^{m-k}_*), \quad \mathcal{C}_T(H^m_*) = \bigcap_{k=0}^m C^k([0,T];H^{m-k}_*)$$

and

$$\mathcal{L}^\infty_T(H^m_*) = \bigcap_{k=0}^m W^{k,\infty}(0,T;H^{m-k}_*),$$

equipped, respectively, with the norms

$$[u]^2_{m,*,T} = \int_0^T |||u(t)|||^2_{m,*}\,dt, \quad |||u|||_{m,*,T} = \sup_{[0,T]} |||u(t)|||_{m,*}.$$

When H^m_* is replaced by the usual Sobolev space H^m we have $\mathcal{L}^2_T(H^m) = H^m(Q_T)$, with the norm $[\cdot]_{m,T}$.

In the sequel, for $j = 0, \ldots, n$, we write as in assumption (C)

$$A_j = \begin{pmatrix} A_j^{I,I} & A_j^{I,II} \\ \\ A_j^{II,I} & A_j^{II,II} \end{pmatrix};$$

accordingly we decompose $u = (u^I, u^{II})$, $F = (F^I, F^{II})$ and so on. Observe that $Pu = (u^I, 0)$. We now introduce the space

$$\mathcal{H}^m = \{u \in H^m_* : \ \partial_1 u^I \in H^{m-1}_*\},$$

with the norm

$$||u||^2_{\mathcal{H}^m} = ||u||^2_{m,*} + ||\partial_1 u^I||^2_{m-1,*}.$$

We set $\mathcal{H}^0 = L^2$. Let also

$$\mathcal{C}_T(\mathcal{H}^m) = \bigcap_{k=0}^{m} C^k([0,T]; \mathcal{H}^{m-k}),$$

with the norm

$$|||u|||_{m,T} = \sup_{[0,T]} |||u(t)|||_m,$$

where $|||u(t)|||^2_m = |||u(t)|||^2_{m,*} + |||\partial_1 u^I|||^2_{m-1,*}$.

Define $\mathcal{C}_T(H^m)$ similarly by using H^{m-k} instead of $\mathcal{H}^{m-k}$.

In this paper, we shall denote by c, c_i, and C different positive constants. The symbol $c(k, s, q)$ means that c depends increasingly at most on the quantities inside the brackets.

3. Main result

Before stating the main result, we consider the compatibility conditions. Given system (1.1), we recursively define $f^{(k)}, k \geq 1$, by formally taking $k-1$ time derivatives of $Lu = F$, solving for $\partial_t^k u$ and evaluating it at initial time $t = 0$. For $k = 0$, let $f^{(0)} = f$. We set

$$|||f|||^2_{m,*} = \sum_{k=0}^{m} ||f^{(k)}||^2_{m-k,*}.$$

The compatibility condition of order $k \geq 0$ for Problem (1.1) is (see [12])

$$M\, f^{(k)} = \partial_t^k G|_{t=0} \text{ on } \Gamma. \tag{3.1}$$

We can now state the main result of the present paper.

Theorem 3.1. *For $T > 0$, let s, m be integers such that $s \geq 2[\frac{n}{2}] + 6$ and $1 \leq m \leq s$. Assume that (A)-(E) hold and that*

$$A_j \in \mathcal{L}_T^\infty(H_*^s),\ j = 0, \dots, n \text{ and } B \in \mathcal{L}_T^\infty(H_*^{s-1}) \text{ if } m \leq s-1, \tag{3.2}$$

$$A_j, B \in \mathcal{L}_T^\infty(H_*^s),\ j = 0, \dots, n \text{ if } m = s. \tag{3.3}$$

Let $(F, G) \in \mathcal{L}_T^2(H_*^m) \times H^m(\Sigma_T)$ and $f^{(k)} \in H_*^{m-k}$, for $k = 0, \dots, m$.

Assume that the compatibility conditions are satisfied up to order $m-1$.

Then Problem (1.1) has a unique solution $u \in \mathcal{C}_T(H_*^m)$ with $Pu_{|\Sigma_T} \in H^m(\Sigma_T)$ and such that, for each t in $[0, T]$,

$$\begin{aligned} &|||u(t)|||_{m,*}^2 + \delta ||Pu||_{H^m(\Sigma_t)}^2 \\ &\quad \leq \{C_1 |||f|||_{m,*}^2 + C_2 [F]_{m,*,t}^2 + \frac{C_1}{\delta} ||G||_{H^m(\Sigma_t)}^2\} e^{C_2 t}, \end{aligned} \tag{3.4}$$

where $C_1 = C(\mu^{-1}, a_0^{-1}, T, |||A_j|||_{s-2,*,T}, |||B|||_{s-2,*,T})$, while $C_2 = C(\mu^{-1}, a_0^{-1}, T, |||A_j|||_{s,*,T}, |||B|||_{s,*,T})$. If $m \leq s-1$, C_2 depends on $|||B|||_{s-1,*,T}$ instead of $|||B|||_{s,*,T}$.

Remark 3.2. We observe that in general the assumptions $F \in \mathcal{L}_T^2(H_*^m)$ and $f \in H_*^m$ only imply that $f^{(k)} \in H_*^{m-2k}$, for $k = 1, \dots, [\frac{m}{2}]$. Thus, we must also assume that $f^{(k)} \in H_*^{m-k}$, for $k = 1, \dots, m$. Note that for $k = m-1$ it is not clear in which sense the compatibility conditions are satisfied. In fact $f^{(m-1)} \in H_*^1$, but for functions in that space the trace on the boundary has no meaning. From Corollary 3.4 we get a further regularity for each $f^{(k)}$ which allows us to analyze the precise sense in which the compatibility conditions are satisfied.

By following the same lines as in Secchi [14] we can prove an additional regularity for the solution of problem (1.1). More precisely we have the following results.

Theorem 3.3. *Assume that conditions of Theorem 3.1 hold; let $u \in \mathcal{C}_T(H_*^m)$ be a solution of Problem (1.1). Then $u \in \mathcal{C}_T(\mathcal{H}^m)$ and, for each $t \in [0, T]$, $M\partial_t^k u(t) \in H^{m-k-1/2}(\Gamma)$, $k = 0, 1, \cdots, m-1$.*

Corollary 3.4. *Assume that the conditions of Theorem 3.3 hold. Then $f^{(k)} \in \mathcal{H}^{m-k}$ and, $Mf^{(k)} \in H^{m-k-1/2}(\Gamma)$, $k = 0, 1, \cdots, m-1$.*

For the proof of Theorem 3.3 and Corollary 3.4 see [14], Theorem 4.2 and Corollary 4.3, respectively.

As a consequence of the previous results we get

Theorem 3.5. *For $T > 0$, let s, m be integers such that $s \geq 2[\frac{n}{2}] + 6$ and $1 \leq m \leq s$. Assume that (A)-(E) hold and that*

$$A_j \in \mathcal{L}_T^\infty(H_*^s),\ j = 0, \dots, n \text{ and } B \in \mathcal{L}_T^\infty(H_*^{s-1}) \text{ if } m \leq s-1, \quad (3.5)$$

$$A_j, B \in \mathcal{L}_T^\infty(H_*^s),\ j = 0, \dots, n \text{ if } m = s. \quad (3.6)$$

Let $(F, G) \in \mathcal{L}_T^2(H_*^m) \times H^m(\Sigma_T)$ and $f^{(k)} \in \mathcal{H}^{m-k}$, for $k = 0, \dots, m-1$.

Assume that the compatibility conditions are satisfied up to order $m - 1$.

Then Problem (1.1) *has a unique solution $u \in \mathcal{C}_T(\mathcal{H}^m)$ with $Pu_{|\Sigma_T} \in H^m(\Sigma_T)$ and such that, for each t in $[0, T]$,*

$$|||u(t)|||_m^2 + \delta ||Pu||_{H^m(\Sigma_t)}^2 \quad (3.7)$$
$$\leq Ce^{Ct}\{|||f|||_{m,*}^2 + |||F(0)|||_{m-1,*}^2 + [F]_{m,*,t}^2 + \frac{1}{\delta}||G||_{H^m(\Sigma_t)}^2\},$$

where $C = C(\mu^{-1}, a_0^{-1}, T, |||A_j|||_{s,*,T}, |||B|||_{s,*,T})$. If $m \leq s - 1$, C depends on $|||B|||_{s-1,*,T}$ instead of $|||B|||_{s,*,T}$.

In order to prove Theorem 3.1 we proceed by finite induction on m. In Section 4 we consider the case $m = 1$, where the existence of a solution becomes reducing to an initial-boundary value problem with homogeneous boundary conditions. In Section 5 we show how to increase the regularity up to the prescribed order $m > 1$. In Section 6 we prove the a priori estimate (3.4).

4. Proof of Theorem 3.1 for m=1

To prove Theorem 3.1 for $m = 1$, we approximate (f, F, G) with regular functions satisfying one further compatibility condition.

Lemma 4.1. *Let (f, F, G) satisfy the assumptions of Theorem* (3.1) *for $m = 1$. Then, there exists $(f_\lambda, F_\lambda, G_\lambda) \in H^3 \times H^3(Q_T) \times H^3(\Sigma_T)$, with $\lambda \in \mathbb{N}$, such that, for $\lambda \to +\infty$,*

$$\begin{array}{lll} f_\lambda \to f & \text{in} & H_*^1, \\ F_\lambda \to F & \text{in} & \mathcal{L}_T^2(H_*^1), \\ G_\lambda \to G & \text{in} & H^1(\Sigma_T), \end{array} \quad (4.1)$$

and the compatibility conditions of order 0 *and* 1 *are satisfied, that is*

$$Mf_\lambda^{(k)} = \partial_t^k G_\lambda|_{t=0} \quad \text{on } \Gamma, \text{ for } k = 0, 1.$$

Proof. For the proof we follow the same lines as in Secchi [14]. Some changes are needed in our case, since we study a non-homogeneous problem.

Let $C^\infty_{(0)}(A)$ be the set of restrictions, to a given set $A \subseteq \mathbb{R}^k$, of functions of $C^\infty_0(\mathbb{R}^k)$. It is well known that $C^\infty_{(0)}(\Omega)$ is dense in H^m_* and in $\mathcal{H}^m$, for $m \geq 1$. Furthermore, $C^\infty_{(0)}([0,T] \times \overline{\Omega})$ is dense in $C_T(H^m_*)$ (see [10], [14]).

From Corollary 3.4 $f \in \mathcal{H}^1$. Hence there exist $(g_\lambda, F_\lambda, G_\lambda) \in H^3 \times H^3(Q_T) \times H^3(\Sigma_T)$ such that

$$\begin{array}{lll} g_\lambda \to f & \text{in} & \mathcal{H}^1, \\ F_\lambda \to F & \text{in} & \mathcal{L}^2_T(H^1_*), \\ G_\lambda \to G & \text{in} & H^1(\Sigma_T). \end{array}$$

We now define the following linear operators

$$B_0 f = f, \quad E_0 F = 0$$

$$B_1 f = -A_0^{-1}\left(\sum_{j=1}^n A_j \partial_j + B\right) f,$$
$$E_1 F = A_0^{-1} F(0),$$

where all matrices are evaluated at time $t = 0$. Hence,

$$f = B_0 f + E_0 F, \qquad f^{(1)} = B_1 f + E_1 F.$$

The main idea is to seek $f_\lambda = g_\lambda - h_\lambda$, with $h_\lambda \in H^3$, $h_\lambda \to 0$ in H^1_* and such that f_λ satisfies one more compatibility condition. More precisely,

$$MB_i h_\lambda = M(B_i g_\lambda + E_i F_\lambda) - \partial_t^i G_\lambda(0) \quad \text{on } \Gamma \text{ for } i = 0, 1. \tag{4.2}$$

Since $M = (I_d, 0)$, we look for $h_\lambda = (k_\lambda, 0)$ with k_λ being a d-vector, a solution of the following trace problem

$$\partial_1^i k_\lambda = b_{i,\lambda} \quad \text{on } \Gamma \text{ for } i = 0, 1,$$

where

$$b_{0,\lambda} = M g_\lambda - G_\lambda(0), \qquad b_{1,\lambda} = M \begin{pmatrix} d_{1,\lambda} \\ 0 \end{pmatrix}$$
$$d_{1,\lambda} = -(A_1^{I,I})^{-1}\Big\{\Big(\sum_{j=2}^n A_j \partial_j + B\Big)\begin{pmatrix} M g_\lambda - G_\lambda(0) \\ 0 \end{pmatrix}$$
$$+ A_0(B_1 g_\lambda + E_1 F_\lambda) - A_0 \begin{pmatrix} \partial_t G_\lambda(0) \\ 0 \end{pmatrix}\Big\}^I$$

where, again, all matrices are evaluated at $t = 0$.

Since $b_{0,\lambda} \in H^{\frac{5}{2}}(\Gamma)$ and $b_{1,\lambda} \in H^{\frac{3}{2}}(\Gamma)$, we can define a linear bounded operator $\mathcal{R} : H^{\frac{5}{2}}(\Gamma) \times H^{\frac{3}{2}}(\Gamma) \longrightarrow H^3$, such that

$$\begin{array}{ll} \partial_1^i \mathcal{R}(b_0, b_1) = b_i \quad \text{on } \Gamma \quad \text{for } i = 0, 1, \\ \partial_1^2 \mathcal{R}(b_0, b_1) = 0 \quad \text{on } \Gamma, \end{array} \tag{4.3}$$

and such that the maps

$$b_0 \longrightarrow \mathcal{R}(b_0, 0), \qquad b_1 \longrightarrow \mathcal{R}(0, b_1), \tag{4.4}$$

are respectively continuous from $H^{\frac{1}{2}}(\Gamma)$ to H^1 and from $H^{\frac{1}{2}}(\Gamma)$ to H^2. The existence of the before-mentioned operator is ensured by Theorem 2.5.7 of [5].

For each $\lambda \in \mathbb{N}$, let $b'_{1,\lambda} \in C_0^\infty(\Gamma)$ such that

$$||b_{1,\lambda} - b'_{1,\lambda}||_{H^{\frac{1}{2}}(\Gamma)} \leq \frac{1}{\lambda}.$$

Take $k_\lambda = k_\lambda^{(1)} + k_\lambda^{(2)} + \omega_\lambda$, where

$$k_\lambda^{(1)} = \mathcal{R}(b_{0,\lambda}, 0) \in H^3, \quad k_\lambda^{(2)} = \mathcal{R}(0, b_{1,\lambda} - b'_{1,\lambda}) \in H^3.$$

Since $g_\lambda \to f$ in $\mathcal{H}^1$, $G_\lambda \to G$ in $H^1(\Sigma_T)$ and $Mf = G|_{t=0}$ on Γ, we get

$$\begin{array}{l} g_\lambda^I \to f^I \quad \text{in } H^1, \\ G_\lambda(0) \to G(0) \quad \text{in } H^{1/2}(\Gamma), \\ b_{0,\lambda} \to 0 \quad \text{in } H^{\frac{1}{2}}(\Gamma). \end{array}$$

Hence, by using the continuity of the maps defined in (4.3) and in (4.4), it follows that $k_\lambda^{(1)} \to 0$ in H^1 and $k_\lambda^{(2)} \to 0$ in H^2.

To conclude, we have to determine $\omega_\lambda \in H^3 \cap H_0^1$ such that

$$\begin{array}{ll} \omega_\lambda \longrightarrow 0 & \text{in } H^1, \\ \partial_1 \omega_\lambda = b'_{1,\lambda} & \text{on } \Gamma, \\ \partial_1^2 \omega_\lambda = 0 & \text{on } \Gamma. \end{array} \tag{4.5}$$

Let $\alpha_\lambda \to +\infty$ be such that $||b'_{1,\lambda}||_{H^1(\Gamma)} \leq \alpha_\lambda$. Set $y = (x_2, \ldots, x_n)$ and define

$$\omega_\lambda(x_1, y) = x_1 \psi(\alpha_\lambda^2 x_1) b'_{1,\lambda}(y),$$

with $\psi = \psi(x_1) \in C_0^\infty(\mathbb{R})$, $\psi = 1$ in a neighborhood of $x_1 = 0$. Then the sequence ω_λ satisfies the properties required in (4.5). In particular, one has that

$$||\omega_\lambda||_{H^1} \leq \frac{c}{\alpha_\lambda} \to 0,$$

where c is a real positive constant independent of λ. Then $h_\lambda \to 0$ in H^1 and satisfies equation (4.2); hence $h_\lambda \to 0$ in H_*^1 and the conclusion follows. $\square$

Let $(f_\lambda, F_\lambda, G_\lambda)$ be as in Lemma 4.1. We consider the following approximate problem

$$\begin{cases} Lu_\lambda = F_\lambda & \text{in} \quad Q_T, \\ Mu_\lambda = G_\lambda & \text{on} \quad \Sigma_T, \\ u_\lambda(x,0) = f_\lambda & \text{in} \quad \Omega. \end{cases} \tag{4.6}$$

Let $w_\lambda \in H^3(Q_T)$ be such that $Mw_\lambda(0) = G_\lambda(0)$, $M\partial_t w_\lambda(0) = \partial_t G_\lambda(0)$ on Γ and let $u_\lambda = v_\lambda + w_\lambda$, with v_λ a solution of

$$\begin{cases} Lv_\lambda = F_\lambda - Lw_\lambda & \text{in} \quad Q_T, \\ Mv_\lambda = 0 & \text{on} \quad \Sigma_T, \\ v_\lambda(x,0) = f_\lambda - w_\lambda(x,0) & \text{in} \quad \Omega. \end{cases} \tag{4.7}$$

Since v_λ satisfies the homogeneous compatibility conditions up to order 1, i.e. $Mv_\lambda^{(k)}|_{t=0} = 0$ on Γ for $k = 0, 1$, Theorem A of [13] ensures that there exists a unique solution of Problem (4.7) such that $v_\lambda \in \mathcal{C}_T(\mathcal{H}^2)$. Therefore there exists a solution $u_\lambda \in \mathcal{C}_T(\mathcal{H}^2)$ of problem (4.6). It remains to show that $u_\lambda \to u$ in $\mathcal{C}_T(H^1_*)$, and that u is the solution of Problem (1.1).

Lemma 4.2. *Let u_λ be the solution of Problem* (4.6). *Then, for any $t \in [0,T]$, the energy-type estimate*

$$\begin{aligned} &|||u_\lambda(t)|||^2_{1,*} + \delta||Pu_\lambda||^2_{H^1(\Sigma_t)} \\ &\quad \le \left\{C_1|||f_\lambda|||^2_{1,*} + C_2[F_\lambda]^2_{1,*,t} + \frac{c}{\delta}||G_\lambda||^2_{H^1(\Sigma_t)}\right\}e^{C_2 t}, \end{aligned} \tag{4.8}$$

holds, where C_i with $i = 1, 2$ are the constants in Theorem 3.1 *and c is a suitable real positive constant independent of λ.*

Proof. We shall denote by $c_i(t)$, for $i = 1, 2, 3$, some real positive functions of time depending on the L^∞–norms of A_j (with $j = 0, \dots, n$) and B and on some of their time and tangential derivatives.

We multiply $(4.6)_1$ by u_λ. By using (E), we obtain

$$\begin{aligned} &\frac{d}{dt}||A_0^{1/2}u_\lambda(t)||^2 + \delta||Pu_\lambda(t)||^2_{L^2(\Gamma)} \\ &\quad \le c_1(t)||u_\lambda(t)||^2 + ||F_\lambda(t)||^2 + \frac{1}{\delta}||G_\lambda(t)||^2_{L^2(\Gamma)}. \end{aligned} \tag{4.9}$$

We now apply the operators ∂_t and ∂_j, for $j = 2, \dots, n$, to $(4.6)_1$ and we multiply the resulting expressions respectively by $\partial_t u_\lambda$ and $\partial_j u_\lambda$. By using (E) again, we get

$$\frac{1}{2}\frac{d}{dt}\Big(||A_0^{1/2}(\partial_t u_\lambda(t))||^2 + \sum_{j=2}^{n}||A_0^{1/2}(\partial_j u_\lambda(t))||^2\Big) \tag{4.10}$$

$$+\frac{\delta}{2}\Big(||P(\partial_t u_\lambda(t))||^2_{L^2(\Gamma)}+\sum_{j=2}^n ||P(\partial_j u_\lambda(t))||^2_{L^2(\Gamma)}\Big)$$
$$\le c_2(t)\Big(||u_\lambda(t)||^2+||\partial_t u_\lambda(t)||^2+\sum_{j=2}^n ||\partial_j u_\lambda(t)||^2\Big)$$
$$+||\partial_t F_\lambda(t)||^2+\sum_{j=2}^n ||\partial_j F_\lambda(t)||^2$$
$$+\frac{c}{\delta}\Big(||\partial_t G_\lambda(t)||^2_{L^2(\Gamma)}+\sum_{j=2}^n ||\partial_j G_\lambda(t)||^2_{L^2(\Gamma)}\Big).$$

We now apply the operator $\sigma(x_1)\partial_1$ to $(4.6)_1$. Since

$$\sigma(x_1)\partial_1(\partial_1)=\partial_1(\sigma(x_1)\partial_1)-\sigma'(x_1)\partial_1,$$

and using $(4.6)_1$ for $A_1\partial_1 u_\lambda$, we get

$$L(\sigma\partial_1 u_\lambda)=\sigma\partial_1 F_\lambda-\Big(\sigma\partial_1 A_0\partial_t+\sum_{j=2}^n \sigma\partial_1 A_j\partial_j\Big)u_\lambda-\partial_1 A_1\sigma\partial_1 u_\lambda-\sigma\partial_1 B u_\lambda$$
$$+\ \sigma'(F_\lambda-A_0\partial_t u_\lambda-\sum_{j=2}^n A_j\partial_j u_\lambda-Bu_\lambda). \tag{4.11}$$

By multiplying (4.11) by $\sigma\partial_1 u_\lambda$, it easily follows that

$$\frac{d}{dt}||A_0^{1/2}(\sigma\partial_1 u_\lambda(t))||^2\le c_3(t)\Big\{||u_\lambda(t)||^2+||\partial_t u_\lambda(t)||^2 \tag{4.12}$$
$$+\sum_{j=2}^n ||\partial_j u_\lambda(t)||^2+||\sigma\partial_1 u_\lambda(t)||^2\Big\}+||\sigma\partial_1 F_\lambda||^2+c\|F_\lambda\|^2.$$

We add (4.9), (4.10), (4.12) and we observe that $\|A_0^{1/2}v\|$ is equivalent to $\|v\|$; integrating on $[0,t]$ the resulting expression and applying the Gronwall lemma we get (4.8). □

We see from (4.1) and (4.8) that, up to a subsequence, u_λ converges to u weakly-star in $\mathcal{L}^\infty_T(H^1_*)$. By passing to the limit for $\lambda\to+\infty$ in Problem (4.6), we find that u is a solution of system (1.1) and obtain (3.4) for $m=1$.

We now show that $u\in\mathcal{C}_T(H^1_*)$. Since $\mathcal{L}^\infty_T(H^1_*)\hookrightarrow C^\alpha([0,T];L^2)$ for all $\alpha>0$, we get that $u\in C^\alpha([0,T];L^2)$. By adapting the approach of Majda [6], we can prove that $u\in C_w([0,T];H^1_*)\cap C^1_w([0,T];L^2)$. In order to show the strong continuity of u, it is sufficient to prove the strong right continuity at time $t = 0$ (see [6], page 44). Since $u \in C^\alpha([0,T];L^2)$, to conclude the proof, we have to show the continuity of time and spatial derivatives only for $t = 0$. For this argument, we follow the same lines as in [14]. Uniqueness is an obvious consequence of linearity and the a priori estimate.

5. Proof of Theorem 3.1 for $\mathbf{m} > \mathbf{1}$

Assume now that Theorem 3.1 holds up to $m-1$. Let $f \in H^m_*$, $F \in \mathcal{L}^2_T(H^m_*)$, $G \in H^m(\Sigma_T)$, with $f^{(k)} \in H^{m-k}_*$, $k = 1, \cdots, m$ and assume also that the compatibility conditions (3.1) hold up to order $m-1$.

By the inductive hypothesis there exists a unique solution u of problem (1.1) such that $u \in \mathcal{C}_T(H^{m-1}_*)$. In order to show that $u \in \mathcal{C}_T(H^m_*)$, it is sufficient to prove that $\partial^\alpha_\star \partial^k_1 u$, with $|\alpha| + 2k = m-1$, belong to $\mathcal{C}_T(H^1_*)$ and that, for m even, $\partial^\alpha_\star \partial^{k+1}_1 u$ are in $C_T(L^2)$ with $|\alpha| + 2k = m-2$.

• **Step 1.** Let us start by considering all the tangential derivatives $\partial^\alpha_\star u$, $|\alpha| = m-1$. Apply the operator $\partial^\alpha_\star$ to $(1.1)_1$ and decompose

$$\partial_1 u = \begin{pmatrix} \partial_1 u^I \\ \partial_1 u^{II} \end{pmatrix}.$$

As in [14], [13], by inverting $A_1^{I,I}$ in $(1.1)_1$, we can write $\partial_1 u^I$ as the sum of tangential derivatives by

$$\partial_1 u^I = \Lambda \partial_\star u + R \tag{5.1}$$

where

$$\Lambda \partial_\star u = (\mathcal{X} - 1)(A_1^{I,I})^{-1} \Big\{ \Big[A_0 \partial_t u + \sum_{j=2}^n A_j \partial_j u \Big]^I + A_1^{I,II} \partial_1 u^{II} \Big\} + \mathcal{X} \partial_1 u^I,$$

$$R = (\mathcal{X} - 1)(A_1^{I,I})^{-1}(Bu - F)^I, \tag{5.2}$$

and $\mathcal{X} = \mathcal{X}(x_1) \in C^\infty(R)$ is equal to zero in a sufficiently small neighborhood of $x_1 = 0$, and equal to 1 for $x_1 > 1$.

Recall that if a matrix A vanishes on Γ, we can write

$$A \partial_1 u = H \sigma(x_1) \partial_1 u, \tag{5.3}$$

where H is a suitable matrix such that $||H||_{s-2,*} \le c||A||_{s,*}$. For a rigorous definition of H and its properties, we refer to Lemmata 7.8 and 7.9 in the Appendix. Since $A_1^{I,II}$ vanishes on Γ, we can consider $A_1^{I,II} \partial_1 u^{II}$ in (5.2) as a tangential derivative. We obtain $\Lambda \in \mathcal{L}^\infty_T(H^{s-2}_*)$.

Hence, we obtain exactly equation (5.3) of Secchi in [13], that is,

$$
\begin{aligned}
&L(\partial_\star^\alpha u) + \sum_{|\alpha'|=|\alpha|-1} \Big(\partial_\star A_0 \partial_t + \sum_{j=2}^n \partial_\star A_j \partial_j\Big) \partial_\star^{\alpha'} u + \sum_{|\alpha'|=|\alpha|-1} \partial_\star A_1 \begin{pmatrix} \Lambda \partial_\star (\partial_\star^{\alpha'} u) \\ 0 \end{pmatrix} \\
&\qquad - \alpha_1 A_1 \begin{pmatrix} \Lambda \partial_\star (\partial_t^{\alpha_0} (x_1\partial_1)^{\alpha_1 - 1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} u) \\ 0 \end{pmatrix} \qquad (5.4) \\
&+ \Big(\sum_{|\alpha'|=|\alpha|-1} \partial_\star A_1 \partial_\star^{\alpha'} - \alpha_1 A_1 \partial_t^{\alpha_0} (x_1\partial_1)^{\alpha_1-1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} \Big) \begin{pmatrix} 0 \\ \partial_1 u^{II} \end{pmatrix} = \overline{F}_\alpha,
\end{aligned}
$$

where

$$
\begin{aligned}
\overline{F}_\alpha &= \partial_\star^\alpha F - \sum_{|\beta|=2}^{|\alpha|} \Big[\Big(\partial_\star^\beta A_0 \partial_t + \sum_{j=2}^n \partial_\star^\beta A_j \partial_j \Big) \partial_\star^{\alpha-\beta} u + \partial_\star^\beta A_1 \partial_\star^{\alpha-\beta} \begin{pmatrix} \Lambda \partial_\star u + R \\ \partial_1 u^{II} \end{pmatrix} \Big] \\
&\quad - \begin{pmatrix} \alpha_1 \\ 2 \end{pmatrix} A_1 \partial_t^{\alpha_0} (x_1\partial_1)^{\alpha_1-2} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} \begin{pmatrix} \Lambda \partial_\star u + R \\ \partial_1 u^{II} \end{pmatrix} \\
&\quad - A_1 \partial_1 \partial_t^{\alpha_0} \Big[(x_1\partial_1 - 1)^{\alpha_1} - (x_1\partial_1)^{\alpha_1} + \alpha_1 (x_1\partial_1 - 1)^{\alpha_1 - 1} \\
&\quad - \begin{pmatrix} \alpha_1 \\ 2 \end{pmatrix} (x_1\partial_1 - 1)^{\alpha_1 - 2} \Big] \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} u - \sum_{|\alpha'|=|\alpha|-1} \partial_\star A_1 \Big[\partial_\star^{\alpha'}, \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} \Big] \partial_\star u \\
&\quad + \alpha_1 A_1 \Big[\partial_t^{\alpha_0} (x_1\partial_1)^{\alpha_1-1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n}, \begin{pmatrix} \Lambda \\ 0 \end{pmatrix} \Big] \partial_\star u - [\partial_\star, B]\, u \\
&\quad - \Big(\sum_{|\alpha'|=|\alpha|-1} \partial_\star A_1 \partial_\star^{\alpha'} - \alpha_1 A_1 \partial_t^{\alpha_0} (x_1\partial_1)^{\alpha_1-1} \partial_2^{\alpha_2} \cdots \partial_n^{\alpha_n} \Big) \begin{pmatrix} R \\ 0 \end{pmatrix}.
\end{aligned}
$$

By (5.3), since for $k = I$ or II both $A_1^{k,II}$ and $\partial_\star A_1^{k,II}$ are equal to zero on Γ, the last term in the left-hand side of (5.4) can be written as a sum of terms

$$H\sigma(x_1)\partial_1 \partial_\star^\gamma u \quad \text{with } |\gamma| = |\alpha| - 1,$$

that is as a sum of tangential derivatives of order α. Hence, equation (5.4) takes the form

$$(L + \overline{B})\partial_\star^\alpha u = \overline{F}_\alpha$$

for a suitable linear operator $\overline{B} \in \mathcal{L}_T^\infty(H_*^{s-3})$. By estimating the products of functions in spaces H_*^m by means of Lemmata 7.4-7.7, we obtain $\overline{F}_\alpha \in \mathcal{L}_T^2(H_*^1)$.

We consider now the problem satisfied by the vector of all the tangential derivatives of order α, by abuse of notation still denoted by $\partial_\star^\alpha u$. It takes the form

$$\begin{cases} (\mathcal{L}+\mathcal{B})\partial_\star^\alpha u = \mathcal{F} & \text{in } Q_T, \\ \mathcal{M}\partial_\star^\alpha u = \partial_\star^\alpha G & \text{on } \Sigma_T, \\ \partial_\star^\alpha u|_{t=0} = \tilde{f} & \text{in } \Omega, \end{cases} \tag{5.5}$$

where

$$\mathcal{L} = \begin{pmatrix} L & & \\ & \ddots & \\ & & L \end{pmatrix}, \qquad \mathcal{M} = \begin{pmatrix} M & & \\ & \ddots & \\ & & M \end{pmatrix},$$

$\mathcal{B} \in \mathcal{L}^\infty_T(H^{s-3}_*)$ is a suitable linear operator and $\mathcal{F}$ is the vector of all right-hand sides $\overline{F}_\alpha$. The initial datum $\tilde{f}$ is the vector of functions $\partial_*^{\alpha'} f^{(\alpha_0)}$ if $\alpha = (\alpha_0, \alpha')$, $\alpha' = (\alpha_1, \cdots, \alpha_n)$.

It is clear that $\mathcal{F} \in \mathcal{L}^2_T(H^1_*)$, $\tilde{f} \in H^1_*$, $\partial_\star^\alpha G \in H^1(\Sigma_T)$. By applying Theorem 3.1 for $m = 1$, we see that problem (5.5) has a unique solution $\partial_\star^\alpha u \in \mathcal{C}_T(H^1_*)$, for all $|\alpha| = m-1$.

• **Step 2.** We apply to the part II of $(1.1)_1$ the operator $\partial_\star^\gamma \partial_1$, with $|\gamma| = m-2$. We obtain equation (5.4) of [13], which we write differently as

$$\Big[(L+\partial_1 A_1)\partial_\star^\gamma + \sum_{|\gamma'|=|\gamma|-1} \Big(\partial_\star A_0 \partial_t + \sum_{j=1}^n \partial_\star A_j \partial_j\Big)\partial_\star^{\gamma'} \\ - \gamma_1 A_1 \partial_1 \partial_t^{\gamma_0}(x_1\partial_1)^{\gamma_1-1}\partial_2^{\gamma_2}\cdots\partial_n^{\gamma_n}\Big]^{II,II} \partial_1 u^{II} = \tilde{\mathcal{G}}, \tag{5.6}$$

where

$$\tilde{\mathcal{G}} = \mathcal{G} - (L+\partial_1 A_1)^{II,I}\Lambda\partial_\star(\partial_\star^\gamma u) - \Big[\Big(\partial_1 A_0 \partial_t + \sum_{j=2}^n \partial_1 A_j \partial_j\Big)\partial_\star^\gamma u\Big]^{II} \\ - \sum_{|\gamma'|=|\gamma|-1} (\partial_\star A_0^{II,I})\Lambda\partial_\star(\partial_t \partial_\star^{\gamma'} u) - \sum_{j=2}^n \partial_\star A_j^{II,I}\Lambda\partial_\star(\partial_j\partial_\star^{\gamma'} u) \\ - \sum_{|\gamma'|=|\gamma|-1} \partial_\star A_1^{II,I}\Lambda\partial_\star(\partial_1\partial_\star^{\gamma'} u) + \gamma_1 A_1^{II,I}\Lambda\partial_\star(\partial_1\partial_t^{\gamma_0}(x_1\partial_1)^{\gamma_1-1}\partial_2^{\gamma_2}\cdots\partial_n^{\gamma_n} u),$$

and $\mathcal{G}$ is exactly the right-hand side of (5.4) in [13].

We can observe that $\tilde{\mathcal{G}}$ contains only tangential derivatives of order at most m. Hence, by using (5.3) again, we can write (5.6) as

$$(\tilde{\mathcal{L}} + \tilde{\mathcal{C}})\partial_\star^\gamma \partial_1 u^{II} = \tilde{\mathcal{G}}, \tag{5.7}$$

where

$$\tilde{\mathcal{L}} = \begin{pmatrix} \tilde{L} & & \\ & \ddots & \\ & & \tilde{L} \end{pmatrix}$$

with

$$\tilde{L} = A_0^{II,II}\partial_t + \sum_{j=1}^{n} A_j^{II,II}\partial_j$$

and $\tilde{\mathcal{C}} \in \mathcal{L}_T^\infty(H_*^{s-2})$ is a suitable linear operator. Observe that the boundary matrix of $\tilde{\mathcal{L}}$ vanishes on Γ. Moreover, by using (5.1), Lemmata 7.4–7.7 and step 1, we get that $\tilde{\mathcal{G}} \in L^2(Q_T)$. By Theorem 2.2 of [1], we find that equation (5.7) has a unique solution $\partial_\star^\gamma \partial_1 u^{II} \in C_T(L^2)$, with $|\gamma| = m-2$. From (5.1) $\partial_\star^\gamma \partial_1 u \in C_T(L^2)$, $|\gamma| = m-2$.

• **Step 3.** The last step is as in Secchi [13], page 867, (ii). However, for convenience of the reader, we provide a sketch of the proof. Suppose that for some fixed k, with $1 \le k < [m/2]$, it has already been shown that $\partial_\star^\alpha \partial_1^h u$ belongs to $C_T(L^2)$, for any h and α such that $h = 1, \cdots, k$, $|\alpha| + 2h \le m$. From (5.1) it immediately follows that $\partial_\star^\alpha \partial_1^{k+1} u^I \in C_T(L^2)$. It remains to prove that $\partial_\star^\alpha \partial_1^{k+1} u^{II} \in C_T(L^2)$. We apply operator $\partial_\star^\alpha \partial_1^{k+1}$, $|\alpha| + 2k = m-2$ to the part II of $(1.1)_1$ and obtain

$$(\tilde{\mathcal{L}} + \tilde{\mathcal{C}})\partial_\star^\alpha \partial_1^{k+1} u^{II} = \tilde{\mathcal{G}}_k,$$

where $\tilde{\mathcal{C}} \in \mathcal{L}_T^\infty(H_*^{s-3})$ is a suitable linear operator and $\tilde{\mathcal{G}}_k \in L^2(Q_T)$. Again by Theorem 2.2 of [1], the solution $\partial_\star^\alpha \partial_1^{k+1} u^{II}$ is in $C_T(L^2)$ for α, k with $|\alpha| + 2k = m-2$. By repeating this procedure we obtain the result for any $k \le [m/2]$, hence $u \in \mathcal{C}_T(H_*^m)$.

6. ENERGY-TYPE ESTIMATE

In this section we shall prove estimate (3.4). This is obtained from the following lemmata.

Lemma 6.1. *Let u be the solution of system* (1.1). *Then $\partial_1 u^I \in \mathcal{C}_T(H_*^{m-1})$ and the estimate*

$$|||\partial_1 u^I(t)|||_{m-1,*} \le c_4(t)\Big(\sum_{|\alpha| \le m} ||\partial_\star^\alpha u(t)|| + |||u^{II}(t)|||_{m,*} + |||F(t)|||_{m-1,*}\Big) \tag{6.1}$$

holds for each $t \in [0,T]$*, where* $c_4(t) = c(\mu^{-1}, |||A_j(t)|||_{s-1,*}, |||B(t)|||_{s-1,*})$ *for* $j = 0, \cdots, n$*; moreover the solution* u *is such that*

$$|||\partial_1 u^I(t)|||_{m-2,*} \leq c_5(t)\Big(\sum_{|\alpha|\leq m-1} ||\partial_\star^\alpha u(t)|| + |||u^{II}(t)|||_{m-1,*} + |||F(t)|||_{m-2,*}\Big) \tag{6.2}$$

for each $t \in [0,T]$*, where* $c_5(t) = c(\mu^{-1}, |||(A_j(t)|||_{s-2,*}, |||B(t)|||_{s-2,*})$ *for* $j = 0, \cdots, n$.

Proof. We write $\partial_1 u^I = \Lambda \partial_\star u + R$ as in (5.1) and apply the operator $\partial_\star^\beta \partial_1^k$ with $|\beta| + 2k \leq m-1$. By using recursively that $\partial_1 u^I$ can always be written in terms of u^{II} or tangential derivatives we obtain (6.1). The proof of (6.2) is similar. □

Lemma 6.2. *Let* u *be the solution of system* (1.1) *and let* α *be a multi-index with* $|\alpha| \leq m$*. Then, for each* $t \in [0,T]$*, the estimate*

$$\begin{aligned} &\frac{d}{dt}\|A_0^{1/2}\partial_\star^\alpha u(t)\|^2 + \delta\|P\partial_\star^\alpha u(t)\|^2_{L^2(\Gamma)} \\ &\quad \leq c(t)(|||u(t)|||^2_{m,*} + |||F(t)|||^2_{m,*}) + \frac{c}{\delta}\|\partial_\star^\alpha G(t)\|^2_{L^2(\Gamma)} \end{aligned}$$

holds, where $c(t) = c\,(|||A_j(t)|||_{s,*}, |||B(t)|||_{s,*})$ *for* $j = 0, \cdots, n$.

Proof. We formally apply $\partial_\star^\alpha$ to $(1.1)_1$ and obtain

$$L(\partial_\star^\alpha u) = F_\alpha, \tag{6.3}$$

where

$$F_\alpha = \partial_\star^\alpha F - [\partial_\star^\alpha, A_0\partial_t]\,u - \sum_{j=1}^n [\partial_\star^\alpha, A_j\partial_j]\,u - [\partial_\star^\alpha, B]\,u.$$

The standard energy estimate for (6.3) yields

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |A_0^{1/2}\partial_\star^\alpha u|^2 + \frac{1}{2}\int_\Gamma (A_\nu \partial_\star^\alpha u, \partial_\star^\alpha u) \leq \int_\Omega (F_\alpha, \partial_\star^\alpha u) + c(t)\|\partial_\star^\alpha u\|^2,$$

where $c(t)$ is a time function depending on the L^∞-norm of the matrices $\partial_t A_0, \partial_j A_j, B$. To conclude the proof we have to estimate $\|F_\alpha\|$.

With a slightly imprecise notation, we write

$$\begin{aligned} &[\partial_\star^\alpha, A_0\partial_t]\,u \approx \partial_\star^{\alpha-1}(\partial_\star A_0 \partial_t u), \\ &[\partial_\star^\alpha, A_j\partial_j]\,u \approx \partial_\star^{\alpha-1}(\partial_\star A_j \partial_j u), \text{ for } j = 1, \cdots, n, \end{aligned}$$

where $\partial_\star^{\alpha-1}$ means $\partial_\star^\beta$ for some multi-index β such that $\beta_i \leq \alpha_i$ and $|\beta| = |\alpha| - 1$. Since $u \in \mathcal{C}_T(H_*^m)$, by Lemma 7.6, we get that

$$\partial_\star A_0 \partial_t u \in \mathcal{C}_T(H_*^{m-1}),$$

$$\partial_\star A_j \partial_j u \in \mathcal{C}_T(H_*^{m-1}), \text{ for } j = 2, \cdots, n.$$

It follows that

$$\| [\partial_\star^\alpha, A_0 \partial_t] u\| \leq c|||A_0(t)|||_{s,*}|||u(t)|||_{m,*},$$

$$\| [\partial_\star^\alpha, A_j \partial_j] u\| \leq c|||A_j(t)|||_{s,*}|||u(t)|||_{m,*} \quad \text{for } j = 2, \cdots, n.$$

When $j = 1$,

$$[\partial_\star^\alpha, A_1 \partial_1] u \approx \partial_\star^{\alpha-1}\left(\partial_\star A_1 \begin{pmatrix} \partial_1 u^I \\ 0 \end{pmatrix}\right) + \partial_\star^{\alpha-1}\left(\partial_\star A_1 \begin{pmatrix} 0 \\ \partial_1 u^{II} \end{pmatrix}\right). \quad (6.4)$$

From Lemma 6.1 we get $\partial_\star A_1 \begin{pmatrix} \partial_1 u^I \\ 0 \end{pmatrix} \in \mathcal{C}_T(H_*^{m-1})$ and

$$\|\partial_\star^{\alpha-1}\left(\partial_\star A_1 \begin{pmatrix} \partial_1 u^I \\ 0 \end{pmatrix}\right)\| \leq c(t)(|||u(t)|||_{m,*} + |||F(t)|||_{m-1,*}),$$

where $c(t)$ is a time function depending on $|||A_j(t)|||_{s,*}$ for $j = 0, \cdots, n$ and $|||B(t)|||_{s-1,*}$. Concerning the second summand in (6.4), we note that both the matrices $\partial_\star A_1^{J,II}$, for $J = I, II$, vanish on Γ. By using (5.3) and Lemma 7.4, we get that $\partial_\star A_1 \begin{pmatrix} 0 \\ \partial_1 u^{II} \end{pmatrix}$ belongs to $\mathcal{C}_T(H_*^{m-1})$. Hence

$$\| [\partial_\star^\alpha, A_1 \partial_1] u\| \leq c(t)(|||u(t)|||_{m,*} + |||F(t)|||_{m-1,*}).$$

Adding the estimate of the other terms it follows that

$$\int_\Omega (F_\alpha, \partial_\star^\alpha u) \leq c(t)(|||u(t)|||_{m,*}^2 + |||F(t)|||_{m,*}^2),$$

where $c(t)$ is a positive time function depending on $|||A_j(t)|||_{s,*}$, for $j = 0, \cdots, n$ and $|||B(t)|||_{s,*}$. The result now follows by using (E). □

Lemma 6.3. *Let u be the solution of system* (1.1)*. Let $k \geq 1$ be an integer and β be a multi-index such that $|\beta| + 2k \leq m$. Then, for each $t \in [0, T]$, $\partial_\star^\beta \partial_1^k u^{II}$ satisfies*

$$\frac{d}{dt}\|(A_0^{II,II})^{1/2}\partial_\star^\beta \partial_1^k u^{II}(t)\|^2 \leq c(t)|||u(t)|||_{m,*}^2 + c|||F(t)|||_{m.*}^2,$$

where $c(t) = c\,(|||A_j(t)|||_{s,*}, |||B(t)|||_{s,*})$ for $j = 0, \cdots, n$.

Proof. We apply $\partial_\star^\beta \partial_1^k$ for $k \geq 1$, $|\beta| + 2k \leq m$, to the part II of $(1.1)_1$. We obtain

$$\left(A_0^{II,II}\partial_t + \sum_{j=1}^n A_j^{II,II}\partial_j + B^{II,II}\right)\partial_\star^\beta \partial_1^k u^{II} = R_{\beta,k}, \quad (6.5)$$

where

$$R_{\beta,k} = \partial_\star^\beta \partial_1^k \Big(F^{II} - A_0^{II,I} \partial_t u^I - \sum_{j=1}^n A_j^{II,I} \partial_j u^I - B^{II,I} u^I \Big) - \left[\partial_\star^\beta \partial_1^k, A_0^{II,II} \partial_t \right] u^{II} - \sum_{j=1}^n \left[\partial_\star^\beta \partial_1^k, A_j^{II,II} \partial_j \right] u^{II} - \left[\partial_\star^\beta \partial_1^k, B^{II,II} \right] u^{II}.$$

We multiply equation (6.5) by $\partial_\star^\beta \partial_1^k u^{II}$, which we denote, for simplicity, by V. By observing that $A_\nu^{II,II}$ vanishes on Γ, we get

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |(A_0^{II,II})^{1/2} V|^2 \leq c(t) \|V\|^2 + \int_\Omega (R_{\beta,k}, V)\,,$$

where $c(t) = c(\|\partial_t A_0^{II,II}(t)\|_{L^\infty}, \|\partial_j A_j^{II,II}(t)\|_{L^\infty}, \|B^{II,II}(t)\|_{L^\infty})$ for $j = 1, \ldots,$ n. We now estimate term by term the L^2-norm of $R_{\beta,k}$. Again from (5.1), it follows from Lemma 7.4 that

$$\left\| \partial_\star^\beta \partial_1^k \left(F^{II} - A_0^{II,I} \partial_t u^I - \textstyle\sum_{j=1}^n A_j^{II,I} \partial_j u^I - B^{II,I} u^I \right) - \left[\partial_\star^\beta \partial_1^k, B^{II,II} \right] u^{II} \right\| \leq |||F(t)|||_{m,*} + \Big(\sum_{j=0}^n |||A_j(t)|||_{s,*} + |||B(t)|||_{s,*} \Big) |||u(t)|||_{m,*}.$$

Moreover,

$$\left[\partial_\star^\beta \partial_1^k, A_0^{II,II} \partial_t\right] u^{II} \approx \partial_\star^{\beta-1} \partial_1^k (\partial_\star A_0^{II,II} \partial_t u^{II}) + \partial_\star^\beta \partial_1^{k-1} (\partial_1 A_0^{II,II} \partial_t u^{II}),$$

$$\left[\partial_\star^\beta \partial_1^k, A_j^{II,II} \partial_j\right] u^{II} \approx \partial_\star^{\beta-1} \partial_1^k (\partial_\star A_j^{II,II} \partial_j u^{II}) + \partial_\star^\beta \partial_1^{k-1} (\partial_1 A_j^{II,II} \partial_j u^{II}).$$

Hence, by using Lemma 7.4 again and (5.3) in case $j = 1$, we get

$$\left\| \left[\partial_\star^\beta \partial_1^k, A_0^{II,II} \partial_t \right] u \right\| + \sum_{j=1}^n \left\| \left[\partial_\star^\beta \partial_1^k, A_j^{II,II} \partial_j \right] u^{II} \right\| \leq \sum_{j=0}^n |||A_j(t)|||_{s,*} |||u(t)|||_{m,*}.$$

By collecting all the above estimates, it follows that

$$\|R_{\beta,k}\| \leq |||F(t)|||_{m,*} + \Big(\sum_{j=0}^n |||A_j(t)|||_{s,*} + |||B(t)|||_{s,*} \Big) |||u(t)|||_{m,*},$$

which yields the thesis. □

Lemma 6.4. *Let u be the solution of system* (1.1). *Then, for each $t \in [0,T]$, the energy-type estimate* (3.4) *holds.*

Proof. From (6.2) we have

$$|||u(t)|||^2_{m,*} = \sum_{|\alpha|\le m} ||\partial^\alpha_\star u(t)||^2 + |||u^{II}(t)|||^2_{m,*} + |||\partial_1 u^I(t)|||^2_{m-2,*} \tag{6.6}$$
$$\le c_5(t)\Big(\sum_{|\alpha|\le m} ||\partial^\alpha_\star u(t)||^2 + |||u^{II}(t)|||^2_{m,*} + |||F(t)|||^2_{m-2,*}\Big),$$

where $c_5(t)$ is the time function appearing in Lemma 6.1.

We add the inequalities appearing in Lemmata 6.2, 6.3 and substitute (6.6). This gives

$$\frac{d}{dt}\Big(\sum_{|\alpha|\le m} ||A_0^{1/2}\partial^\alpha_\star u(t)||^2 + \sum_{|\beta|+2k\le m} ||(A_0^{II,II})^{1/2}\partial^\beta_\star \partial_1^k u^{II}(t)||^2\Big)$$
$$+ \delta \sum_{|\alpha|\le m} ||\partial^\alpha_\star Pu(t)||^2_{L^2(\Gamma)}$$
$$\le c(t)\Big(\sum_{|\alpha|\le m} ||\partial^\alpha_\star u(t)||^2 + |||u^{II}(t)|||^2_{m,*} + |||F(t)|||^2_{m,*}\Big)$$
$$+ \frac{c}{\delta}\sum_{|\alpha|\le m} ||\partial^\alpha_\star G(t)||^2_{L^2(\Gamma)},$$

where $c(t) = c(\mu^{-1}, |||A_j(t)|||_{s,*}, |||B(t)|||_{s,*})$ for $j = 0,\cdots,n$ is a suitable time function. We integrate in time, use $A_0 \ge a_0 I$ and get

$$\sum_{|\alpha|\le m} ||\partial^\alpha_\star u(t)||^2 + |||u^{II}(t)|||^2_{m,*} \tag{6.7}$$
$$\le C_1|||f|||^2_{m,*} + C_2([F]^2_{m,*,t} + [u]^2_{m,*,t}) + \frac{c}{\delta}[G]^2_{H^m(\Sigma_t)},$$

where the constants C_1 and C_2 depend on $|||A_j, B|||_{s-2,*T}$ and $|||A_j, B|||_{s,*T}$, respectively. We also have

$$|||F(t)|||_{m-2,*} \le |||F(0)|||_{m-2,*} + t^{1/2}[F]_{m,*,t} \le C_1|||f|||_{m,*} + t^{1/2}[F]_{m,*,t},$$

that we substitute in (6.2). By adding the estimate thus obtained to (6.7) an application of Gronwall's lemma gives

$$|||u(t)|||^2_{m,*} + \delta||Pu||^2_{H^m(\Sigma_t)} \le \Big\{C_1|||f|||^2_{m,*} + C_2[F]^2_{m,*,t} + \frac{C_1}{\delta}[G]^2_{H^m(\Sigma_t)}\Big\}e^{C_2 t},$$

which concludes the proof. □

Proof of Theorem 3.5. The existence of a solution of problem (1.1) in $\mathcal{C}_T(\mathcal{H}^m)$ comes from Theorem 3.1, Theorem 3.3 and Corollary 3.4. Hence it remains to prove the energy-type estimate (3.7). We observe that from (6.1) we have

$$\begin{aligned}|||u(t)||_m^2 &= |||u(t)|||_{m,*}^2 + |||\partial_1 u^I(t)|||_{m-1,*}^2 \\ &\le \sum_{|\alpha|\le m} ||\partial_\star^\alpha u(t)||^2 + |||u^{II}(t)|||_{m,*}^2 + |||\partial_1 u^I(t)|||_{m-1,*}^2 \qquad (6.8) \\ &\le c_4(t)\Big(\sum_{|\alpha|\le m} ||\partial_\star^\alpha u(t)||^2 + |||u^{II}(t)|||_{m,*}^2 + |||F(t)|||_{m-1,*}^2\Big),\end{aligned}$$

where $c_4(t)$ is the time function appearing in (6.1). We estimate the right-hand side by means of (3.4) and

$$|||F(t)|||_{m-1,*} \le |||F(0)|||_{m-1,*} + t^{1/2}[F]_{m,*,t},$$

and finally obtain the result.

7. APPENDIX

Recall that P is the orthogonal projection onto $(\ker A_\nu)^\perp$. The following lemma asserts that, when $\partial\Omega$ is compact, the strictly dissipative condition (E) is equivalent to the strict positivity of A_ν. More precisely:

Lemma 7.1. *Let Ω be a bounded open set with C^∞ boundary Γ and A_ν be a continuous $N \times N$ matrix of $(x,t) \in \Sigma_T$. Assume that A_ν satisfies assumption* (C). *Let $M = M(x)$ be a given $N \times N$ matrix defined on Γ. Assume that M has constant rank $d \le r$ everywhere on Γ. Then condition*

$$(A_\nu u, u) > 0,\ \forall u\ \in \ker M,\ Pu \ne 0, \qquad (7.1)$$

is equivalent to

$$\exists\, \delta > 0 :\ (A_\nu u, u) \ge \delta |Pu|^2 - \frac{1}{\delta}|Mu|^2\ \forall\ u \in \mathbb{R}^N. \qquad (7.2)$$

Proof. We may assume without loss of generality that M is a self-adjoint operator. If it is not the case we may replace M with M^*M. Observe that $Mu = 0$ if and only if $M^*Mu = 0$.

(7.1) is an immediate consequence of (7.2). Assume now that (7.1) holds. To prove the result we consider the following cases.

(i) Let $u \in \ker M$ be such that $Pu \ne 0$. Let $S = \{u \in \ker M : |Pu| = 1\}$. Then $(A_\nu u, u) \ge \delta_1 = \min_{u\in S,(x,t)\in\Sigma_T} (A_\nu u, u) > 0$, for each $u \in S$, $(x,t) \in \Sigma_T$. By using the bilinearity of the scalar product and by

$$\Big(A_\nu \frac{u}{|Pu|}, \frac{u}{|Pu|}\Big) \ge \delta_1, \quad \forall\, u \in \ker M,\ Pu \ne 0,$$

we get the thesis with any $\delta \le \delta_1$.

(ii) If $u \in \ker M, Pu = 0$, then $u \in \ker A_\nu$ and (7.2) is trivially verified.

(iii) Now we consider $u \notin \ker M$. We write $u = Pu + (I - P)u$, with $(I-P)u \in \ker A_\nu$. By using the symmetry of A_ν we get

$$(A_\nu u, u) = (A_\nu Pu, Pu).$$

Let Q be the orthogonal projection onto $(\ker M)^\perp$. Since M is assumed to be a self-adjoint operator we have $MQ = QM = M$. Let K be the pseudo-inverse of M. We have $KM = MK = Q$. From the hypothesis $\ker A_\nu \subseteq \ker M$ we get $PQ = Q$, $MP = M$. For more details we refer to Secchi, [14].

We write $Pu = QPu + (I-Q)Pu$, hence again by the symmetry of A_ν we get

$$\begin{aligned}(A_\nu Pu, Pu) = (A_\nu QPu, QPu) &+ 2(A_\nu QPu, (I-Q)Pu) \\ &+ (A_\nu (I-Q)Pu, (I-Q)Pu).\end{aligned}$$

We observe that $(I-Q)Pu \in \ker M$. By denoting $|||B|||$ the norm of a matrix B, by using (i) and (ii) and Young's inequality we get for any $\delta_2 \le \delta_1$

$$\begin{aligned}(A_\nu Pu, Pu) &\ge \delta_2 |(I-Q)Pu|^2 + 2(A_\nu QPu, (I-Q)Pu) + (A_\nu QPu, QPu) \\ &\ge \delta_2 |(I-Q)Pu|^2 - \frac{\delta_2}{2}|(I-Q)Pu|^2 - \frac{c}{\delta_2}|A_\nu QPu|^2 - |||A_\nu||| |QPu|^2 \\ &\ge \frac{\delta_2}{4}|Pu|^2 - c(\delta_2, |||A_\nu|||)|QPu|^2 \\ &= \frac{\delta_2}{4}|Pu|^2 - c(\delta_2, |||A_\nu|||)|KMPu|^2 \\ &= \frac{\delta_2}{4}|Pu|^2 - c(\delta_2, |||A_\nu|||)|KMu|^2 \\ &\ge \frac{\delta_2}{4}|Pu|^2 - c(\delta_2, |||A_\nu|||)|||K|||^2|Mu|^2 = \delta|Pu|^2 - \frac{1}{\delta}|Mu|^2\end{aligned}$$

for a suitable $\delta > 0$. □

We recall some fundamental results about function spaces H^m_*. For the proof we refer to Secchi [19], [13] and [14].

In the sequel, we denote by $C^\infty_{(0)}(\Omega)$ the set of restrictions to Ω of functions of $C^\infty_0(\mathbb{R}^n)$.

Lemma 7.2. *$C^\infty_{(0)}(\Omega)$ is dense in H^m_*, $m \ge 1$.*

Lemma 7.3. *$C^\infty_{(0)}(\Omega)$ is dense in $\mathcal{H}^m$.*

Lemma 7.4. *Let $s \geq 2[\frac{n}{2}]+3$, $m = 0, \cdots, s$. Consider functions $u \in H^m_*$ and $v \in H^s_*$. Then $uv \in H^m_*$ and $\|uv\|_{m,*} \leq c\|u\|_{m,*}\|v\|_{s,*}$.*

Lemma 7.5. *Let $s \geq 2[\frac{n}{2}]+4$, and $m = 0, \cdots, s$. If $u \in \mathcal{H}^m$ and $v \in H^s_*$, then $uv \in \mathcal{H}^m$. Moreover,*

$$\|uv\|_{m,*} + \|\partial_1 (uv)^I\|_{m-1,*} \leq c(\|u\|_{m,*} + \|\partial_1 u^I\|_{m-1,*})\|v\|_{s,*}.$$

Lemma 7.6. *Let $s \geq 2\left[\frac{n}{2}\right]+4$, and $m = 0, \cdots, s-1$. If $u \in \mathcal{L}^2_T(H^m_*)$ and $v \in \mathcal{L}^\infty_T(H^{s-1}_*)$, then $uv \in \mathcal{L}^2_T(H^m_*)$ and $[uv]_{m,*,T} \leq c[u]_{m,*,T}|||v|||_{s-1,*,T}$. If $u \in \mathcal{C}_T(H^m_*)$ and $v \in \mathcal{C}_T(H^s_*)$, then $uv \in \mathcal{C}_T(H^m_*)$ and for every $t \in [0,T]$*

$$|||uv(t)|||_{m,*} \leq c|||u(t)|||_{m,*}|||v(t)|||_{s,*}.$$

If $u \in \mathcal{L}^2_T(H^s_*)$ and $v \in \mathcal{L}^\infty_T(H^s_*)$, then $uv \in \mathcal{L}^2_T(H^s_*)$ and

$$[uv]_{s,*,T} \leq c[u]_{s,*,T}|||v|||_{s,*,T}.$$

Lemma 7.7. *Let $s \geq 2\left[\frac{n}{2}\right]+5$, $m = 0, \cdots, s-2$. Assume that $f \in H^m_*$ is such that $f^{(k)} \in H^{m-k}_*$, for $k = 0, \cdots, m$, and that $v \in \mathcal{C}_T(H^{s-2}_*)$. Then,*

$$|||v(0)f|||_{m,*} \leq c|||v(0)|||_{s-2,*}|||f|||_{m,*}.$$

If $f \in H^{s-1}_*$ is such that $f^{(k)} \in H^{s-1-k}_*$, for $k = 1, \cdots, s-1$, and $v \in \mathcal{C}_T(H^{s-1}_*)$, then

$$|||v(0)f|||_{s-1,*} \leq c|||v(0)|||_{s-1,*}|||f|||_{s-1,*}.$$

Lemma 7.8. *Let $\sigma = 2\left[\frac{n}{2}\right]+4$ and let A be a matrix-valued function such that $A \in H^\sigma_*(\mathbb{R}^n_+)$ and $A = 0$ if $x_1 = 0$. Then, for each sufficiently regular vector-valued function u*

$$\|A\partial_1 u\| \leq c\|A\|_{\sigma,*}\|x_1\partial_1 u\|. \tag{7.3}$$

Proof. Let

$$H(x_1, x') = (x_1)^{-1}\int_0^{x_1} \partial_1 A(y, x')dy.$$

Then, $A\partial_1 u = Hx_1\partial_1 u$. By recursively applying Hardy's inequality

$$\int_0^\infty |f(x)/x|^2 dx \leq 4\int_0^\infty |f'(x)|^2 dx,$$

which is satisfied by any $f \in H^1(\mathbb{R}_+)$ such that $f(0) = 0$, it follows that

$$\|H\|_l \leq c_l\|\partial_1 A\|_l, \quad \text{for } l = 0, 1, \ldots.$$

By a Sobolev embedding, we have

$$|H|_\infty \leq c\|H\|_{[\frac{n}{2}]+1} \leq c\|A\|_{\sigma,*},$$

which gives (7.3). □

Lemma 7.9. *Let $s \geq 2[\frac{n}{2}]+6$. Let $A \in H^s_*(\mathbb{R}^n_+)$ be a matrix-valued function such that $A = 0$ if $x_1 = 0$ and let H be defined as in the proof of Lemma 7.8. Then,*

$$\|H\|_{s-2,*} \leq c\|A\|_{s,*}.$$

Lemma 7.10. *Let $\tau = 2[\frac{n}{2}] + 6$, and let $A \in H^\tau_*(\mathbb{R}^n_+)$ be a matrix-valued function such that $A = 0$ if $x_1 = 0$. Then, for each sufficiently smooth vector-valued function u*

$$\|\partial^\alpha_\star A\partial_1 u\| \leq c\|A\|_{\tau,*}||x_1\partial_1 u\| \qquad \text{if } \|\alpha| \leq 2,$$
$$|||\partial^\alpha_\star A\partial_1 u|||_{1,*} \leq c\|A\|_{\tau,*}|||x_1\partial_1 u|||_{1,*} \quad \text{if } |\alpha| \leq 1.$$

REFERENCES

[1] H. Beirão da Veiga, *Boundary-value problems for a class of first order partial differential equations in Sobolev spaces and applications to the Euler flow*, Rend. Sem. Mat. Univ. Padova, 79 (1988), 247–273.

[2] H. Friedrich, and G. Nagy, *The initial boundary value problem for Einstein's vacuum field equation*, Commun. Math. Phys., 201 (1999), 619–655.

[3] H.O. Kreiss, *Initial Boundary Value Problems for Hyperbolic Systems*, Comm. Pure Appl. Math., XXIII (1970), 277–298.

[4] Chen Shuxing, *On the initial-boundary value problem for quasilinear symmetric hyperbolic system with characteristic boundary*, Chinese Ann. Math., 3 (1982), 223–232.

[5] L. Hörmander, "Linear Partial Differential Operators," Springer, Berlin, 1969.

[6] A. Majda, " Compressible flow and systems of conservation laws in several space variables," Appl. Math. Sciences, 53, Springer Verlag, New York, 1984.

[7] A. Majda, *The stability of multi-dimensional shock fronts*, Mem. Amer. Math. Soc., 41 (275), (1983).

[8] A. Majda and S. Osher, *Initial-boundary value problem for hyperbolic equations with uniformly characteristic boundary*, Comm. Pure Appl. Math., 28 (1975), 607–675.

[9] G. Métivier, *The block structure condition for symmetric hyperbolic systems,* Bull. London Math. Soc., 32 (2000), 689–702.

[10] M. Ohno, Y. Shizuta, and T. Yanagisawa, *The initial boundary value problem for linear symmetric hyperbolic systems with boundary characteristic of constant multiplicity*, J. Math. Kyoto Univ. (JMKYAZ), 35 (1995), 143–210.

[11] J. Rauch, *Symmetric positive systems with boundary characteristic of constant multiplicity*, Trans. Amer. Math. Soc., 291 (1985), 167–187.

[12] J. Rauch and J. Massey, *Differentiability of solutions to hyperbolic initial-boundary value problems*, Trans. Amer. Math. Soc., 189 (1974), 303–318.

[13] P. Secchi, *Linear Symmetric Hyperbolic System with Characteristic Boundary*, Math. Meth. Appl. Sci., 18 (1995), 855–870.

[14] P. Secchi, *The initial boundary value problem for linear symmetric hyperbolic system with characteristic boundary of constant multiplicity*, Differential and Integral equations, 9 (1996), 671–700.

[15] P. Secchi, *Well-posedness for a mixed problem for the equations of ideal magneto-hydrodynamics*, Archiv Math. (Basel), 64 (1995), 237–245.

[16] P. Secchi, *Well-posedness of Characteristic Symmetric Hyperbolic Systems*, Arch. Rational Mech. Anal., 134 (1996), 155–197.

[17] P. Secchi, *Characteristic symmetric hyperbolic systems with dissipation. Global existence and asymptotics*, Math. Methods Appl. Sci., 20 (1997), 207–216.

[18] P. Secchi, *Inflow-outflow problems for inviscid compressible fluids*, Comm. Appl. Anal., 2 (1998), 81–110.

[19] P. Secchi, *Some properties of anisotropic Sobolev spaces*, Arch. Math., 75 (2000), 207–216.

[20] P. Secchi, *An initial boundary value problem in ideal magneto-hydrodynamics*, NoDEA, 9 (2001), 195–217.

[21] B. Szilágyi and J. Winicour, *Well-posed initial-boundary evolution in general relativity* Phys. Rev. D, 3 68 (2003), no. 4, 041501, 5 pp.

[22] T. Yanagisawa and A. Matsumura, *The fixed boundary value problems for the equations of ideal magneto-hydrodynamics with a perfectly conducting wall condition*, Commun. Math. Phys., 136 (1991), 119–140.