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TIME BEHAVIOR FOR A CLASS OF NONLINEAR BEAM EQUATIONS

C. Buriol
Department of Mathematics, Federal University of Santa Maria
Faixa de Camobi, km 9, 97105-900, Santa Maria, RS, Brazil

G. Perla Menzala
National Laboratory of Scientific Computation LNCC/MCT, Rua Getulio Vargas 333, Quitandinha, Petropolis 25651-070, RJ, Brasil and Institute of Mathematics, Federal University of Rio de Janeiro, RJ, P.O. 68530, Rio de Janeiro, RJ, Brazil

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Abstract. We consider a class of nonlinear beam equations in the whole space $\mathbb{R}^n$. Using previous important work due to Levandovsky and Strauss we prove that, locally, the H^1-norm of a strong solution approaches zero as $t \to +\infty$ as long as the spatial dimension $n \geq 6$. The problem remains open for dimensions $1 \leq n \leq 5$.

1. Introduction

We consider the nonlinear evolution equation

$$u_{tt} + \Delta^2 u + u - M\Big(\int_{\mathbb{R}^n} |\nabla u|^2\,dx\Big)\Delta u = 0 \tag{1.1}$$

in $\mathbb{R}^n \times (0, +\infty)$. Here Δ denotes the usual Laplace operator, $\Delta^2 = \Delta\Delta$ is the biharmonic operator, ∇u denotes the spatial gradient of u and $M(s)$ is a nonlinear smooth function satisfying suitable assumptions which will be given later. Dimensions one and two are relevant from the point of view of physics and engineering because in those situations model (1.1) is considered a good approximation for describing nonlinear vibrations of beams or plates (see [3], [4]). The term $M\left(\int_{\mathbb{R}^n} |\nabla u|^2\,dx\right)\Delta u$ is usually known as Timoshenko's type nonlinearity.

In this article we are concerned with asymptotic properties of the solution of the Cauchy problem associated with (1.1). The main result says that,

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locally, the H^1-norm of a strong solution u of (1.1) approaches zero as $t \to +\infty$, as long as the dimension n is bigger than or equal to six.

Recently, S. Levandosky and W. Strauss [8] studied the nonlinear equation

$$u_{tt} + \Delta^2 u + f(u) = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, +\infty), \tag{1.2}$$

where f was assumed to satisfy the condition

$$sf(s) \geq 2F(s) \geq c_0 s^2 \tag{1.3}$$

for all $s \in \mathbb{R}$ and some positive constant c_0 where $F(s)$ is such that $F' = f$ and $F(0) = 0$. They proved that for $n \geq 5$ the quantity $G(u) = \frac{1}{r}\,[uf(u) - 2F(u)] + \frac{1}{r^3}\,|\nabla u|^2$, where $r = |x|$ satisfies the estimate

$$\int_0^{+\infty} \int_{\mathbb{R}^n} G(u) dx dt \leq C\, E(0)$$

for some positive constant C where $E(0)$ is given by

$$E(0) = \frac{1}{2} \int_{\mathbb{R}^n} [u_0^2 + |\Delta u_1|^2]\, dx + \int_{\mathbb{R}^n} F(u_0)\, dx.$$

Here $\{u_0, u_1\}$ denotes the initial data at $t = 0$ for equation (1.2). For $n \geq 6$ they also proved the estimate

$$\int_0^{+\infty} \int_{\mathbb{R}^n} \frac{1}{r^5}\, u^2\, dx dt \leq C\, E(0).$$

The main tool used in [8] was a radial multiplier introduced by Cathleen Morawetz in the late sixties. The corresponding identities using this multiplier with equation (1.2) unfortunately do not help for lower dimensions $(1 \leq n \leq 4)$.

Observe that in model (1.1) the nonlocal nonlinearity does not satisfy condition (1.3). Even so, we use the basic idea of [8] to obtain the corresponding identities and prove a slightly better result for the solutions of (1.1), namely, for any bounded region Ω of $\mathbb{R}^n$ we show that

$$||u(\,\cdot\,, t)||_{H^1(\Omega)} \longrightarrow 0 \quad \text{as} \quad t \to +\infty \quad \text{ if } n \geq 6$$

and

$$||\nabla u(\,\cdot\,, t)||_{L^2(\Omega)} \longrightarrow 0 \quad \text{as} \quad t \to +\infty \quad \text{ if } n \geq 5$$

There is a close relationship between solutions of (1.1) and solutions of the nonlinear Schrodinger-type equation

$$iw_t + \Delta w + w - M\Big(\int_{\mathbb{R}^n} |\text{Re}\nabla w|^2\, dx\Big)\text{Re}\; w = 0 \tag{1.4}$$

in $\mathbb{R}^n \times (0, +\infty)$. In fact, if w solves (1.4) with initial data $w(x, 0) = \varphi(x) + i\psi(x)$, then u solves (1.1) with initial data $u(x, 0) = \varphi(x)$, $u_t(x, 0) = -\Delta\psi(x)$ and conversely. This observation was used in [2] in order to show local smoothing effects of the solutions of (1.1) and also in [6] with a similar nonlinearity but with a dissipative character.

In Section 2 for sake of completeness we describe the function spaces where we shall consider the global solutions of problem (1.1). In Section 3 we use the identities obtained by S. Levandosky and W. Strauss in [8] properly adapted to our case and show the (local) asymptotic stability as $t \to +\infty$.

Our notation throughout the article is standard: $H^m(\mathbb{R}^n)$ denotes the Sobolev space of order m in $\Omega = \mathbb{R}^n$. Given a function $f\colon \mathbb{R}^n \to \mathbb{R}$ we denote by $\hat{f}$ the Fourier transform of f and by $\overline{\hat{f}}$ its complex conjugate. For $\xi, \eta \in \mathbb{R}^n$ $\xi \bullet \eta$ means the usual inner product in $\mathbb{R}^n$ and $|\xi| = \left(\sum_{j=1}^n \xi_j^2\right)^{1/2}$ whenever $\xi = (\xi_1, \xi_2, \dots, \xi_n)$.

2. Well posedness

In this section we briefly describe for sake of completeness the function spaces where we will consider the Cauchy problem for equation (1.1). Let $X = H^2(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ with the inner product

$$\langle (u, v), (\tilde{u}, \tilde{v}) \rangle_X = \langle u, \tilde{u} \rangle_{H^2(\mathbb{R}^n)} + \langle v, \tilde{v} \rangle_{L^2(\mathbb{R}^n)} .$$

We rewrite the linear part of (1.1), that is, (1.1) with $M \equiv 0$ as a first-order system

$$\frac{dU}{dt} = AU \tag{2.1}$$

where $U(t) = \begin{pmatrix} u \\ v \end{pmatrix}$, $A = \begin{bmatrix} 0 & I \\ -I - \Delta^2 & 0 \end{bmatrix}$. The domain of A denoted by $\mathcal{D}(A)$ is given by $H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$. It is well known that A generates a C_0 semigroup of operators in X (see for instance [11]). From now on we will assume that the function $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

$$M \in C^1, \quad M(s) \geq 0 \quad \forall\, s \in \mathbb{R}^+. \tag{2.2}$$

Lemma 2.1. *Let $N\colon H^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ be given by*

$$N(u) = M\Big(\int_{\mathbb{R}^n} |\nabla u|^2\, dx\Big)\Delta u,$$

where M satisfies (2.2). *Then, N is a locally Lipschitz function.*

Proof. Let $u, v \in H^2(\mathbb{R}^n)$ and $\rho > 0$ such that

$$||u||_{H^2} \leq \rho, \quad ||v||_{H^2} \leq \rho.$$

We denote by $\hat{u}$ and $\hat{v}$ the Fourier transform of u and v respectively. Clearly,

$$\begin{aligned}\left|\widehat{N(u)} - \widehat{N(v)}\right| &= \left|M\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)|\xi|^2\,\hat{u} - M\Big(\int_{\mathbb{R}^n} |\nabla v|^2 dx\Big)|\xi|^2\,\hat{v}\right| \\ &= \Big|M\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)|\xi|^2(\hat{u}-\hat{v}) \\ &\quad + \Big[M\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big) - M\Big(\int_{\mathbb{R}^n} |\nabla v|^2 dx\Big)\Big]|\xi|^2\,\hat{v}\Big|. \qquad (2.3)\end{aligned}$$

We use the mean value theorem in (2.3) to deduce that

$$\begin{aligned}\left|\widehat{N(u)} - \widehat{N(v)}\right| &\leq M\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)|\xi|^2\,|\hat{u}-\hat{v}| \\ &+ \max_{0\leq s\leq\rho^2} |M'(s)|\Big|\int_{\mathbb{R}^n} |\xi|^2\big(|\hat{u}|^2 - |\hat{v}|^2\big)d\xi\Big||\xi|^2|\hat{v}|. \qquad (2.4)\end{aligned}$$

Observe that Holder's inequality implies that

$$\begin{aligned}\Big|\int_{\mathbb{R}^n} |\xi|^2\big(|\hat{u}|^2 - |\hat{v}|^2\big)d\xi\Big|^2 &\leq \Big(\int_{\mathbb{R}^n} |\xi|^2\big(|\hat{u}|^2 + |\hat{v}|\big)^2\,d\xi\Big)\Big(\int_{\mathbb{R}^n} |\xi|^2|\hat{u}-\hat{v}|^2\,d\xi\Big) \\ &\leq \big[||u||_{H^1}^2 + ||v||_{H^1}^2\big]||u-v||_{H^1}^2. \qquad (2.5)\end{aligned}$$

Consequently, using (2.4) and (2.5) we deduce

$$||N(u) - N(v)||_{L^2}^2 \leq$$
$$\Big\{2\Big[\max_{0\leq s\leq\rho^2} M(s)\Big]^2 + 4\max_{0\leq s\leq\rho^2} |M'(s)|^2||v||_{H^2}^2\Big[||u||_{H^1}^2 + ||v||_{H^1}^2\Big]\Big\}||u-v||_{H^2}^2$$

which proves Lemma 2.1.

Theorem 2.2. (Global existence). *Suppose $M(s)$ satisfies (2.2) and $(u_0, u_1) \in H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$. Then, equation (1.1) admits a unique global strong solution $u = u(x,t)$ such that $u(x,0) = u_0(x)$, $u_t(x,0) = u_1(x)$ and*

$$u \in C(\mathbb{R}^+; H^4) \cap C^1(\mathbb{R}^+; H^2) \cap C^2(\mathbb{R}^+; L^2).$$

Proof. We rewrite equation (1.1) as a first-order system

$$\frac{dU}{dt} = AU + \widetilde{N}(U),$$

where $\widetilde{N}(U) = \begin{pmatrix} 0 \\ N(u) \end{pmatrix}$ and $N(u)$ is as in Lemma 2.1. It follows that $\widetilde{N}(U)$ is also locally Lipschitz and takes $\mathcal{D}(A)$ into itself. This shows local existence for a solution of (1.1) with initial data $\{u_0, u_1\}$. We can consider the local existence in the maximal interval of existence $[0, T_{\max})$. To show that $T_{\max} = +\infty$ we need a priori estimates. First, in any interval $0 \leq t \leq T < T_{\max}$ we consider the quantity (total energy) $E(t)$ given by

$$E(t) = \int_{\mathbb{R}^n} \left[u_t^2 + |\Delta u|^2 + u^2\right] dx + \widetilde{M}\Big(\int_{\mathbb{R}^n} |\nabla u|^2\, dx\Big), \tag{2.6}$$

where $\widetilde{M}(s) = \displaystyle\int_0^s M(\tau)\, d\tau$. We claim

$$E(t) = E(0) \quad \forall\, t \in [0, T]. \tag{2.7}$$

In fact, multiplying equation (1.1) by u_t and integrating over all of $\mathbb{R}^n$ gives us the identity

$$\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}^n} \left[u_t^2 + (\Delta u)^2 + u^2\right] dx + M\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)\int_{\mathbb{R}^n} \nabla u_t \bullet \nabla u\, dx = 0$$

or

$$\frac{1}{2}\frac{d}{dt}\left\{\int_{\mathbb{R}^n} \left[u_t^2 + (\Delta u)^2 + u^2\right] dx + \widetilde{M}\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)\right\} = 0,$$

which proves (2.7). In particular, it follows that

$$||u||_{H^2}^2 \leq E(0), \quad ||u_t||_{L^2}^2 \leq E(0). \tag{2.8}$$

In order to obtain estimates for u in $H^4(\mathbb{R}^n)$ and u_t in $H^2(\mathbb{R}^n)$ we can use Fourier analysis to deduce that in the interval $[0, T]$ the identity

$$\hat{u}_{tt} + \big(|\xi|^4 + 1\big)\hat{u} + M\Big(\int_{\mathbb{R}^n} |\xi|^2\, |\hat{u}|^2\, d\xi\Big)|\xi|^2\, \hat{u} = 0 \tag{2.9}$$

holds, where $\hat{u} = \hat{u}(\xi, t)$ denotes the Fourier transform of u in the spatial variable $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. We consider the functional

$$G(\xi, t) = \frac{1}{2}|\xi|^4\Big\{|\hat{u}_t|^2 + \big(1 + |\xi|^4\big)|\hat{u}|^2 + M\Big(\int_{\mathbb{R}^n} |\xi|^2|\hat{u}|^2 d\xi\Big)|\xi|^2|\hat{u}|^2\Big\}. \tag{2.10}$$

Let us take the derivative of G with respect to t and use (2.9) to obtain

$$\frac{d}{dt}G(\xi, t) = M'\Big(\int_{\mathbb{R}^n} |\xi|^2\, |\hat{u}|^2\, d\xi\Big)\Big(\int_{\mathbb{R}^n} |\xi|^2\, \mathrm{Re}\big\{\hat{u}\, \overline{\hat{u}_t}\big\} d\xi\Big)|\xi|^6\, |\hat{u}|^2. \tag{2.11}$$

From (2.8), our assumptions on M and Holder's inequality we deduce that

$$\begin{aligned}
&\frac{d}{dt}\,G(\xi,t) \le \max_{0\le s\le E(0)} |M'(s)|\Big\{\int_{\mathbb{R}^n} |\xi|^2|\hat{u}|\,|\bar{\hat{u}}_t|\,d\xi\Big\}|\xi|^6|\hat{u}|^2\\
&\le \max_{0\le s\le E(0)} |M'(s)|\Big(\int_{\mathbb{R}^n} |\xi|^4|\hat{u}|^2\,d\xi\Big)^{\frac12}\Big(\int_{\mathbb{R}^n} |\hat{u}_t|^2 d\xi\Big)^{\frac12}\frac12[|\xi|^8|\hat{u}|^2+|\xi|^4|\hat{u}|^2]\\
&\le \max_{0\le s\le E(0)} |M'(s)|\Big(\int_{\mathbb{R}^n} (\Delta u)^2 dx\Big)^{\frac12}\Big(\int_{\mathbb{R}^n} u_t^2\,dx\Big)^{\frac12} G(\xi,t)\\
&\le \max_{0\le s\le E(0)} |M'(s)|\,E(0)\,G(\xi,t). \qquad (2.12)
\end{aligned}$$

Using Gronwall's inequality it follows from (2.12) that

$$G(\xi,t) \le G(\xi,0)\exp(CT) \qquad (2.13)$$

for any $0 \le t \le T < T_{\max}$ where $C = \max_{0\le s\le E(0)} |M'(s)|E(0)$. Integration of inequality (2.13) over $\mathbb{R}^n$ gives us the estimate

$$\begin{aligned}
&\frac12\int_{\mathbb{R}^n} |\xi|^4\left[|\hat{u}_t|^2+(1+|\xi|^4)|\hat{u}|^2\right]d\xi + M\Big(\int_{\mathbb{R}^n} |\xi|^2|\hat{u}|^2 d\xi\Big)\frac12\int_{\mathbb{R}^n} |\xi|^6|\hat{u}|^2 d\xi\\
&\le \frac12\Big\{||u_1||^2_{H^2} + 2||u_0||^2_{H^4} + ||u_0||^2_{H^3} M\Big(\int_{\mathbb{R}^n} |\nabla u_0|^2 dx\Big)\Big\}\exp(CT). \qquad (2.14)
\end{aligned}$$

Observe that (2.14) gives us a bound for u in H^4. In fact,

$$\begin{aligned}
||u||^2_{H^4} &= \int_{\mathbb{R}^n} \big(1+|\xi|^2\big)^4|\hat{u}|^2\,d\xi \le 8\int_{\mathbb{R}^n} \big(1+|\xi|^8\big)|\hat{u}|^2\,d\xi\\
&\le 8\int_{\mathbb{R}^n} |\hat{u}|^2\,d\xi + 16\,C_0\,\exp(CT) \le 8\,E(0) + 16\,C_0\,\exp(CT), \qquad (2.15)
\end{aligned}$$

where

$$C_0 = \frac12\Big\{||u_1||^2_{H^2} + 2||u_0||^2_{H^4} + ||u_0||^2_{H^3} M\Big(\int_{\mathbb{R}^n} |\nabla u_0|^2 dx\Big)\Big\}.$$

Similarly,

$$\begin{aligned}
||u_t||^2_{H^2} &= \int_{\mathbb{R}^n} \big(1+|\xi|^2\big)^2|\hat{u}_t|^2\,d\xi \le 2\int_{\mathbb{R}^n} \big(1+|\xi|^4\big)|\hat{u}_t|^2\,d\xi\\
&\le 2E(0) + 4C_0\,\exp(CT).
\end{aligned}$$

Finally, multiplying equation (1.1) by u_{tt}, integrating over $\mathbb{R}^n$ and using the above estimates we obtain for any $\delta > 0$

$$\int_{\mathbb{R}^2} u_{tt}^2 dx = -\int_{\mathbb{R}^n} u_{tt}\,\Delta^2 u\,dx - \int_{\mathbb{R}^n} u_{tt} u\,dx + M\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)\int_{\mathbb{R}^n} \Delta u\,u_{tt}\,dx$$

$$\le \frac{1}{2\delta}\int_{\mathbb{R}^n}(\Delta^2 u)^2 dx + \frac{1}{2\delta}\int_{\mathbb{R}^n} u^2\, dx + \delta\int_{\mathbb{R}^n} u_{tt}^2 dx$$
$$+\Big[\max_{0\le s\le E(0)} M(s)\Big]^2 \frac{1}{2\delta}\int_{\mathbb{R}^n}(\Delta u)^2 dx + \frac{\delta}{2}\int_{\mathbb{R}^n} u_{tt}^2 dx.$$

Hence,

$$\Big(1-\frac{3\delta}{2}\Big)\int_{\mathbb{R}^n} u_{tt}^2\, dx \le \frac{1}{2\delta}\Big[1+\begin{pmatrix}\max M(s)\\ 0\le s\le E(0)\end{pmatrix}^2\Big]E(0) + \frac{1}{2\delta}\,||u||_{H^4}^2;$$

taking $\delta < \frac{2}{3}$ and using (2.15) we obtain a bound for $||u_{tt}||_{L^2}$. This completes the proof of global existence. Uniqueness is standard and follows using Gronwall's inequality.

3. Local asymptotic stability

In this section we use Morawetz's multiplier to obtain estimates for solutions of equation (1.1). As we mentioned in Section 1, S. Levandosky and W. Strauss in [8] used such multipliers for nonlinear beam equations with a class of nonlinearities different than ours. Clearly, by density it is sufficient to do all calculations with smooth initial data with compact support.

A calculus-type lemma will be very useful in what follows.

Lemma 3.1. *Let $f \in C^1(\mathbb{R}^n)$, $F = (F_1, F_2, \dots, F_n) \in \big[C^1(\mathbb{R}^n)\big]^n$ and $p, q \ge 0$. Assume that*

a) $f \in L^1(\mathbb{R}^n)$, $\operatorname{div}\Big(\frac{F(x)}{|x|^q}\Big) \in L^1(\mathbb{R}^n)$,

b) $|F(x)\bullet x| \le |f(x)|\,|x|^p \quad \forall x \in \mathbb{R}^n$.

Then

$$\int_{\mathbb{R}^n}\operatorname{div}\Big(\frac{F(x)}{|x|^q}\Big)dx = 0$$

provided $-1 \le q-p < n-2$ with $n > 3$.

Proof. By a) we know that there exists a sequence $\{M_j\}_{j=1}^{+\infty}$ $M_j \to +\infty$ such that

$$\int_{|x|=M_j} |f(x)|\, d\Gamma \to 0 \quad \text{as} \quad M_j \to +\infty.$$

Here the integral $\int_{|x|=M} |f|\, d\Gamma$ means the surface integral of $|f|$ over the surface $\{x\in\mathbb{R}^n : |x| = M\}$. Using the divergence theorem and assumptions a) and b) we have

$$\Big|\int_{\mathbb{R}^n}\operatorname{div}\Big(\frac{F(x)}{|x|^q}\Big)dx\Big| = \Big|\lim_{\substack{\varepsilon\to 0^+\\ M_j\to+\infty}}\int_{\varepsilon\le|x|\le M_j}\operatorname{div}\Big(\frac{f(x)}{|x|^q}\Big)dx\Big|$$

$$= \Big| \lim_{M_j \to +\infty} \int_{|x|=M_j} \frac{F(x) \bullet x}{|x|^q \, M_j} \, d\Gamma - \lim_{\varepsilon \to 0^+} \int_{|x|=\varepsilon} \frac{F(x) \bullet x}{|x|^q \, \varepsilon} \, d\Gamma \Big|$$
$$\leq \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_{|x|=\varepsilon} \frac{|F(x) \bullet x|}{|x|^q} \, d\Gamma + \lim_{M_j \to +\infty} \frac{1}{M_j^{q-p+1}} \int_{|x|=M_j} |f(x)| d\Gamma$$
$$\leq \max_{|x| \leq 1} |f(x)| \lim_{\varepsilon \to 0^+} \varepsilon^{n-2+p-q} = 0,$$

which proves Lemma 3.1. □

Next, we want to use Morawetz's multiplier and with this purpose it is convenient to use the same notation as in [8]:

$$x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n, \quad |x| = r = \Big(\sum_{j=1}^n x_j^2 \Big)^{1/2},$$
$$u_r = \frac{x}{r} \bullet \nabla u, \quad Bu = \left(\frac{\partial}{\partial r} + \frac{(n-1)}{2r} \right) u = u_r + \frac{n-1}{2r} \, u$$
$$u_i = \frac{\partial u}{\partial x_i}, \quad u_{ii} = \frac{\partial^2 u}{\partial x_i^2}, \quad u_{iijj} = \frac{\partial^4 u}{\partial x_i^2 \partial x_j^2} \ \ (1 \leq i, j \leq n)$$
$$S_{ij} u = \frac{x_i}{r} \Big[(u_r)_j - \frac{x_j}{r} \, (u_r)_r \Big] + \Big(u_i - \frac{x_i}{r} \, u_r \Big)_j \tag{3.1}$$

and

$$Pu = \frac{2}{r} \Big\{ \sum_{i=1}^n \sum_{j=1}^n \big(S_{ij} u \big)^2 - \sum_{i=1}^n \Big(\sum_{j=1}^n \frac{x_j}{r} \, S_{ij} u \Big)^2 \Big\}. \tag{3.2}$$

Lemma 3.2. *Let $n \geq 5$, $\{u_0, u_1\} \in H^4(\mathbb{R}^n) \times H^2(\mathbb{R}^n)$ and $\{u, u_t\}$ be the global strong solution of* (1.1) *with initial data $\{u_0, u_1\}$. Let M be as in* (2.2). *If $n > 5$, then the identity*

$$\begin{aligned} 0 = {} & \frac{d}{dt} \Big\{ \int_{\mathbb{R}^n} u_t \, Bu \, dx \Big\} + \frac{1}{2} \, (n-1)(n-3) \int_{\mathbb{R}^n} \frac{u_r^2}{r^3} \, dx \\ & + \frac{1}{2} \, (n^2 + 2n - 19) \int_{\mathbb{R}^n} \frac{1}{r^3} \left\{ |\nabla u|^2 - u_r^2 \right\} dx + \int_{\mathbb{R}^n} Pu \, dx \\ & + \frac{3}{4} \, (n-1)(n-3)(n-5) \int_{\mathbb{R}^n} \frac{u^2}{r^5} \, dx \\ & + M \Big(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \Big) \int_{\mathbb{R}^n} \frac{1}{r} \ \left[|\nabla u|^2 - u_r^2 \right] dx \\ & + \frac{1}{4} \, M \Big(\int_{\mathbb{R}^n} |\nabla u|^2 \, dx \Big) (n-1)(n-3) \int_{\mathbb{R}^n} \frac{u^3}{r^3} \, dx \end{aligned} \tag{3.3}$$

holds. Furthermore, if $n = 5$ we have the identity

$$
\begin{aligned}
0 = {} & \frac{d}{dt}\left\{\int_{\mathbb{R}^5} u_t\, Bu\, dx\right\} + 4\int_{\mathbb{R}^5} \frac{u_r^2}{r^3}\, dx \\
& + 8\int_{\mathbb{R}^5} \frac{1}{r^3}\left[|\nabla u|^2 - u_r^2\right] dx + \int_{\mathbb{R}^5} Pu\, dx + \\
& + 16\pi^2[u(0,t)]^2 + M\left(\int_{\mathbb{R}^5} |\nabla u|^2 dx\right)\int_{\mathbb{R}^5} \frac{1}{r}\left[|\nabla u|^2 - u_r^2\right] dx + \\
& + 2M\left(\int_{\mathbb{R}^5} |\nabla u|^2\, dx\right)\int_{\mathbb{R}^5} \frac{u^2}{r^3}\, dx. \qquad (3.4)
\end{aligned}
$$

In (3.3) *and* (3.4), *Pu is given as in* (3.2) (*with $S_{ij}u$ as in* (3.1)).

Proof. As we mentioned above it is sufficient to do the calculations with solutions u of (1.1) which are smooth and small enough as $|x| \to +\infty$. If $n > 5$, multiplying the linear part of equation (1.1) by Bu we get

$$
(u_{tt} + u + \Delta^2 u)Bu = \frac{d}{dt}\{u_t Bu\} + \frac{1}{2}\,\mathrm{div}\left(\frac{x}{r}\,u^2 - \frac{x}{r}\,u_t^2\right) + \Delta^2 u\, Bu. \qquad (3.5)
$$

After long calculations, in [8] it was shown that

$$
\begin{aligned}
\Delta^2 u\, Bu = {} & \frac{1}{2}(n-1)(n-3)\frac{u_r^2}{r^3} + \frac{n^2+2n-19}{2r^3}\left[|\nabla u|^2 - u_r^2\right] \\
& + (n-1)(n-3)(n-5)\frac{3u^2}{4r^5} + Pu + \mathrm{div}\, Q, \qquad (3.6)
\end{aligned}
$$

where

$$
\begin{aligned}
Q = {} & -\frac{n-9}{2r^2}\, u_r \nabla u + Bu(\nabla u)_r - \frac{(n-1)}{2r^3}\, u\nabla u + \frac{3(n-1)}{2r^4}\, x u u_r \\
& - \frac{3}{4r^3}(n-1)(n-3)xu^2 + Bu\,\nabla(\Delta u) - \frac{1}{r}\nabla\left(|\nabla u|^2\right) \\
& - \frac{5}{2r^3}|\nabla u|^2 x - \Delta u\, B(\nabla u) + \frac{1}{r}\nabla u\,\Delta u + \frac{x}{r^2}\left(|\nabla u|^2\right)_r \\
& + \frac{x}{2r}(\Delta u)^2 - \frac{1}{r^2}\left[x \bullet (\nabla u)_r\right]\nabla u - \frac{2x}{r^3}\, u_r^2\,.
\end{aligned}
$$

Substitution of (3.6) into (3.5), integration over $\mathbb{R}^n$ and using Lemma 3.1 give us the first five terms on the right-hand side of (3.3). It remains only to consider the term $\int_{\mathbb{R}^n}\left\{-M\left(\int_{\mathbb{R}^n}|\nabla u|^2 dx\right)\Delta u\right\} Bu\, dx$. Clearly, it is sufficient to study $\int_{\mathbb{R}^n} \Delta u\, Bu\, dx$. Using the notation given before Lemma 3.2, straightforward calculations give us the identity

$$\begin{aligned}
\Delta u\, Bu &= \Delta u\Big(u_r + \frac{n-1}{2r}\, u\Big) \\
&= \frac{1}{2} \sum_{i,k=1}^{n} \Big[u_i^2 \Big(\frac{x_k}{r}\Big)_k\Big] - \sum_{i,k=1}^{n} \Big[u_i u_k \Big(\frac{x_k}{r}\Big)_i\Big] \\
&+ \frac{(n-1)}{4}\, u^2 \sum_{i=1}^{n} \Big(\frac{1}{r}\Big)_{ii} - \frac{(n-1)}{2r} \sum_{i=1}^{n} u_i^2 - \sum_{i,k=1}^{n} \Big(\frac{1}{2}\, u_i^2\, \frac{x_k}{r}\Big)_k \\
&+ \frac{(n-1)}{2} \sum_{i=1}^{n} \Big(\frac{1}{r}\, u u_i\Big)_i - \frac{(n-1)}{4} \sum_{i=1}^{n} \Big[\Big(\frac{1}{r}\Big)_i\, u^2\Big]_i \\
&+ \sum_{i,k=1}^{n} \Big(\frac{x_k}{r}\, u_i u_k\Big)_i = \frac{(n-1)}{2r}\, |\nabla u|^2 + \frac{u_r^2}{r} - \frac{1}{r}\, |\nabla u|^2 \\
&- \frac{(n-1)(n-3)}{4r^3}\, u^2 - \frac{(n-1)}{2r}\, |\nabla u|^2 \\
&+ \operatorname{div} \Big\{\nabla u\, Bu + \frac{x}{2r}\, \Big[\frac{(n-1)u^2}{2r^2} - |\nabla u|^2\Big]\Big\}. \qquad (3.7)
\end{aligned}$$

Now we claim that

$$\int_{\mathbb{R}^n} \operatorname{div} \Big\{\nabla u\, Bu + \frac{x}{2r}\, \Big[\frac{(n-1)u^2}{2r^2} - |\nabla u|^2\Big]\Big\}\, dx = 0. \qquad (3.8)$$

In order to prove (3.8) we use Lemma 3.1. Let us work for example with $[\frac{(n-1)u^2}{2r^2}]\frac{x}{2r}$ · All other terms are treated in a similar way. In this case we take $F(x) = \frac{(n-1)}{4}\, u^2 x$, $f(x) = \frac{(n-1)}{4}\, u^2$, $p = 2$ and $q = 3$. Clearly,

$$\int_{\mathbb{R}^n} |f(x)|\, dx = \frac{(n-1)}{4} \int_{\mathbb{R}^n} u^2\, dx \le \frac{(n-1)}{4}\, E(0) < +\infty$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^n} \Big| \operatorname{div} \Big(\frac{(n-1)u^2 x}{4|x|^3}\Big) \Big| dx = \int_{\mathbb{R}^n} \Big| \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \Big(\frac{(n-1)}{4r^3}\, u^2 x_i\Big) \Big| dx \\
&= \frac{(n-1)}{4} \int_{\mathbb{R}^n} \Big| \sum_{i=1}^{n} \Big\{\Big(\frac{1}{r^3} - \frac{3x_i^2}{r^5}\Big) u^2 + 2u u_i\, \frac{x_i}{r^3}\Big\} \Big| dx \\
&\le \frac{(n-1)}{4} \int_{\mathbb{R}^n} \Big\{(n-3)\, \frac{u^2}{r^3} + \frac{1}{r^2}\, (u^2 + |\nabla u|^2)\Big\}\, dx. \qquad (3.9)
\end{aligned}$$

Each one of the integrals on the right-hand side of (3.9) is finite. For example the integral $\int_{\mathbb{R}^n} \frac{u^2}{r^3}\,dx$ can be estimated as follows

$$\int_{\mathbb{R}^n} \frac{u^2}{r^3}\,dx = \int_{|x|\le 1} \frac{u^2}{r^3}\,dx + \int_{|x|\ge 1} \frac{u^2}{r^3}\,dx$$
$$\le \begin{pmatrix} \max u^2 \\ |x|\le 1 \end{pmatrix} \int_{|x|\le 1} \frac{dx}{r^3} + E(0) < +\infty$$

because $n > 5$. Similarly,

$$\int_{\mathbb{R}^n} \frac{1}{r^2}\,(u^2 + |\nabla u|^2)dx < +\infty.$$

Using Lemma 3.1 we conclude that (3.8) holds. Next, we integrate identity (3.7) to obtain

$$\int_{\mathbb{R}^n} \Delta u\, Bu\, dx = \int_{\mathbb{R}^n} \frac{1}{r}\,\left[u_r^2 - |\nabla u|^2\right] dx - \frac{(n-1)(n-3)}{4}\int_{\mathbb{R}^n} \frac{u^2}{r^3}\,dx. \quad (3.10)$$

Identity (3.10) together with our previous discussion on the linear part of (1.1) (see (3.5) and (3.6)) proves identity (3.3). As observed in [8] when $n = 5$ the term $(n-1)(n-3)(n-5)\frac{3u^2}{4r^5}$ has to be replaced by $u^2\Delta^2(\frac{1}{r}) = u^2\,16\pi^2\,\delta(x)$ and after integration in $\mathbb{R}^n$ we have the term $16\pi^2[u(0,t)]^2$ which appears in (3.4).

Lemma 3.3. *Let $n \ge 5$. Under the assumptions of Lemma 3.2 the following estimates*

$$\int_0^{+\infty}\int_{\mathbb{R}^n} M\Big(\int_{\mathbb{R}^n} |\nabla u|^2 dx\Big)\frac{u^2}{r^3}\,dxdt \le C_1\,E(0) \quad (3.11)$$

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \frac{u_r^2}{r^3}\,dxdt \le C_2\,E(0) \quad (3.12)$$

and

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \frac{1}{r^3}\,\left[|\nabla u|^2 - u_r^2\right] dxdt \le C_3\,E(0) \quad (3.13)$$

hold. Here the constants c_j $(1 \le j \le 3)$ can be chosen as $c_1 = \frac{4}{(n-1)(n-3)}$, $c_2 = \frac{1}{2}\,c_1$, $c_3 = \frac{2}{n^2+2n-19}$ and $E(t)$ is given by (2.6). Furthermore, if $n \ge 6$ we also have the estimate

$$\int_0^{+\infty}\int_{\mathbb{R}^n} \frac{u^2}{r^5}\,dxdt \le \frac{C_1}{3(n-5)}\,E(0). \quad (3.14)$$

Proof. Let $n \geq 5$. Clearly,

$$0 \leq (u_t \pm Bu)^2 = u_t^2 \pm 2u_t\,Bu + (Bu)^2$$

thus,

$$\begin{aligned} 2|u_t\,Bu| &\leq u_t^2 + (Bu)^2 = u_t^2 + u_r^2 + \frac{(n-1)}{r}\,uu_r + \frac{(n-1)^2}{4r^2}\,u^2 \\ &= u_t^2 + u_r^2 + \operatorname{div}\Big(\frac{(n-1)u^2x}{2r^2}\Big) - \frac{(n-1)(n-2)}{2r^2}\,u^2 \\ &+ \frac{(n-1)^2}{4r^2}\,u^2 \leq u_t^2 + u_r^2 + \operatorname{div}\Big(\frac{(n-1)}{2r^2}\,u^2x\Big) \end{aligned} \tag{3.15}$$

because $\frac{uu_r}{r} = \operatorname{div}\Big(\frac{u^2x}{2r^2}\Big) - \frac{(n-2)}{2r^2}\,u^2$ and $n \geq 5$. Integration over $\mathbb{R}^n$ of inequality (3.15) implies that

$$\Big|\int_{\mathbb{R}^n} u_t\,Bu\,dx\Big| \leq \frac{1}{2}\int_{\mathbb{R}^n}\big(u_t^2 + |\nabla u|^2\big)dx \leq \frac{1}{2}\,E(0) \tag{3.16}$$

because by Lemma 3.1 we have $\int_{\mathbb{R}^n} \operatorname{div}\Big(\frac{(n-1)}{2r^2}\,u^2x\Big)\,dx = 0$. Next, we integrate identity (3.3) (or (3.4)) over a time interval $[0,T]$. Since $Pu \geq 0$ (by Schwarz's inequality) we obtain the inequality

$$\begin{aligned} &\frac{1}{4}\,(n-1)(n-3)\int_0^T\int_{\mathbb{R}^n} M\Big(\int_{\mathbb{R}^n}|\nabla u|^2\,dx\Big)\,\frac{u^2}{r^3}\,dxdt \\ &+ \frac{1}{2}\,(n-1)(n-3)\int_0^T\int_{\mathbb{R}^n}\frac{u_r^2}{r^3}\,dxdt \\ &+ \frac{1}{2}\,(n^2+2n-19)\int_0^T\int_{\mathbb{R}^n}\frac{1}{r^3}\big[|\nabla u|^2 - u_r^2\big]dxdt \\ &\leq \Big|\int_{\mathbb{R}^n} u_t\,Bu\,dx\Big|\Big|_{t=0}^{t=T} \leq E(0) \end{aligned}$$

because of (3.16). Letting $T \to +\infty$ we conclude (3.11)-(3.13) if $n \geq 5$.
If $n \geq 6$ we also have from (3.3)

$$\frac{3}{4}\,(n-1)(n-3)(n-5)\int_0^T\int_{\mathbb{R}^n}\frac{u^2}{r^5}\,dxdt \leq E(0)$$

which proves (3.14).

Observation. If $n \geq 5$ as a consequence of (3.12) and (3.13) we deduce the validity of the estimate

$$\int_0^{+\infty} \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{r^3} \, dxdt \leq (c_2 + c_3) E(0). \tag{3.17}$$

Theorem 3.4. *Let u be the global solution of equation (1.1) obtained in Theorem 2.2 and let $\Omega \subseteq \mathbb{R}^n$ be a bounded region. Then, if $n \geq 6$ the solution u satisfies*

$$||u(\,\cdot\,, t)||_{H^1(\Omega)} \longrightarrow 0 \quad \textit{as} \quad t \to +\infty.$$

If $n = 5$ we also have that

$$||\nabla u(\,\cdot\,, t)||_{L^2(\Omega)} \longrightarrow 0 \quad \textit{as} \quad t \to +\infty.$$

Proof. Let us fix $\rho > 0$ such that $\Omega \subseteq B_\rho(0) = \{x \in \mathbb{R}^n : |x| < \rho\}$; if $n \geq 6$ Lemma 3.3 implies that for any $\delta > 0$ we have

$$\begin{aligned} \frac{1}{\rho^5} \int_0^{+\infty} \int_{\delta \leq |x| \leq \rho} u^2 \, dxdt &= \frac{1}{\rho^5} \int_0^{+\infty} \int_{\delta \leq |x| \leq \rho} |x|^5 \, \frac{u^2}{|x|^5} \, dxdt \\ \leq \int_0^{+\infty} \int_{\delta \leq |x| \leq \rho} \frac{u^2}{r^5} \, dxdt &\leq \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{u^2}{r^5} \, dxdt \leq \frac{c_1}{3(n-5)} \, E(0). \end{aligned}$$

Letting $\delta \to 0$ we deduce that

$$\int_0^{+\infty} \int_{|x| \leq \rho} u^2 \, dxdt \leq \frac{c_1}{3(n-5)} \, E(0) \rho^5 < +\infty. \tag{3.18}$$

Also

$$\Big| \frac{d}{dt} \int_{|x| \leq \rho} u^2 \, dx \Big| = 2 \Big| \int_{|x| \leq \rho} u u_t \, dx \Big| \leq 2 \, E(0) < +\infty. \tag{3.19}$$

Now, we use a calculus-type lemma "Let $f \in C^1(\mathbb{R}^+)$, $f(t) \geq 0$ such that $f \in L^1(\mathbb{R}^+)$ and $\left| \frac{d}{dt} f(t) \right|$ is bounded for all $t \geq 0$, then $f(t) \to 0$ as $t \to +\infty$." Thus, (3.18) and (3.19) imply that

$$\int_{|x| \leq \rho} u^2 \, dx \longrightarrow 0 \quad \text{as} \quad t \to +\infty.$$

Consequently,

$$\int_\Omega u^2 \, dx \longrightarrow 0 \quad \text{as} \quad t \to +\infty. \tag{3.20}$$

Now, let $n \geq 5$ and $\rho_1 > \rho > 0$. Choose $\delta > 0$ and use (3.17) to deduce that

$$\frac{1}{\rho_1^3} \int_0^{+\infty} \int_{\delta \leq |x| \leq \rho_1} |\nabla u|^2 \, dxdt \leq \int_0^{+\infty} \int_{\delta \leq |x| \leq \rho_1} \frac{|\nabla u|^2}{r^3} \, dxdt$$

$$\leq \int_0^{+\infty} \int_{\mathbb{R}^n} \frac{|\nabla u|^2}{r^3}\, dxdt \leq (c_2 + c_3)E(0).$$

Letting $\delta \to 0$ we deduce that for any $\rho_1 > \rho$ we have

$$\int_0^{+\infty} \int_{|x|\leq\rho_1} |\nabla u|^2\, dxdt \leq (c_2 + c_3)E(0)\rho_1^3. \tag{3.21}$$

Next, let r_1 and r_2 be such that $r_2 > r_1$ and $r_2 \geq \rho_1 \geq r_1 > \rho$ and consider the auxiliary function

$$G(t) = \int_{r_1}^{r_2} \Big(\int_{|x|\leq\rho_1} |\nabla u|^2\, dx\Big) d\rho_1\,. \tag{3.22}$$

We claim that $G(t) \to 0$ as $t \to +\infty$. In fact, since we are working with smooth functions we can interchange the order of integration and use (3.21) to deduce that

$$\begin{aligned} \int_0^{+\infty} G(t)dt &= \int_0^{+\infty} \int_{r_1}^{r_2} \Big(\int_{|x|\leq\rho_1} |\nabla u|^2\, dx\Big) d\rho_1\, dt \qquad (3.23)\\ &\leq \int_{r_1}^{r_2} (c_1 + c_2)E(0)\rho_1^3\, d\rho_1 = (c_2 + c_3)E(0)\, \frac{r_2^4 - r_1^4}{4} < +\infty. \end{aligned}$$

Now we prove that $\left|\frac{dG(t)}{dt}\right|$ is bounded. In fact, using Green's identity we obtain

$$\begin{aligned} &\Big|\frac{dG(t)}{dt}\Big| = 2\Big|\int_{r_1}^{r_2} \int_{|x|\leq\rho_1} \nabla u \bullet \nabla u_t\, dx\, d\rho_1\Big| \\ &= 2\Big|\int_{r_1}^{r_2} \Big\{\int_{|x|=\rho_1} u_t\, \nabla u \bullet \frac{x}{\rho_1}\, d\Gamma - \int_{|x|\leq\rho_1} u_t\, \Delta u\, dx\Big\} d\rho_1\Big| \\ &\leq 2\int_{r_1}^{r_2} \Big(\frac{1}{2}\int_{|x|\leq\rho_1} [u_t^2 + (\Delta u)^2] dx\Big) d\rho_1 + 2\int_{r_1}^{r_2} \int_{|x|=\rho_1} |u_t|\, |\nabla u|\, d\Gamma\, d\rho_1 \\ &\leq (r_2 - r_1)E(0) + 2\int_{r_1\leq|x|\leq r_2} |u_t|\, |\nabla u|\, dx \\ &\leq (r_2 - r_1)E(0) + E(0) < +\infty. \qquad (3.24) \end{aligned}$$

From (3.23) and (3.24) we deduce that $G(t) \to 0$ as $t \to +\infty$. Since $\rho_1 > \rho$, then

$$G(t) = \int_{r_1}^{r_2} \int_{|x|\leq\rho_1} |\nabla u|^2\, dx\, d\rho_1 \geq (r_2 - r_1) \int_{|x|\leq\rho} |\nabla u|^2\, dx.$$

Thus,

$$\int_{|x|\leq\rho} |\nabla u|^2 \, dx \leq \frac{G(t)}{r_2 - r_1} \longrightarrow 0 \quad \text{as} \quad t \to +\infty.$$

Consequently,

$$\int_{\Omega} |\nabla u|^2 \, dx \longrightarrow 0 \quad \text{as} \quad t \to +\infty, \tag{3.25}$$

Clearly, (3.20) and (3.25) complete the proof of Theorem 3.4.

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