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# RESOLVENT KERNEL ESTIMATES NEAR THRESHOLDS

Matania Ben-Artzi
Institute of Mathematics, Hebrew University
Jerusalem 91904, Israel

Yves Dermenjian and Anton Monsef
Université de Provence, CMI
39, Rue Joliot Curie, 13453 Marseille cedex 13, France

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**Abstract.** The paper deals with the spectral structure of the operator $H = -\nabla \cdot b\nabla$ in $\mathbb{R}^n$ where $b$ is a stratified matrix-valued function. Using a partial Fourier transform, it is represented as a direct integral of a family of ordinary differential operators $H_p$, $p \in \mathbb{R}^n$. Every operator $H_p$ has two thresholds and the kernels are studied in their (spectral) neighborhoods, uniformly in compact sets of $p$ . As in [3], such estimates lead to a limiting absorption principle for $H$. Furthermore, estimates of the resolvent of $H$ near the bottom of its spectrum ("low energy" estimates) are obtained.

## 1. Introduction

In [3], estimates of resolvent kernels near thresholds have been given. Such estimates are essential in the study of the acoustic propagator $H_0 = -c^2(y)\rho(y)\nabla.\frac{1}{\rho(y)}\nabla$ where $\nabla = \nabla_{x,y}$, $(x,y) \in \mathbb{R}^{n-1} \times \mathbb{R}$, $n \geq 2$, with $\rho \in C^1(\mathbb{R})$ and $c$ piecewise continuous.

In this paper, we get the same estimates for a more general operator

$$H = -\nabla \cdot b\nabla, \tag{1.1}$$

where $\nabla = \nabla_{x,y}$, $(x,y) \in \mathbb{R}^{n-1}\times\mathbb{R}$, $n \geq 2$, $b(x,y) = b(y)$, $b \in L^\infty(\mathbb{R}, M(n,n))$ ($M(n,n)$ is the algebra of complex square matrices of order $n$). We suppose that there exist positive constants $y_c, c_+, c_-$ and $c$, $c_+ < c_-$, such that

$$\begin{cases} b(y) = c_\pm^2 Id \text{ if } \pm y \geq y_c, \\ b(y) \text{ is a Hermitian matrix for } a.e.\ y, \\ \sum_{j,l} p_j\overline{p}_l b_{jl} \geq c|p|^2,\ \forall p \in \mathbb{C}^n,\ \text{for } a.e.\ y. \end{cases} \tag{1.2}$$

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We first reduce the study of $H$ to a study of a direct integral of operators $H_p$ acting in $L^2(\mathbb{R})$ (see [13], Chapter XIII). The positive selfadjoint operator $H$ acting in $L^2(\mathbb{R}^n)$ is associated with the continuous, coercive and Hermitian form $a : H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to \mathbb{C}$ defined by

$$a(u,v) = \sum_{j,l} \int_{\mathbb{R}^n} b_{jl}(y)\partial_j u \partial_l \overline{v} dx\, dy.$$

If we take the partial Fourier transform with respect to the $x$-coordinates ($\hat{u}(p,y) = \frac{1}{(2\pi)^{\frac{n-1}{2}}} \int_{\mathbb{R}^{n-1}} e^{-ip.x} u(x,y)dx$), we define a Hermitian form $\hat{a}$ on $\hat{V}$ such that $a(u,v) = \hat{a}(\hat{u},\hat{v})$ where

$$\begin{aligned}\hat{a}(\hat{u},\hat{v}) = &\int_{\mathbb{R}^n} b_{nn}(y)\partial_n \hat{u} \partial_n \overline{\hat{v}} dpdy - \int_{\mathbb{R}^n} (i \sum_{j<n} p_j b_{nj}) \overline{\hat{v}} \partial_n \hat{u} dpdy \\ &+ \int_{\mathbb{R}^n} (i \sum_{j<n} p_j b_{jn}) \hat{u} \partial_n \overline{\hat{v}} dpdy + \int_{\mathbb{R}^n} (\sum_{j,l<n} p_j p_l b_{jl}) \hat{u} \overline{\hat{v}} dpdy\end{aligned}$$

and

$$\hat{V} := \{u \in L^2(\mathbb{R}^n); \int [(1+|p|^2)|u(p,y)|^2 + |\frac{\partial u(p,y)}{\partial y}|^2] dydp < +\infty\}.$$

The form $\hat{a}$ defines a positive selfadjoint operator $\hat{H}$ in $L^2(\mathbb{R}^n; dpdy)$ that is unitarily equivalent to the operator $H$. As $L^2(\mathbb{R}^n)$ is unitarily equivalent to $\int_{\mathbb{R}^{n-1}}^{\oplus} L^2(\mathbb{R})dp$, we are led to define in this last space the following Hermitian form

$$\tilde{a}(\phi,\psi) = \int_{\mathbb{R}^{n-1}} a_p(\phi(p,.), \psi(p,.))dp$$

where

$$\begin{aligned}a_p(f,g) = &\int_{\mathbb{R}} b_{nn}(y) f' \overline{g}' dy - \int_{\mathbb{R}} (i \sum_{j<n} p_j b_{nj}) \overline{g} f' dy \\ &+ \int_{\mathbb{R}} (i \sum_{j<n} p_j b_{jn}) f \overline{g}' dy + \int_{\mathbb{R}} (\sum_{j,l<n} p_j p_l b_{jl}) f \overline{g} dy\end{aligned}$$

is a Hermitian, coercive form in $H^1(\mathbb{R})$ (satisfying $a_p(f,f) \geq \alpha[|p|^2\|f\|^2 + \|f'\|^2]$, where $\alpha > 0$) which defines a selfadjoint operator $H_p$,

$$H_p f = -(b_{nn}f')' - i \sum_{j<n} p_j b_{nj} f' - i \sum_{j<n} p_j (b_{jn} f)' + \sum_{j,l<n} p_j p_l b_{jl} f. \qquad (1.3)$$

The form $\tilde{a}$ defines, since $p \to (H_p+i)^{-1}$ is measurable, a selfadjoint operator $\tilde{H} = \int_{\mathbb{R}^{n-1}}^{\oplus} H_p dp$.

If

$$F : L^2(\mathbb{R}^n; dpdy) \to \int_{\mathbb{R}^{n-1}}^{\oplus} L^2(\mathbb{R}) dp$$

denotes $F(f)(p) = f(p,.)$, we see that

$$\hat{a}(\hat{u}, \hat{v}) = \int_{\mathbb{R}^{n-1}} a_p(F(\hat{u})(p), F(\hat{v})(p)) dp;$$

i.e., $\hat{a}(\hat{u}, \hat{v}) = \tilde{a}(F(\hat{u}), F(\hat{v}))$. In other words, the operators $\hat{H}$ and $\tilde{H}$ are unitarily equivalent and $\hat{H} \sim \int_{\mathbb{R}^{n-1}}^{\oplus} H_p dp$. Note in particular that $H_p$ is positive, with essential spectrum $[c_+^2|p|^2, \infty)$. All its possible eigenvalues are located in $[0, c_+^2|p|^2]$, where $|p|^2 = \sum_{j<n} p_j^2$.

Here we study the behavior of the resolvent kernel associated with $H_p$, especially near the points $\{c_\pm^2|p|^2\}$.

If we denote by $\Phi_p(\xi, \eta; z)$ the resolvent kernel associated with $(H_p - zI)^{-1}$, we obtain identical results to [3].

**Theorem 1.** *We fix $L > 0$, $a > 0$ and $0 < \beta < \pi$. For all $p$, we denote*

$$I_p = (c_+^2|p|^2, c_+^2|p|^2 + a)$$

*and*

$$\Omega_p^+ = \{z \in \mathbb{C} \quad ; z - c_+^2|p|^2 = re^{i\theta}, \quad 0 < r < a \quad 0 < \theta < \beta\}.$$

*Then, there exists a constant $C > 0$, depending only on $L$, $a$ and $\beta$ such that*

$$|\Phi_p(\xi, \eta; z)| \leq C\Big(|z - c_+^2|p|^2|^{-\frac{1}{2}} + |z - c_-^2|p|^2|^{-\frac{1}{2}}\Big) \tag{1.4}$$

*for $z \in \Omega_p^+$, $p \in [0, L]^{n-1}$, $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}$. Furthermore, the function $\Phi_p(\xi, \eta; z)$ can be extended continuously to $\mathbb{R} \times \mathbb{R} \times I_p$ and (1.4) is valid for $z \in I_p \setminus \{c_-^2|p|^2\}$.*

**Theorem 2.** *We fix $a > 0$ and $0 < l < L$ and let $I_p$ be as in Theorem 1 with $p \in [l, L]^{n-1}$. Then, there exists a constant $C > 0$, depending only on $l$, $L$ and $a$ such that, for $\lambda \in I_p \setminus \{c_-^2 \sum_{j<n} p_j^2\}$,*

$$\Big|\frac{\partial}{\partial \lambda} \Phi_p(\xi, \eta; k)\Big| \leq C\Big[|\lambda - c_+^2|p|^2|^{-\frac{3}{2}} + |\lambda - c_-^2|p|^2|^{-\frac{3}{2}}\Big](1 + |\xi| + |\eta|). \tag{1.5}$$

Since this work is close to [3], we refrain from giving detailed proofs for parts that are similar to [3] and will elaborate on points which are rather different from that paper, as for example Proposition 8. In particular, Theorems 1,2 lead to a "limiting absorption principle" for the operator $H$ which we describe next.

For $\sigma \in \mathbb{R}$ we define the weighted space $L^{2,\sigma}(\mathbb{R}^n)$ by its norm

$$\|f\|^2_{L^{2,\sigma}(\mathbb{R}^n)} = \int_{\mathbb{R}^n} (1+|x|^2)^\sigma |f(x)|^2 dx.$$

For $z \in \mathbb{C} \setminus \mathbb{R}$ let $R(z) = (H-z)^{-1}$ be the resolvent operator. We have the following theorem.

**Theorem 3.** *The limits*

$$R^\pm(\lambda) = \lim_{\varepsilon \to 0+} R(\lambda \pm i\varepsilon), \qquad \lambda \in \mathbb{R} \setminus \{0\}, \tag{1.6}$$

*exist in the uniform operator topology of $B(L^{2,\sigma}(\mathbb{R}^n), L^{2,-\sigma}(\mathbb{R}^n))$, for every $\sigma > 1$. (We denote by $B(X,Y)$ the space of linear bounded operators from $X$ to $Y$.) Fix $0 < \tau < \pi$ and let $\Lambda_\tau = \{z \quad ; 0 \le \arg z \le \tau\}$. Then we have the following estimates for the resolvent.*

- *For any $n \ge 2$*

$$\|R(z)\|_{B(L^{2,\sigma}(\mathbb{R}^n), L^{2,-\sigma}(\mathbb{R}^n))} = O(log|z|), \qquad as \quad z \to 0 \quad in \quad \Lambda_\tau, \quad \sigma > 1.$$

- *For any $n \ge 4$*

$$\|R(z)\|_{B(L^{2,\sigma}(\mathbb{R}^n), L^{2,-\sigma}(\mathbb{R}^n))} = O(1), \qquad as \quad z \to 0 \quad in \quad \Lambda_\tau, \quad \sigma > \frac{3}{2}.$$

*Indeed, in this case $R(z)$ can be extended continuously to $\overline{\mathbb{C}^\pm}$ in the operator topology.*

*Analogous estimates hold in the lower half-plane.*

The "limiting absorption principle" (LAP), as expressed in the last theorem, is a basic tool in the study of spectral properties (and eigenfunction expansions) of a variety of families of self-adjoint operators of mathematical physics. Our estimates of the "fiber kernels" imply the LAP for the operator $H$ in a straightforward manner. Since this derivation is identical to that applied to the acoustic operator $H_0$ (see [3, Appendix and Corollary in Section 1]), we shall not repeat it here. In addition, note that we obtain estimates for the resolvent kernel of $H$ near the bottom of the continuous spectrum, the so-called "low energy estimates". We refer to [8, 9] for a derivation of such estimates by Kuroda's method and to [4, 7, 10] , where the LAP for the acoustic propagator $H_0$ is obtained by the Mourre commutator method. In these cited papers the reader may find some additional background material concerning the LAP. Also, in [8, 9] these estimates are used in order to study the long-time behavior of the related wave equation, as well as dissipative systems.

When the matrix $b(y)$ is smooth we can differentiate in (1.1) so that we obtain, in the strip $-y_c \le y \le y_c$, the operator $-\sum_{i,j=1}^{n} b_{ij}(y)\partial_i\partial_j$, perturbed by a first-order ("drift") operator with coefficients that do not decay in the $x$-directions. We refer to [11, 12] where the LAP is derived (away from the bottom of the essential spectrum) for a Helmholtz operator with potential (zero-order) perturbation which does not decay at infinity.

We note that there are very few works in the existing literature concerning the spectral properties of multi-dimensional operators of the form (1.1). In the case $n = 1$ (namely, no $x$ coordinates) we refer to [1, 3, 5, 6] and references therein. However, in our case a major difficulty is the need to estimate uniformly a family of operators of the form (1.3). On the other hand, it is not clear how to extend the other approaches mentioned above (for the acoustic propagator) to the case at hand. The detailed study of the "fibers" $H_p$ allows us also to conclude that the resolvent kernel of $H$ can be analytically extended to a certain sector of the lower half-plane (in the spectral parameter). In addition, its associated spectral measure is (weakly) differentiable, with a derivative which is Hölder-continuous in the operator norm of the weighted $L^2$ spaces that are used in the formulation of the LAP.

The paper is organized as follows. In the next section we establish several propositions which, as has already been mentioned, allow us to apply the method of [3]. Then in Section 3 we give a brief survey of the proofs of Theorems 1, 2. As mentioned above, the proof of Theorem 3 is based on these theorems and is identical to the proof of the analogous LAP theorem in the Appendix of [3].

## 2. Some basic propositions

The equation $H_p f = zf$ can be written in the following form

$$-(b_{nn}f')' - (i\sum_{j<n} p_j b_{nj})f' - i\sum_{j<n} p_j(b_{jn}f)' + (\sum_{j,l<n} p_j p_l b_{jl})f = zf, \quad (2.1)$$

where, for $\pm y \ge y_c$, we have $b_{ii} = c_\pm^2$ and $b_{ij} = 0$ for all $i \ne j$. Following [3], we suppose, without loss of generality, $c_+ < c_-$. We denote $\mathbb{C}^+ = \{w \in \mathbb{C}; \mathrm{Im}\ w > 0\}$ and introduce $k = (zc_+^{-2} - |p|^2)^{\frac{1}{2}}$. We shall always take $\mathrm{Im}(z^{\frac{1}{2}}) \ge 0$ for $z \in \mathbb{C}$ and $z^{\frac{1}{2}} \ge 0$ if $z \ge 0$. The Jost functions $\varphi$, $\psi$ which are solutions of $(H_p - z)f = 0$ are given by their asymptotic behavior for large $|y|$, where $\varphi, \psi \in L^2_{loc}(\mathbb{R})$,

$$\begin{aligned}\varphi(y;k,p) &= \exp(iky),\ y \geq y_c,\\ \psi(y;k,p) &= \exp[-i((k^2+|p|^2)c_+^2c_-^{-2} - |p|^2)^{\frac{1}{2}}y],\ y \leq -y_c.\end{aligned} \tag{2.2}$$

Denoting $\tilde{k} = \tilde{k}(p) = (c_-^2c_+^{-2} - 1)^{\frac{1}{2}}|p|$, the function $\psi$ can also be written as

$$\psi(y;k,p) = \exp(-ic_+c_-^{-1}(k^2 - \tilde{k}^2)^{\frac{1}{2}}y),\ y \leq -y_c. \tag{2.3}$$

In order to preserve continuity in $k$, in the upper half-plane, we make an exception to the above rule concerning roots of positive real numbers and we take $\arg[(k^2-\tilde{k}^2)^{\frac{1}{2}}] = \pi$ if $k < -\tilde{k}$.

**Definition 1.** *The Wronskian of two solutions $u(y), v(y)$ of $H_pf = zf$, is defined by $[u,v](y) = b_{nn}(y)(u'v - uv')(y)$.*

**Proposition 1.** *The Green's function for equation* (2.1) *is given, for $k \in \mathbb{C}^+$, by*

$$\Phi_p(\xi,\eta;k) = \begin{cases} \dfrac{\varphi(\xi;k,p)\psi(\eta;k,p)}{[\varphi,\psi](\eta;k,p)} & \text{if} \quad \xi \geq \eta, \\ \dfrac{\varphi(\eta;k,p)\psi(\xi;k,p)}{[\varphi,\psi](\eta;k,p)} & \text{if} \quad \xi \leq \eta. \end{cases} \tag{2.4}$$

**Remarks.** (a) We use the same notation as in [3]. Let us note that the function $\Phi_p(\xi,\eta;k)$ is the kernel of the resolvent $-(H_p - zI)^{-1}$.

(b) We have introduced the Wronskian as in [3], but now the Wronskian is only a continuous function of $y$ not necessarily constant as is usual for a Sturm-Liouville operator with real coefficients. See Proposition 3 for details.

(c) We remark that we can write the Wronskian (as a function of $y$) in the form

$$[u,v] = (b_{nn}u' + i\sum_{j<n} p_jb_{jn}u)v - (b_{nn}v' + i\sum_{j<n} p_jb_{jn}v)u.$$

**Proposition 2.** (*i*) *The functions $\varphi$, $\psi$, $b_{nn}\varphi' + i\sum_{j<n} p_jb_{jn}\varphi$ and $b_{nn}\psi' + i\sum_{j<n} p_jb_{jn}\psi$ are continuous in $\mathbb{R}\times\overline{\mathbb{C}^+}\times[0,L]^{n-1}$, belong* (*as functions of $y$*) *to $W^{1,\infty}_{loc}(\mathbb{R})$, and for any fixed $(y,p)$ in $\mathbb{R}\times[0,L]^{n-1}$, are analytic in $k \in \mathbb{C}^+$.*

(*ii*) *For any fixed $(y,k) \in \mathbb{R}\times\overline{\mathbb{C}^+}$, the functions $\varphi$ and $b_{nn}\varphi' + i\sum_{j<n} p_jb_{jn}\varphi$ are analytic in $p$.*

**Note:** The functions $\varphi', \psi'$ in Claim 2.1 of [3] are replaced by $b_{nn}\varphi' + i\sum_{j<n} p_jb_{jn}\varphi$ and $b_{nn}\psi' + i\sum_{j<n} p_jb_{jn}\psi$. So, the usual Cauchy data $(\varphi(x_0), \varphi'(x_0))$ are replaced by $(\varphi(x_0),\ (b_{nn}\varphi' + i\sum_{j<n} p_jb_{jn}\varphi)(x_0))$. Also note that the functions $\varphi', \psi'$ are in $L^2_{loc}(\mathbb{R})$.

While the form of $H_p$ is not completely standard, the proof can be carried out using standard techniques, such as expressing the functions $\varphi, \psi$ as solutions of the corresponding integral equation. We therefore omit the details.

**Proposition 3.** *For all $u, v \in W^{1,\infty}_{loc}(\mathbb{R})$ solutions of* (2.1), *the Wronskian $\eta \to [u,v](\eta; k, p)$ (where $k = (zc_+^{-2} - |p|^2)^{\frac{1}{2}}$) is continuous in $\eta$ and*

$$[u,v](\eta; k,p) = [u,v](y_c; k,p)\exp(-2i\int_{y_c}^{\eta} \frac{1}{b_{nn}}\mathrm{Re}(\sum_{j<n} p_j b_{nj})dt).$$

*Furthermore, the Wronskian of the Jost functions $\varphi, \psi$ (defined in* (2.2)), *is continuous in $(y,k,p) \in \mathbb{R} \times \overline{\mathbb{C}^+} \times [0,L]^{n-1}$.*

**Proof.** The function $\eta \to [u,v](\eta; k,p)$ belongs to $L^{\infty}_{loc}(\mathbb{R})$. Let us take its weak derivative. For each $\theta \in C_0^{\infty}(\mathbb{R})$, we write

$$< [u,v]', \theta >= - < [u,v], \theta' > .$$

So we have

$$< [u,v], \theta' >= \int \Big\{(b_{nn}u' + i\sum_{j<n} p_j b_{jn} u)v\theta' - (b_{nn}v' + i\sum_{j<n} p_j b_{jn} v)u\theta'\Big\}dt$$

and we can write, since $u, v \in W^{1,\infty}_{loc}(\mathbb{R})$,

$$\begin{aligned} < [u,v], \theta' >= \int \Big\{&(b_{nn}u' + i\sum_{j<n} p_j b_{jn} u)(v\theta)' - (b_{nn}u' + i\sum_{j<n} p_j b_{jn} u)v'\theta \\ &-(b_{nn}v' + i\sum_{j<n} p_j b_{jn} v)(u\theta)' + (b_{nn}v' + i\sum_{j<n} p_j b_{jn} v)u'\theta\Big\}dt. \end{aligned}$$

Using Proposition 2 and (2.1), this yields

$$< [u,v]', \theta >= \int \Big\{(i\sum_{j<n} p_j b_{nj} + i\sum_{j<n} p_j b_{jn})(v'u - u'v)\Big\}\theta dt$$

or, in other words,

$$[u,v]' = -\frac{1}{b_{nn}}(i\sum_{j<n} p_j b_{nj} + i\sum_{j<n} p_j b_{jn})[u,v].$$

Since the right-hand side is in $L^{\infty}_{loc}(\mathbb{R})$, we see that $[u,v] \in W^{1,\infty}_{loc}(\mathbb{R})$ and is a continuous function. Similarly, we use Proposition 2 to show that the Wronskian $[\varphi, \psi]$ is continuous in $(y,k,p) \in \mathbb{R} \times \overline{\mathbb{C}^+} \times [0,L]^{n-1}$. $\square$

**Proposition 4.** *The Jost functions $\varphi$, $\psi$ are linearly independent (hence $[\varphi,\psi](\eta;k,p)\neq 0$, for all $\eta\in\mathbb{R}$) for every $(k,p)\in\overline{\mathbb{C}^+}\times[0,L]^{n-1}$, $\mathrm{Re}(k)\neq 0$.*

**Proof.** See the proof of Claim 2.2 in [3] for $k\notin\mathbb{R}\setminus\{0\}$.

Now, we consider $k\in\mathbb{R}$ and $k\neq 0$. First, we recall that if $k<-\tilde{k}$, we have $\arg(k^2-\tilde{k}^2)^{\frac{1}{2}}=\pi$. Then, if $k$ is real, we have,

$$\begin{aligned} [\varphi,\overline{\varphi}](\eta;k,p) &= 2ikc_+^2,\ \eta\geq y_c,\\ [\psi,\overline{\psi}](\eta;k,p) &= \begin{cases} 0, & |k|\leq\tilde{k},\ \eta\leq -y_c,\\ -2ic_+c_-(k^2-\tilde{k}^2)^{\frac{1}{2}}, & |k|>\tilde{k},\ \eta\leq -y_c.\end{cases}\end{aligned}\tag{2.5}$$

In [3], it was possible to compare directly $[\varphi,\overline{\varphi}](\eta;k,p)$ with $[\psi,\overline{\psi}](\eta;k,p)$ to get a contradiction since the Wronskian $[\varphi,\overline{\varphi}](\eta;k,p)$ is a function independent of $\eta$. But in our case, the Wronskian is not a constant function (see Proposition 3) and, moreover, if we have $Hf=zf$ then, for real $z$, we get $\overline{H}\ \overline{f}=z\overline{f}$ where $\overline{H}\neq H$.

To get around this difficulty, we define the following function

$$W(\varphi,\overline{\varphi})=(b_{nn}\varphi'+i\sum_{j<n}p_jb_{jn}\varphi)\overline{\varphi}-(b_{nn}\overline{\varphi}'-i\sum_{j<n}p_jb_{nj}\overline{\varphi})\varphi,\tag{2.6}$$

where $\varphi$ and $\overline{\varphi}$ are solutions, respectively, of $H\varphi=z\varphi$ and $\overline{H}\overline{\varphi}=z\overline{\varphi}$ since $z$ is real. Since $b_{nn}\varphi'+i\sum_{j<n}p_jb_{jn}\varphi$ and $b_{nn}\overline{\varphi}'-i\sum_{j<n}p_jb_{nj}\overline{\varphi}$ are in $W^{1,\infty}_{loc}(\mathbb{R})$ (see Proposition 2), we get

$$\begin{aligned}(W(\varphi,\overline{\varphi}))' &= (b_{nn}\varphi'+i\sum_{j<n}p_jb_{jn}\varphi)'\overline{\varphi}+(b_{nn}\varphi'+i\sum_{j<n}p_jb_{jn}\varphi)\overline{\varphi}'\\ &\quad -(b_{nn}\overline{\varphi}'-i\sum_{j<n}p_jb_{nj}\overline{\varphi})'\varphi-(b_{nn}\overline{\varphi}'-i\sum_{j<n}p_jb_{nj}\overline{\varphi})\varphi'.\end{aligned}$$

Using (2.1), we get

$$(W(\varphi,\overline{\varphi}))'=(\sum_{j,l<n}p_jp_lb_{jl}-\sum_{j,l<n}p_jp_lb_{lj})\varphi\overline{\varphi}=0.$$

Thus, $W(\varphi,\overline{\varphi})(\eta;k,p)=K(k,p)$. For $y_\epsilon=y_c+\epsilon$, $\epsilon>0$, we have

$$W(\varphi,\overline{\varphi})(y_\epsilon;k,p)=[\varphi,\overline{\varphi}](y_\epsilon;k,p),$$

and, in the same way,

$$W(\psi,\overline{\psi})(-y_\epsilon;k,p)=[\psi,\overline{\psi}](-y_\epsilon;k,p).$$

If we have $\varphi=\alpha\psi$ with $\alpha\in\mathbb{C}$, we get $W(\varphi,\overline{\varphi})(\eta;k,p)=|\alpha|^2W(\psi,\overline{\psi})(\eta;k,p)$. In view of (2.5), we get a contradiction since $k\neq 0$ and $\alpha\neq 0$. $\square$

**Proposition 5.** *Let $\Gamma \subset \overline{\mathbb{C}^+} \setminus \{0\}$ be a compact set such that $\pm\tilde{k}(p) \notin \Gamma$ for all $p \in [0, L]^{n-1}$, then we have*

$$\sup_{\substack{-\infty<\eta\leq\xi<\infty \\ k\in\Gamma, p\in[0,L]^{n-1}}} |\varphi(\xi; k, p)\psi(\eta; k, p)| < \infty. \tag{2.7}$$

**Proof.** In view of (2.2) and (2.3), the boundedness in (2.7) is clear if $\eta \leq y_c$, $\xi \geq -y_c$. For $y_c \leq \eta \leq \xi$, we consider a solution $\varphi_1(\eta; k, p)$ of (2.1) such that $\varphi_1(\eta; k, p) = \exp(-ik\eta)$ for $\eta \geq y_c$. Then $\varphi$ and $\varphi_1$ are linearly independent in $k \in \Gamma, p \in [0, L]^{n-1}$ and we have $\psi(\eta; k, p) = \alpha(k, p)\varphi(\eta; k, p) + \beta(k, p)\varphi_1(\eta; k, p)$. In view of Proposition 3 the functions $\beta(k, p)$ and $\alpha(k, p)$ are continuous in $(k, p) \in \Gamma \times [0, L]^{n-1}$. The boundedness for $y_c \leq \eta \leq \xi$ now follows from the boundedness of the product $\varphi_1(\eta; k, p)\varphi(\xi; k, p)$. For $\eta \leq \xi \leq -y_c$, it follows in exactly the same way since $k^2 - \tilde{k}^2$ does not vanish for $(k, p) \in \Gamma \times [0, L]^{n-1}$. □

**Proposition 6.** *For any fixed $p \in [0, L]^{n-1}$, the Jost functions $\varphi, \psi$ as well as $b_{nn}\varphi' + i\sum_{j<n} p_j b_{jn}\varphi$, $b_{nn}\psi' + i\sum_{j<n} p_j b_{jn}\psi$ can be extended analytically to the disk $\{k; |k| < \tilde{k}(p)\}$. In particular, the Wronskian $[\varphi, \psi](\eta; k, p)$ is also analytic in this disk. Furthermore, if $p = 0$, these functions can be extended analytically to all of $\mathbb{C}$.*

**Proof.** In view of Equations (2.2),(2.3) the analyticity of the functions related to $\varphi$ in the disk $\{k; |k| < \tilde{k}(p)\}$ is clear for $y > y_c$ while the functions related to $\psi$ are analytic for $y < -y_c$. Then, the analyticity for all other values of $y$ follows by applying standard theorems to Equation (2.1), since $z$ is analytic in $k$ in the disk. The analyticity of the Wronskian follows from Remark (c) preceding Proposition 2. □

**Proposition 7.** *Fix $\delta > 0$. Then there exists a positive constant $\gamma > 0$ such that, for all $\eta \in \mathbb{R}$,*

$$|[\varphi, \psi](\eta; k, p)| \geq \gamma|k|, \;\; k \in [-\delta, \delta], \;\; p \in [0, L]^{n-1}. \tag{2.8}$$

**Proof.** We write, for real $k \neq 0$,

$$\psi(\eta; k, p) = \alpha(k, p)\varphi(\eta; k, p) + \beta(k, p)\varphi_1(\eta; k, p), \tag{2.9}$$

where $\alpha$ and $\beta$ are continuous functions on $([-\delta, \delta] \setminus \{0\}) \times [0, L]^{n-1}$ and $\varphi_1(\eta; k, p)$ is defined in the proof of Proposition 5. We will use the function $W(\varphi, \overline{\varphi})$ defined by (2.6). We know that

$$W(\psi, \overline{\psi})(\eta; k, p) = -2ic_+c_-(k^2 - \tilde{k}^2)^{\frac{1}{2}}.(1 - \chi_k), \;\; \eta \leq -y_c,$$

where $\chi_k = 1$ if $k \in [-\tilde{k}, \tilde{k}]$ and 0 otherwise (in [3], a misprint gives $(k^2 - \tilde{k}^2)$ instead of $(k^2 - \tilde{k}^2)^{\frac{1}{2}}$). Furthermore, we have

$$\begin{aligned} W(\psi, \overline{\psi}) &= |\alpha|^2 W(\varphi, \overline{\varphi}) + \alpha(k,p)\overline{\beta}(k,p) W(\varphi, \overline{\varphi}_1) \\ &\quad + \beta(k,p)\overline{\alpha}(k,p) W(\varphi_1, \overline{\varphi}) + |\beta|^2 W(\varphi_1, \overline{\varphi}_1). \end{aligned}$$

Since $z$ is real, the functions $\overline{\varphi}$ and $\overline{\varphi}_1$ are solutions of $\overline{H}\ \overline{f} = z\overline{f}$; so we get, for $\eta \geq y_c$, $W(\varphi, \overline{\varphi}_1) = [\varphi, \overline{\varphi}_1](\eta) = [\varphi, \varphi](\eta) = 0$ and, in the same way, $W(\varphi_1, \overline{\varphi}) = 0$. For $\eta \geq y_c$, we have $W(\varphi_1, \overline{\varphi}_1)(\eta) = [\varphi_1, \overline{\varphi}_1](\eta) = [\overline{\varphi}, \varphi](\eta)$ which gives $W(\varphi_1, \overline{\varphi}_1)(\eta) = -[\varphi, \overline{\varphi}](\eta) = -W(\varphi, \overline{\varphi})(\eta)$. As $[\varphi, \overline{\varphi}] = 2ikc_+^2$, for $\eta \geq y_c$ (see (2.5)) we get the following equality

$$-2ic_+c_-(k^2 - \tilde{k}^2)^{\frac{1}{2}}.(1 - \chi_k) = 2ikc_+^2(|\alpha|^2 - |\beta|^2)$$

which yields

$$|\alpha(k,p)| \leq |\beta(k,p)|,\ \forall k \in [-\delta, \delta] \setminus \{0\},\ \forall p \in [0, L]^{n-1}. \tag{2.10}$$

Note that, by Proposition 3,

$$|[\varphi, \psi](\eta; k, p)| = |[\varphi, \psi](y_c; k, p)| = 2|k|c_+^2|\beta(k,p)|. \tag{2.11}$$

So, it is sufficient to prove (2.8) for $\eta = y_c$. Suppose that there exists a sequence $\{k_m\}$ in $[-\delta, \delta] \backslash \{0\}$ and $\{p_m\}$ in $[0, L]^{n-1}$ such that $\beta(k_m, p_m) \to 0$. Necessarily $\alpha(k_m, p_m) \to 0$. Since the sequences $k_m$ and $p_m$ are bounded, we have, passing to a subsequence (indexed again by $m$), $k_m \to K$ and $p_m \to q$ with $K \in [-\delta, \delta]$. We get, by continuity, $\psi(\eta; K, q) = 0$ for all $\eta \geq y_c$ which implies that the function $\psi$ vanishes identically, which contradicts (2.3). Thus,

$$|\beta(k,p)| > 0,\ \forall k \in [-\delta, \delta] \setminus \{0\},\ \forall p \in [0, L]^{n-1},$$

which concludes the proof in view of (2.11). □

Combining this result with Propositions 3, 4, 6 we obtain the following corollary.

**Corollary 1.** *Define the function* $\Upsilon_p : k \mapsto [\varphi, \psi](0; k, p)$*, for* $\{(k,p) \in (\mathbb{C}^+ \cup \{|k| < \tilde{k}(p)\}) \times (0, L]^{n-1}\}$*. Then it is continuous on its domain and analytic in* $k$ *for each fixed* $p$*. Furthermore, for every* $0 < l < L$*, there exists a constant* $d_{l,L}$ *such that*

$$|\frac{d}{dk}\Upsilon_p(k)| \geq \frac{\gamma}{2}, \qquad \textit{for } |k| < d_{l,L}, \quad \textit{uniformly in } p \in [l, L]^{n-1}. \tag{2.12}$$

We note that in view of Propositions 3, 4 the zeros of the function $[\varphi, \psi](\eta; k, p)$ are identical to the zeros of $\Upsilon_p(k)$, are independent of $\eta$ and, for $\mathrm{Im}\,(k) \geq 0$, satisfy $\mathrm{Re}\,(k) = 0$. By the above corollary there is at most a finite number of such zeros and we denote them by $k = i\zeta_j$

$$|p| \geq \zeta_1(p) \geq \cdots \geq \zeta_J(p) \geq 0, \tag{2.13}$$

where $J = J(p)$ is the total number, including multiplicity. We set $J = 0$ if there are no zeros.

**Proposition 8.** *There exists a constant $M > 0$ such that $J(p) \leq M$ for all $p \in [0, L]^{n-1}$.*

**Proof.** The proof is carried out in two steps. First we prove the result for the case $n = 2$ by a different argument than that of [3]. Then, we do the general case.

**First step: case $n = 2$.** We introduce a function $p \mapsto \hat{J}(p)$ of integer values, continuous from the left, non-decreasing, and such that $J(p) \leq \hat{J}(p)$ for all $p \in (0, L]$.

For each $p \in (0, L]$, we choose a real number $r(p)$ and construct a domain $\Theta_p \subset \mathbb{C}$, consisting of two half-disks and a rectangle:

$$\begin{aligned}\Theta_p := \{z \in \mathbb{C}; |z| < r(p),\ \mathrm{Im}(z) \leq 0\} \cup \{z \in \mathbb{C}; |z - ip| < r(p),\ \mathrm{Im}(z) \geq p\} \\ \cup \{z \in \mathbb{C}; |\mathrm{Re}(z)| < r(p),\ 0 \leq \mathrm{Im}(z) \leq p\}.\end{aligned}$$

We denote $\gamma_p = \partial\Theta_p$. Let us fix $\beta_L,\ 0 < \beta_L < 1$, such that $\Upsilon_p$ does not vanish on $\gamma_L$ corresponding to $r(L) = (1+\beta_L)\min(\frac{L}{2}, \frac{\tilde{k}(L)}{2})$ which is possible by analyticity of $\Upsilon_L$.

From $p = L$, we let $p$ decrease. Let $r(p) = (1+\beta_L)\min(\frac{p}{2}, \frac{\tilde{k}(p)}{2})$, we stop at $\hat{p}$ when $\Upsilon_{\hat{p}}$ vanishes on $\gamma_{\hat{p}}$. By the above corollary there exists $\hat{\beta},\ 0 < \hat{\beta} < \beta_L$, such that $r(\hat{p}) = (1+\hat{\beta})\min(\frac{\hat{p}}{2}, \frac{\tilde{k}(\hat{p})}{2})$ and $\Upsilon_{\hat{p}}$ does not vanish on $\gamma_{\hat{p}}$ associated to the new value of $r(\hat{p})$. It is clear that $\Theta_{p_1} \subset \Theta_{p_2}$ if $p_1 \leq p_2$. Observe that in view of the estimate (2.12) such a point $\hat{p}$ is necessarily isolated, so that we can indeed continue the procedure down to $p = 0$.

Now, let

$$\hat{J}(p) := \frac{1}{2i\pi}\int_{\gamma_p} \frac{\Upsilon_p'(k)}{\Upsilon_p(k)} dk,$$

a function which gives the number of zeros of $\Upsilon_p$ included in $\Theta_p$. This function $\hat{J}$ is well defined on $(0, L]$, it has integer values, satisfying $J(p) \leq \hat{J}(p)$ and, by construction, it is clear that $\hat{J}(p_1) \leq \hat{J}(p_2)$ if $p_1 \leq p_2$.

**Second step: the general case.** Now, we are in the hypercube $Q := [0, L]^{n-1} \setminus \{0\}$ and suppose the function $p \mapsto J(p)$ is not bounded in $Q$. So, there exists a sequence $(p^l)_l \subset Q$ such that $J(p^l) \to +\infty$ if $l \to +\infty$. For a sub-sequence, keeping the same notation, we can suppose $p^l \to \bar{p}$. Suppose $\bar{p} \neq 0$. We construct, as in the first step, a domain $\Theta_{\bar{p}}$ which contains all roots of the form $i\zeta_j(\bar{p})$, $0 < j \leq J(\bar{p})$. Let $\hat{J}(\bar{p})$ be the number of roots of $\Upsilon_{\bar{p}}$ in $\Theta_{\bar{p}}$. Due to Corollary 1 there exists a neighborhood of $\bar{p}$ in $Q$ such that $\hat{J}(p)$ is constant in this neighborhood, in contradiction to the assumption $J(p^l) \to +\infty$. So, we need to treat the case $\bar{p} = 0$. We can suppose that the sequence $(|p^l|)_l$ tends to zero in a non-increasing way. We define a polygonal line connecting the points $p^l$ down to zero.

Clearly, along this line we can carry out the analysis of Step 1 in order to show that $\hat{J}(p)$ decreases with $|p|$, and in particular the sequence $J(p^l)$ is bounded. □

**Remark.** We can also prove this result in a different way: Use the notion of sub-analytic sets to prove that the set $E = \{p \in [0, L]^{n-1}; [\varphi, \psi](\eta; 0, p) \neq 0\}$ has a finite number of pathwise connected components.

## 3. A brief survey of the proofs of Theorems 1,2

We outline here the method of proof of Theorems 1 and 2, following [3].

**Outline of proof of Theorem 1.** Let $L_1 = L\sqrt{n-1} + 1$, so that $|p| < L_1$ if $p \in [0, L]^{n-1}$. Let $\delta = 2(L_1 + \tilde{k}(L, L, ..., L))$. Let $\Gamma_\delta = \overline{\mathbb{C}^+} \cap \{|k| \leq \delta\}$.

• In view of our choice of $\delta$, Equation (2.4), Proposition 5 and Corollary 1, the kernel $\Phi_p(\xi, \eta; k)$ is uniformly bounded in $(\xi, \eta) \in \mathbb{R} \times \mathbb{R}$ and $k \in \partial\Gamma_\delta \cap \{|k| = \delta\}$. In fact, by symmetry we only consider $\eta \leq \xi$.

• Inspecting the asymptotic behavior of the Jost functions $\varphi, \psi$ (and in particular Equation (2.2)) and using Proposition 7 we conclude also that $k\Phi_p(\xi, \eta; k)$ is uniformly bounded for $y_c \leq \eta \leq \xi$ and $-\delta \leq k \leq \delta$.

• In the case $\eta \leq \xi \leq -y_c$ and $-\delta \leq k \leq \delta$ we need to express $\varphi$ in terms of $\psi$ and $\psi_1$, where the latter is the conjugate of the former for $\eta < -y_c$ (in analogy with $\varphi_1$ in the proof of Proposition 7).

• The difficulty is only near the threshold $k = \tilde{k}$, where $\psi = \psi_1$. However, the pair $\psi, \frac{\psi - \psi_1}{(k^2 - \tilde{k}^2)^{\frac{1}{2}}}$ is a basis (since $k$ is real) for the solutions of $(H_p - z)f = 0$ and, expressing $\varphi$ in this basis we obtain that $k(k^2 - \tilde{k}^2)^{\frac{1}{2}}\Phi_p(\xi, \eta; k)$ is uniformly bounded for $\eta \leq \xi \leq -y_c$ and $-\delta \leq k \leq \delta$.

• Since all other possibilities for $\eta \le \xi$ can be treated directly by the asymptotic and continuity properties of the Jost functions we conclude that

$$\sup_{k\in\partial\Gamma_\delta,(\xi,\eta)\in\mathbb{R}\times\mathbb{R}} |k(k^2-\tilde{k}^2)^{\frac{1}{2}}\Phi_p(\xi,\eta;k)| < \infty. \tag{3.1}$$

• By analyticity (Proposition 2 ) and the fact that $\Phi_p(\xi,\eta;k)$ has at most finitely many poles, all located on the imaginary axis (Proposition 8) we conclude that $k(k^2-\tilde{k}^2)^{\frac{1}{2}}\Phi_p(\xi,\eta;k)$ is uniformly bounded in every sector $\Gamma_\delta \cap \{0 \le \arg k \le \theta < \frac{\pi}{2}\}$.

• In view of the definitions $k = (zc_+^{-2} - |p|^2)^{\frac{1}{2}}$ and $\tilde{k} = \tilde{k}(p) = (c_-^2 c_+^{-2} - 1)^{\frac{1}{2}}|p|$ the last point is equivalent to the statement of the theorem.

**Outline of proof of Theorem 2.** We take $0 < \delta < \frac{1}{2}\tilde{k}(l,...,l)$, so that by Proposition 6 the Jost functions and the Wronskian are analytic in the full disk $B_\delta = \{|k| \le \delta\}$ for all $p \in [l,L]^{n-1}$. In what follows we only consider these values of $p$.

• The derivative (with respect to $k$) of $[\varphi,\psi](\eta;k,p)$ is bounded by analyticity, so the inequality (2.8) holds for all $k \in B_\delta^\varepsilon = B_\delta \cap \{|\arg k| < \varepsilon\}$, for some $0 < \varepsilon < \frac{\pi}{2}$.

• By inspection of the asymptotic properties of the Jost functions we get

$$|k\Phi_p(\xi,\eta;k)| \le C\exp[(-\mathrm{Im}k)^+(\xi^+ + \eta^+)], \quad (\xi,\eta)\in\mathbb{R}\times\mathbb{R},\ k\in B_\delta^\varepsilon, \tag{3.2}$$

where $b^+ = \max(b,0)$. Observe that for $\arg k < 0$ the function $\Phi_p(\xi,\eta;k)$ is not the resolvent kernel but is the analytic extension of the kernel from the upper half-plane.

• Let $0 < k_0 < \frac{\delta}{2}$ and let $\Lambda_0 \subseteq B_\delta^\varepsilon$ be the circle of radius $(2+\xi^+ + \eta^+)^{-1}k_0\sin\varepsilon$, centered at $k_0$. By the previous point $k\Phi_p(\xi,\eta;k)$ is bounded on $\Lambda_0$, with a bound which is independent of its variables. By the Cauchy formula we conclude

$$|\frac{\partial}{\partial k}(k\Phi_p(\xi,\eta;k))_{k=k_0}| \le Ck_0^{-1}(1+\xi^+ + \eta^+), \qquad (\xi,\eta)\in\mathbb{R}\times\mathbb{R}. \tag{3.3}$$

• Using $k = (zc_+^{-2} - |p|^2)^{\frac{1}{2}}$ and noting the estimate (3.2) we obtain the first term in the right-hand side of (1.5), in a neighborhood of $c_+^2|p|^2$.

• Similar considerations are applied in the neighborhood of $c_-^2|p|^2$, thus obtaining the second term in the right-hand side of (1.5).

## REFERENCES

[1] M. Ben-Artzi, *On spectral properties of the acoustic propagator in a layered band*, J. Diff. Eqs. 136 (1997), 115–135.

[2] M. Ben-Artzi, Y. Dermenjian, and J. C. Guillot, *Acoustic waves in perturbed stratified fluids : A spectral theory*, Comm. PDE, 14 (1989), 479–517.

[3] M. Ben-Artzi, Y. Dermenjian, and J. C. Guillot, *Analyticity properties and estimates of resolvent kernels near thresholds*, Comm. PDE, 25 (2000), 1753–1770.

[4] A. Boutet de Monvel-Berthier, and D. Manda, *Spectral and scattering theory for wave propagation in perturbed stratified media*, J. Math. Anal. Appl., 191 (1995), 137–167.

[5] A. Cohen and T. Kappeler, *Scattering and inverse scattering for steplike potentials in the Schrödinger equation*, Indiana Univ. Math. J., 34 (1985), 127–180.

[6] E. Croc and Y. Dermenjian, *Analyse spectrale d'une bande acoustique multistratifiée, Partie I: Principe d'absorption limite pour une stratification simple*, SIAM J. Math. Anal., 26 (1995), 880–924.

[7] S. DeBièvre and W. Pravica, *Spectral analysis for optical fibers and stratified fluids I: The limiting absorption principle*, J. Func. Anal., 98 (1991), 404–436.

[8] M. Kadowaki, *Low and high energy resolvent estimates for wave propagation in stratified media and their applications*, J. Diff. Eqs., 179 (2002), 246–277.

[9] M. Kadowaki, *Resolvent estimates and scattering states for dissipative systems*, Publ. RIMS, Kyoto Univ., 38 (2002), 191–209.

[10] K. Kikuchi and H. Tamura, *The limiting amplitude principle for acoustic propagators in perturbed stratified fluids*, J. Diff. Eqs., 93 (1991), 260–282.

[11] B. Perthame and L. Vega, *Morrey-Campanato estimates for Helmholtz equations*, J. Func. Anal., 164 (1999), 340–355.

[12] B. Perthame and L. Vega, *Energy decay and Sommerfeld condition for Helmholtz equation with variable index at infinity*, Preprint (2002).

[13] M. Reed, B. Simon, "Methods of Modern Mathematical Physics," Vol. IV, Academic Press. 1978