Differential and Integral Equations Volume 19, Number 2 (2006), 211–222

LOW REGULARITY GLOBAL SOLUTIONS OF THE DIRAC-KLEIN-GORDON EQUATIONS IN ONE SPACE DIMENSION

Nikolaos Bournaveas and Dominic Gibbeson
University of Edinburgh, School of Mathematics
JCMB, King's Buildings, Edinburgh EH9 3JZ, UK

(Submitted by: Tohru Ozawa)

Abstract. We prove global existence for the Dirac-Klein-Gordon equations in one space dimension with $\psi \in L^2$ (charge class) and $\phi \in H^{1/4}$. This improves the global existence result of Fang [7] by $1/4+\epsilon$ derivatives in ϕ. The proof relies on bilinear estimates for solutions of the Dirac equation and a decomposition of the spinor field into 'left' and 'right' spinors.

1. Introduction

We study global low regularity solutions of the Dirac-Klein-Gordon equations in one space dimension with Yukawa interaction:

$$i\mathcal{D}\psi = (M - \mathrm{g}\phi)\psi \tag{1.1a}$$

$$\Box\phi = \mathrm{g}\overline{\psi}\psi - m^2\phi. \tag{1.1b}$$

We prescribe initial data at $t = 0$:

$$\psi(0,\cdot) = g\ ,\ \phi(0,\cdot) = f\ ,\ \phi_t(0,\cdot) = h. \tag{1.2}$$

Here $\psi(t,x)$ is a 2-spinor[1] field and $\phi(t,x)$ is a scalar field defined on $[0,\infty)\times\mathbb{R}$. The Dirac operator is defined by $\mathcal{D}\psi = \gamma^0\partial_t\psi + \gamma^1\partial_x\psi$ where $\gamma^0 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0\end{smallmatrix}\right)$ and $\gamma^1 = \left(\begin{smallmatrix} i & 0 \\ 0 & -i\end{smallmatrix}\right)$. We define $\overline{\psi} = \psi^\dagger\gamma^0$, where $\psi^\dagger$ is the complex conjugate transpose of ψ. The wave operator is defined by $\Box = \partial_t^2 - \partial_x^2$. Finally, $M \geq 0$, $m \geq 0$ and g, are constants.

Accepted for publication: September 2005.
AMS Subject Classifications: 35L05, 35B30, 35B45.
Supported by the EU network HARP (HPRP-CT-2001-00273-HARP)..

[1]The spinor dimension is $s = 2^{\left[\frac{n+1}{2}\right]}$ where $n+1$ is the dimension of spacetime. Thus, in $1+1$ and $2+1$ dimensions we have 2-spinors and in $3+1$ dimensions we have 4-spinors. See [14] for more.

This system was first studied by Chadam and Glassey in [4] and [5] where the global existence was proved and the asymptotic behavior was studied of classical solutions with $\psi(t) \in H^1$, $\phi(t) \in H^1$ using energy estimates and certain invariants of the system (see also the more recent work [14] on invariants). This existence result was improved in [2] (see [9] for a different proof) to global existence with $\psi(t) \in L^2$, $\phi(t) \in H^1$. The proof of local existence in [2] relies on a null form estimate of Klainerman and Machedon for solutions of the wave equation which is adapted to the setting of the Dirac equation. Once local existence has been established, global existence is derived as a consequence of conservation of charge. This approach has also been used more recently in [13] to study the nonlinear Dirac equation.

Fang [7] has achieved the following improvement to the results of [2] and [9]:

Theorem 1. (*Fang* [7]) *Let* $g \in L^2(\mathbb{R})$, $f \in H^{1/2+\epsilon}(\mathbb{R})$, *and* $h \in H^{-1/2+\epsilon}(\mathbb{R})$, *where* $\epsilon > 0$. *Then there exists a unique global solution* (ϕ, ψ) *to the Dirac-Klein-Gordon equations* (1.1) *with initial data* (1.2) *with*

$$\begin{aligned}&\psi \in C^0\left([0,\infty), L^2(\mathbb{R})\right)\\&\phi \in C^0\big([0,\infty), H^{1/2+\epsilon}(\mathbb{R})\big) \cap C^1\big([0,\infty), H^{-1/2+\epsilon}(\mathbb{R})\big).\end{aligned}$$

In this paper we first sketch a simpler proof of Fang's result and then improve it to $\psi(t) \in L^2$, $\phi(t) \in H^{1/4}$.

Theorem 2. (*Global Existence*) *Consider initial data* $g \in L^2(\mathbb{R})$, $f \in H^r(\mathbb{R})$, $h \in H^{r-1}(\mathbb{R})$, *where* $\frac{1}{4} \leq r < \frac{1}{2}$. *There exists a unique global solution* (ϕ, ψ) *to the Dirac-Klein-Gordon equations* (1.1) *with initial data as in* (1.2) *with*

$$\begin{aligned}&\phi \in C^0\left([0,\infty), H^r(\mathbb{R})\right) \cap C^1\left([0,\infty), H^{r-1}(\mathbb{R})\right)\\&\psi \in C^0\left([0,\infty), L^2(\mathbb{R})\right).\end{aligned}$$

There are three main ingredients in our proof. The first is a new bilinear estimate for solutions of the Dirac equation with quadratic right-hand sides which are products of a solution of the wave equation and a solution of the Dirac equation (Proposition 1). The second ingredient is a decomposition of the spinor field. It corresponds to the decomposition of a 4-spinor field in three space dimensions into left and right spinors as in Klainerman [11] and Bournaveas [1] which reduces the four-dimensional Dirac equation into two Pauli-type equations. Here, the one-dimensional Dirac equation is decomposed into two simple-transport equations. This makes it much easier to handle the Fourier integral operators that come up in the bilinear estimates of Proposition 1. In the context of (1.1) this decomposition is as follows: we write first $\psi = \binom{\psi_1}{\psi_2}$ and then introduce the two complex-valued scalar fields $u = i\psi_1 + \psi_2$ and $v = i\psi_1 - \psi_2$. We define the simple transport operators $\partial_\pm = \partial_t \pm \partial_x$. Then the system (1.1) becomes

$$\partial_+ u = -(M - \mathrm{g}\phi)v \tag{1.3a}$$

$$\partial_- v = (M - \mathrm{g}\phi)u \tag{1.3b}$$

$$\Box\phi = \mathtt{g} Im(\overline{u}v) - m^2\phi. \tag{1.3c}$$

The third ingredient of our proof is the so called 'two-step' iteration [12]. To prove an existence theorem one usually defines a sequence $(\psi^{(n)}, \phi^{(n)})$ of approximate solutions and shows that it converges to a limit that solves the equations. An induction argument is involved which shows that certain estimates for the $(n-1)$-th iterate imply the same estimates for the n-th iterate. In our case this doesn't work. It can be shown that the information contained in the estimates for $\|\psi^{(n-1)}(t)\|_{L^2}$ and $\|\phi^{(n-1)}(t)\|_{H^{1/4}}$ is not enough to prove the same estimates for $\|\psi^{(n)}(t)\|_{L^2}$ and $\|\phi^{(n)}(t)\|_{H^{1/4}}$. A similar situation arises in the study of wave maps type problems in [12]. To deal with it we use the 'two-step' technique of [12]. The $n-th$ iterates $(\psi^{(n)}, \phi^{(n)})$ solve equations with right-hand sides containing $(\psi^{(n-1)}, \phi^{(n-1)})$ which themselves solve another system of equations. Taking this into account we express $(\psi^{(n)}, \phi^{(n)})$ in terms of both $(\psi^{(n-1)}, \phi^{(n-1)})$ and $(\psi^{(n-2)}, \phi^{(n-2)})$ and use the estimates for the $(n-1)$-th and $(n-2)$-th iterates to derive the estimates for the n-th iterate.

2. Bilinear Estimates

The proof of Theorem 2 uses the following estimate:

Proposition 1. *Fix initial data*

$$\zeta_0 \in L^2(\mathbb{R})\ ,\ \psi_0 \in H^{-1/4}(\mathbb{R})\ ,\ \phi_0 \in H^{1/4}(\mathbb{R})\ ,\ \phi_1 \in H^{-3/4}(\mathbb{R})$$

and right-hand sides

$$G \in L^1([0,T]; H^{-1/4}(\mathbb{R}))\ ,\ F \in L^1([0,T]; H^{-3/4}(\mathbb{R})).$$

Let the 2-spinor field ζ solve

$$\mathcal{D}\zeta = i\phi\psi \quad , \quad \zeta(0,\cdot) = \zeta_0, \tag{2.1}$$

where the scalar field ϕ *and the 2-spinor field* ψ *solve*

$$\mathcal{D}\psi = G \ , \ \psi(0,\cdot) = \psi_0 \tag{2.2}$$

$$\Box\phi = F \ , \ \phi(0,\cdot) = \phi_0 \ , \ \phi_t(0,\cdot) = \phi_1. \tag{2.3}$$

Then, for each $t \in [0,T]$*, we have*

$$\|\zeta(t)\|_{L^2(\mathbb{R})} \le \|\zeta_0\|_{L^2} + C(t)\Big[\|\psi_0\|_{H^{-1/4}(\mathbb{R})} + \int_0^t \big\|G(t',\cdot)\big\|_{H^{-1/4}(\mathbb{R})}\,dt'\Big]\cdot$$
$$\times\Big[\|\phi_0\|_{H^{1/4}(\mathbb{R})} + \|\phi_1\|_{H^{-3/4}(\mathbb{R})} + \int_0^t \big\|F(t',\cdot)\big\|_{H^{-3/4}(\mathbb{R})}\,dt'\Big]. \tag{2.4}$$

Proof. We consider the special case in which ζ_0, F, G and ϕ_0 vanish, as the result in the general case can easily be derived from the estimate in the special case. In other words we now have

$$\mathcal{D}\zeta = i\phi\psi \quad , \quad \zeta(0,\cdot) = 0, \tag{2.5}$$

where

$$\mathcal{D}\psi = 0 \ , \ \psi(0,\cdot) = \psi_0 \tag{2.6}$$

$$\Box\phi = 0 \ , \ \phi(0,\cdot) = 0 \ , \ \phi_t(0,\cdot) = \phi_1. \tag{2.7}$$

Write $\psi = \binom{\psi_1}{\psi_2}$, $\zeta = \binom{\zeta_1}{\zeta_2}$ and define the complex-valued scalar fields

$$U = i\zeta_1 + \zeta_2,\ V = i\zeta_1 - \zeta_2 \ \ ; \ \ u = i\psi_1 + \psi_2,\ \mathrm{v} = i\psi_1 - \psi_2.$$

The Dirac equations in (2.5) and (2.6) then reduce to simple transport equations

$$\partial_+ U = \phi\mathrm{v} \ , \ U(0) = 0 \tag{2.8}$$

$$\partial_- V = -\phi u \ , \ V(0) = 0, \tag{2.9}$$

where u and v solve

$$\partial_+ u = 0 \ , \ u(0) = u_0 := i\psi_1(0) + \psi_2(0) \tag{2.10}$$

$$\partial_- \mathrm{v} = 0 \ , \ \mathrm{v}(0) = \mathrm{v}_0 := i\psi_1(0) - \psi_2(0) \tag{2.11}$$

and ϕ solves (2.7). We show

$$\|U(t)\|_{L^2(\mathbb{R})} \le C(t)\,\|\phi_1\|_{H^{-3/4}(\mathbb{R})}\,\|\mathrm{v}_0\|_{H^{-1/4}(\mathbb{R})}\,. \tag{2.12}$$

The corresponding estimate for V follows similarly and together they imply the desired estimate for ζ. We have the following representation for U.

$$\widehat{U}(t,\xi) = \frac{i}{2}\int_{-\infty}^{\infty}\Big[e^{-it\eta}\,\frac{\sin t(\xi-\eta)}{\xi-\eta} - \frac{\sin t\xi}{\xi}\Big]\,\frac{\widehat{\phi_1}(\eta)}{\eta}\,\widehat{\mathrm{v}_0}\,(\xi-\eta)\,d\eta. \tag{2.13}$$

Define

$$A(\eta)=\frac{\widehat{\phi_1}(\eta)}{(1+|\eta|)^{3/4}}\ ,\ B(\eta)=\frac{\widehat{\mathrm{v}_0}\,(\eta)}{(1+|\eta|)^{1/4}}.$$

Let $H(\xi)$ be a test function and define the bilinear form

$$\mathcal{Q}[A,B](t):=\iint\Big[e^{-it\eta}\,\frac{\sin t(\xi-\eta)}{\xi-\eta}-\frac{\sin t\xi}{\xi}\Big]\cdot \tag{2.14}$$

$$\times\,\frac{(1+|\eta|)^{3/4}(1+|\xi-\eta|)^{1/4}}{\eta}\,A(\eta)B(\xi-\eta)H(\xi)\,d\xi d\eta. \tag{2.15}$$

By duality, it suffices to show that

$$|\,\mathcal{Q}[A,B](t)\,|\le c(t)\,\|A\|_{L^2(\mathbb{R})}\,\|B\|_{L^2(\mathbb{R})}\,\|H\|_{L^2(\mathbb{R})}\,. \tag{2.16}$$

We may assume that $A,B,H\ge 0$. Write $\mathcal{Q}[A,B](t)=\mathcal{Q}_1(t)+\mathcal{Q}_2(t)$ where

$$\mathcal{Q}_1(t)=\iint_{|\eta|\ge 1}\cdots\,d\xi d\eta\ ,\ \mathcal{Q}_2(t)=\iint_{|\eta|<1}\cdots\,d\xi d\eta.$$

Consider first $|\eta|\ge 1$. Using

$$\Big|\frac{\sin(t\,|\xi-\eta|)}{|\xi-\eta|}\Big|\le\frac{c(t)}{1+|\xi-\eta|},\qquad \Big|\frac{\sin(t\,|\xi|)}{|\xi|}\Big|\le\frac{c(t)}{1+|\xi|},$$

and $|\eta|\simeq 1+|\eta|$ we have

$$\mathcal{Q}_1(t)\le c(t)\,[\mathcal{Q}_{11}+\mathcal{Q}_{12}]\,,$$

where

$$\mathcal{Q}_{11}=\iint\frac{A(\eta)B(\xi-\eta)H(\xi)}{(1+|\xi-\eta|)^{3/4}(1+|\eta|)^{1/4}}\,d\xi d\eta \tag{2.17}$$

$$\mathcal{Q}_{12}=\iint\frac{(1+|\xi-\eta|)^{1/4}}{(1+|\xi|)(1+|\eta|)^{1/4}}A(\eta)B(\xi-\eta)H(\xi)\,d\xi d\eta. \tag{2.18}$$

To estimate $\mathcal{Q}_{11}$ we change variables $\xi\mapsto\xi+\eta$. We then have

$$\begin{aligned}\mathcal{Q}_{11}&=\iint\frac{A(\eta)B(\xi)H(\xi+\eta)}{(1+|\xi|)^{3/4}(1+|\eta|)^{1/4}}\,d\xi d\eta\\&=\iint_{|\xi|\ge|\eta|}\cdots\,d\xi d\eta+\iint_{|\xi|<|\eta|}\cdots\,d\xi d\eta.\end{aligned} \tag{2.19}$$

For the first integral in (2.19) we have $(1+|\xi|)^{3/4}(1+|\eta|)^{1/4}\ge(1+|\eta|)$, therefore,

$$\iint_{|\xi|\ge|\eta|}\le\iint\frac{A(\eta)B(\xi)H(\xi+\eta)}{1+|\eta|}\,d\xi d\eta$$

$$
\begin{aligned}
&= \int \frac{A(\eta)}{1+|\eta|} \int B(\xi) H(\xi+\eta) d\xi d\eta \\
&\le \int \frac{A(\eta)}{1+|\eta|} \Big(\int |B(\xi)|^2 d\xi \Big)^{1/2} \Big(\int |H(\xi+\eta)|^2 d\xi \Big)^{1/2} d\eta \\
&= \|B\|_{L^2(\mathbb{R})} \|H\|_{L^2(\mathbb{R})} \int \frac{A(\eta)}{1+|\eta|} d\eta \\
&\le \|B\|_{L^2(\mathbb{R})} \|H\|_{L^2(\mathbb{R})} \Big(\int |A(\eta)|^2 d\eta \Big)^{1/2} \Big(\int \frac{1}{(1+|\eta|)^2} d\eta \Big)^{1/2} \\
&\le C \|B\|_{L^2(\mathbb{R})} \|H\|_{L^2(\mathbb{R})} \|A\|_{L^2(\mathbb{R})} .
\end{aligned}
$$

The estimate for the second integral in (2.19) is similar. To estimate $\mathcal{Q}_{12}$ in (2.18) we use $\frac{(1+|\xi-\eta|)^{1/4}}{(1+|\xi|)(1+|\eta|)^{1/4}} \le \frac{1}{(1+|\xi|)^{3/4}(1+|\eta|)^{1/4}} + \frac{1}{1+|\xi|}$ and work as above.

Next we estimate $\mathcal{Q}_2(t)$. We make use of the fact that the symbol in the brackets in (2.15) cancels the singularity at $\eta = 0$. We have:

$$
\begin{aligned}
m(\xi,\eta;t) &:= \Big| \Big[e^{-it\eta} \frac{\sin t(\xi-\eta)}{\xi-\eta} - \frac{\sin t\xi}{\xi} \Big] \frac{(1+|\eta|)^{3/4}(1+|\xi-\eta|)^{1/4}}{\eta} \Big| \\
&\le c(t).
\end{aligned}
\tag{2.20}
$$

Indeed, writing $e^{-it\eta} = 1 - 2\sin^2 \frac{t\eta}{2} - i \sin t\eta$ and using the fact that $(1+|\eta|)^{3/4}$ is now bounded, we have

$$
m(\xi,\eta;t) \le C \sum_{k=1}^{3} m_k(\xi,\eta;t), \tag{2.21}
$$

where

$$
m_1(\xi,\eta;t) = \Big| \frac{\sin t(\xi-\eta)}{\xi-\eta} - \frac{\sin t\xi}{\xi} \Big| \frac{(1+|\xi-\eta|)^{1/4}}{|\eta|} \tag{2.22}
$$

$$
m_2(\xi,\eta;t) = \Big| \sin \frac{t\eta}{2} \Big|^2 \frac{|\sin t(\xi-\eta)|}{|\xi-\eta|} \frac{(1+|\xi-\eta|)^{1/4}}{|\eta|} \tag{2.23}
$$

$$
m_3(\xi,\eta;t) = |\sin t\eta| \frac{|\sin t(\xi-\eta)|}{|\xi-\eta|} \frac{(1+|\xi-\eta|)^{1/4}}{|\eta|}. \tag{2.24}
$$

For m_1 we use the fact that $g(x) = \frac{\sin(tx)}{x}$ satisfies $|g'(x)| \le \frac{c(t)}{(1+|x|)}$ and the mean value theorem to get that, for some $\theta \in [0,1]$, we have

$$
\begin{aligned}
|m_1| &= \frac{|g(\xi-\eta) - g(\xi)|}{|\eta|} (1+|\xi-\eta|)^{1/4} \\
&\le \left| g'(\theta\xi + (1-\theta)(\xi-\eta)) \right| (1+|\xi-\eta|)^{1/4}
\end{aligned}
$$

$$\leq \frac{c(t)}{1+|\xi-(1-\theta)\eta|}(1+|\xi-\eta|)^{1/4}.$$

Recall that $|\eta| \leq 1$, therefore $1+|\xi-(1-\theta)\eta| \approx (1+|\xi|)$ and also $(1+|\xi-\eta|)^{1/4} \approx (1+|\xi|)^{1/4}$. Therefore,

$$m_1(\xi,\eta;t) \leq \frac{c(t)}{(1+|\xi|)^{3/4}} \leq c(t).$$

The terms m_2 and m_3 are estimated similarly. This completes the proof of (2.20). Using that estimate we can bound $\mathcal{Q}_2(t)$ as follows:

$$\begin{aligned}
|\mathcal{Q}_2(t)| &\leq c(t)\int_{|\eta|\leq 1} A(\eta)\int_{\mathbb{R}} B(\xi-\eta)H(\xi)d\xi d\eta \\
&\leq c(t)\int_{|\eta|\leq 1} A(\eta)\Big(\int_{\mathbb{R}} B(\xi-\eta)^2 d\xi\Big)^{1/2}\Big(\int_{\mathbb{R}} H(\xi)^2 d\xi\Big)^{1/2} d\eta \\
&\leq \|B\|_{L^2(\mathbb{R})}\|H\|_{L^2(\mathbb{R})}\int_{|\eta|\leq 1} A(\eta)d\eta \\
&\leq C\|A\|_{L^2(\mathbb{R})}\|B\|_{L^2(\mathbb{R})}\|H\|_{L^2(\mathbb{R})}, \qquad (2.25)
\end{aligned}$$

where $C = |\{|\eta| \leq 1\}|^{1/2}$. This completes the proof of the proposition. □

3. Global existence

3.1. **Proof of Theorem 1.** In this section we give a proof of Theorem 1 which is simpler than the one in [7]. It is based on the observation that the proof in [2] can be adapted to deal with $\phi \in H^{1/2+\epsilon}$.

Proof of Theorem 1. We only present the relevant a priori estimates for local existence. Global existence is a consequence of conservation of charge, see [2, 7, 9] for details.

Let (ϕ,ψ) be a solution of (1.1). For simplicity of exposition we take $M = m = 0$ and $\mathrm{g} = 1$. Define

$$X(T) = \sup_{0\leq t\leq T}\left[\|\psi(t)\|_{L^2} + \|\phi(t)\|_{H^{1/2+\epsilon}} + \|\partial_t\phi(t)\|_{H^{-1/2+\epsilon}}\right]$$

and $D_0 = \|\psi(0)\|_{L^2} + \|\phi(0)\|_{H^{1/2+\epsilon}} + \|\partial_t\phi(0)\|_{H^{-1/2+\epsilon}}$. The charge estimate gives

$$\begin{aligned}
\sup_{0\leq t\leq T}\|\psi(t)\|_{L^2} &\leq C\Big(D_0 + \int_0^T \|\mathcal{D}\psi(t)\|_{L^2}\,dt\Big) \\
&\leq C\Big(D_0 + \int_0^T \|\phi(t)\psi(t)\|_{L^2}\,dt\Big)
\end{aligned}$$

$$\leq C\Big(D_0 + \int_0^T \|\phi(t)\|_{L^\infty} \|\psi(t)\|_{L^2}\, dt\Big)$$

$$\leq C\Big(D_0 + \int_0^T \|\phi(t)\|_{H^{1/2+\epsilon}} \|\psi(t)\|_{L^2}\, dt\Big) \leq C\big(D_0 + TX(T)^2\big). \tag{3.1}$$

To estimate $\|\phi(t)\|_{H^{1/2+\epsilon}}$ we write $\phi = \phi_L + \phi_N$ where ϕ_L is the solution of the IVP: $\Box\phi_L = 0$, $\phi_L(0) = f$, $\partial_t\phi_L(0) = h$, and ϕ_N is the solution of the IVP:

$$\Box\phi_N = \overline{\psi}\psi \tag{3.2a}$$

$$\phi_N(0) = 0\ ,\ \ \partial_t\phi_N(0) = 0. \tag{3.2b}$$

The generalized energy estimate gives

$$\sup_{0\leq t\leq T} [\|\phi_L(t)\|_{H^{1/2+\epsilon}} + \|\partial_t\phi_L(t)\|_{H^{-1/2+\epsilon}}] \leq C(T)D_0. \tag{3.3}$$

Combining Lemmata 1 and 3 of [2], we have

$$\left\|\overline{\psi}\psi\right\|_{L^2([0,T]\times\mathbb{R})} \leq C(T)\Big(D_0 + \int_0^T \|\mathcal{D}\psi(t)\|_{L^2}\, dt\Big)^2.$$

By the estimates that lead to (3.1) we have

$$\int_0^T \|\mathcal{D}\psi(t)\|_{L^2}\, dt \leq C(T)TX(T)^2.$$

Therefore,

$$\left\|\overline{\psi}\psi\right\|_{L^2([0,T]\times\mathbb{R})} \leq C(T)\left(D_0 + TX(T)^2\right)^2 \tag{3.4}$$

hence also

$$\int_0^T \|\overline{\psi(t)}\psi(t)_{L^2}\|dt \leq T^{1/2}\left\|\overline{\psi}\psi\right\|_{L^2([0,T]\times\mathbb{R})} \leq C(T)(D_0 + TX(T)^2)^2. \tag{3.5}$$

This estimate says that the right-hand side of equation (3.2a) is in $L^1([0,T];$ $L^2(\mathbb{R}))$, therefore, by the linear theory, $\phi_N(t) \in H^1$ and $\partial_t\phi_N(t) \in L^2$ (i.e., ϕ_N is smoother than ϕ_L) and moreover (energy estimate)

$$\sup_{0\leq t\leq T} [\|\phi_N(t)\|_{H^1} + \|\partial_t\phi_N(t)\|_{L^2}] \leq C(T)\int_0^T \|\Box\phi_N(t)\|_{L^2}\, dt \tag{3.6}$$

$$\leq C(T)T^{1/2}(D_0 + TX(T)^2)^2. \tag{3.7}$$

Hence, a fortiori,

$$\sup_{0\leq t\leq T} [\|\phi_N(t)\|_{H^{1/2+\epsilon}} + \|\partial_t\phi_N(t)\|_{H^{-1/2+\epsilon}}] \leq C(T)T^{1/2}(D_0 + TX(T)^2)^2. \tag{3.8}$$

Estimates (3.1), (3.3) and (3.8) allow us to bootstrap $X(T)$. □

3.2. **Proof of Theorem 2.**

Proof. We shall first prove local existence and then derive global existence as a consequence of conservation of charge. For simplicity of exposition we set $M = m = 0$([2]) and g = 1. Define an iteration scheme as follows: start with $\psi^{(-1)} \equiv 0$, $\phi^{(-1)} \equiv 0$, and define inductively $(\psi^{(n)}, \phi^{(n)})$ to be the solution of the initial-value problems

$$\mathcal{D}\psi^{(n)} = i\phi^{(n-1)}\psi^{(n-1)}\ ,\ \psi^{(n)}(0,\cdot) = g \tag{3.9}$$

$$\Box\phi^{(n)} = \overline{\psi^{(n-1)}}\psi^{(n-1)}\ ,\ \phi^{(n)}(0,\cdot) = f\ ,\ \phi_t^{(n)}(0,\cdot) = h. \tag{3.10}$$

Notice that $\psi^{(0)}$ and $\phi^{(0)}$ solve the linear homogeneous Cauchy problems

$$\mathcal{D}\psi^{(0)} = 0,\ \psi^{(0)}(0) = g$$

and

$$\Box\phi^{(0)} = 0,\ \phi^{(0)}(0) = f,\ \phi_t^{(0)}(0) = h.$$

Fix $T > 0$, to be determined later, and define

$$X_n(T) = \sup_{0\le t\le T}\left(\left\|\psi^{(n)}(t)\right\|_{L^2} + \left\|\phi^{(n)}(t)\right\|_{H^r} + \left\|\phi_t^{(n)}(t)\right\|_{H^{r-1}}\right) \tag{3.11}$$

$$\Delta_n(T) = \sup_{0\le t\le T}\Big(\left\|\psi^{(n)}(t) - \psi^{(n-1)}(t)\right\|_{L^2} + \left\|\phi^{(n)}(t) - \phi^{(n-1)}(t)\right\|_{H^r} + \left\|\phi_t^{(n)}(t) - \phi_t^{(n-1)}(t)\right\|_{H^{r-1}}\Big). \tag{3.12}$$

Recall that $r \in [\frac{1}{4}, \frac{1}{2})$. We claim that, if T is sufficiently small, then for all n we have:

$$X_n(T) \le M \tag{3.13}$$

$$\Delta_n(T) \le \frac{1}{2}\Delta_{n-1}(T), \tag{3.14}$$

where M, to be determined later, is a constant independent of n and depending only on the quantity $D_0 = \|g\|_{L^2} + \|f\|_{H^r} + \|h\|_{H^{r-1}}$. We shall only present the proof of (3.13) as the proof of (3.14) is similar. We use induction on n. The estimate is trivial for $n = -1$. For $n = 0$ it follows easily from charge and generalized energy estimates that

$$X_0(T) \le c\cdot(1+T)D_0 \le cD_0,$$

[2] $M\phi\psi$ and $m^2\phi$ can easily be estimated as lower-order terms in the right-hand sides of the equations.

where c is an absolute constant and we have assumed, as we may, that $T \leq 1$. We show that the cases $n-2$ and $n-1$ of (3.13) imply the case n. Proposition 1 applied to equation (3.9) gives:

$$\sup_{0\leq t\leq T} \|\psi^{(n)}(t)\|_{L^2} \leq C(T)\Big[\|g\|_{H^{-1/4}} + \int_0^T \|\mathcal{D}\psi^{(n-1)}(t)\|_{H^{-1/4}}dt\Big] \quad (3.15a)$$

$$\times \Big[\, \|f\|_{H^{1/4}} + \|h\|_{H^{-3/4}} + \int_0^T \big\|\Box\phi^{(n-1)}(t)\big\|H^{-3/4}\, dt\Big]. \quad (3.15b)$$

The generalized energy estimate applied to equation (3.10) gives:

$$\sup_{0\leq t\leq T} \Big[\big\|\phi^{(n)}(t)\big\|_{H^r} + \big\|\phi_t^{(n)}(t)\big\|_{H^{r-1}}\Big]$$
$$\leq C(T)\Big[\, \|f\|_{H^r} + \|h\|_{H^{r-1}} + \int_0^T \Big\|\Box\phi^{(n)}(t)\Big\|_{H^{r-1}}\, dt\Big]. \quad (3.16)$$

We estimate the integral terms in the right-hand sides of the last two inequalities. To estimate the integral term in (3.15a) we use the following fractional 'Leibnitz' rule[3]:

$$\|uv\|_{H^{-1/4}(\mathbb{R})} \leq c\,\|u\|_{H^{1/4}(\mathbb{R})}\,\|v\|_{L^2(\mathbb{R})}$$

and the inductional hypothesis to see that for each $t \in [0,T]$ we have

$$\begin{aligned}\Big\|\mathcal{D}\psi^{(n-1)}(t)\Big\|_{H^{-1/4}} &= \Big\|\phi^{(n-2)}(t)\psi^{(n-2)}(t)\Big\|_{H^{-1/4}} \\ &\leq c\,\Big\|\phi^{(n-2)}(t)\Big\|_{H^{1/4}}\,\Big\|\psi^{(n-2)}(t)\Big\|_{L^2} \\ &\leq c\,\Big\|\phi^{(n-2)}(t)\Big\|_{H^r}\,\Big\|\psi^{(n-2)}(t)\Big\|_{L^2} \quad (\text{since } r \geq 1/4) \\ &\leq c\,X_{n-2}(T)^2 \leq c\,M^2,\end{aligned}$$

therefore,

$$\int_0^T \Big\|\mathcal{D}\psi^{(n-1)}(t)\Big\|_{H^{-1/4}}\, dt \leq cTM^2. \quad (3.17)$$

[3] $\|uv\|_{H^{-1/4}} \leq C\,\|uv\|_{L^{4/3}} \leq C\,\|u\|_{L^4}\,\|v\|_{L^2} \leq C\,\|u\|_{H^{1/4}}\,\|v\|_{L^2}$. See [6] for more on fractional Leibnitz rules.

Using the estimate[4] $\|uv\|_{H^{r-1}} \leq C\|u\|_{L^2}\|v\|_{L^2}$ and the inductional hypothesis we see that for each $t \in [0,T]$, we have

$$\left\|\Box\phi^{(n)}(t)\right\|_{H^{r-1}} = \left\|\overline{\psi^{(n-1)}(t)}\psi^{(n-1)}(t)\right\|_{H^{r-1}} \leq C\left\|\psi^{(n-1)}(t)\right\|_{L^2}^2$$
$$\leq CX_{n-1}(T)^2 \leq CM^2,$$

therefore,

$$\int_0^T \left\|\Box\phi^{(n)}(t)\right\|_{H^{r-1}} dt \leq CTM^2. \tag{3.18}$$

Using (3.17) and (3.18) into (3.15) and (3.16) we obtain

$$X_n(T) \leq C(T)\left[D_0 + TM^2\right]^2 + C(T)\left[D_0 + TM^2\right].$$

Since we are assuming that $T \leq 1$ we can bound all the constants $C(T)$ by an absolute constant and obtain

$$X_n(T) \leq c(2D_0^2 + 2TM^4) + c(D_0 + TM^2)$$
$$\leq c'(D_0^2 + D_0) + c''T(M^3 + M)M,$$

where the lower case c's are absolute constants. Take $M = 2c'(D_0^2 + D_0)$ and assume that T is small enough so that $c''T(M^3+M) \leq \frac{1}{2}$. Then $X_n(T) \leq M$ and (3.13) is proved. The proof of (3.14) is similar and local existence then follows by standard arguments.

Next we show that the solution exists globally in time. It suffices to show that if the solution exists in $[0,T^*)$ where $T^* < \infty$, then

$$\sup_{0\leq t<T^*}\left[\|\psi(t)\|_{L^2} + \|\phi(t)\|_{H^r} + \|\phi_t(t)\|_{H^{r-1}}\right] < \infty. \tag{3.19}$$

Indeed, by conservation of charge, $\|\psi(t)\|_{L^2}$ remains bounded. On the other hand, using the energy estimate, and working as above we have

$$\sup_{0\leq t<T^*}\left[\|\phi(t)\|_{H^r} + \|\phi_t(t)\|_{H^{r-1}}\right] \leq C(T^*)\Big[D_0 + \int_0^{T^*}\left\|\overline{\psi}\psi(t)\right\|_{H^{r-1}} dt\Big]$$
$$\leq C(T^*)\Big[D_0 + \int_0^{T^*}\|\psi(t)\|_{L^2}^2\, dt\Big] \leq C(T^*, D_0).$$

4

$$\|uv\|_{H^{r-1}}^2 \simeq \int \frac{\left|\int \hat{u}(\eta)\hat{v}(\xi-\eta)d\eta\right|^2}{(1+|\xi|)^{2(1-r)}}d\xi \leq \int \frac{\left(\int |\hat{u}(\eta)|^2 d\eta\right)\left(\int |\hat{v}(\xi-\eta)|^2 d\eta\right)}{(1+|\xi|)^{2(1-r)}}d\xi$$
$$\leq \|\hat{u}\|_{L^2}^2\|\hat{v}\|_{L^2}^2 \int \frac{1}{(1+|\xi|)^{2(1-r)}}d\xi \leq C\|u\|_{L^2}^2\|v\|_{L^2}^2 \quad (\text{since } r < 1/2)$$

This completes the proof of global existence. Uniqueness follows from similar estimates. □

Remarks. 1) We have insisted on taking $\psi(t,\cdot) \in L^2$ so that global existence can be derived. If one is only interested in the lowest possible exponents r and s for which the Cauchy problem (1.1) is locally (in time) well posed in the spaces $\psi(t) \in H^s$ and $\phi(t) \in H^r$ then the critical values are $s = -1$, $r = -1/2$. Fang [7] has shown local well posedness with $s = -1/4 + \epsilon$, $r = 1/2 + \epsilon$.

2) Theorems 1 and 2 leave open the case $\psi(t) \in L^2$, $\phi(t) \in H^{1/2}$.

3) The results of this paper consist of part of [10].

REFERENCES

[1] N. Bournaveas, *Local existence for the Maxwell-Dirac equations in three space dimensions,* Comm. Partial Differential Equations, 21 (1996), 693–720.

[2] N. Bournaveas, *A new proof of global existence for the Dirac Klein-Gordon equations in one space dimension,* J. Funct. Anal., 173 (2000), 203–213.

[3] M. Beals and M. Bézard, *Low regularity local solutions for field equations,* Comm. Partial Differential Equations, 21 (1996), 79–124.

[4] J. Chadam, *Global solutions of the Cauchy problem for the (classical) coupled Maxwell-Dirac equations in one space dimension,* J. Funct. Anal., 13 (1973), 173-184.

[5] J. Chadam and R. Glassey, *On certain global solutions of the Cauchy problem for the (classical) coupled Klein-Gordon-Dirac equations in one and three space dimensions,* Arch. Rational Mech. Anal., 54 (1974), 223–237.

[6] Jean-Yves Chemin, "Perfect Incompressible Fluids," Oxford University Press, 1998

[7] Yung-fu Fang, *On the Dirac-Klein-Gordon equations in one space dimension,* Differential Integral Equations, 17 (2004), 1321–1346.

[8] Yung-fu Fang, *Low regularity solutions for Dirac-Klein-Gordon equations in one space dimension,* Electron. J. Differential Equations, No. 102 (2004), 19.

[9] Yung-fu Fang, *A direct proof of global existence for the Dirac-Klein-Gordon equations in one space dimension,* Taiwanese J. Math., 8 (2004), 33–41.

[10] D. Gibbeson, *Low regularity solutions of nonlinear wave equations,* PhD Thesis, University of Edinburgh, 2004.

[11] S Klainerman, *Mathematical theory of classical fields and general relativity,* Mathematical physics, X (Leipzig, 1991), 213–236, Springer, Berlin, 1992.

[12] S. Klainerman and M. Machedon, *On the regularity properties of a model problem related to wave maps,* Duke Math. J., 87 (1997), 553–589.

[13] S. Machihara, *One dimensional Dirac equation with quadratic nonlinearities,* Discrete Contin. Dyn. Syst., 13 (2005), 277–290.

[14] Tohru Ozawa and Kazuyuki Yamauchi, *Structure of Dirac matrices and invariants for nonlinear Dirac equations,* Differential Integral Equations, 17 (2004), 971–982.