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REGULARITY OF NAVIER-STOKES EQUATIONS

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Abstract. The purpose of this article is to show that a weak solution to the Navier-Stokes equations (NSE) in an infinite channel or in a spherical domain is actually regular up to the boundary if one component of the velocity is smooth enough. As a novelty, the 3D NSE are viewed as a perturbation of the 2D NSE in a sense to be explained below.

Introduction

The evolution of a viscous, incompressible fluid is governed by the Navier-Stokes equations:

$$\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p = f, \tag{0.1}$$

$$\operatorname{div} u = 0, \tag{0.2}$$

$$u|_{t=0} = u_0. \tag{0.3}$$

As usual, u and p denote the unknown velocity and pressure, f is the external body force, and ν is the viscosity coefficient. In the case when the fluid is filling the whole space or just a bounded, smooth region of $\mathbb{R}^3$, it is well known that there exists a weak solution. We recall that $u \in L^\infty(0,T;L^2) \cap L^2(0,T;H^1)$, with $\operatorname{div} u = 0$, is said to be a weak solution if it satisfies (0.1)-(0.3) in the sense of distributions.

Much attention has been devoted lately to the problem of regularization for the solutions to the Navier-Stokes equations. The typical regularization theorem states that a weak solution which is "slightly smoother" is in fact as regular as the data allow. "Slightly smoother" means that either a component of the velocity, or the gradient of a component of the velocity, or the pressure, have some additional regularity. Several criteria of regularity have been formulated. For a fairly complete list of the existing results we refer the reader to [15].

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As an example, we mention the result classical by now due to J. Serrin [16] who derived a regularity criterion under the assumption that the weak solution belongs to $L^p(0,T;L^q)$ for

$$\frac{2}{p}+\frac{3}{q}<1, \quad 2\leq p<\infty,\ 3<q\leq\infty.$$

Other regularity criteria have been obtained by assuming "good behavior" of the derivatives of the velocity (see [4],[9],[12],[15]), or of the vorticity (see [9]), or of vorticity direction $\omega/|\omega|$ (see [10],[7]).

Recently, Neustupa, Novotny, and Penel [13] obtained an interior regularity criterion in the case of the Navier-Stokes equations in a bounded, smooth domain $\Omega\subset\mathbb{R}^3$ by assuming additional regularity on one component of the velocity only. More precisely, they proved that a suitable weak solution u (in the sense of Caffarelli, Kohn, Nirenberg, see [8]) for which

$$u_3\in L^p(0,T;L^q), \quad 2/p+3/q\leq 1/2, \quad 4\leq q\leq\infty, \quad 6<q\leq\infty, \tag{0.4}$$

in a sub-domain $D\subset\Omega\times(0,T)$ of the time-space domain is locally regular in D, that is, regular on any compact set included in D. The main idea of their proof is that the third component of the velocity controls the third component of the vorticity. Then, knowing that the third components of the velocity and of the vorticity are regular, they prove that u is regular in D.

Our aim in this article is to extend the regularity result of [13] up to the boundary in the case of some special domains, namely an infinite channel or a spherical domain, under the assumption of smoothness of a suitable component of u (that orthogonal to the boundary.)

Here we consider the Navier-Stokes equations in both an infinite channel and a spherical domain. We prove that if the wall-normal component (respectively the radial component) of the velocity is smooth, then the whole velocity is smooth, up to the boundary. Instead of using the vorticity equation as in [13], we employ the equations of motion for u with the difference that we view them as a perturbation in the wall-normal direction (respectively in the radial direction) of the two-dimensional problem in the plane (respectively on the sphere.) We first estimate the tangential derivatives of u, and then the whole gradient. The proof requires more work in the case of the spherical domain. We need to find an appropriate multiplier that would produce a satisfactory estimate for the tangential derivatives of u, and the estimates in general are more difficult. The case of Navier-Stokes equations in general curvilinear coordinates will be addressed in a separate article.

The article is organized as follows. In the first section we consider the Navier-Stokes equations in an infinite channel. In section 2 we treat the case of a spherical domain.

1. THE NSE IN AN INFINITE CHANNEL

In this section we consider the Navier-Stokes equations in an infinite channel $\Omega_\infty = \mathbb{R}^2 \times (0, L_3)$.

$$\partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla p = f \qquad \text{in } \Omega_\infty, \tag{1.1}$$

$$\nabla \cdot u = 0 \qquad \text{in } \Omega_\infty. \tag{1.2}$$

The velocity u vanishes at the boundary $\partial\Omega_\infty$, i.e., at $x_3 = 0$ and at $x_3 = L_3$:

$$u = 0 \qquad \text{on } \partial\Omega_\infty. \tag{1.3}$$

We also assume periodicity in x_1 (with period L_1) and in x_2 (with period L_2) for all involved functions. Equations (1.1)-(1.2) are supplemented with the initial condition

$$u = u_0 \qquad \text{at } t = 0. \tag{1.4}$$

We set $\Omega = (0, L_1) \times (0, L_2) \times (0, L_3)$, and introduce the natural function spaces

$$\begin{aligned} H = \{u = (u_1, u_2, u_3) \in L^2(\Omega) : \operatorname{div} u = 0,\ u_3 = 0 \text{ on } \partial\Omega, \\ u_1 \text{ is } L_1\text{-periodic in } x_1,\ u_2 \text{ is } L_2\text{-periodic in } x_2\}, \\ V = \{u = (u_1, u_2, u_3) \in H^1(\Omega) : \operatorname{div} u = 0,\ u = 0 \text{ on } \partial\Omega, \\ u \text{ is } L_1\text{-periodic in } x_1 \text{ and } L_2\text{-periodic in } x_2\}. \end{aligned}$$

We also introduce the Stokes operator

$$Au = P(-\Delta u), \quad u \in D(A),$$

where P is the orthogonal projector of $(L^2(\Omega))^3$ onto the space H, and $D(A) = (H^2(\Omega))^3 \cap V$.

Below is the main result of this section.

Theorem 1.1. *Let $u \in L^\infty(0, T; H) \cap L^2(0, T; V)$ be any weak solution to the Navier-Stokes equations with $f \in L^2(0, T; H)$ and with initial condition $u_0 \in V$. Assume that there exist two constants C_1 and C_2 such that $|\nabla u_3(t)|_{L^2} \leq C_1$, and $|\Delta u_3(t)|_{L^2} \leq C_2$ for almost every $0 \leq t \leq T$. Then u is the strong solution on $[0, T]$.*

Remark 1.2. We do not need to assume any suitability property of the weak solution u as in e.g. [13], and not even the usual energy inequality (see [11].) Note however that since $u_0 \in V$, the solution is already known to be smooth on some "small" interval $[0, T*)$ (see below).

Proof of Theorem 1.1. The theorem is proved in two steps. First we show that the tangential derivatives $\partial_1 u$ and $\partial_2 u$ are bounded in $L^2(\Omega)$, then we prove that the whole gradient of the velocity is in $L^\infty(0, T; L^2(\Omega))$, thus showing that $\partial_3 u$ is also uniformly bounded in $L^2(\Omega)$. The idea is to treat the Navier-Stokes equations in the channel Ω_∞ as a perturbation in the vertical direction x_3 of the two-dimensional Navier-Stokes equations in the horizontal plane $x_3 = 0$. For this purpose we split the velocity into

$$u = v + u_3 e_3, \tag{1.5}$$

then (1.1)-(1.2) is written as a system:

$$\partial_t v + (v \cdot \nabla_2)v + u_3 \partial_3 v - \nu \Delta_2 v - \nu \partial_3^2 v + \nabla_2 p = f_h, \tag{1.6}$$

$$\partial_t u_3 + (v \cdot \nabla_2)u_3 + u_3 \partial_3 u_3 - \nu \Delta_2 u_3 - \nu \partial_3^2 u_3 + \partial_3 p = f_3, \tag{1.7}$$

$$\operatorname{div}_2 v = -\partial_3 u_3, \tag{1.8}$$

where Δ_2, ∇_2, and div_2 are the two-dimensional Laplacian, gradient, and divergence operators in the horizonal plane $x_3 = 0$, and $f_h = f - f_3 e_3$ is the horizontal component of f.

Next we proceed with the energy estimates. In what follows we assume that the solution $u = (v, u_3)$ is as regular as needed. At the end of the proof we show how to use these estimates for the actual (given) weak solution. In order to obtain an energy-like equation for $\nabla_2 u$ in $L^2(\Omega)$, we multiply equations (1.1) by $-\Delta_2 u$. Note that since $\operatorname{div}(\Delta_2 u) = \Delta_2(\operatorname{div} u) = 0$ and $\Delta_2 u_3 = 0$ on $\partial\Omega_\infty$, $\Delta_2 u$ belongs to the space H, which will make the pressure term disappear. The following energy equation will emerge:

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}|\nabla_2 u|_{L^2}^2 + \nu(|\Delta_2 u|_{L^2}^2 + |\partial_3 \nabla_2 u|_{L^2}^2) &= -\int_\Omega f \Delta_2 u + \int_\Omega (v \cdot \nabla_2) v \, \Delta_2 v \\ &+ \int_\Omega u_3 \, \partial_3 v \, \Delta_2 v + \int_\Omega (v \cdot \nabla_2) u_3 \, \Delta_2 u_3 + \int_\Omega u_3 \, \partial_3 u_3 \, \Delta_2 u_3. \end{aligned} \tag{1.9}$$

We estimate the terms on the right-hand side of (1.9) in the usual way, using Hölder's inequality and the assumption of regularity for u_3. One term requires more attention, and that is $\int_\Omega (v \cdot \nabla_2) v \, \Delta_2 v$. We express it using the following formulas from vector calculus:

$$-\Delta_2 v = \operatorname{curl}_2(\operatorname{curl}_2 v) - \nabla_2(\operatorname{div}_2 v), \tag{1.10}$$

$$(v\cdot\nabla_2)v = \frac{1}{2}\nabla_2|v|^2 - v\times\operatorname{curl}_2 v, \tag{1.11}$$

where curl_2 is the usual curl in two dimensions (of a scalar or of a vector, depending on the argument.) Therefore we find that

$$\begin{aligned} I = \int_\Omega (v\cdot\nabla_2)v\,\Delta_2 v &= -\int_\Omega \operatorname{div}_2((v\cdot\nabla_2)v)\operatorname{div}_2 v \\ &\quad -\int_\Omega (\frac{1}{2}\nabla_2|v|^2 - v\times\operatorname{curl}_2 v)\operatorname{curl}_2(\operatorname{curl}_2 v). \end{aligned} \tag{1.12}$$

First note that

$$\operatorname{div}_2((v\cdot\nabla_2)v) = \sum_{i,j=1,2}\partial_j v_i\partial_i v_j + v\cdot\nabla_2(\operatorname{div}_2 v).$$

Obviously, the following term vanishes:

$$\int_\Omega \frac{1}{2}\nabla_2|v|^2\operatorname{curl}_2(\operatorname{curl}_2 v) = 0.$$

Furthermore, we can simplify

$$\begin{aligned} \int_\Omega (v\times\operatorname{curl}_2 v)\operatorname{curl}_2(\operatorname{curl}_2 v) &= \int_\Omega \operatorname{curl}_2(v\times\operatorname{curl}_2 v)\operatorname{curl}_2 v \\ &= -\int_\Omega [\partial_1(v_1\operatorname{curl}_2 v) + \partial_2(v_2\operatorname{curl}_2 v)]\operatorname{curl}_2 v \\ &= -\int_\Omega \operatorname{div}_2 v\,|\operatorname{curl}_2 v|^2 - \int_\Omega [v_1\partial_1(\operatorname{curl}_2 v) + v_2\partial_2(\operatorname{curl}_2 v)]\ \operatorname{curl}_2 v \\ &= -\frac{1}{2}\int_\Omega \operatorname{div}_2 v\,|\operatorname{curl}_2 v|^2. \end{aligned}$$

Therefore, I can be expressed as

$$\begin{aligned} I &= -\int_\Omega \Big(\sum_{i,j=1,2}\partial_j v_i\partial_i v_j + v\cdot\nabla_2(\operatorname{div}_2 v) + \frac{1}{2}\,|\operatorname{curl}_2 v|^2\Big)\operatorname{div}_2 v \\ &= \int_\Omega \Big(\sum_{i,j=1,2}\partial_j v_i\partial_i v_j - v\cdot\nabla_2(\partial_3 u_3) + \frac{1}{2}\,|\operatorname{curl}_2 v|^2\Big)\partial_3 u_3, \end{aligned} \tag{1.13}$$

and then it can be bounded as follows

$$|I| \le (|\nabla_2 v|_{L^4}^2 + |\operatorname{curl}_2 v|_{L^4}^2)|\partial_3 u_3|_{L^2} + |v|_{L^6}|\nabla_2\partial_3 u_3|_{L^2}|\partial_3 u_3|_{L^3} \tag{1.14}$$

$$
\begin{aligned}
&\leq c\,(|\Delta_2 v|_{L^2} + |\partial_3 \nabla_2 v|_{L^2})^{3/2} |\nabla_2 v|_{L^2}^{1/2} |\partial_3 u_3|_{L^2} \\
&\quad + |\partial_3 \nabla_2 v|_{L^2} |\nabla v|_{L^2} |\nabla u_3|_{L^2}^{1/2} ||\Delta u_3|_{L^2}^{1/2} \\
&\leq \frac{\nu}{4}(|\Delta_2 v|_{L^2}^2 + |\partial_3 \nabla_2 v|_{L^2}^2) + (\frac{c}{\nu^3}|\nabla u_3|_{L^2}^4 + \frac{c}{\nu}|\Delta u_3|_{L^2}|\nabla u_3|_{L^2})|\nabla u|_{L^2}^2.
\end{aligned}
$$

Here c is a numerical constant that depends on Ω only, and which might vary from one line to another.

The remaining terms on the right-hand side of (1.9) are bounded as follows:

$$
\begin{aligned}
&|-\int_\Omega f \Delta_2 u + \int_\Omega u_3\, \partial_3 v\, \Delta_2 v + \int_\Omega v \cdot \nabla_2 u_3\, \Delta_2 u_3 + \int_\Omega u_3\, \partial_3 u_3\, \Delta_2 u_3| \qquad (1.15)\\
&\quad \leq \frac{\nu}{4}(|\Delta_2 u|_{L^2}^2 + |\partial_3 \nabla_2 v|_{L^2}^2) + \frac{c}{\nu}|f|_{L^2}^2 \\
&\qquad + \frac{c}{\nu}|u_3|_{L^\infty}^2(|\partial_3 v|_{L^2}^2 + |\partial_3 u_3|_{L^2}^2) + \frac{c}{\nu}|\nabla_2 u_3|_{L^3}^2 |v|_{L^6}^2 \\
&\qquad \text{(using the Agmon and Sobolev inequalities)} \\
&\quad \leq \frac{\nu}{4}(|\Delta_2 u|_{L^2}^2 + |\partial_3 \nabla_2 v|_{L^2}^2) + \frac{c}{\nu}|f|_{L^2}^2 + \frac{c}{\nu}|\nabla u_3|_{L^2}|\Delta u_3|_{L^2}|\nabla u|_{L^2}^2.
\end{aligned}
$$

By taking into account (1.14), (1.15), and the assumption on u_3, we end up with the following inequality:

$$
\begin{aligned}
&\frac{d}{dt}|\nabla_2 u|_{L^2}^2 + \nu(|\Delta_2 u|_{L^2}^2 + |\partial_3 \nabla_2 u|_{L^2}^2) \qquad (1.16)\\
&\qquad \leq \frac{c}{\nu}|f|_{L^2}^2 + \big(\frac{c}{\nu^3}|\nabla u_3|_{L^2}^4 + \frac{c}{\nu}|\Delta u_3|_{L^2}|\nabla u_3|_{L^2}\big)|\nabla u|_{L^2}^2 \\
&\qquad \leq \frac{c}{\nu}|f|_{L^2}^2 + \big(\frac{c}{\nu^3}C_1^4 + \frac{c}{\nu}C_1 C_2\big)|\nabla u|_{L^2}^2.
\end{aligned}
$$

Since u is a weak solution to the Navier-Stokes equations, we can invoke the classical $L^\infty(H)$–$L^2(V)$ estimates. We have

$$
|u(t)|_{L^2}^2 + \nu \int_0^t |\nabla u(\tau)|_{L^2}^2\, d\tau \leq |u_0|_{L^2}^2 + \frac{c}{\nu}|f|_{L^2(0,T;H)}^2, \quad 0 \leq t \leq T. \qquad (1.17)
$$

Thanks to the above estimate and to (1.16) we obtain that for every $0 \leq t \leq T$,

$$
\begin{aligned}
&|\nabla_2 u(t)|_{L^2}^2 + \nu \int_0^T (|\Delta_2 u(t)|_{L^2}^2 + |\partial_3 \nabla_2 u(t)|_{L^2}^2) \leq |\nabla_2 u_0|_{L^2}^2 + \frac{c}{\nu}|f|_{L^2(0,T;H)}^2 \\
&\qquad + \big(\frac{c}{\nu^3}C_1^4 + \frac{c}{\nu}C_1 C_2\big)\big(\frac{|u_0|_{L^2}^2}{\nu} + \frac{c}{\nu^2}|f|_{L^2(0,T;H)}^2\big). \qquad (1.18)
\end{aligned}
$$

For the second part of the proof, we write the Navier-Stokes equations (1.1) as

$$\partial_t u + (v \cdot \nabla_2)u + u_3 \partial_3 u - \nu \Delta u + \nabla p = f. \tag{1.19}$$

Since $\nabla_2 u$ and u_3 are bounded in L^2, the above equation could be seen as a linear equation. We multiply it by Au and integrate over Ω to obtain

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|A^{1/2}u|_{L^2}^2 + \nu|Au|_{L^2}^2 &= \int_\Omega [f - (v\cdot\nabla_2)u - u_3\partial_3 u]\, Au \\
&\le |f|_{L^2}|Au|_{L^2} + |v|_{L^\infty}|\nabla_2 u|_{L^2}|Au|_{L^2} + |u_3|_{L^\infty}|\partial_3 u|_{L^2}|Au|_{L^2} \\
&\le |f|_{L^2}|Au|_{L^2} + c|\nabla_2 u|_{L^2}|A^{1/2}u|_{L^2}^{1/2}|Au|_{L^2}^{3/2} + c|u_3|_{L^\infty}|A^{1/2}u|_{L^2}|Au|_{L^2} \\
&\le \frac{\nu}{2}|Au|_{L^2}^2 + \frac{c}{\nu}|f|_{L^2}^2 + \big(\frac{c}{\nu^3}|\nabla_2 u|_{L^2}^4 + \frac{c}{\nu}|\Delta u_3|_{L^2}|\nabla u_3|_{L^2}\big)|A^{1/2}u|_{L^2}^2.
\end{aligned}$$

The following inequality emerges:

$$\begin{aligned}
\frac{d}{dt}|A^{1/2}u|_{L^2}^2 + \nu|Au|_{L^2}^2 \le \frac{c}{\nu}|f|_{L^2}^2 \\
+ \big(\frac{c}{\nu^3}|\nabla_2 u|_{L^2}^4 + \frac{c}{\nu}|\Delta u_3|_{L^2}|\nabla u_3|_{L^2}\big)|A^{1/2}u|_{L^2}^2.
\end{aligned} \tag{1.20}$$

Using (1.18) and (1.18) in (1.20), we find that there exists a constant C_3 which depends on $|u_0|_V$, $|f|_{L^2(0,T;H)}$, ν, C_1, and C_2, such that

$$|A^{1/2}u(t)|_{L^2}^2 + \nu\int_0^t |Au(\tau)|_{L^2}^2\, d\tau \le C, \quad 0 \le t \le T. \tag{1.21}$$

As we said, these calculations assumed that u was as smooth as needed. Since u_0 is in V, it is known that there exists a unique strong solution to the Navier-Stokes equations (1.1)–(1.2) with initial condition u_0 on some interval $[0, T^*)$, $0 < T^* \le T$. Assume that $[0, T^*)$ is the maximal interval on which the strong solution exists. Our weak solution coincides with the strong solution on its interval of existence $[0, T^*)$; then $u \in L^\infty(0, T^*-\delta; V) \cap L^2(0, T^*-\delta; D(A))$, for every $\delta > 0$. The calculations that we performed above, leading to (1.20)–(1.21), are valid on $[0, T^* - \delta]$, for every $\delta > 0$ (using of course the assumptions of Theorem 1.1). As $t \to T^* - 0$, (1.21) shows that ∇u remains bounded in $L^\infty(0, T^*; L^2)$ independently of T^*, and therefore we obtain a contradiction with the maximality of $T^* < T$. We conclude that u is a strong solution on $[0, T]$.

2. THE NSE IN A SPHERICAL DOMAIN

In this section we consider the Navier-Stokes equations for an incompressible fluid filling a spherical domain $\Omega = \{(x,y,z) \in \mathbb{R}^3 : R_1 < (x^2+y^2+z^2)^{1/2} < R_2\}$. We will assume Dirichlet boundary conditions

$$u = 0 \quad \text{on } \partial\Omega. \tag{2.1}$$

We will use spherical coordinates r, θ, φ. Let $\mathbf{e_r}$, $\mathbf{e}_\theta$, and $\mathbf{e}_\varphi$ be the unit vectors parallel to the coordinate lines and in the directions of increase of r, θ, and φ respectively.

We recall some useful formulas in spherical coordinates. Let $h = h(r,\theta,\varphi)$ be a scalar function, and let $u = u_r\mathbf{e_r} + u_\theta\mathbf{e}_\theta + u_\varphi\mathbf{e}_\varphi$ be a vector field. Then

$$\nabla h = \frac{\partial h}{\partial r}\,\mathbf{e_r} + \frac{1}{r}\frac{\partial h}{\partial \theta}\,\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial h}{\partial \varphi}\,\mathbf{e}_\varphi, \tag{2.2}$$

$$\Delta h = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial h}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial h}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 h}{\partial\varphi^2}, \tag{2.3}$$

$$\begin{aligned}(u\cdot\nabla)u = \big(u\cdot\nabla u_r - \frac{u_\theta^2+u_\varphi^2}{r}\big)\mathbf{e_r} + \big(u\cdot\nabla u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2\cot\theta}{r}\big)\mathbf{e}_\theta \\ + \big(u\cdot\nabla u_\varphi + \frac{u_r u_\varphi}{r} + \frac{u_\theta u_\varphi\cot\theta}{r}\big)\mathbf{e}_\varphi,\end{aligned} \tag{2.4}$$

$$\operatorname{div} u = \frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}, \tag{2.5}$$

$$\begin{aligned}\Delta u = &\left(\Delta u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right)\mathbf{e_r} \\ &+ \left(\Delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi} - \frac{u_\theta}{r^2\sin^2\theta}\right)\mathbf{e}_\theta \\ &+ \left(\Delta u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right)\mathbf{e}_\varphi.\end{aligned} \tag{2.6}$$

In Riemannian geometry, the Laplacian of a vector is not uniquely defined, contrarily to the flat case. But in this article, and in classical mechanics in general, Δu is defined by $\Delta u = \operatorname{curl}(\operatorname{curl} u) - \operatorname{grad}(\operatorname{div} u)$.

We recall that

$$\mathbf{e_r} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta), \qquad \mathbf{e}_\theta = (\cos\theta\cos\varphi, \cos\theta\sin\varphi, -\sin\theta),$$
$$\mathbf{e}_\varphi = (-\sin\varphi, \cos\varphi, 0).$$

Obviously,

$$\frac{\partial \mathbf{e_r}}{\partial r} = \frac{\partial \mathbf{e}_\theta}{\partial r} = \frac{\partial \mathbf{e}_\varphi}{\partial r} = 0. \tag{2.7}$$

A less known formula is that of the gradient of a vector field. If $u = u_r \mathbf{e_r} + u_\theta \mathbf{e}_\theta + u_\varphi \mathbf{e}_\varphi$, then ∇u is given by:

$$\nabla u = \begin{bmatrix} \dfrac{\partial u_r}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_r}{\partial \theta} - \dfrac{u_\theta}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_r}{\partial \varphi} - \dfrac{u_\varphi}{r} \\[2ex] \dfrac{\partial u_\theta}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_\theta}{\partial \theta} + \dfrac{u_r}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\theta}{\partial \varphi} - \dfrac{u_\varphi \cot\theta}{r} \\[2ex] \dfrac{\partial u_\varphi}{\partial r} & \dfrac{1}{r}\dfrac{\partial u_\varphi}{\partial \theta} & \dfrac{1}{r\sin\theta}\dfrac{\partial u_\varphi}{\partial \varphi} + \dfrac{u_r}{r} + \dfrac{u_\theta \cot\theta}{r} \end{bmatrix}. \tag{2.8}$$

Green's identity remains valid in spherical coordinates. If u and v are two regular vector fields on Ω such that v vanishes on the boundary of Ω, then

$$(-\Delta u, v)_{L^2} = (\nabla u, \nabla v)_{L^2}, \tag{2.9}$$

where Δ and ∇ are defined by (2.6) and (2.8), respectively, and $(\cdot,\cdot)_{L^2}$ is the usual inner product on $L^2(\Omega)$.

The Navier-Stokes equations in scalar form become (see [3]):

$$\frac{\partial u_r}{\partial t} + u\cdot\nabla u_r - \frac{u_\theta^2}{r} - \frac{u_\varphi^2}{r} + \frac{\partial p}{\partial r} = \tag{2.10}$$
$$f_r + \nu\left(\Delta u_r - \frac{2u_r}{r^2} - \frac{2}{r^2\sin\theta}\frac{\partial(u_\theta\sin\theta)}{\partial\theta} - \frac{2}{r^2\sin\theta}\frac{\partial u_\varphi}{\partial\varphi}\right),$$

$$\frac{\partial u_\theta}{\partial t} + u\cdot\nabla u_\theta + \frac{u_r u_\theta}{r} - \frac{u_\varphi^2\cot\theta}{r} + \frac{1}{r}\frac{\partial p}{\partial\theta} = \tag{2.11}$$
$$f_\theta + \nu\left(\Delta u_\theta + \frac{2}{r^2}\frac{\partial u_r}{\partial\theta} - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\varphi}{\partial\varphi} - \frac{u_\theta}{r^2\sin^2\theta}\right),$$

$$\frac{\partial u_\varphi}{\partial t} + u\cdot\nabla u_\varphi + \frac{u_r u_\varphi}{r} + \frac{u_\theta u_\varphi\cot\theta}{r} + \frac{1}{r\sin\theta}\frac{\partial p}{\partial\varphi} = \tag{2.12}$$
$$f_\varphi + \nu\left(\Delta u_\varphi + \frac{2}{r^2\sin\theta}\frac{\partial u_r}{\partial\varphi} + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi}{r^2\sin^2\theta}\right),$$

$$\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sin\theta\, u_\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} = 0. \tag{2.13}$$

Before stating the main result of the section, we introduce the function spaces for this problem. Let (see [17])

$$H = \{u \in L^2(\Omega) : \operatorname{div} u = 0,\ u_r = 0 \text{ on } \partial\Omega\}, \tag{2.14}$$
$$V = \{u \in H^1(\Omega) : \operatorname{div} u = 0,\ u = 0 \text{ on } \partial\Omega\}. \tag{2.15}$$

Theorem 2.1. *Assume that $u_0 \in V$ and $f \in L^2(0,T;H)$. Let $u \in L^\infty(0,T;H) \cap L^2(0,T;V)$ be* **any** *weak solution to the Navier-Stokes equations* (2.10)–(2.13) *with external force f and with initial condition u_0. Assume that there exist two constants C_1 and C_2 such that $|u_r(t)|_{W^{1,\infty}} \le C_1$, and $|\partial_t u_r(t)|_{L^2} \le C_2$ for almost every $0 \le t \le T$. Then u is a strong solution on $[0,T]$.*

Remark 2.2. The same remark as Remark 1.2 applies.

Proof of Theorem 2.1. The proof is similar to that from the Cartesian case. First we prove that the tangential derivatives are smooth. We then complete the proof by showing that the full gradient of u remains bounded in the L^2 norm.

We use the idea already applied to the case of an infinite channel. We express the Navier-Stokes equations (2.10)-(2.13) in the spherical domain Ω as a perturbation in the radial direction of the Navier-Stokes equations on the sphere.

But first we need to introduce some notation and recall a few formulas for the sphere. These can be obtained from the corresponding formulas for spherical coordinates by suppressing all components and derivatives in the radial direction $\mathbf{e_r}$. Let $h = h(\theta,\varphi)$ be a scalar function, and let $v = v_\theta \mathbf{e}_\theta + v_\varphi \mathbf{e}_\varphi$ be a vector tangent to the sphere of arbitrary radius r centered about the origin. Then

$$\nabla_2 h = \frac{1}{r}\frac{\partial h}{\partial \theta}\,\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial h}{\partial\varphi}\,\mathbf{e}_\varphi, \tag{2.16}$$
$$\Delta_2 h = \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial h}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 h}{\partial\varphi^2}, \tag{2.17}$$
$$\operatorname{curl}_2 h = \frac{1}{r\sin\theta}\frac{\partial h}{\partial\varphi}\mathbf{e}_\theta - \frac{1}{r}\frac{\partial h}{\partial\theta}\mathbf{e}_\varphi, \tag{2.18}$$
$$(v\cdot\nabla_2)v = \big(v\cdot\nabla_2 v_\theta - \frac{v_\varphi^2\cot\theta}{r}\big)\mathbf{e}_\theta + \big(v\cdot\nabla_2 v_\varphi + \frac{v_\theta v_\varphi\cot\theta}{r}\big)\mathbf{e}_\varphi, \tag{2.19}$$
$$\operatorname{div}_2 v = \frac{1}{r\sin\theta}\frac{\partial(v_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi}, \tag{2.20}$$

$$\operatorname{curl}_2 v = \frac{1}{r\sin\theta}\Big[\frac{\partial(v_\varphi \sin\theta)}{\partial\theta} - \frac{\partial v_\theta}{\partial\varphi}\Big] \tag{2.21}$$

$$\Delta_2 v = \left(\Delta_2 v_\theta - \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\varphi}{\partial\varphi} - \frac{v_\theta}{r^2\sin^2\theta}\right)\mathbf{e}_\theta \tag{2.22}$$
$$+\left(\Delta_2 v_\varphi + \frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial\varphi} - \frac{v_\varphi}{r^2\sin^2\theta}\right)\mathbf{e}_\varphi.$$

We recall that Δ_2 commutes with div_2 and with curl_2. Also the gradient of v is given by (compare to (2.8)):

$$\nabla_2 v = \begin{bmatrix} \frac{1}{r}\frac{\partial v_\theta}{\partial\theta} & \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial\varphi} - \frac{v_\varphi\cot\theta}{r} \\ \frac{1}{r}\frac{\partial v_\varphi}{\partial\theta} & \frac{1}{r\sin\theta}\frac{\partial v_\varphi}{\partial\varphi} + \frac{v_\theta\cot\theta}{r} \end{bmatrix}. \tag{2.23}$$

We now split f and u into $f = f_r\mathbf{e_r} + f_\tau$ and $u = u_r\mathbf{e_r} + v$, with $v = u_\tau$, and write (2.10)–(2.13) as a system for the tangential velocity v and the radial velocity u_r:

$$\frac{\partial v}{\partial t} + (v\cdot\nabla_2)v + u_r\frac{\partial v}{\partial r} + u_r\frac{v}{r} + \nabla_2 p \tag{2.24}$$
$$= f_\tau + \nu\left(\Delta_2 v + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial v}{\partial r}) + \frac{2}{r}\nabla_2 u_r\right),$$

$$\frac{\partial u_r}{\partial t} + v\cdot\nabla_2 u_r + u_r\frac{\partial u_r}{\partial r} - \frac{|v|^2}{r} + \frac{\partial p}{r} \tag{2.25}$$
$$= f_r + \nu\left(\Delta_2 u_r + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u_r}{\partial r}) + \frac{2}{r}\frac{\partial}{\partial r}(ru_r)\right),$$

$$\operatorname{div}_2 v = -\frac{1}{r^2}\frac{\partial(r^2u_r)}{\partial r}. \tag{2.26}$$

In the derivation of (2.24)–(2.25) we took into account relation (2.7) and the incompressibility condition (2.13).

In order to obtain an estimate for $\nabla_2 u$, we need to multiply equations (2.24), (2.25) by a quantity Lu chosen such that:
(i) the leading part of the operator L is Δ_2, and
(ii) $Lu \in H$ if $u \in H^2(\Omega)\cap V$.
Notice that condition (ii) is essential for the elimination of the pressure.

We now proceed with the construction of the operator L. By taking into account the definitions of ∇_2 and Δ_2 given by (2.16) and (2.17), respectively, we easily deduce that if $h = h(r,\theta,\varphi)$ and $\psi = \psi(r)$ are two given scalar functions, then

$$\nabla_2(\psi(r)h) = \psi(r)\nabla_2 h, \tag{2.27}$$

$$\Delta_2(\psi(r)h) = \psi(r)\Delta_2 h, \tag{2.28}$$

$$\Delta_2(\frac{\partial h}{\partial r}) = \frac{\partial}{\partial r}(\Delta_2 h) + \frac{2}{r}\Delta_2 h. \tag{2.29}$$

Consider then, for any $p \in H^1(\Omega)$:

$$\begin{aligned}
(\Delta_2 v, \nabla_2 p)_{L^2} &= -(\operatorname{div}_2(\Delta_2 v), p)_{L^2} = -(\Delta_2(\operatorname{div}_2 v), p)_{L^2} \\
&= (\Delta_2(\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r}), p)_{L^2} = (\frac{1}{r^2}\Delta_2(\frac{\partial(r^2 u_r)}{\partial r}), p)_{L^2} \\
&= (\frac{1}{r^2}\frac{\partial}{\partial r}\Delta_2(r^2 u_r) + \frac{2}{r}\Delta_2 u_r, p)_{L^2} \\
&= \int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} \frac{\partial}{\partial r}\Delta_2(r^2 u_r)\, p\, \sin\theta\, dr d\theta d\varphi + (\Delta_2(\frac{2u_r}{r}), p)_{L^2} \\
&= -\int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} r^2 \Delta_2 u_r \frac{\partial p}{\partial r} \sin\theta\, dr d\theta d\varphi - (\nabla_2(\frac{2u_r}{r}), \nabla_2 p) L^2 \\
&= -(\Delta_2 u_r, \frac{\partial p}{\partial r})_{L^2} - (\nabla_2(\frac{2u_r}{r}), \nabla_2 p)_{L^2},
\end{aligned}$$

which yields the following identity:

$$(\Delta_2 v + \nabla_2(\frac{2u_r}{r}), \nabla_2 p)_{L^2} + (\Delta_2 u_r, \frac{\partial p}{\partial r})_{L^2} = 0. \tag{2.30}$$

A natural choice for the operator L is then

$$Lu = \Delta_2 v + \frac{2}{r}\nabla_2 u_r + (\Delta_2 u_r)\mathbf{e_r}. \tag{2.31}$$

It is obvious that $(Lu)_r = Lu \cdot \mathbf{e_r} = \Delta_2 u_r = 0$ on $\partial\Omega$ if $u_r = 0$ on $\partial\Omega$. Also (2.30) implies that $\operatorname{div}(Lu) = 0$, and therefore $Lu \in H$.

By taking into account formulas (2.6), (2.22), and (2.5), it is a simple exercise to check that Lu can be also expressed as

$$\begin{aligned}
Lu &= \Delta u - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) - \frac{2}{r^2}\frac{\partial}{\partial r}(r u_r)\mathbf{e_r} \\
&= \Delta_s u - \frac{2}{r^2}\frac{\partial}{\partial r}(r u_r)\mathbf{e_r} = (\Delta_2 u_r)\mathbf{e_r} + (\Delta_s u)_\theta \mathbf{e}_\theta + (\Delta_s u)_\varphi \mathbf{e}_\varphi,
\end{aligned} \tag{2.32}$$

where $\Delta_s u$ stands for

$$\Delta_s u = \Delta u - \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) = [\Delta_2 u_r + \frac{2}{r}\frac{\partial}{\partial r}(ru_r)]\mathbf{e_r} + \Delta_2 v + \frac{2}{r}\nabla_2 u_r. \quad (2.33)$$

We also define the associated gradient $\nabla_s u$ by

$$\nabla_s u = \begin{bmatrix} 0 & \frac{1}{r}\frac{\partial u_r}{\partial\theta} - \frac{u_\theta}{r} & \frac{1}{r\sin\theta}\frac{\partial u_r}{\partial\varphi} - \frac{u_\varphi}{r} \\ 0 & \frac{1}{r}\frac{\partial u_\theta}{\partial\theta} + \frac{u_r}{r} & \frac{1}{r\sin\theta}\frac{\partial u_\theta}{\partial\varphi} - \frac{u_\varphi\cot\theta}{r} \\ 0 & \frac{1}{r}\frac{\partial u_\varphi}{\partial\theta} & \frac{1}{r\sin\theta}\frac{\partial u_\varphi}{\partial\varphi} + \frac{u_r}{r} + \frac{u_\theta\cot\theta}{r} \end{bmatrix}. \quad (2.34)$$

Notice that a Green-type identity is true for the operators Δ_s and ∇_s defined by (2.33) and (2.34). Indeed if u and v are two regular vector fields on Ω such that v vanishes on the boundary of Ω, then

$$\begin{aligned}(-\Delta_s u, v)_{L^2} &= (-\Delta u + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}), v)_{L^2} \quad (2.35)\\ &= (\nabla u, \nabla v)_{L^2} + \int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})\, v\, \sin\theta dr d\theta d\varphi \\ &= (\nabla u, \nabla v)_{L^2} - \int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi}\frac{\partial u}{\partial r}\frac{\partial v}{\partial r}\, r^2\sin\theta dr d\theta d\varphi \\ &= (\nabla u, \nabla v)_{L^2} - (\frac{\partial u}{\partial r}, \frac{\partial v}{\partial r})_{L^2} = (\nabla_s u, \nabla_s v)_{L^2}.\end{aligned}$$

We are now in position to derive an estimate for $\nabla_s u$, the tangential derivative of the three-dimensional velocity u. We multiply the equations of motion (2.24)-(2.25) by Lu, and integrate the result over Ω. We will compute separately each term.

It turns out that it is more convenient to use the alternative definition (2.32) of Lu in the computations of the terms containing time derivatives and second order derivatives. We simplify

$$\begin{aligned}-(\partial_t u, Lu)_{L^2} &= (\partial_t u, -\Delta_s u + \frac{2}{r^2}\frac{\partial}{\partial r}(ru_r)\mathbf{e_r})_{L^2} \quad (2.36)\\ &= \frac{1}{2}\frac{d}{dt}|\nabla_s u|_{L^2}^2 + 2(\partial_t u_r, \frac{1}{r}\frac{\partial u_r}{\partial r} + \frac{u_r}{r^2})_{L^2} \\ &= \frac{1}{2}\frac{d}{dt}|\nabla_s u|_{L^2}^2 + \frac{d}{dt}\Big|\frac{u_r}{r}\Big|_{L^2}^2 + 2\big(\partial_t u_r, \frac{1}{r}\frac{\partial u_r}{\partial r}\big)_{L^2}.\end{aligned}$$

The contribution of the viscous term is:

$$
\begin{aligned}
(\Delta u, Lu)_{L^2} &= (\Delta_s u + \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u}{\partial r}\big), \Delta_s u - \frac{2}{r^2}\frac{\partial}{\partial r}(ru_r)\mathbf{e_r})_{L^2} \qquad (2.37)\\
&= |\Delta_s u|^2_{L^2} + \big(\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u}{\partial r}\big), \Delta_s u\big)_{L^2}\\
&- \big((\Delta_s u)_r, \frac{2}{r^2}\frac{\partial}{\partial r}(ru_r)\big)_{L^2} - \big(\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u_r}{\partial r}\big), \frac{2}{r^2}\frac{\partial}{\partial r}(ru_r)\big)_{L^2}.
\end{aligned}
$$

In order to treat the second term in the right-hand side of (2.37) we need the following result.

Lemma 2.3. *Let u be a vector field defined in Ω.*

(i) The commutator of ∇_s and $\dfrac{\partial}{\partial r}$ is given by

$$\nabla_s\big(\frac{\partial u}{\partial r}\big) - \frac{\partial}{\partial r}(\nabla_s u) = \frac{1}{r}\nabla_s u. \qquad (2.38)$$

(ii) If $h = h(r)$ is a scalar function, then

$$\nabla_s(h(r)u) = h(r)\nabla_s u. \qquad (2.39)$$

The proof is immediate. Notice that $\nabla_s u$ can be written as

$$
\nabla_s u = \frac{1}{r}Du, \quad \text{where } D = \begin{bmatrix} 0 & \dfrac{\partial u_r}{\partial \theta} - u_\theta & \dfrac{1}{\sin\theta}\dfrac{\partial u_r}{\partial \varphi} - u_\varphi \\ 0 & \dfrac{\partial u_\theta}{\partial \theta} + u_r & \dfrac{1}{\sin\theta}\dfrac{\partial u_\theta}{\partial \varphi} - u_\varphi\cot\theta \\ 0 & \dfrac{\partial u_\varphi}{\partial \theta} & \dfrac{1}{\sin\theta}\dfrac{\partial u_\varphi}{\partial \varphi} + u_r + u_\varphi\cot\theta \end{bmatrix}.
$$

Therefore,

$$\frac{\partial}{\partial r}(\nabla_s u) = \frac{\partial}{\partial r}\big(\frac{1}{r}Du\big) = \frac{1}{r}D\big(\frac{\partial u}{\partial r}\big) - \frac{1}{r^2}Du = \nabla_s\big(\frac{\partial u}{\partial r}\big) - \frac{1}{r}\nabla_s u.$$

For part (ii) we have

$$\nabla_s(h(r)u) = \frac{1}{r}D(h(r)u) = \frac{h(r)}{r}Du = h(r)\nabla_s u. \qquad \square$$

We will use the previous lemma, Green's identity (2.35), and the fact that $\nabla_s u$ vanishes on $\partial\Omega$ to simplify the second term in the right-hand side of (2.37):

$$\big(\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u}{\partial r}\big), \Delta_s u\big)_{L^2} = -\big(\nabla_s\big(\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})\big), \nabla_s u\big)_{L^2}$$

$$= -\big(\frac{1}{r^2}\nabla_s\big(\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})\big), \nabla_s u\big)_{L^2}$$

$$= -\int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} \nabla_s\big(\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})\big)\cdot\nabla_s u\,\sin\theta drd\theta d\varphi$$

$$= -\int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} \left[\frac{\partial}{\partial r}\nabla_s\big(r^2\frac{\partial u}{\partial r}\big) + \frac{1}{r}\nabla_s\big(r^2\frac{\partial u}{\partial r}\big)\right]\cdot\nabla_s u\,\sin\theta drd\theta d\varphi$$

$$= \int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} \nabla_s\big(r^2\frac{\partial u}{\partial r}\big)\cdot\frac{\partial}{\partial r}(\nabla_s u)\,\sin\theta drd\theta d\varphi$$

$$- \int\int\int \nabla_s\big(\frac{\partial u}{\partial r}\big)\cdot\nabla_s u\, r\sin\theta drd\theta d\varphi$$

$$= \int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} \nabla_s\big(\frac{\partial u}{\partial r}\big)\cdot\Big(r^2\frac{\partial}{\partial r}(\nabla_s u) - r\nabla_s u\Big)\sin\theta drd\theta d\varphi$$

$$= \int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} \Big(\frac{\partial}{\partial r}(\nabla_s u) + \frac{1}{r}\nabla_s u\Big)\cdot\Big(\frac{\partial}{\partial r}(\nabla_s u) - \frac{1}{r}\nabla_s u\Big)r^2\sin\theta drd\theta d\varphi$$

$$= \big|\frac{\partial}{\partial r}(\nabla_s u)\big|_{L^2}^2 - \big|\frac{1}{r}\nabla_s u\big|_{L^2}^2.$$

Thus,

$$\big(\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u}{\partial r}\big), \Delta_s u\big)_{L^2} = \big|\frac{\partial}{\partial r}(\nabla_s u)\big|_{L^2}^2 - \big|\frac{1}{r}\nabla_s u\big|_{L^2}^2. \tag{2.40}$$

Notice here that the L^2 norms of $\nabla_s(\frac{\partial u}{\partial r})$ and $\frac{\partial}{\partial r}(\nabla_s u)$ are equal

$$\big|\nabla_s\big(\frac{\partial u}{\partial r}\big)\big|_{L^2} = \big|\frac{\partial}{\partial r}(\nabla_s u)\big|_{L^2}, \tag{2.41}$$

since

$$\big|\nabla_s\big(\frac{\partial u}{\partial r}\big)\big|_{L^2}^2 - \big|\frac{\partial}{\partial r}(\nabla_s u)\big|_{L^2}^2 = \big|\frac{\partial}{\partial r}(\nabla_s u) + \frac{1}{r}\nabla_s u\big|_{L^2}^2 - \big|\frac{\partial}{\partial r}(\nabla_s u)\big|_{L^2}^2$$

$$= \big|\frac{1}{r}\nabla_s u\big|_{L^2}^2 + 2\big(\frac{\partial}{\partial r}(\nabla_s u), \frac{1}{r}\nabla_s u\big)_{L^2}$$

$$= \big|\frac{1}{r}\nabla_s u\big|_{L^2}^2 + 2\int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} \frac{\partial}{\partial r}(\nabla_s u)\cdot\nabla_s u\, r\sin\theta drd\theta d\varphi$$

$$= \big|\frac{1}{r}\nabla_s u\big|_{L^2}^2 + \int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} \frac{\partial}{\partial r}|\nabla_s u|^2 r\sin\theta drd\theta d\varphi$$

$$= \big|\frac{1}{r}\nabla_s u\big|_{L^2}^2 - \int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} |\nabla_s u|^2\sin\theta drd\theta d\varphi = 0.$$

We take into account relation (2.40) and estimate the remaining terms in (2.37) using the Cauchy-Schwarz and Young inequalities to find

$$\begin{aligned}(\Delta u, Lu)_{L^2} \geq & |\Delta_s u|^2_{L^2} + \big|\frac{\partial}{\partial r}(\nabla_s u)\big|^2_{L^2} - \big|\frac{1}{r}\nabla_s u\big|^2_{L^2} \\ & - 2\,|\Delta_s u|_{L^2}\big|\frac{1}{r^2}\frac{\partial}{\partial r}(ru_r)\big|_{L^2} - 2\,\big|\frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial u_r}{\partial r}\big)\big|_{L^2}\big|\frac{1}{r^2}\frac{\partial}{\partial r}(ru_r)\big|_{L^2} \\ \geq & \frac{1}{2}|\Delta_s u|^2_{L^2} + \big|\frac{\partial}{\partial r}(\nabla_s u)\big|^2_{L^2} - \big|\frac{1}{r}\nabla_s u\big|^2_{L^2} \\ & - 3\,\big|\frac{1}{r^2}\frac{\partial}{\partial r}(ru_r)\big|^2_{L^2} - \big|\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u_r}{\partial r})\big|^2_{L^2}.\end{aligned} \tag{2.42}$$

In order to estimate the nonlinear terms we will use the expression (2.31) for Lu. We recall that we are multiplying equation (2.24) by $\Delta_2 v + 2r^{-1}\nabla_2 u_r$, the tangential component of Lu, and equation (2.25) by $\Delta_2 u_r$, the radial component of Lu, respectively. The contribution of the nonlinear term is

$$\begin{aligned}NLT = & ((v\cdot\nabla_2)v + u_r\frac{\partial v}{\partial r} + u_r\frac{v}{r}, \Delta_2 v + \frac{2}{r}\nabla_2 u_r)_{L^2} \\ & + (v\cdot\nabla_2 u_r + u_r\frac{\partial u_r}{\partial r} - \frac{|v|^2}{r}, \Delta_2 u_r)_{L^2}\end{aligned} \tag{2.43}$$

The terms which contain u_r will be bounded in the usual way, using the Hölder and Sobolev inequalities as well as the assumptions on the regularity of u_r. Similarly to the Cartesian case, the only term that requires special attention is $((v\cdot\nabla_2)v, \Delta_2 v)_{L^2}$. The following relations remain valid in the case of spherical coordinates:

$$-\Delta_2 v = \operatorname{curl}_2(\operatorname{curl}_2 v) - \nabla_2(\operatorname{div}_2 v), \tag{2.44}$$

$$(v\cdot\nabla_2)v = \frac{1}{2}\nabla_2|v|^2 - v\times\operatorname{curl}_2 v. \tag{2.45}$$

We now derive an expression for $((v\cdot\nabla_2)v, \Delta_2 v)_{L^2}$ similar to (1.13) previously obtained in the case of Cartesian coordinates. We have

$$((v\cdot\nabla_2)v, \Delta_2 v)_{L^2} = (\frac{1}{2}\nabla_2|v|^2 - v\times\operatorname{curl}_2 v, \nabla_2(\operatorname{div}_2 v) - \operatorname{curl}_2(\operatorname{curl}_2 v))_{L^2}. \tag{2.46}$$

First it can be easily checked that

$$(v\times\operatorname{curl}_2 v, \operatorname{curl}_2(\operatorname{curl}_2 v))_{L^2} = -(\frac{1}{2}\operatorname{div}_2 v, |\operatorname{curl}_2 v|^2)_{L^2}. \tag{2.47}$$

Another term in (2.46) is

$$\begin{aligned}(\frac{1}{2}\nabla_2|v|^2, \nabla_2(\operatorname{div}{}_2 v))_{L^2} &= -(\operatorname{div}{}_2(\frac{1}{2}\,\nabla_2|v|^2), \operatorname{div}{}_2 v)_{L^2}\\ &= -\frac{1}{2}(\Delta_2|v|^2, \operatorname{div}{}_2 v)_{L^2}.\end{aligned}$$

Straightforward calculations lead to the following formula:

$$\frac{1}{2}\Delta_2|v|^2 = v_\theta\Delta_2 v_\theta + v_\varphi\Delta_2 v_\varphi + |\nabla_2 v_\theta|^2 + |\nabla_2 v_\varphi|^2.$$

Unfortunately this is not very satisfactory due to the presence of the Laplace operator acting on scalars instead of the Laplace operator acting on vectors. An alternative formula is

$$\frac{1}{2}\Delta_2|v|^2 = v\cdot\Delta_2 v + |\nabla_2 v|^2 + \frac{|v|^2}{r}. \tag{2.48}$$

The last non-vanishing term from (2.46) is

$$\begin{aligned}&-(v\times\operatorname{curl}{}_2 v, \nabla_2(\operatorname{div}{}_2 v))_{L^2} = (\operatorname{div}{}_2(v\times\operatorname{curl}{}_2 v), \operatorname{div}{}_2 v)_{L^2}\\ &= (|\operatorname{curl}{}_2 v|^2, \operatorname{div}{}_2 v)_{L^2} + (\frac{v_\varphi}{r}\frac{\partial(\operatorname{curl}{}_2 v)}{\partial\theta} - \frac{v_\theta}{r\sin\theta}\frac{\partial(\operatorname{curl}{}_2 v)}{\partial\varphi}, \operatorname{div}{}_2 v)_{L^2}.\end{aligned}$$

Therefore, we found that

$$\begin{aligned}&((v\cdot\nabla_2)v, \Delta_2 v)_{L^2} = -(v\cdot\Delta_2 v + |\nabla_2 v|^2 + \frac{|v|^2}{r}, \operatorname{div}{}_2 v)_{L^2}\\ &+ (|\operatorname{curl}{}_2 v|^2, \operatorname{div}{}_2 v)_{L^2} + (\frac{v_\varphi}{r}\frac{\partial(\operatorname{curl}{}_2 v)}{\partial\theta} - \frac{v_\theta}{r\sin\theta}\frac{\partial(\operatorname{curl}{}_2 v)}{\partial\varphi}, \operatorname{div}{}_2 v)_{L^2}.\end{aligned}$$

Now we use the incompressibility condition to write

$$\operatorname{div}{}_2 v = -\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r}.$$

We apply Hölder's inequality and bound NLT by

$$\begin{aligned}|NLT| \le \big(&|v|_{L^2}|\Delta_2 v|_{L^2} + |\nabla_2 v|^2_{L^2} + \big|\frac{|v|^2}{r}\big|_{L^2} + \frac{1}{2}|\operatorname{curl}{}_2 v|^2_{L^2}\big)\big|\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r}\big|_{L^\infty}\\ &+ (|\operatorname{curl}{}_2 v|^2_{L^2} + |v|_{L^2}|\nabla_2(\operatorname{curl}{}_2 v)|_{L^2})\big|\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r}\big|_{L^\infty}\\ &+ |v|_{L^2}|\nabla_2 v|_{L^2}\big|\frac{2}{r}\nabla_2 u_r\big|_{L^\infty} + |u_r|_{L^\infty}(\big|\frac{\partial v}{\partial r}\big|_{L^2} + \big|\frac{v}{r}\big|_{L^2})|\Delta_s u|_{L^2}\\ &+ (|v|_{L^6}|\nabla_2 u_r|_{L^3} + |u_r|_{L^6}\big|\frac{\partial u_r}{\partial r}\big|_{L^3} + \big|\frac{|v|^2}{r}\big|_{L^2})|\Delta_2 u_r|_{L^2}.\end{aligned} \tag{2.49}$$

From the very definitions of curl_2, ∇_s, and Δ_s, it follows that

$$|\operatorname{curl}_2 v|_{L^2} \leq 2|\nabla_s u|_{L^2}, \tag{2.50}$$

$$|\Delta_2 v|_{L^2} \leq |\Delta_s u|_{L^2} + \big|\frac{2}{r}\nabla_2 u_r\big|_{L^2}, \tag{2.51}$$

$$|\Delta_2 u_r|_{L^2} \leq |\Delta_s u|_{L^2} + \big|\frac{2}{r}\frac{\partial}{\partial r}(r u_r)\big|_{L^2}. \tag{2.52}$$

Also the L^2 norm of $\nabla(\operatorname{curl}_2 v)$ is dominated by the dissipation and u_r. Indeed we have

$$\begin{aligned}
|\nabla(\operatorname{curl}_2 v)|^2_{L^2} &= -(\Delta(\operatorname{curl}_2 v), \operatorname{curl}_2 v)_{L^2} \qquad (2.53)\\
&= -(\Delta_2(\operatorname{curl}_2 v) + \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial}{\partial r}(\operatorname{curl}_2 v)\big), \operatorname{curl}_2 v)_{L^2}\\
&= -(\operatorname{curl}_2(\Delta_2 v) + \frac{1}{r^2}\frac{\partial}{\partial r}\big(r^2\frac{\partial}{\partial r}(\operatorname{curl}_2 v)\big), \operatorname{curl}_2 v)_{L^2}\\
&= -(\Delta_2 v, \operatorname{curl}_2(\operatorname{curl}_2 v))_{L^2}\\
&\quad + \int_{R_1}^{R_2}\int_0^{\pi}\int_0^{2\pi} r^2\big|\frac{\partial}{\partial r}(\operatorname{curl}_2 v)\big|^2 \sin\theta\, dr d\theta d\varphi\\
&= (\Delta_2 v, \Delta_2 v + \nabla_2(\operatorname{div}_2 v))_{L^2} + \big|\frac{\partial}{\partial r}(\operatorname{curl}_2 v)\big|^2_{L^2}\\
&\leq |\Delta_2 v|^2_{L^2} + \big|\frac{\partial}{\partial r}(\operatorname{curl}_2 v)\big|^2_{L^2} + |\Delta_2 v|_{L^2}\big|\nabla_2\big(\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r}\big)\big|_{L^2}\\
&\leq |\Delta_2 v|^2_{L^2} + 2\big|\frac{\partial}{\partial r}(\nabla_s u)\big|^2_{L^2} + |\Delta_2 v|_{L^2}\big|\nabla_2\big(\frac{1}{r^2}\frac{\partial(r^2 u_r)}{\partial r}\big)\big|_{L^2}.
\end{aligned}$$

We take into account the assumption on u_r and the above relations, we apply the Sobolev and Young inequalities, and bound NLT by

$$|NLT| \leq \frac{\nu}{4}|\Delta_s u|^2_{L^2} + C_3|\nabla u|^2_{L^2} + C_4, \tag{2.54}$$

where C_3 and C_4 are two constants which depend only on C_1, ν, and R_1.

Therefore, (2.36), (2.42), and (2.54), lead to the following estimate (the analog of (1.16)):

$$\begin{aligned}
&\frac{d}{dt}|\nabla_s u|^2_{L^2} + \frac{d}{dt}\big|\frac{u_r}{r}\big|^2_{L^2} + \frac{\nu}{2}\big(|\Delta_s u|^2_{L^2} + \big|\frac{\partial}{\partial r}(\nabla_s u)\big|^2_{L^2}\big) \qquad (2.55)\\
&\quad\leq \frac{c}{\nu}|f|^2_{L^2} + 4\,|\partial_t u_r|_{L^2}\big|\frac{1}{r}\frac{\partial u_r}{\partial r}\big|_{L^2} + 2\nu\big|\frac{1}{r}\nabla_s u\big|^2_{L^2}\\
&\quad + 6\nu\,\big|\frac{1}{r^2}\frac{\partial}{\partial r}(r u_r)\big|^2_{L^2} + 2\nu\,\big|\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u_r}{\partial r})\big|^2_{L^2} + C_3|\nabla u|^2_{L^2} + C_4
\end{aligned}$$

$$\leq C_5 |\nabla u|_{L^2}^2 + C_6,$$

where C_5 and C_6 are constants that depend only on C_1, C_2, ν, $|f|_{L^2(0,T;H)}$, and R_1, the interior radius. We use the fact that u, being a weak solution to the Navier-Stokes equations, belongs to $L^\infty(0,T;H) \cap L^2(0,T;V)$, and therefore we obtain from (2.55) the following estimate:

$$|\nabla_s u(t)|_{L^2} \leq C \quad \text{ a.e. } 0 < t < T, \tag{2.56}$$

where C depends on C_1, C_2, ν, $|f|_{L^2(0,T;H)}$, R_1, and T.

For the second part of the proof, we multiply the equations of motion (2.24)–(2.25) by $Au = P(-\Delta u)$ and integrate over Ω. But first we need to rewrite the nonlinear term in a suitable way. By taking into account formulas (2.4), (2.7), and (2.34), we notice that the nonlinear term can be written as

$$(u \cdot \nabla)u = u_r \frac{\partial u}{\partial r} + (v \cdot \nabla_s)u. \tag{2.57}$$

We obtain the following estimate:

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}|A^{1/2}u|_{L^2}^2 + \nu |Au|_{L^2}^2 &= (f, Au)_{L^2} - (u_r \frac{\partial u}{\partial r} + (v \cdot \nabla_s)u, Au)_{L^2} \\
&\leq |f|_{L^2}|Au|_{L^2} + \big(|u_r|_{L^\infty} \big|\frac{\partial u}{\partial r}\big|_{L^2} + |v|_{L^\infty}|\nabla_s u|_{L^2}\big)|Au|_{L^2} \\
&\leq |f|_{L^2}|Au|_{L^2} + c\,(|u_r|_{L^\infty}|A^{1/2}u|_{L^2} + |\nabla_s u|_{L^2}|A^{1/2}u|_{L^2}^{1/2}|Au|_{L^2}^{3/2}.
\end{aligned}$$

and therefore

$$\frac{d}{dt}|A^{1/2}u|_{L^2}^2 + \nu |Au|_{L^2}^2 \leq \frac{c}{\nu}|f|_{L^2}^2 + (\frac{c}{\nu}|u_r|_{L^\infty}^2 + \frac{c}{\nu^3}|\nabla_s u|_{L^2}^4)|A^{1/2}u|_{L^2}^2. \tag{2.58}$$

As in the previous case, we use the fact that $|\nabla_s u|_{L^2}$ and $|u_r|_{L^\infty}$ are bounded to derive that $u \in L^\infty(0,T;V) \cap L^2(0,T;D(A))$. Then we repeat the argument from the case of the channel. Let $[0, T^*)$ be the maximal interval on which the strong solution u with initial condition u_0 exists. The estimates obtained previously are valid on the interval $[0, T^* - \delta]$ for every $\delta > 0$. We let $t \to T^* - 0$, and deduce that ∇u remains bounded in $L^\infty(0, T^*; L^2)$ independently of T^*, therefore we have a contradiction with the maximality of T^*.

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