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DEGREE THEORETIC METHODS IN THE STUDY OF POSITIVE SOLUTIONS FOR NONLINEAR HEMIVARIATIONAL INEQUALITIES

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Abstract. In this paper we study the existence of positive solutions for nonlinear elliptic problems driven by the p-Laplacian differential operator and with a nonsmooth potential (hemivariational inequalities). The hypotheses, in the case $p = 2$ (semilinear problems), incorporate in our framework of analysis the so-called asymptotically linear problems. The approach is degree theoretic based on the fixed-point index for nonconvex-valued multifunctions due to Bader [3].

1. Introduction

The goal of this paper is to use degree theoretic methods to study the problem of existence of positive solutions for nonlinear hemivariational inequalities driven by the p-Laplacian differential operator. So the problem under consideration is the following:

$$\left\{ \begin{array}{l} -\text{div}(\|Dx(z)\|^{p-2}Dx(z)) \in \partial j(z, x(z)) \text{ a.e. on } Z \\ x|_{\partial Z} = 0, \ 1 < p < \infty. \end{array} \right\} \tag{1.1}$$

Here $Z \subseteq \mathbb{R}^N$ is a bounded domain with a C^2 boundary ∂Z and the potential function $j(z, x)$ is measurable in $z \in Z$ and only locally Lipschitz and in general nonsmooth in $x \in \mathbb{R}$. By $\partial j(z, x)$ we denote the generalized subdifferential of the locally Lipschitz function $x \to j(z, x)$ (see Section 2).

The hypotheses on the potential $j(z, x)$ are such that when $p = 2$ (semilinear problems), the analytical framework of this work incorporates the so-called "asymptotically linear problems", which since the appearance of the pioneering work of Amann-Zehnder [1], have attracted the interest of many

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researchers. The question of existence of positive solutions for such problems was investigated by Zhou [19] for $p = 2$ and by Fan-Zhao-Huang [6], Huang [11], Li-Zhou [13] and Zhang-Li-Liu-Feng [18] when $p > 1$. In all these works $j(z, \cdot) \in C^1(\mathbb{R})$ and $\frac{d}{dx} j(z,x) = f(z,x)$ with $f \in C(\overline{Z} \times \mathbb{R})$. Moreover, the asymptotic conditions on the "slopes" $\frac{f(z,x)}{x^{p-1}}$ as $x \to +\infty$ and as $x \to 0^+$ are more restrictive than ours. In all the aforementioned works the approach is variational based on the mountain pass theorem. Recently the question of existence of positive solutions for nonlinear hemivariational inequalities driven by the p-Laplacian was addressed by Motreanu-Papageorgiou [14]. In that paper the asymptotic inequalities at $+\infty$ and 0^+ have opposite direction than the ones used here. This made the Euler functional of their problem coercive. In contrast the Euler functional of problem (1.1) in this paper is indefinite. The method of proof of Motreanu-Papageorgiou [14] was variational based on the nonsmooth critical-point theory. Our work here complements that of Motreanu-Papageorgiou [14]. In addition we present a totally different approach based on the fixed-point index for multifunctions, introduced recently by Bader [3]. To our knowledge in the past only Goeleven-Motreanu [9] and Goeleven-Motreanu-Panagiotopoulos [8] have used degree theoretic methods to study certain semilinear eigenvalue problems for hemivariational inequalities.

Hemivariational inequalities are a new type of variational expression, which arise naturally in problems of mechanics and engineering, when one wants to consider more realistic laws of multivalued and nonmonotone nature. Then the corresponding energy functional is nonsmooth and nonconvex. Concrete applications of hemivariational inequalities can be found in the book of Naniewicz-Panagiotopoulos [15].

2. Mathematical Background

As we already mentioned our degree theoretic approach is based on the fixed-point index for multifunctions introduced recently by Bader [3]. So briefly let us mention how this fixed-point index is defined.

First let us recall that, if V, Z are Hausdorff topological spaces and $G : V \to 2^Z \setminus \{\emptyset\}$ is a multifunction, then we say that G is upper semicontinuous (usc for short), if for every $C \subseteq Z$ nonempty and closed, the set $G^-(C) = \{v \in V : G(v) \cap C \neq \emptyset\}$ is closed in V. If Z is regular and G has closed values, then the upper semicontinuity of G implies that $GrG = \{(v,z) \in V \times Z : z \in G(v)\}$ (the graph of G) is closed in $V \times Z$. The converse is true if G is locally compact, namely for every $v \in V$, we can find a neighborhood U of

v such that $\overline{G(U)}$ is compact in Z. For details we refer to Hu-Papageorgiou [12].

Let X, Y be two Banach spaces and let $C \subseteq X$, $D \subseteq Y$ be nonempty, closed and convex sets. In what follows by (D, w) we denote the set D furnished with the relative weak topology of Y. We consider multifunctions $G : C \to 2^C \setminus \{\emptyset\}$ which admit the following decomposition

$$G = K \circ N, \tag{2.1}$$

where $N : C \to 2^D \setminus \{\emptyset\}$ is an upper semicontinuous multifunction into (D, w) with weakly compact and convex values and $K : (D, w) \to C$ is a sequentially continuous map, i.e., if $y_n \overset{w}{\to} y$ in D, then $K(y_n) \to K(y)$ in C with the norm topology. We also assume that the multifunction G is compact, namely it maps bounded sets onto relatively compact sets. Let $\mathcal{D}$ be the family of (G, U, C), where G and C are as above and $U \subseteq C$ is a nonempty, bounded, relatively open set such that $\mathrm{Fix}(G) \cap \partial U = \emptyset$, where $\mathrm{Fix}(G) = \{x \in C : x \in G(x)\}$ (the set of fixed points of G). Then on $\mathcal{D}$ we can define a fixed-point index $i_C : \mathcal{D} \to \mathbb{Z}$, which has properties analogous to those of the usual fixed-point index (see for example Zeidler [17]). We only mention how we define the homotopies in this setting. Suppose that $G = K \circ N$ and $F = S \circ L$. We say that G and F are homotopic if the following is true: "There exists an upper semicontinuous multifunction $H : [0,1] \times C \to 2^D \setminus \{\emptyset\}$ (D always equipped with the relative weak topology of Y) which has weakly compact convex values, such that $H(0, \cdot) = N$ and $H(1, \cdot) = L$ and a continuous map $u : [0,1] \times (D, w) \to C$ such that $u(0, \cdot) = K$ and $u(1, \cdot) = S$". Then the homotopy invariance of the fixed-point index i_C says that if $\Psi(t, x) = u(t, H(t, x))$, Ψ is compact and $x \notin \Psi(t, x)$ for all $(t, x) \in [0,1] \times \partial U$, then $i_C(G, U, C) = i_C(F, U, C)$. We emphasize that the multifunctions which have the form (2.1), and on which i_C is defined, need not have convex values. This makes the fixed-point index i_C a valuable tool in many situations. For further details on this fixed-point index, we refer to Bader [3].

We will also use some basic things about the spectrum of the negative p-Laplacian with Dirichlet boundary conditions, i.e., of $(-\triangle_p, W_0^{1,p}(Z))$. Let $m \in L^\infty(Z)_+$, $m \neq 0$. Here $L^\infty(Z)_+ = \{m \in L^\infty(Z) : m(z) \geq 0 \text{ almost everywhere on } Z\}$. We consider the following nonlinear weighted eigenvalue problem:

$$\left\{ \begin{array}{l} -\mathrm{div}(\|Dx(z)\|^{p-2} Dx(z)) = \widehat{\lambda} m(z) |x(z)|^{p-2} x(z) \text{ a.e. on } Z \\ x|_{\partial Z} = 0, \ \widehat{\lambda} \in \mathbb{R}. \end{array} \right\} \tag{2.2}$$

Problem (2.2) has a smallest eigenvalue denoted by $\widehat{\lambda}_1(m)$, which is positive, isolated and simple (i.e., the corresponding eigenspace is one dimensional). It has a variational characterization given by

$$\widehat{\lambda}_1(m) = \inf\left[\frac{\|Dx\|_p^p}{\int_Z m|x|^p dz} : x \in W_0^{1,p}(Z),\ x \neq 0\right]. \tag{2.3}$$

The infimum in (2.3) is realized at a corresponding eigenfunction u_1 which belongs in $C_0^1(\overline{Z})$. Moreover, u_1 does not change sign and so we may say that $u_1(z) \geq 0$ for all $z \in Z$. In fact by virtue of the nonlinear strong maximum principle we have $u_1(z) > 0$ for all $z \in Z$. If $u \in W_0^{1,p}(Z)$ is an eigenfunction corresponding to an eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1(m)$, then $u \in C_0^1(\overline{Z})$ and u must change sign. Here $C_0^1(\overline{Z}) = \{u \in C^1(\overline{Z}) : u(z) = 0 \text{ for all } z \in \partial Z\}$. If $m_1, m_2 \in L^\infty(Z)_+$ are two weight functions such that $m_1(z) \leq m_2(z)$ almost everywhere on Z with strict inequality on a set of positive measure, then $\widehat{\lambda}_1(m_1) > \widehat{\lambda}_1(m_2)$, i.e., we have strict monotonicity of the principal eigenvalue $\widehat{\lambda}_1(m)$ on the weight function m. If $m \equiv 1$, then we write $\widehat{\lambda}_1(1) = \lambda_1$. For further details we refer to Anane [2] and Denkowski-Migorski-Papageorgiou [5].

Finally if E is a Banach space and $\varphi : E \to \mathbb{R}$ a locally Lipschitz function, the generalized directional derivative of φ at $x \in E$ in the direction $h \in E$, denoted by $\varphi^0(x;h)$, is defined

$$\varphi^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{\varphi(x' + \lambda h) - \varphi(x')}{\lambda}.$$

It is easy to check that $h \to \varphi^0(x;h)$ is sublinear and continuous, and so it is the support function of a nonempty, convex and w^*-compact set $\partial\varphi(x) \subseteq E^*$ defined by

$$\partial\varphi(x) = \{x^* \in E^* : \langle x^*, h\rangle_E \leq \varphi^0(x;h) \text{ for all } h \in E\}.$$

Here by $\langle \cdot,\cdot\rangle_E$ we denote the duality brackets for the pair (E, E^*). The multifunction $x \to \partial\varphi(x)$ is the generalized subdifferential of φ.

In what follows for any Banach space E by $P_{wkc}(E)$ we will denote the space of all nonempty, weakly compact and convex subsets of E.

3. Positive Solutions

The hypotheses on the nonsmooth potential $j(z,x)$ are the following:

$H(j)$ $j: Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $0 \in \partial j(z,0)$ almost everywhere on Z, $\partial j(z,x) \subseteq \mathbb{R}_+$ almost everywhere on Z for all $x \geq 0$ and

(i) for all $x \in \mathbb{R}$, $z \to j(z,x)$ is measurable;
(ii) for almost all $z \in Z$, $x \to j(z,x)$ is locally Lipschitz;
(iii) for almost all $z \in Z$, all $x \in \mathbb{R}$, and all $u \in \partial j(z,x)$, we have

$$|u| \leq \alpha(z) + c|x|^{p-1} \text{ with } \alpha \in L^\infty(Z)_+,\ c > 0;$$

(iv) there exists $\theta \in L^\infty(Z)_+$ such that $\theta(z) \geq \lambda_1$ almost everywhere on Z, the inequality is strict on a set of positive measure and

$$\liminf_{x \to +\infty} \frac{u}{x^{p-1}} \geq \theta(z) \text{ uniformly for almost all } z \in Z$$

and all $u \in \partial j(z,x)$;

(v) there exists $\eta \in L^\infty(Z)_+$ such that $\eta(z) \leq \lambda_1$ almost everywhere on Z, the inequality is strict on a set of positive measure and

$$\limsup_{x \to 0^+} \frac{u}{x^{p-1}} \leq \eta(z) \text{ uniformly for almost all } z \in Z$$

and all $u \in \partial j(z,x)$.

Remark 3.1. Hypotheses (iv) and (v) are nonuniform nonresonance conditions at $+\infty$ and at 0^+ respectively and are more general than the corresponding asymptotic conditions in the papers mentioned in the Introduction. In those works (all dealing with a smooth potential), the asymptotic limits exist and are constants different from $\lambda_1 > 0$. A simple nonsmooth locally Lipschitz function satisfying hypotheses $H(j)$ is given by $j(z,x) = \min\{\frac{1}{r}|x|^r, \frac{\theta(z)}{p}|x|^p\}$ with $1 < p < r < +\infty$ and $\theta \in L^\infty(Z)_+$, $0 < \lambda_1 \leq \theta(z)$ almost everywhere on Z with the last inequality being strict on a set of positive measure.

We start by examining the following auxiliary problem:

$$\left\{ \begin{array}{l} -\mathrm{div}(\|Dx(z)\|^{p-2}Dx(z)) = h(z) \text{ a.e. on } Z \\ x|_{\partial Z} = 0,\ h \in L^{p'}(Z),\ \frac{1}{p} + \frac{1}{p'} = 1. \end{array} \right\} \tag{3.1}$$

Proposition 3.2. *For every $h \in L^{p'}(Z)$ ($\frac{1}{p} + \frac{1}{p'} = 1$), problem* (3.1) *has a unique solution $V(h) \in W_0^{1,p}(Z)$ and the map $h \to V(h)$ is completely continuous from $L^{p'}(Z)$ into $W_0^{1,p}(Z)$; i.e. if $h_n \xrightarrow{w} h$ in $L^{p'}(Z)$, then $V(h_n) \to V(h)$ in $W_0^{1,p}(Z)$ as $n \to +\infty$.*

Proof. Let $\langle \cdot, \cdot \rangle$ denote the duality brackets for the pair $(W_0^{1,p}(Z), W^{-1,p'}(Z) = W_0^{1,p}(Z)^*)$ and consider the nonlinear operator $A : W_0^{1,p}(Z) \to W^{-1,p'}(Z)$ defined by

$$\langle A(x), y \rangle = \int_Z \|Dx(z)\|^{p-2} (Dx(z), Dy(z))_{\mathbb{R}^N} \, dz \ \text{ for all } \ x, y \in W_0^{1,p}(Z).$$

It easy to check that A is demicontinuous and strictly monotone, hence maximal monotone (see Denkowski-Migorski-Papageorgiou [5], page 37) Also we have

$$\langle A(x), x \rangle = \|Dx\|_p^p$$

and so by Poincare's inequality, we infer that A is weakly coercive. But a weakly coercive maximal monotone operator is surjective (see Denkowski-Migorski-Papageorgiou [5], page 49). So given $h \in L^{p'}(Z) \subseteq W^{-1,p'}(Z)$, we can find $x \in W_0^{1,p}(Z)$ such that

$$\begin{aligned} &A(x) = h \ \text{ in } \ W^{-1,p'}(Z), \\ \Rightarrow &-\operatorname{div}\left(\|Dx(z)\|^{p-2} Dx(z)\right) = h(z) \ \text{ a.e. on } Z, \ x|_{\partial Z} = 0. \end{aligned}$$

Therefore, $x \in W_0^{1,p}(Z)$ solves problem (3.1) and by virtue of the strict monotonicity of A this solution is unique. So we can define the solution map $V : L^{p'}(Z) \to W_0^{1,p}(Z)$. We will show that V is completely continuous. To this end let $h_n \xrightarrow{w} h$ in $L^{p'}(Z)$ and set $x_n = V(h_n)$. We have

$$\begin{aligned} &A(x_n) = h_n \ \text{ for all } \ n \geq 1, \\ \Rightarrow &\langle A(x_n), x_n \rangle = \|Dx_n\|_p^p = \int_Z h_n x_n dz \leq c_1 \|h_n\|_{p'} \|Dx_n\|_p \\ &\text{for some } c_1 > 0, \text{ all } n \geq 1 \text{ (by Poincare's inequality)}, \\ \Rightarrow &\|Dx_n\|_p^{p-1} \leq c_1 \|h_n\|_{p'} \leq c_2 \ \text{ for some } \ c_2 > 0, \text{ all } n \geq 1, \\ \Rightarrow &\{x_n\}_{n \geq 1} \subseteq W_0^{1,p}(Z) \ \text{ is bounded (by Poincare's inequality)}. \end{aligned}$$

So by passing to a suitable subsequence if necessary, we may assume that

$$x_n \xrightarrow{w} x \text{ in } W_0^{1,p}(Z) \ \text{ and } \ x_n \to x \text{ in } L^p(Z) \text{ as } n \to \infty.$$

We have

$$\langle A(x_n), x_n - x \rangle = \int_Z h_n(x_n - x) dz \to 0 \text{ as } n \to \infty. \tag{3.2}$$

But A being maximal monotone, it is generalized pesudomonotone (see Denkowski-Migorski-Papageorgiou [5], page 59) and so from (3.2) it follows that

$$\langle A(x_n), x_n\rangle \to \langle A(x), x\rangle \ \Rightarrow\ \|Dx_n\|_p \to \|Dx\|_p.$$

Recall that $Dx_n \xrightarrow{w} Dx$ in $L^p(Z, \mathbb{R}^N)$ and $L^p(Z, \mathbb{R}^N)$ is uniformly convex. So by the Kadec-Klee property, we have $Dx_n \to Dx$ in $L^p(Z, \mathbb{R}^N)$. Therefore $x_n \to x$ in $W_0^{1,p}(Z)$ and we have

$$A(x) = h; \ \text{ i.e., } \ x = V(h).$$

Finally, from Urysohn's criterion for convergent sequences, we conclude that for the original sequence we have $V(h_n) \to V(h)$ in $W_0^{1,p}(Z)$ and so V is completely continuous. $\square$

We introduce the Lipschitz continuous truncation function $\tau : \mathbb{R} \to \mathbb{R}_+$ defined by

$$\tau(x) = \begin{cases} 0 & \text{if } \ x \le 0 \\ x & \text{if } \ x > 0. \end{cases}$$

Let $j_1(z, x) = j(z, \tau(x))$. Evidently for all $x \in \mathbb{R}$ $z \to j_1(z, x)$ is measurable and for almost all $z \in Z$, $x \to j_1(z, x)$ is locally Lipschitz. Moreover, from the nonsmooth chain rule (see for example Denkowski-Migorski-Papageorgiou [4], page 611), we have

$$\partial j_1(z, x) \subseteq \begin{cases} \{0\} & \text{if } \ x < 0 \\ \operatorname{conv}\{r\partial j(z, 0) : r \in [0, 1]\} & \text{if } \ x = 0 \\ \partial j(z, x) & \text{if } \ x > 0. \end{cases} \tag{3.3}$$

We introduce the multifunction $G : W_0^{1,p}(Z) \to 2^{L^{p'}(Z)}$ defined by

$$G(x) = S^{p'}_{\partial j_1(\cdot, x(\cdot))} = \{u \in L^{p'}(Z) : u(z) \in \partial j_1(z, x(z)) \text{ a.e. on } Z\}.$$

Proposition 3.3. *If hypotheses $H(j)(i), (ii)$ and (iii) hold, then the multifunction G has values in $P_{wkc}(L^{p'}(Z))$ and it is usc from $W_0^{1,p}(Z)$ into $L^{p'}(Z)_w$ (here by $L^{p'}(Z)_w$ we denote the Lebesgue space $L^{p'}(Z)$ endowed with the weak topology).*

Proof. Let $x \in W_0^{1,p}(Z)$. By definition (see Section 2), for every $y \in \mathbb{R}$ we have

$$j_1^0(z, x(z); y) = \limsup_{\substack{v \to x(z) \\ \lambda \downarrow 0}} \frac{j_1(z, v + \lambda y) - j_1(z, v)}{\lambda}$$

$$= \inf_{\varepsilon,\delta>0} \sup_{\substack{|v-x(z)|<\delta \\ 0<\lambda<\varepsilon}} \frac{j_1(z,v+\lambda y)-j_1(z,v)}{\lambda}$$

$$= \inf_{n,m\geq 1} \sup_{\substack{|v-x(z)|<\frac{1}{n} \\ 0<\lambda<\frac{1}{m}, \\ v,\lambda \text{ rationals}}} \frac{j_1(z,v+\lambda y)-j_1(z,v)}{\lambda}. \qquad (3.4)$$

Since $z \to j_1(z,x)$ is measurable and for almost all $z \in Z$ $x \to j_1(z,x)$ is locally Lipschitz, it follows that the function $(z,x) \to j_1(z,x)$ is $\mathcal{L}(Z) \times B(\mathbb{R})$ measurable, with $\mathcal{L}(Z)$ being the Lebesgue σ-field of Z and $B(\mathbb{R})$ the Borel σ-field of $\mathbb{R}$. So from (3.4) it follows that $z \to j_1^0(z,x(z);y)$ is Lebesgue measurable. On the other hand we know that $y \to j_1^0(z,x(z);y)$ is continuous. Recall that

$$\mathrm{Gr}\partial j_1(\cdot,x(\cdot)) = \{(z,u) \in \mathbb{R} : uy \leq j^0(z,x(z);y) \text{ for all } y \in \mathbb{R}\}.$$

If $\{y_n\}_{n\geq 1}$ is an enumeration of the rationals in $\mathbb{R}$, then

$$\mathrm{Gr}\partial j_1(\cdot,x(\cdot)) = \bigcap_{n\geq 1}\{(z,u) \in Z\times\mathbb{R} : uy_n \leq j_1^0(z,x(z);y_n)\} \in \mathcal{L}(Z)\times B(\mathbb{R}).$$

Invoking the Yankov-von Neumann-Aumann selection theorem (see Denkowski-Migorski-Papageorgiou [4], p. 432) and because of hypothesis $H(j)(iii)$, we infer that

$$G(x) \in P_{wkc}(L^{p'}(Z)) \text{ for all } x \in W_0^{1,p}(Z).$$

By virtue of hypothesis $H(j)(iii)$, the multifunction G is locally compact from $W_0^{1,p}(Z)$ into $L^{p'}(Z)$. Since weakly compact sets in $L^{p'}(Z)$ furnished with the relative weak topology are metrizable, to show the desired upper semicontinuity from $W_0^{1,p}(Z)$ into $L^{p'}(Z)_w$ it suffices to show that $\mathrm{Gr}G = \{(x,u) \in W_0^{1,p}(Z)\times L^{p'}(Z) : u \in G(x)\}$ is sequentially closed in $W_0^{1,p}(Z) \times L^{p'}(Z)_w$. So let $(x_n,u_n) \in \mathrm{Gr}G$ $n \geq 1$ and suppose that $x_n \to x$ in $W_0^{1,p}(Z)$ and $u_n \xrightarrow{w} u$ in $L^{p'}(Z)$. We may assume that $x_n(z) \to x(z)$ almost everywhere on Z. For every $n \geq 1$ we have

$$u_n(z) \in \partial j_1(z,x_n(z)) \text{ a.e. on } Z.$$

From Denkowski-Migorski-Papageorgiou [4], page 484 and since the multifunction $y \to \partial j_1(z,y)$ has a closed graph, in the limit as $n\to\infty$, we have

$$u(z) \in \partial j_1(z,x(z)) \text{ a.e. on } Z,$$
$$\Rightarrow (x,u) \in \mathrm{Gr}G.$$

This proves the desired upper semicontinuity of G. $\square$

In what follows let

$$C_+ = \{x \in W_0^{1,p}(Z) : x(z) \geq 0 \text{ a.e. on } Z\}.$$

This is the positive cone in the ordered Banach space $W_0^{1,p}(Z)$. The set C_+ is a retract of $W_0^{1,p}(Z)$. If $x \in C_+$ and $y \in (V \circ G)(x)$, then $A(y) = u$ with $u \in G(x)$. Use as a test function $-y^- \in W_0^{1,p}(Z)$. Then $\|Dy^-\|_p^p = \int_Z u(-y^-)dz \leq 0$ since $\partial j_1(z, x(z)) \subseteq \mathbb{R}_+$ almost everywhere on Z. Hence $y^- = 0$ and so $y \in C_+$. Therefore $(N \circ G)(C_+) \subseteq C_+$. So we can consider the fixed-point index $i_{C_+}(V \circ G, B_\rho^+)$, where $B_\rho^+ = B_\rho \cap C_+$ with $B_\rho = \{x \in W_0^{1,p}(Z) : \|x\| < \rho\},\ \rho > 0$ (see Bader [3]).

Proposition 3.4. *If hypotheses $H(j)$ hold, then there exists $\rho_0 >$ such that $i_{C_+}(V \circ G, B_\rho^+) = 1$ for all $0 < \rho \leq \rho_0$.*

Proof. Let $K : L^p(Z) \to L^{p'}(Z)$ be the bounded, continuous, strictly monotone map defined by

$$K(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot).$$

We consider the following admissible multivalued homotopy

$$h_1(\beta, x) = V\left(\beta\eta K(x) + (1-\beta)G(x)\right) \text{ for all } (\beta, x) \in [0,1] \times C_+.$$

We will show that there exists $\rho_0 > 0$ such that

$$x \notin h_1(\beta, x) \text{ for all } \beta \in [0,1] \text{ and all } x \in C_+ \text{ with } \|x\| = \rho,\ 0 < \rho \leq \rho_0. \tag{3.5}$$

We argue indirectly. Suppose that (3.5) is not true. Then we can find $\{x_n\}_{n\geq 1} \subseteq C_+$ such that

$$x_n \to 0 \text{ in } W_0^{1,p}(Z),\ x_n \neq 0, \text{ and } A(x_n) = \beta_n\eta K(x_n) + (1-\beta_n)u_n$$

with $u_n \in G(x_n)$ for all $n \geq 1$ and $\beta_n \to \beta \in [0,1]$ (recall the definition of V, see Proposition 3.2). Set $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. Exploiting the $(p-1)$-homogeneity of A and K, we have

$$A(y_n) = \beta_n\eta K(y_n) + (1-\beta_n)\frac{u_n}{\|x_n\|^{p-1}}.$$

First suppose that by passing to a subsequence if necessary, we have $\beta_n < 1$ for all $n \geq 1$. Since $A(x_n) = \beta_n\eta K(x_n) + (1-\beta_n)u_n$, we have

$$-\text{div}(\|Dx_n(z)\|^{p-2}Dx_n(z)) = \beta_n\eta(z)|x_n(z)|^{p-2}x_n(z) + (1-\beta_n)u_n(z)$$

almost everywhere on Z. From Stampacchia's theorem (see for example Denkowski-Migorski-Papageorgiou [4], page 349), we know that $Dx_n(z) = 0$ almost everywhere on $\{x_n = 0\}$. So we obtain

$$(1-\beta_n)u_n(z) = 0 \text{ a.e. on } \{x_n = 0\},$$
$$\Rightarrow u_n(z) = 0 \text{ a.e. on } \{x_n = 0\} \text{ (since } 0 \leq \beta_n < 1).$$

Combining this with hypothesis $H(j)(iii)$ and (3.3), we can say that

$$|u_n(z)| \leq c_3 \left(|x_n(z)|^{p-1} + |x_n(z)|^{\tau-1}\right) \tag{3.6}$$

almost everywhere on Z, for some $c_3 > 0$, all $n \geq 1$.

Here $\tau = p + \gamma$, with $\gamma = \frac{p^*-p}{p^*}(p-1)$ if $p < N$ (p^* is the Sobolev critical exponent; i.e., $p^* = \frac{Np}{N-p}$) and $\frac{1}{p'} < \gamma < 1$ if $p \geq N$. From (3.6) we have

$$\frac{|u_n(z)|}{\|x_n\|^{p-1}} \leq c_3 \left(|y_n(z)|^{p-1} + |x_n(z)|^{\tau-p}|y_n(z)|^{p-1}\right) \text{ a.e. on } Z,$$
$$\Rightarrow \int_Z \frac{|u_n(z)|^{p'}}{\|x_n\|^p} dz \leq c_3\|y_n\|_p^p + c_3 \int_Z |x_n(z)|^{(\tau-p)p'}|y_n(z)|^p dz \tag{3.7}$$
$$\text{(since } p-1 = \tfrac{p}{p'}).$$

Note that if $p < N$, then $|y_n|^p \in L^{\frac{p^*}{p}}(Z)$ (Sobolev embedding theorem) and from the choice of τ in this case we have

$$|x_n(z)|^{(\tau-p)p'} = |x_n(z)|^{\gamma p'} = |x_n(z)|^{\frac{p^*-p}{p^*}p} \text{ (since } p' = \tfrac{p}{p-1}).$$

Moreover, $|x_n|^{\frac{p^*-p}{p^*}p} \in L^{\frac{p^*}{p^*-p}}(Z)$. Set $\mu = \frac{p^*}{p}$ and $\mu' = \frac{p^*}{p^*-p}$. We have $\frac{1}{\mu} + \frac{1}{\mu'} = 1$ and so using Hölder's inequality, we obtain

$$\int_Z |x_n(z)|^{(\tau-p)p'}|y_n(z)|^p dz \leq \|x_n\|_p^{\frac{p}{\mu'}} \|y_n\|_{p^*}^{\frac{p^*}{\mu}}. \tag{3.8}$$

Returning to (3.7), using (3.8) and recalling that $\|y_n\| = 1$ for all $n \geq 1$ and that $W_0^{1,p}(Z)$ is embedded continuously into $L^{p^*}(Z)$, we infer that $\{\frac{u_n}{\|x_n\|^{p-1}}\}_{n\geq 1} \subseteq L^{p'}(Z)$ is bounded when $p < N$.

If $p \geq N$, then from the Sobolev embedding theorem we have that $|y_n|^p \in L^{(\frac{1}{\gamma})'}(Z)$ ($\gamma + \frac{1}{(\frac{1}{\gamma})'} = 1$) and also $|x_n|^{(\tau-p)p'} = |x_n|^{\gamma p'} \in L^{\frac{1}{\gamma}}(Z)$. Therefore, again via Hölder's inequality, we obtain

$$\int_Z |x_n(z)|^{(\tau-p)p'}|y_n(z)|^p dz \leq \|x_n\|_{p'}^{\gamma p'} \|y_n\|_{p(\frac{1}{\gamma})'}^p. \tag{3.9}$$

Returning to (3.7), using (3.9) and recalling that $\|y_n\| = 1$ for all $n \geq 1$ and that $W_0^{1,p}(Z)$ is embedded continuously in $L^r(Z)$ for all $1 \leq r < \infty$, we infer that $\{\frac{u_n}{\|x_n\|^{p-1}}\}_{n\geq 1} \subseteq L^{p'}(Z)$ is bounded when $p \geq N$.

Therefore, for all $1 < p < \infty$, $\{\frac{u_n}{\|x_n\|^{p-1}}\}_{n\geq 1} \subseteq L^{p'}(Z)$ is bounded. Hence by passing to a suitable subsequence if necessary, we may assume that

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} f \text{ in } L^{p'}(Z).$$

Given $\varepsilon > 0$ and $n \geq 1$, consider the set

$$C_{\varepsilon,n} = \{z \in Z : x_n(z) > 0,\ \frac{u_n(z)}{x_n(z)^{p-1}} \leq \eta(z) + \varepsilon\}.$$

Since $x_n \to 0$ in $W_0^{1,p}(Z)$, we may assume (at least for a subsequence) that $x_n(z) \to 0^+$ almost everywhere on Z (recall that $x_n \in C_+$ for all $n \geq 1$). Then by virtue of hypothesis $H(j)(v)$ we have that

$$\chi_{C_{\varepsilon,n}}(z) \to 1 \text{ a.e. on } Z.$$

Note that

$$\left\| \left(1 - \chi_{C_{\varepsilon,n}}\right) \frac{u_n}{\|x_n\|^{p-1}} \right\|_{p'} \to 0,$$

$$\Rightarrow \chi_{C_{\varepsilon,n}} \frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} f \text{ in } L^{p'}(Z) \text{ as } n \to \infty.$$

From the definition of the set $C_{\varepsilon,n}$, we have

$$\begin{aligned} &\chi_{C_{\varepsilon,n}}(z) \frac{u_n(z)}{\|x_n\|^{p-1}} = \chi_{C_{\varepsilon,n}}(z) \frac{u_n(z)}{x_n(z)^{p-1}} y_n(z)^{p-1} \\ &\leq \chi_{C_{\varepsilon,n}}(z)(\eta(z)+\varepsilon) y_n(z)^{p-1} \text{ a.e. on } Z \text{ (since } y_n \geq 0 \text{ for all } n \geq 1). \end{aligned}$$

Passing to the limit as $n \to \infty$, we obtain

$$f(z) \leq (\eta(z) + \varepsilon)\, y(z)^{p-1} \text{ a.e. on } Z.$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \downarrow 0$ and so

$$f(z) \leq \eta(z) y(z)^{p-1} \text{ a.e. on } Z. \tag{3.10}$$

Recall that for all $n \geq 1$

$$A(y_n) = \beta_n \eta K(y_n) + (1 - \beta_n) \frac{u_n}{\|x_n\|^{p-1}}, \tag{3.11}$$

$$\Rightarrow \langle A(y_n), y_n - y \rangle = \beta_n \int_Z \eta |y_n|^{p-2} y_n (y_n - y) dz$$

$$+ (1-\beta_n)\int_Z \frac{u_n}{\|x_n\|^{p-1}}(y_n - y)dz,$$
$$\Rightarrow \lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0.$$

Then as in the proof of Proposition 3.2, exploiting the generalized pseudomonotonicity of A and the Kadec-Klee property of $L^p(Z,\mathbb{R}^N)$, we deduce that $y_n \to y$ in $W_0^{1,p}(Z)$. Therefore, if we pass to the limit as $n \to \infty$ in (3.11), we obtain

$$\begin{aligned} A(y) &= \beta\eta K(y) + (1-\beta)f \\ &\leq \beta\eta K(y) + (1-\beta)\eta K(y) \text{ in } W^{-1,q}(Z) \text{ (see (3.10))}, \\ \Rightarrow A(y) &\leq \eta K(y) \text{ in } W^{-1,q}(Z). \end{aligned} \tag{3.12}$$

Using as a test function $y \in C_+$, from (3.12) we have

$$\|D(y)\|_p^p \leq \int_Z \eta|y|^p dz \leq \lambda_1\|y\|_p^p \text{ (see hypothesis } H(j)(v)), \tag{3.13}$$
$$\Rightarrow y = 0 \text{ or } y = u_1.$$

If $y = 0$, then $y_n \to 0$ in $W_0^{1,p}(Z)$, a contradiction to the fact that $\|y_n\| = 1$ for all $n \geq 1$.

If $y = u_1$, then from the first inequality in (3.13), since $u_1(z) > 0$ for all $z \in Z$ and recalling the hypothesis on $\eta \in L^\infty(Z)_+$ (see $H(j)(v)$), we obtain

$$\|D(y)\|_p^p < \lambda_1\|y\|_p^p$$

which contradicts the variational characterization of $\lambda_1 > 0$ (see (2.3)).

Therefore, if $\beta_n < 1$ for all $n \geq 1$, then (3.5) holds.

Now suppose that $\beta_n = 1$ for all $n \geq n_0 \geq 1$. Then

$$A(y_n) = \eta K(y_n) \;\Rightarrow\; \langle A(y_n), y_n - y\rangle = \int_Z \eta|y_n|^{p-2}y_n(y_n - y)dz$$
$$\Rightarrow \lim_{n\to\infty}\langle A(y_n), y_n - y\rangle = 0.$$

As above, from this it follows that $y_n \to y$ in $W_0^{1,p}(Z)$. Then in the limit as $n \to \infty$ we obtain

$$A(y) = \eta K(y)$$

from which arguing as before we reach a contradiction to the variational characterization of $\lambda_1 > 0$. Therefore again (3.5) holds.

Exploiting the homotopy invariance of the fixed-point index i_{C_+}, we have

$$i_{C_+}\left(h_1(0,\cdot), B_\rho^+\right) = i_{C_+}\left(h_1(1,\cdot), B_\rho^+\right)$$

where $B_\rho^+ = B_\rho \cap C_+,\ B_\rho = \{x \in W_0^{1,p}(Z) : \|x\| < \rho\}$

$$\Rightarrow i_{C_+}\left(V \circ G, B_\rho^+\right) = i_{C_+}\left(V \circ (\eta K), B_\rho^+\right) \text{ for all } 0 < \rho \leq \rho_0. \quad (3.14)$$

We will compute $i_{C_+}\left(V \circ (\eta K), B_\rho^+\right)$. For this purpose we consider the admissible homotopy $h_2 : [0,1] \times C_+ \to W_0^{1,p}(Z)$ defined by

$$h_2(\beta, x) = \beta \left(V \circ (\eta K)\right)(x) \text{ for all } (\beta, x) \in [0,1] \times C_+.$$

We claim that for all $\rho > 0$, all $x \in \partial B_\rho^+$ and all $\beta \in [0,1]$, we have $x \neq h_2(\beta, x)$. Indeed, if $x = h_2(\beta, x)$, then $A(x) = \beta^{p-1}\eta K(x)$ and so

$$\left\{ \begin{array}{l} -\mathrm{div}(\|Dx(z)\|^{p-2}Dx(z)) = \beta^{p-1}\eta(z)|x(z)|^{p-2}x(z) \text{ a.e. on } Z \\ x|_{\partial Z} = 0 \end{array} \right\}. \quad (3.15)$$

Due to the strict monotonicity of the principal eigenvalue of the weighted eigenvalue problem on the weight (see Section 2), we have

$$\widehat{\lambda}_1(\beta^{p-1}\eta) > \widehat{\lambda}_1(\lambda_1) = 1.$$

So from (3.15) it follows that $x \equiv 0$, a contradiction to the fact that $\|x\| = \rho > 0$. Hence for all $\rho > 0$, all $x \in \partial B_\rho^+$ and all $\beta \in [0,1]$, we have $x \neq h_2(\beta, x)$. Once more via the homotopy invariance of the fixed-point index i_{C_+}, we obtain

$$i_{C_+}\left(V \circ (\eta K), B_\rho^+\right) = i_{C_+}\left(0, B_\rho^+\right) = 1 \text{ for all } \rho > 0. \quad (3.16)$$

From (3.14) and (3.16), we have that

$$i_{C_+}\left(V \circ G, B_\rho^+\right) = 1 \text{ for all } 0 < \rho \leq \rho_0.$$

□

We will prove an analogous result for large balls.

Proposition 3.5. *If hypotheses $H(j)$ hold, then there exists $R_0 > 0$ such that $i_{C_+}\left(V \circ G, B_R^+\right) = 0$ for all $R \geq R_0$.*

Proof. Given $g \in L^\infty(Z)_+$, with $g(z) \geq \theta(z)$ almost everywhere on Z we consider the following admissible homotopy

$$h_3(\beta, x) = V \circ \left(\beta g K + (1-\beta) G\right)(x) \text{ for all } (\beta, x) \in [0,1] \times C_+.$$

We will show that we can choose $R_0 > 0$ and $g \in L^\infty(Z)_+$ such that

$$x \notin h_3(\beta, x) \text{ for all } \beta \in [0,1], \text{ all } x \in \partial B_R^+ \text{ and all } R \geq R_0. \quad (3.17)$$

We argue indirectly. Suppose that we can find $\{x_n\}_{n \geq 1} \subseteq C_+$ such that

$$A(x_n) = \beta_n g K(x_n) + (1-\beta_n)u_n \text{ with } u_n \in G(x_n)$$

$$\text{and } \|x_n\| \to \infty,\ \beta_n \to \beta \in [0,1].$$

We set $y_n = \dfrac{x_n}{\|x_n\|}$, $n \geq 1$. We may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(Z) \text{ and } y_n \to y \text{ in } L^p(Z).$$

We have

$$A(y_n) = \beta_n g K(y_n) + (1-\beta_n)\frac{u_n}{\|x_n\|^{p-1}},\ n \geq 1.$$

By virtue of hypothesis $H(j)(iii)$

$$\frac{|u_n(z)|}{\|x_n\|^{p-1}} \leq \frac{\alpha(z)}{\|x_n\|^{p-1}} + c|y_n(z)|^{p-1} \text{ a.e. on } Z, \tag{3.18}$$

$$\Rightarrow \left\{\frac{u_n}{\|x_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'}(Z) \text{ is bounded.}$$

Hence we may assume that

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h \text{ in } L^{p'}(Z).$$

For every $\varepsilon > 0$ and $n \geq 1$ we introduce the set

$$D_{\varepsilon,n} = \left\{z \in Z : x_n(z) > 0,\ \frac{u_n(z)}{x_n(z)^{p-1}} \geq \theta(z) - \varepsilon\right\}.$$

Note that $x_n(z) \to +\infty$ almost everywhere on $\{y > 0\}$. So by virtue of hypothesis $H(j)(iv)$ we have that $\chi_{D_{\varepsilon,n}}(z) \to 1$ almost everywhere on $\{y > 0\}$. Observe that

$$\left\|\left(1-\chi_{D_{\varepsilon,n}}\right)\frac{u_n}{\|x_n\|^{p-1}}\right\|_{L^{p'}(\{y>0\})} \to 0,$$

$$\Rightarrow \chi_{D_{\varepsilon,n}}\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} h \text{ in } L^{p'}(\{y>0\}). \tag{3.19}$$

We have

$$\chi_{D_{\varepsilon,n}}(z)\frac{u_n(z)}{\|x_n\|^{p-1}} = \chi_{D_{\varepsilon,n}}(z)\frac{u_n(x)}{x_n(z)^{p-1}}y_n(z)^{p-1}$$
$$\geq \chi_{D_{\varepsilon,n}}(z)\left(\theta(z)-\varepsilon\right)y_n(z)^{p-1} \text{ a.e. on } Z \text{ (recall } y_n \geq 0).$$

Passing to the limit as $n \to \infty$, we obtain

$$h(z) \geq \left(\theta(z)-\varepsilon\right)y(z)^{p-1} \text{ a.e. on } \{y > 0\} \text{ (see (3.19)).}$$

Since $\varepsilon > 0$ was arbitrary, we let $\varepsilon \downarrow 0$ to obtain

$$h(z) \geq \theta(z)y(z)^{p-1} \text{ a.e. on } \{y > 0\}.$$

On the other hand from (3.18) it is clear that $h(z) = 0$ almost everywhere on $\{y = 0\}$. Since $y \in C_+$, we have that $Z = \{y > 0\} \cup \{y = 0\}$ and so finally we can say that

$$h(z) \geq \theta(z) y(z)^{p-1} \ \text{ a.e. on } Z. \tag{3.20}$$

Also by virtue of hypotheses $H(j)(iii)$ and (v) and also because of (3.3), we have

$$h(z) \leq c_4 y(z)^{p-1} \ \text{ a.e. on } Z \text{ for some } \ c_4 > 0. \tag{3.21}$$

From (3.20) and (3.21) it follows that

$$h(z) = g_0(z) y(z)^{p-1} \ \text{ a.e. on } \ Z, \text{ with } g_0 \in L^\infty(Z)_+, \ g_0(z) \geq \theta(z) \ \text{ a.e. on } Z. \tag{3.22}$$

Recall that

$$A(y_n) = \beta_n g K(y_n) + (1 - \beta_n) \frac{u_n}{\|x_n\|^{p-1}}, \ \ n \geq 1. \tag{3.23}$$

As before acting with the test function $y_n - y$ and exploiting the generalized pseudomonotonicity of A and the Kadec-Klee property of $L^p(Z, \mathbb{R}^N)$, we obtain that $y_n \to y$ in $W_0^{1,p}(Z)$. So if we pass to the limit as $n \to \infty$ in (3.23), we obtain

$$A(y) = \beta g K(y) + (1 - \beta) g_0 K(y) \ \ (\text{see } (3.22)).$$

Set $\widehat{g}_0 = \beta g + (1 - \beta) g_0$. We have

$$A(y) = \widehat{g}_0 K(y),$$

$$\Rightarrow -\operatorname{div}\left(\|Dy(z)\|^{p-2} Dy(z)\right) = \widehat{g}_0(z) |y(z)|^{p-2} y(z) \ \text{ a.e. on } \ Z, \ y|_{\partial Z} = 0. \tag{3.24}$$

Because $\widehat{g}_0(z) \geq \theta(z) \geq \lambda_1$ almost everywhere on Z and the last inequality is strict on a set of positive measure, we have that

$$\widehat{\lambda}_1(\widehat{g}_0) < \widehat{\lambda}_1(\lambda_1) = 1.$$

Then from (3.24) it follows that $y \in C_+ \backslash \{0\}$ can not be the principal eigenfunction of the weighted eigenvalue problem with weight $\widehat{g}_0 \in L^\infty(Z)_+$ and so y must change sign, a contradiction. Therefore (3.17) is true. Exploiting the homotopy invariance of the fixed-point index i_{C_+}, we have

$$i_{C_+}\left(V \circ G, B_R^+\right) = i_{C_+}\left(V \circ (\widehat{g}_0 K), B_R^+\right) \ \text{ for all } \ R \geq R_0. \tag{3.25}$$

Next we will compute $i_{C_+}\left(V \circ (\widehat{g}_0 K), B_R^+\right)$. To this end let $h \in L^q(Z)_+$, $h \neq 0$, and consider the admissible homotopy

$$h_4(\beta, x) = V \circ ((\widehat{g}_0 K)(x) + \beta h) \ \text{ for all } \ (\beta, x) \in [0,1] \times C_+.$$

We claim that $x \neq h_4(\beta, x)$ for all $\beta \in [0,1]$, all $x \in \partial B_R^+$, and all $R > 0$. Indeed, if $x = h_4(\beta, x)$, then we have

$$\left\{ \begin{array}{l} -\mathrm{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = \widehat{g}_0(z)|x(z)|^{p-2}x(z) + \beta h(z) \text{ a.e. on } Z \\ x|_{\partial Z} = 0, \;\; 0 \leq \beta \leq 1. \end{array} \right\} \tag{3.26}$$

Because $\widehat{\lambda}_1(g_0) < 1$, by virtue of Proposition 4.1 of Godoy-Gossez-Paczka [8], we know that (3.26) has no solutions in $C_+\backslash\{0\}$, a contradiction since $x \in C_+\backslash\{0\}$. Therefore

$$i_{C_+}\left(V \circ ((g_0K) + h), B_R^+\right) = 0 \text{ for all } R > 0.$$

So from the homotopy invariance of the fixed-point index i_{C_+}, it follows that

$$i_{C_+}\left(V \circ (g_0K), B_R^+\right) = 0 \text{ for all } R > 0. \tag{3.27}$$

From (3.25) and (3.27), we obtain

$$i_{C_+}\left(V \circ G, B_R^+\right) = 0 \text{ for all } R \geq R_0.$$

□

Now we are ready for the existence theorem concerning positive solutions for problem (1.1).

Theorem 3.6. *If hypotheses $H(j)$ hold, then there exists a solution $x \in C_0^1(\overline{Z})$ of (1.1) such that $x(z) > 0$ for all $z \in Z$ and $\frac{\partial x}{\partial n}(z) < 0$ for all $z \in \partial Z$.*

Proof. Let $0 < \rho \leq \rho_0$ and $R \geq R_0$. Because of the additivity property of the fixed-point index i_{C_+}, we have

$$\begin{aligned} &i_{C_+}\left(V \circ G, B_R^+\right) = i_{C_+}\left(V \circ G, B_\rho^+\right) + i_{C_+}\left(V \circ G, B_R^+\backslash\overline{B}_\rho^+\right), \\ \Rightarrow &0 = 1 + i_{C_+}\left(V \circ G, B_R^+\backslash\overline{B}_\rho^+\right) \text{ (see Propositions (3.4) and (3.5))}, \\ \Rightarrow &i_{C_+}\left(V \circ G, B_R^+\backslash\overline{B}_\rho^+\right) = -1. \end{aligned}$$

This means that we can find $x \in B_R^+\backslash\overline{B}_\rho^+$ (hence $x \neq 0$, $x \geq 0$) such that

$$\begin{aligned} &A(x) = u \text{ with } u \in G(x) \\ \Rightarrow &-\mathrm{div}\left(\|Dx(z)\|^{p-2}Dx(z)\right) = u(z) \text{ a.e. on } Z,\; x|_{\partial Z} = 0, \end{aligned}$$

where $u(z) \in \partial j_1(z, x(z)) = \partial j(z, x(z))$ almost everywhere on Z (see (3.3). Therefore $x \in W_0^{1,p}(Z)$ is a positive solution of problem (1.1). Moreover, from nonlinear regularity theory (see for example Gasinski-Papageorgiou [7], pages 115-116), we have that $x \in C_0^1(\overline{Z})$.

Finally, since $u(z) \geq 0$ almost everywhere on Z (see hypothesis $H(j)$) we can apply the strict maximum principle of Vazquez [16] and conclude that

$$x(z) > 0 \ \text{ for all } \ z \in Z \ \text{ and } \ \frac{\partial x}{\partial n}(z) \ \text{ for all } \ z \in \partial Z.$$

□

Remark 3.7. The conclusion of Theorem 3.6 implies that $x \in intC_0^1(\overline{Z})_+$.

The above analysis suggests that we can have a multiplicity result if the asymptotic conditions in $H(j)(iv)$ and (v) are symmetric at $\pm\infty$ and at $0^{\pm}$. So the new hypotheses on the nonsmooth potential are the following:

$H(j)'$ $j : Z \times \mathbb{R} \to \mathbb{R}$ is a function such that $0 \in \partial j(z, 0)$ almost everywhere on Z, for almost all $z \in Z$, all $x \in \mathbb{R}$ and all $u \in \partial j(z, x)$ we have $ux \geq 0$, hypotheses $H(j)(i), (ii), (iii)$ hold and

(iv) there exists $\theta \in L^\infty(Z)_+$ such that $\theta(z) \geq \lambda_1$ almost everywhere on Z, the inequality is strict on a set of positive measure and

$$\liminf_{|x| \to +\infty} \frac{u}{|x|^{p-2}x} \geq \theta(z) \text{ uniformly for almost all } z \in Z$$

and all $u \in \partial j(z, x)$;

(v) there exists $\eta \in L^\infty(Z)_+$ such that $\eta(z) \leq \lambda_1$ almost everywhere on Z with strict inequality on a set of positive measure and

$$\limsup_{|x| \to 0} \frac{u}{|x|^{p-2}x} \leq \eta(z) \text{ uniformly for almost all } z \in Z$$

and all $u \in \partial j(z, x)$.

Then we have the following multiplicity result.

Theorem 3.8. *If hypotheses $H(j)'$ hold, then problem* (1.1) *has at least two nontrivial solutions $x_0, x_1 \in C_0^1(\overline{Z})$ such that $x_0(z) < 0 < x_1(z)$ for all $z \in \overline{Z}$ and $\frac{\partial x_1}{\partial n}(z) < 0 < \frac{\partial x_0}{\partial n}(z)$ for all $z \in \partial Z$.*

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