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LARGE DATA WAVE OPERATOR FOR THE GENERALIZED KORTEWEG-DE VRIES EQUATIONS

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Abstract. We consider the generalized Korteweg-de Vries equations :

$$u_t + (u_{xx} + u^p)_x = 0, \quad t, x \in \mathbb{R},$$

for $p \in (3, \infty)$. Let $U(t)$ be the associated linear group. Given V in the weighted Sobolev space $H^{2,2} = \{f \in L^2 : (1+|x|)^2(1-\partial_x^2)f\|_{L^2} < \infty\}$, possibly large, we construct a solution $u(t)$ of the generalized Korteweg-de Vries equation such that :

$$\lim_{t\to\infty} \|u(t) - U(t)V\|_{H^1} = 0.$$

We also prove uniqueness of such a solution in an adequate space. In the L^2-critical case ($p = 5$), this result can be improved to any possibly large function V in L^2 (with convergence in L^2).

1. Introduction

1.1. **Review of known results.** We consider the generalized Korteweg-de Vries equation

$$u_t + (u_{xx} + u^p)_x = 0, \quad t, x, \in \mathbb{R}, \tag{1.1}$$

where $p \geq 2$. The case $p = 2$ corresponds to the original equation introduced by Korteweg and de Vries [6] in the context of shallow water waves. For both $p = 2$ and $p = 3$, this equation has many applications to physics; see for example Miura [14], Lamb [7].

There are two formally conserved quantities for solutions to (1.1)

$$\int u^2(t) = \int u^2(0) \qquad (L^2 \text{ mass}), \tag{1.2}$$

$$E(u(t)) = \frac{1}{2}\int u_x^2(t) - \frac{1}{p+1}\int u^{p+1}(t) = E(u(0)) \qquad (\text{energy}). \tag{1.3}$$

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The local Cauchy problem for (1.1) has been intensively studied by many authors. Kenig, Ponce and Vega [5] proved the following existence and uniqueness result in $H^1(\mathbb{R})$: for $u_0 \in H^1(\mathbb{R})$, there exist $T = T(\|u_0\|_{H^1}) > 0$ and a solution $u \in C([0,T], H^1(\mathbb{R}))$ to (1.1) satisfying $u(0) = u_0$, which is unique in some class $Y_T \subset C([0,T], H^1(\mathbb{R}))$. For such a solution, one has conservation of mass and energy. Moreover, if T_1 denotes the maximal time of existence for u, then either $T_1 = +\infty$ (global solution) or $T_1 < \infty$ and $\|u(t)\|_{H^1} \to \infty$ as $t \uparrow T_1$ (blow-up solution).

For $p = 2$ and $p = 3$, equation (1.1) is completely integrable, and thus has very special features. The inverse scattering transform method allows us to solve the Cauchy problem in an appropriate space (for example if $\|(1+|x|^2)^5 u_0\|_{C^4} < \infty$) and to find the asymptotic behavior of solutions as $t \to \pm\infty$; see for example Schuur [16], Eckhaus and Schuur [2], Miura [14]. However, if $p \neq 2$ or 3, the inverse scattering transform method does no longer apply, and the description of solutions as $t \to +\infty$ in the non-integrable case is an open problem. Let us recall some results which are not based on the inverse scattering transform method.

In the case $2 \le p < 5$, all solutions in H^1 are global and uniformly bounded due to the conservations laws and the Gagliardo-Nirenberg inequality

$$\forall v \in H^1(\mathbb{R}), \qquad \int |v|^{p+1} \le C(p) \Big(\int v^2 \Big)^{\frac{p+3}{4}} \Big(\int v_x^2 \Big)^{\frac{p-1}{4}}.$$

The case $p = 5$ is L^2-critical, in the sense that the mass remains unaffected by scaling. If

$$u_t + (u_{xx} + u^5)_x = 0, \quad t, x, \in \mathbb{R}, \tag{1.4}$$

then $u_\lambda(t,x) = \lambda^{1/6} u(\lambda t, \lambda^{1/3} x)$ is also a solution to (1.4), and $\|u_\lambda\|_{L^2} = \|u\|_{L^2}$. In this case, the local existence result of [5] is improved to initial data in L^2 (instead of H^1). However, existence of finite time blow-up solutions was proved by Merle [13] and Martel and Merle [10]. Therefore $p = 5$ also appears as a critical exponent for the long-time behavior of solutions to (1.1).

Another problem which was studied by many authors is scattering for small initial data in an appropriate functional space; see for example [17], [15], [1], [4]. Let us recall the result of Hayashi and Naumkin [4]. Introduce the following weighted Sobolev spaces

$$H^{s,m} = \{ \phi \in \mathcal{S}' : \|\phi\|_{H^{s,m}} = \|(1+|x|^2)^{m/2}(1-\partial_x^2)^{s/2}\phi\|_{L^2} < \infty \}. \tag{1.5}$$

Let $p > 3$. Given u_0 small enough in $H^{1,1}$, the outcoming solution $u(t)$ is global in time, and there is scattering, in the sense that there exists a function $V \in L^2$ so that $\|u(t) - U(t)V\|_{L^2} \to 0$, where $U(t)$ denotes the linear operator for the KdV equation; i.e., $v(t) = U(t)V$ satisfies $v_t + v_{xxx} = 0$, $v(0) = V$. This is the description of solutions with initial data around 0 (in $H^{1,1}$).

Now, there exist also special explicit travelling wave solutions called solitons. If Q denotes the unique solution (up to translation) of

$$Q > 0, \quad Q \in H^1(\mathbb{R}), \quad Q_{xx} + Q^p = Q, \quad \text{i.e.,} \quad Q(x) = \left(\frac{p+1}{2\cosh^2(\frac{p-1}{2}x)}\right)^{\frac{1}{p-1}},$$

then for $c > 0$, the soliton $R_{c,x_0} = c^{\frac{1}{p-1}} Q(\sqrt{c}(x - x_0 - ct))$ is a solution to (1.1). We should notice that for $p > 3$, solitons do not appear in the small data analysis of [4], as $\|R_{c,x_0}\|_{H^{1,1}} \geq c_0$, for some uniform constant $c_0 > 0$. Indeed

$$\int (R_{c,x_0})_x^2 = c^{\frac{p+3}{2(p-1)}} \int Q_x^2, \quad \text{and} \quad \int (xR_{c,x_0})^2 = c^{\frac{7-3p}{2(p-1)}} \int (xQ)^2.$$

For $p > 3$, $\frac{p+3}{2(p-1)} > 0$ but $\frac{7-3p}{2(p-1)} < 0$, so that if $\|R_{c,x_0}\|_{H^1} \to 0$, then $c \to 0$ and thus $\|R_{c,x_0}\|_{H^{0,1}} \to \infty$. (Notice that $H^{1,1}$ is not sharp from this point of view.)

Description of solutions around a sum of decoupled solitons is available, Martel and Merle [9], Martel, Merle and Tsai [11] proved stability in H^1 and asymptotic stability (in $L^2(x \geq ct)$ for $c > 0$) of a sum of decoupled solitons, in the subcritical case $2 \leq p < 5$ (in the critical case $p = 5$, one has blow-up around a soliton [10]).

Our goal is to construct solutions to (1.1) with a given linear asymptotic behavior, that is, the construction of a wave operator with respect to the free KdV operator $U(t)$. This problem is reciprocal to (linear) scattering for small initial data.

Let $p > 3$, and $V \in H^{2,2}$ (without smallness assumptions). We construct a solution $u(t)$ to (1.1), defined for large enough times, and such that $u(t) - U(t)V \to 0$ in H^1 as $t \to \infty$. Furthermore, $u(t)$ is unique in an adequate space.

In the L^2-critical case ($p = 5$), we obtain an optimal result, in the sense that for $V \in L^2$, we construct a solution $u(t)$ to (1.4) such that $u(t) - U(t)V \to 0$ in L^2. Again V need not be small in L^2.

The philosophy underlying these results is the following: the tools needed to prove global existence for *small data* can be applied successfully to construct solutions with a given linear profile, *small or large*.

In the case of non-linear Schrödinger equations, there are many results concerning the construction of wave operators. For a review, see e.g. Ginibre and Velo [3].

1.2. **Statement of the results.** There are two main results, one in the general case $p > 3$ which uses the framework of Hayashi and Naumkin [4], and one in the critical case $p = 5$, using the framework of Kenig, Ponce and Vega [5].

Let $p > 3$. Fix once and for all the three constants

$$\gamma \in (0, \min\{1/2, (p-3)/3\}), \quad \alpha = \tfrac{1}{2} - \gamma \in (0, 1/2) \text{ and } \delta = \frac{p-3-2\gamma}{3} > 0. \tag{1.6}$$

Following the framework of Hayashi and Naumkin [4], we will use the notation $D = \partial_x = \frac{\partial}{\partial x}$ for partial differentiation with respect to the space variable x, and

$$D^\alpha \phi = \mathcal{F}^{-1} \xi^\alpha e^{-(i\pi/2)(1+\alpha)} \hat{\phi}.$$

As in [4], we will use the following two operators

$$J^t \phi = U(t) x U(-t) \phi = (x - 3t\partial_x^2)\phi, \text{ and } I^t \phi = x\phi + 3t \int_{-\infty}^x \partial_t \phi(t, y) dy.$$

We write J^t and I^t to emphasize that we will always consider norms at a fixed time t although J^t and I^t are space-time operators.

Our working spaces will be defined through the time dependent M_0^t norm

$$\mathcal{H}_t = \{\phi \in L^2(\mathbb{R}) : M_0^t(\phi) = \|\phi\|_{H^1} + \|DJ^t\phi\|_{L^2} + \|D^\alpha J^t \phi\|_{L^2} < \infty\}.$$

J^t only appears in the norm, since it is convenient to do linear estimates (see [4], Lemma 2.3). But we introduced I^t because it is easier to handle when doing energy methods estimates. Notice that M_0^0 is very similar to $\|\cdot\|_{H^{1,1}}$.

Different positive constants might be denoted by the same letter C.

We now state our result:

Theorem 1 (Large data wave operator)**.** *Let $p > 3$, and $V \in H^{2,2}$. Then*

1. *There exist $T_0 = T_0(\|V\|_{H^{2,2}}) \geq 1$ and a unique $u \in C([T_0, \infty), \mathcal{H}_t)$ solution to (1.1) so that $M_0^t(u(t) - U(t)V) \to 0$ as $t \to \infty$.*
2. *Moreover, there exists a constant C independent of time and V, so that*

$$\forall t \geq T_0, \quad M_0^t(u(t) - U(t)V) \leq C(1 + \|V\|_{H^{2,2}}^p) t^{-\delta}.$$

In particular, $\|u(t) - U(t)V\|_{H^1} \leq Ct^{-\delta}$.

Remark. The scattering result of Hayashi and Naumkin [4] is the following: for small initial data in $H^{1,1}$, the associated solution $u(t)$ is global in time, satisfies $\|u(t)\|_{L^\infty} \leq Ct^{-1/3}$ (linear decay rate), and there exists a scattering function V such that $u(t) - U(t)V \to 0$ in L^2 as $t \to \infty$.

Our point here is the reciprocal problem; we construct a solution u with a given scattering state $U(t)V$. We do not need any smallness assumption; however some integration by parts do not work as well as in [4], so we need to assume $V \in H^{2,2}$, with basically a convergence result in $H^{1,1}$.

This result can be extended to a more general non-linearity. For example:

Corollary 1. *Let us consider the equation*

$$u_t + (u_{xx} + f(u))_x = 0, \tag{1.7}$$

where $f : \mathbb{R} \to \mathbb{R}$ *is* C^1*, with* $f(0) = 0$ *and* $f'(x) = O(x^{p-1})$ *as* $x \to 0$ *(and* $p > 3$*). Given* $V \in H^{2,2}$*, there exist* $T_0 = T_0(\|V\|_{H^{2,2}}) \geq 1$ *and a unique solution* $u \in C([T_0, \infty), \mathcal{H}_t)$ *to* (1.7) *so that* $M_0^t(u(t) - U(t)V) \to 0$ *as* $t \to \infty$*. Moreover, there exists a constant* C *independent of time and* V*, so that*

$$\forall t \geq T_0, \quad M_0^t(u(t) - U(t)V) \leq C(1 + \|V\|_{H^{2,2}}^p)t^{-\delta}.$$

The proof of the corollary follows from that of Theorem 1. In particular, although our proof is done for the focusing power case $f(x) = x^p$, it is also true for the defocusing case $f(x) = -|x|^{p-1}x$.

In the critical case $p = 5$, we prove an analogous result for $V \in L^2$:

Theorem 2 (Wave operator in the critical case)**.** *Let* $p = 5$*. For any* $V \in L^2$*, there exist* $T_0 = T_0(V) \in \mathbb{R}$ *and* $u \in C^0([T_0, \infty), L^2)$ *solution to the critical KdV equation* (1.4)*, so that* $\|u(t) - U(t)V\|_{L^2} \to 0$*.* u *is unique in the class* $\{u : u \in L_t^\infty L_x^2 \cap L_x^5 L_t^{10}$ *and* $\partial_x u \in L_x^\infty L_t^2\}$*.*

Remark. The theorem is true if one weakens the hypothesis to V being such that $\|U(t)V\|_{L_x^5 L_t^{10}} + \|\partial_x U(t)V\|_{L_x^\infty L_t^2} < \infty$. As previously, our proof extends to the defocusing case $u_t + (u_{xx} - u^5)_x = 0$.

The proofs of the two results strongly rely on the scattering analysis of [4] and [5]. However, the arguments developed in each case are of completely different nature; this is why the proofs will be presented in a separate way. Section 2 gives the strategy of the proofs, Section 3 is devoted to the proof of Theorem 1, and Section 4, to that of Theorem 2.

2. Strategy of the proofs.

2.1. **The general case $p > 3$.** We shall always consider $w(t) = u(t) - U(t)V$. Thus, we are looking for w satisfying the equivalent conditions

$$w_t + w_{xxx} + (w + U(t)V)_x^p = 0, \quad M_0^t(w(t)) \to 0 \quad \text{as } t \to \infty. \tag{2.1}$$

To construct w, we will use the following approximation scheme. Let $(S_n)_{n\in\mathbb{N}}$ be an increasing sequence in $\mathbb{R}$ such that $S_n \to \infty$ as $n \to \infty$. Define $w_n(t)$ as a solution to

$$w_{nt} + w_{nxxx} + (w_n + U(t)V)^p_x = 0, \quad w_n(S_n) = 0, \tag{2.2}$$

defined on a maximal interval of the form $(I_n, S_n]$; it corresponds to a solution u_n to (1.1) with initial condition $u_n(S_n) = U(S_n)V$. Since $V \in H^{2,2}$, $U(S_n)V \in H^2$, so u_n exists and is the unique H^2 solution; the same is true for w_n.

Our method is then to prove that in fact

(1) for $I_n \leq T_0$ independent of n, we can define w_n on an interval $[T_0, S_n]$ whose lower bound T_0 is fixed.
(2) We have uniform (in n) decay estimates for w_n on the interval $[T_0, S_n]$.
 (Of course, if $3 < p < 5$, there is global existence in H^1, and thus $I_n = -\infty$ is automatic.) To prove this, we will make an intensive use of the tools developed by Hayashi and Naumkin; this is the heart of the proof, and it is done in Proposition 1.
(3) We then prove that the sequence $w_n(t)$ converges to a certain $w(t)$ (in $C^0_t L^2_x$).
(4) By weak limit, we improve the regularity of $w(t)$, to conclude that w is a strong solution to (2.1).

This does the existence part of the theorem. For the uniqueness part, we study again the L^2 difference of a solution with the one we constructed, and show, with a Gronwall-type argument, that these two solutions coincide where they are defined.

This scheme of proof is very similar to that of [8] and [12]. In [8], Martel also studied the problem of constructing solutions with a given asymptotic behavior; there it is proved that given a sum of solitons $R(t)$, there exists a unique solution $u(t)$ to (1.1) with $2 \leq p \leq 5$ such that $\|u(t) - R(t)\|_{H^1} \to 0$ as $t \to \infty$, and furthermore, the convergence takes place in all the Sobolev spaces H^s with an exponential decay. We remark that Martel deeply used ideas of the stability for a sum of decoupled solitons [11].

2.2. **The L^2-critical case ($p = 5$).** In the critical case, we do not need $H^{1,1}$ regularity. Since Kenig, Ponce and Vega [5] obtained global existence for small data in L^2, we use their setting.

The proof goes in this way: denoting as usual $w(t) = u(t) - U(t)V$, we write formally that an eventual w should solve a fixed-point problem with the condition $w(t) \to 0$ as $t \to \infty$.

The fixed-point problem is in fact very similar to that of the Cauchy problem, so that we can reuse the linear estimates proved by Kenig, Ponce and Vega [5] for their global existence theorem for small data.

The fact that $w(t) \to 0$ is our smallness condition, which allows us to have a contracting map, and thus a fixed point.

3. Proof of Theorem 1, $p > 3$

We start by recalling the linear estimates of [4] which will be used throughout the proof.

3.1. **Linear estimates.** Let $p > 3$. Recall our notation: three fixed constants $\gamma \in (0, \min\{(p-3)/3, 1/2\})$, $\alpha = 1/2 - \gamma$, $\delta = (p-3-2\gamma)/3 > 0$, the operator $J^t\phi = x\phi - 3t\partial_x^2\phi = U(t)xU(-t)\phi$, and our working norm:

$$M_0^t(\phi) = \|\phi\|_{H^1} + \|D^\alpha J^t\phi\|_{L^2} + \|DJ^t\phi\|_{L^2}.$$

First a few remarks on M_0^t. Of course $M_0^0(\phi) \le C\|\phi\|_{H^{1,1}}$. Second, note that $J^tU(t)V = U(t)xV$ (and $U(t)$ is an H^s isometry), so that if $V \in H^{1,1}$, we have the uniform control in t

$$M_0^t(U(t)V) \le C\|V\|_{H^{1,1}}. \tag{3.1}$$

We now recall the linear results obtained in [4] (Lemma 2.2), in a slightly improved form.

Lemma 1. *Let $t > 0$ and ϕ be a function so that $M_0^t(\phi)$ is bounded. Then for $r > 4$*

$$\|\phi\|_{L^r} \le \frac{C}{(1+t)^{1/3-1/(3r)}} M_0^t(\phi).$$

And one also has the pointwise inequalities

$$|\phi(x)| \le \frac{CM_0^t(\phi)}{(1+t)^{1/3}} \left(1 + \left|\frac{x}{t^{1/3}}\right|\right)^{-\frac{1}{4}}, \quad |\phi_x(x)| \le \frac{CM_0^t(\phi)}{t^{2/3}} \left(1 + \left|\frac{x}{t^{1/3}}\right|\right)^{\frac{1}{4}}.$$

As a simple consequence, for $V \in H^{1,1}$, we have similar decay estimates on $U(t)V$.

Proof. See [4], Lemma 2.2 and its proof (especially inequalities (2.16), (2.17), and (2.18). For completeness, the proof is given in Appendix A.

We will also need the polarized version of Lemma 2.3 of [4]:

Lemma 2. *Let $h, k : \mathbb{R} \to \mathbb{R}$. Then the following inequalities are valid if their right-hand side is bounded*

$$\begin{aligned} \|D^\alpha k^p\|_{L^2} &\le C\|k^{p-1}\|_{L^2}(\|kk_x\|_{L^\infty}^{1/2} + \|k\|_{L^\infty}^{3\gamma}\|kk_x\|_{L^\infty}^{(1-3\gamma)/2}), \\ \|D^\alpha |k|^{p-1}h_x\|_{L^2} &\le C(\|D^\alpha h\|_{L^2} + \|h_x\|_{L^2})(\|k\|_{L^\infty}^{p-3}\|kk_x\|_{L^\infty} \end{aligned}$$

$$+\|k\|_{L^\infty}^{p-3-2\gamma}\|k\|_{L^2}^{2\gamma}\|kk_x\|_{L^\infty}+\|k\|_{L^\infty}^{p-3+2\gamma}\|kk_x\|_{L^\infty}^{1-\gamma}).$$

Proof. See [4], Lemma 2.3 and its proof (the case $\sigma = 0$).

3.2. **Uniform estimates on w_n.** Recall that w_n is the solution to the problem

$$w_{nt} + w_{nxxx} + (w_n + U(t)V)^p_x = 0, \quad w_n(S_n) = 0, \tag{2.2}$$

where $S_n \to \infty$ is an increasing sequence of times. Using estimates developed in [4], we have the following proposition for w_n.

Proposition 1 (Uniform estimates on w_n). *Let $p > 3$. There exists $T_0 \geq 1$ independent of n, so that for all $n \in \mathbb{N}$ such that $S_n \geq T_0$, we have $w_n \in C([T_0, S_n], H^1(\mathbb{R}))$, and*

$$\forall t \in [T_0, S_n], \quad M_0^t(w_n(t)) \leq C(1 + \|V\|^p_{H^{2,2}})t^{-\delta} \tag{3.2}$$

where $\delta = (p - 3 + 2\gamma)/3 > 0$ and C are independent of n and V.

Of course Proposition 1 is the heart of the proof of Theorem 1.

Proof. Define $I_n^* \geq 1$ minimal so that

$$\forall t \in [I_n^*, S_n], \quad M_0^t(w_n(t)) \leq 1. \tag{3.3}$$

Lemma 3. *Suppose we can prove*

$$\forall t \in [I_n^*, S_n], \quad M_0^t(w_n(t)) \leq C(1 + \|V\|^p_{H^{2,2}})t^{-\delta}.$$

Then $I_n^* \leq T_0$ independent of n and Proposition 1 holds true,

$$\forall t \in [T_0, S_n], \quad M_0^t(w_n(t)) \leq C(1 + \|V\|^p_{H^{2,2}})t^{-\delta}.$$

Proof of Lemma 3. This follows from a continuity argument. First, due to Theorem 3, proved in Appendix B, $t \mapsto M_0^t(u_n(t))$ is upper semi-continuous, since $M_0^t(w_n(S_n)) = 0$, $I_n^* < S_n$.

Let $T_0 \geq 1$ be such that $C(1 + \|V\|^p_{H^{2,2}})T_0^{-\delta} \leq 1$. If $I_n^* > T_0$, $C(1 + \|V\|^p_{H^{2,2}})I_n^{*-\delta} < 1$, our hypothesis gives that

$$\forall t \in [I_n^*, S_n], \quad M_0^t(w_n(t)) \leq C(1 + \|V\|^p_{H^{2,2}})t^{-\delta} \leq C(1 + \|V\|^p_{H^{2,2}})I_n^{*-\delta} < 1.$$

By our upper semi-continuity argument (again Theorem 3), there would be a $t' < I_n^*$ so that for $t \in [t', S_n]$, $M_0^t(w_n(t)) \leq 1$. This contradicts the minimality of I_n^*, and so $I_n^* \leq T_0$.

The decay estimate follows immediately. □

So it is enough to prove

$$\forall t \in [I_n^*, S_n], \quad M_0^t(w_n(t)) \leq C(1 + \|V\|^p_{H^{2,2}})t^{-\delta},$$

assuming (3.3). Recall

$$M_0^t(\phi) = \|\phi\|_{L^2} + \|D\phi\|_{L^2} + \|D^\alpha J^t \phi\|_{L^2} + \|DJ^t\phi\|_{L^2}.$$

We will now estimate each one of the 4 norms involved. Let us denote $K = 1 + \|V\|_{H^{2,2}}$. We will successively prove that for $t \in [I_n^*, S_n]$, we have

(*i*) $\|w_n\|_{L^2} \le CK^p t^{-(p-3)/3}$;
(*ii*) $\|w_n\|_{H^1} \le CK^p t^{-(p-3)/3}$;
(*iii*) $\|D^\alpha I^t w_n\|_{L^2} \le CE^p t^{-\delta}$;
(*iv*) $\|I^t w_{nx}\|_{L^2} \le CK^p t^{-(p-3)/3}$;
(*v*) $M_0^t(w(t)) \le CK^p t^{-\delta}$.

Note that we first do estimates on I^t (step (*iii*) and (*iv*)), since it is easier to handle when doing energy methods estimates. In step (*v*) we shall go back to estimates on J^t.

Before we do this, we have to notice that Lemma 1 applies for $t \in [I_n^*, S_n]$ to give

$$|w_n(t,x)| \le Ct^{-1/3}\Big(1+\frac{|x|}{t^{1/3}}\Big)^{-\frac14}, \quad |w_{nx}(t,x)| \le Ct^{-2/3}\Big(1+\frac{|x|}{t^{1/3}}\Big)^{\frac14}.$$

Then we can add up $U(t)V$ in these estimates to get

$$|(U(t)V + w_n(t))(x)| \le CKt^{-1/3}\Big(1+\frac{|x|}{t^{1/3}}\Big)^{-\frac14}, \tag{3.4}$$

$$|(U(t)V + w_n(t))_x(x)| \le CKt^{-2/3}\Big(1+\frac{|x|}{t^{1/3}}\Big)^{\frac14}, \tag{3.5}$$

$$\|U(t)V + w_n(t)\|_{L^r} \le CKt^{-1/3-1/(3r)}, \quad r > 4. \tag{3.6}$$

Proof of (*i*)**.** Let us multiply (2.2) by w_n and integrate in x

$$\frac12 \frac{d}{dt}\int w_n^2 = -\int (U(t)V + w_n)_x^p w_n = -p\int (U(t)V+w_n)_x (U(t)V+w_n)^{p-1} w.$$

So (after simplification by $\|w_n\|_{L^2}$)

$$\frac{d}{dt}\|w_n\|_{L^2} \le \|U(t)V + w_n\|_{L^2}\|(U(t)V+w_n)(U(t)V+w_n)_x\|_{L^\infty} \times \|U(t)V+w_n\|_{L^\infty}^{p-3}.$$

For $t \in [I_n^*, S_n]$, we get

$$\frac{d}{dt}\|w_n\|_{L^2} \le C(K + \|w_n(t)\|_{L^2}) \cdot \frac{K^2}{t} \cdot \frac{K^{p-3}}{t^{(p-3)/3}} \le CK^p t^{-\frac{p}{3}}.$$

Thus, by integration in time on $[t, S_n]$, using $w_n(S_n) = 0$, this gives

$$\|w_n(t)\|_{L^2} \leq C(p)K^p \left(t^{-(p-3)/3} - S_n^{-(p-3)/3}\right) \leq CK^p t^{-(p-3)/3}. \qquad (3.7)$$

Proof of (ii)**.** Let us differentiate (2.2) with respect to x

$$w_{nxt} + w_{nxxxx} + (U(t)V + w_n)^p_{xx} = 0.$$

Now, multiply by w_{nx} and integrate in x. After an integration by parts, we get

$$\frac{1}{2}\frac{d}{dt}\int w_{nx}^2 = \int (U(t)V + w_n)^p_x w_{nxx}$$
$$= p\int w_{nxx}(U(t)V + w_n)_x (U(t)V + w_n)^{p-1}.$$

The point is to rule out w_{nxx}, a term that we cannot control; for this we will split the second term and do integrations by parts. First

$$2\int w_{nxx}w_{nx}(U(t)V + w_n)^{p-1} = -p\int w_{nx}^2 (U(t)V + w_n)_x (U(t)V + w_n)^{p-2},$$

which we can easily control. Indeed

$$p\left|\int w_{nxx}w_{nx}(U(t)V + w_n)^{p-1}\right|$$
$$\leq p\|w_{nx}\|^2_{L^2}\|(U(t)V + w_n)_x(U(t)V + w_n)\|_{L^\infty}\|U(t)V + w_n\|^{p-3}_{L^\infty}$$
$$\leq CK^{p-1}\|w_{nx}(t)\|^2_{L^2}t^{\frac{p}{3}} \leq CK^p\|w_{nx}\|_{L^2}t^{\frac{p}{3}}$$

(as $\|w_n\|_{H^1} \leq M_0^t(w(t)) \leq 1$). For the second term

$$\int w_{nxx}(U(t)V)_x(U(t)V + w_n)^{p-1} = -\int w_{nx}(U(t)V)_{xx}(U(t)V + w_n)^{p-1}$$
$$-\int w_{nx}(U(t)V)_x(U(t)V + w_n)^{p-2}(U(t)V + w_n)_x.$$

The first integral would be troublesome (with the double derivative on $U(t)V$); this is why we made the hypothesis $V \in H^{2,2}$ (in [4], this phenomenon is avoided because the integration by parts works better). The second integral is fine. So we can do the estimate (note that we use the L^∞ decay estimate (3.7) on $(U(t)V)_{xx} = (U(t)V_x)_x$, as $V_x \in H^{1,1}$): for all $t \in [I_n^*, S_n]$

$$p\left|\int w_{nxx}(U(t)V)_x(U(t)V + w_n)^{p-1}\right|$$

$$\leq C\|w_{nx}\|_{L^2}\|(U(t)V+w_n)(U(t)V_x)_x\|_{L^\infty}\|U(t)V+w_n\|_{L^2}\|U(t)V+w_n\|_{L^\infty}^{p-3}$$
$$+C\|w_{nx}\|_{L^2}\|(U(t)V)_x\|_{L^2}\|(U(t)V+w_n)(U(t)V+w_n)_x\|_{L^\infty}\|U(t)V+w_n\|_{L^\infty}^{p-3}$$
$$\leq CK^p\|w_{nx}\|_{L^2}t^{-p/3}$$

(we used estimate (3.7), and $\|w_n\|_{H^1} \leq 1$). So finally, after simplifying by $\|w_{nx}\|_{L^2}$

$$\frac{d}{dt}\|w_{nx}(t)\|_{L^2} \leq CK^p t^{-p/3}.$$

After integration between t and S_n ($w_n(S_n)=0$)

$$\|w_{nx}(t)\|_{L^2} \leq CK^p\left(t^{-(p-3)/3} - S_n^{-(p-3)/3}\right).$$

And so, we neglect the S_n term, and when adding up with (3.7), we obtain

$$\|w_n(t)\|_{H^1} \leq C(p)K^p t^{-(p-3)/3}. \tag{3.8}$$

Proof of (iii)**.** We now turn to the estimates on $I^t w_n$. Let us denote $L = \partial_t + \partial_{xxx}$ the linear KdV operator, and recall some commutation relations of the different operators involved . Recall the definition of the dilation operator $I^t\phi = x\phi + 3t\int_{-\infty}^x \phi_t dx$ and of $J^t\phi = x\phi - 3t\partial_{xx}\phi$. Then

$$I^t\phi - J^t\phi = 3t\int_{-\infty}^x L\phi dx.$$

We have the following commutation relations

$$[L, J^t] = 0, \quad [L, I^t]\phi = 3\int_{-\infty}^x L\phi dx, \quad [J^t, \partial_x] = [I^t, \partial_x] = -Id.$$

Again notice that

$$I^tU(t)V - J^tU(t)V = 3t\int_{-\infty}^x LU(t)Vdx = 0$$

so that

$$\|D^\alpha I^tU(t)V\|_{L^2} + \|DI^tU(t)V\|_{L^2} \leq C\|V\|_{H^{1,1}}.$$

Let $f:\mathbb{R}\to\mathbb{R}$ be a C^1 function and ϕ be such that $f'(\phi)I^t\phi_x$ has a sense. Then

$$I^t(f(\phi)_x) = xf(\phi)_x + 3tf(\phi)_t = xf'(\phi)\phi_x + 3tf'(\phi)\phi_t = f'(\phi)I^t\phi_x.$$

We will use this formula for $f(x) = x^p$ and $\phi = U(t)V + w_n(t)$.

We compute

$$\begin{aligned} LI^tw_n &= I^tLw_n + 3\int_{-\infty}^x Lw_n = -I^t(U(t)V+w_n)_x^p - 3(U(t)V+w_n)^p \\ &= -p(U(t)V+w_n)^{p-1}I^t(U(t)V+w_n)_x - 3(U(t)V+w_n)^p \end{aligned}$$

$$= \quad -p(U(t)V+w_n)^{p-1}(I^t(U(t)V+w_n))_x - (3-p)(U(t)V+w_n)^p.$$

Apply the operator D^α, multiply by $D^\alpha I^t w_n$ and integrate in space (recall that $[D^\alpha, L] = 0$ and $(L\phi,\phi) = \frac{1}{2}\frac{d}{dt}\int \phi^2$)

$$\frac{d}{dt}\|D^\alpha I^t w_n\|_{L^2}^2 \leq \|D^\alpha I^t w_n\|_{L^2}\|D^\alpha(U(t)V+w_n)^{p-1}(I^t(U(t)V+w_n))_x\|_{L^2} + C\|D^\alpha I^t w_n\|_{L^2}\|D^\alpha(U(t)V+w_n)^p\|_{L^2}. \quad (3.9)$$

We now apply Lemma 2 with $k = U(t)V + w_n$ and $h = I^t(U(t)V+w_n)$. So that (along with the linear estimates (3.4), (3.5), (3.6))

$$\|kk_x\|_{L^\infty} \leq CK^2t^{-1}, \ \ \|k\|_{L^2} \leq CK, \quad \|k\|_{L^\infty} \leq CKt^{-\frac{1}{3}},$$
$$\|k^{p-1}\|_{L^2} = \|k\|_{L^{2(p-1)}}^{p-1} \leq CK^{p-1}t^{(p-1)(-\frac{1}{3}+\frac{1}{6(p-1)})} \leq CK^{p-1}t^{-(2p-3)/6},$$

and

$$\|D^\alpha h\|_{L^2} \leq CK, \quad \|h_x\|_{L^2} \leq CK.$$

We can compute

$$\|D^\alpha(U(t)V+w_n)^{p-1}(I^t(U(t)V+w_n))_x\|_{L^2}$$
$$\leq CK^p(t^{-(p-3)/3+1} + t^{-(p-3-2\gamma)/3+1} + t^{-(p-3+2\gamma)/3-1+\gamma}) \leq CK^pt^{-(p-2\gamma)/3}$$

(as $t \geq I_n^* \geq 1$; the point being $(p-2\gamma)/3 > 1$). And for the second term

$$\begin{aligned} \|D^\alpha(U(t)V+w_n)^p\|_{L^2} &\leq CK^p(t^{-(2p-3)/6-1/2} + t^{-(2p-3)/6-\gamma-(1-3\gamma)/2}) \\ &\leq CK^pt^{-p/3+\gamma/2}. \end{aligned}$$

Finally, after simplification by $\|D^\alpha I^t w_n\|_{L^2}$, (3.9) gives

$$\frac{d}{dt}\|D^\alpha I^t w_n\|_{L^2} \leq CK^pt^{-(p-2\gamma)/3}.$$

And as before, we integrate on $[t, S_n]$

$$\|D^\alpha I^t w_n(t)\|_{L^2} \leq CK^pt^{-(p-3-2\gamma)/3}. \quad (3.10)$$

Proof of *(iv)***.** Again, we compute

$$\begin{aligned} LI^t(w_{nx}) &= I^tLw_{nx} + 3Lw_n = -I^t(U(t)V+w_n)^p_{xx} - 3(U(t)V+w_n)^p_x \\ &= -(I^t(U(t)V+w_n)^p_x)_x - 2(U(t)V+w_n)^p_x \\ &= -p(U(t)V+w_n)^{p-1}(I^t(U(t)V+w_n)_x)_x \\ &\quad -p(p-1)(U(t)V+w_n)^{p-2}(U(t)V+w_n)_xI^t(U(t)V+w_n)_x \\ &\quad -2p(U(t)V+w_n)^{p-1}(U(t)V+w_n)_x. \quad (3.11) \end{aligned}$$

We want to multiply by I^tw_{nx} and integrate in space. There is essentially one troublesome term, the double derivative one. We split it into

$$(U(t)V+w_n)^{p-1}(I^t(U(t)V+w_n)_x)_x \\ = (U(t)V+w_n)^{p-1}(I^tU(t)V_x)_x + (I^t w_{nx})_x). \quad (3.12)$$

The second term will have only first order terms after integration by parts, which is fine

$$\begin{aligned}
&\left|\int (U(t)V+w_n)^{p-1}(I^t w_{nx})_x I^t w_{nx}\right| \\
&= \frac{(p-1)}{2}\left|\int (U(t)V+w_n)^{p-2}(U(t)V+w_n)_x (I^t w_{nx})^2\right| \\
&\leq C\|(U(t)V+w_n)(U(t)V+w_n)_x\|_{L^\infty}\|U(t)V+w_n\|_{L^\infty}^{p-3}\|I^t w_{nx}\|_{L^2}^2 \\
&\leq CK^{p-1}t^{-p/3}\|I^t w_{nx}\|_{L^2}^2.
\end{aligned}$$

However the first term on the right-hand side of (3.12) requires extra regularity on V. Indeed

$$(I^tU(t)V_x)_x = (J^tU(t)V_x)_x = (U(t)xV_x)_x,$$

and $(xV_x \in H^{1,1})$, $M_0^t(U(t)(xV_x)) \leq \|V\|_{H^{2,2}}$ so that by Lemma 1

$$|(I^tU(t)V_x)_x(x)| \leq \frac{C\|V\|_{H^{2,2}}}{t^{2/3}}\left(1+\left|\frac{x}{t^{1/3}}\right|\right)^{\frac{1}{4}}.$$

We can now estimate (essentially in the same way as for the H^1 estimate)

$$\begin{aligned}
&\left|\int (U(t)V+w_n)^{p-1}(I^tU(t)V_x)_x I^t w_{nx}\right| \\
&\quad\leq \|U(t)V+w_n\|_{L^\infty}^{p-3}\|U(t)V+w_n\|_{L^2} \\
&\qquad\quad \|(U(t)V+w_n)(I^tU(t)V_x)_x\|_{L^\infty}\|I^t w_{nx}\|_{L^2} \\
&\quad\leq CK^{p-1}t^{p/3}\|I^t w_{nx}\|_{L^2}.
\end{aligned}$$

The remaining two terms in (3.11) are simpler and can be treated directly (after multiplication by $I^t w_{nx}$ and integration in x)

$$\begin{aligned}
&\left|\int (U(t)V+w_n)^{p-2}(U(t)V+w)_x I^t(U(t)V+w_n)_x I^t w_{nx}\right| \\
&\quad\leq \|U(t)V+w_n\|_{L^\infty}^{p-3}\|(U(t)V+w_n)(U(t)V+w_n)_x\|_{L^\infty} \\
&\qquad\quad \|I^t(U(t)V+w_n)_x\|_{L^2}\|Iw_{nx}\|_{L^2} \\
&\quad\leq CK^p t^{p/3}\|I^t w_{nx}\|_{L^2}, \\
&\left|\int (U(t)V+w_n)^{p-2}(U(t)V+w_n)_x I^t w_{nx}\right| \\
&\quad\leq \|U(t)V+w_n\|_{L^\infty}^{p-3}\|(U(t)V+w_n)(U(t)V+w_n)_x\|_{L^\infty} \\
&\qquad\quad \|U(t)V+w_n\|_{L^2}\|I^t w_{nx}\|_{L^2} \\
&\quad\leq CK^p t^{p/3}\|I^t w_{nx}\|_{L^2}.
\end{aligned}$$

Hence, (after simplifying by $\|I^t w_{nx}\|_{L^2}$) we get

$$\frac{d}{dt}\|I^t w_{nx}\|_{L^2} \leq CK^{p-1}(K + \|I^t w_{nx}\|_{L^2})t^{p/3} \leq CK^p t^{-p/3}.$$

And Gronwall's lemma (between t and S_n) gives

$$\|I^t w_{nx}(t)\|_{L^2} \leq CK^p t^{-(p-3)/3}. \tag{3.13}$$

Proof of (v)**.** We now have to go back to

$$J^t w_n(t) = I^t w_n(t) - 3t\int_{-\infty}^{x} Lw_n(t)dx' = I^t w_n(t) + 3t(U(t)V + w_n(t))^p.$$

Using the commutation relations, we get

$$\begin{aligned}\|D^\alpha J^t w_n\|_{L^2} + \|DJ^t w_n\|_{L^2} &\leq \|D^\alpha I^t w_n\|_{L^2} + 3t\|D^\alpha(U(t)V + w_n(t))^p\|_{L^2} \\ &+ \|I^t w_{nx}\|_{L^2} + \|w_n\|_{L^2} + 3pt\|(U(t)V + w_n(t))^{p-1}(U(t)V + w_n(t))_x\|_{L^2}.\end{aligned}$$

Now, using again Lemma 2 with $k = U(t)V + w_n(t)$

$$\begin{aligned}\|D^\alpha(U(t)V + w_n(t))^p\| &\leq C\|k^{p-1}\|_{L^2}(\|kk_x\|_{L^\infty}^{1/2} + \|k\|_{L^\infty}^{3\gamma}\|kk_x\|_{L^\infty}^{(1-3\gamma)/2}) \\ &\leq CK^p t^{-p/3+\gamma/2}.\end{aligned}$$

And as usual

$$\begin{aligned}&\|(U(t)V + w_n)^{p-1}(U(t)V + w_n)_x\|_{L^2} \\ &\leq \|U(t)V + w_n\|_{L^\infty}^{p-3}\|(U(t)V + w_n)(U(t)V + w_n)_x\|_{L^\infty}\|U(t)V + w_n\|_{L^2} \\ &\leq CK^p t^{p/3}.\end{aligned}$$

So using these last two inequalities, along with (3.7), (3.10) and (3.13), we get

$$\|D^\alpha J^t w_n\|_{L^2} + \|DJ^t w_n\|_{L^2} \leq CK^p t^{(p-3+2\gamma)/3}. \tag{3.14}$$

Recall $\delta = (p - 3 - 2\gamma)/3 > 0$. So adding up (3.8) and (3.14) gives

$$\forall t \in [I_n^*, S_n], \quad M_0^t(w_n(t)) \leq CK^p t^{-\delta}. \qquad \square$$

3.3. Construction and uniqueness of u.

Proof of Theorem 1. *Existence of u.* Proposition 1 provides us with a sequence $w_n(t)$ of solutions to (2.2) satisfying uniform estimates in n

$$\forall t \in [T_0, S_n], \quad M_0^t(w_n(t)) \leq CK^p t^{-\delta}.$$

In particular, for $t \in [T_0, S_n]$, estimates (3.4) and (3.5) are valid.

Let us prove that for all $k \in \mathbb{N}$, $(w_n)_{n\geq k}$ is a convergent sequence in $C^0([T_0, S_k], L^2(\mathbb{R}))$. For this we show that $(w_n)_{n\geq k}$ is a Cauchy sequence in $C^0([T_0, S_k], L^2(\mathbb{R}))$. Let us consider $v_{n,m} = w_n - w_m$ in L^2. Without loss of generality, we can suppose that $m > n \geq k$. First $\|v_{n,m}(S_n)\|_{L^2} \leq CK^p S_n^{-(p-3)/3}$ (see (3.7)).

$v_{n,m}$ satisfies (we denote $v = v_{n,m}$ for simplicity in the computations)

$$v_t + v_{xxx} + (U(t)V + w_n)^p_x - (U(t)V + w_m)^p_x = 0. \tag{3.15}$$

We multiply by v and integrate in x

$$\frac{1}{2}\frac{d}{dt}\int v^2 = \int v((U(t)V + w_n)^p_x - (U(t)V + w_m)^p_x).$$

Now, for any functions ϕ and ψ

$$\phi^p_x - \psi^p_x = p\phi^{p-1}\phi_x - p\psi^{p-1}\psi_x = p\phi^{p-1}(\phi - \psi)_x + p(\phi^{p-1} - \psi^{p-1})\psi_x,$$

and

$$|\phi^{p-1} - \psi^{p-1}| \leq C|\psi - \phi|(|\phi|^{p-2} + |\psi|^{p-2}).$$

So that with $\phi = U(t)V + w_n$, $\psi = U(t)V + w_m$ (after integration by parts on the first term)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int v^2 = -\frac{p-1}{2}\int v^2(U(t)V + w_n)^{p-2}(U(t)V + w_n)_x \\ + \int v(U(t)V + w_n)^{p-1} - (U(t)V + w_m)^{p-1})(U(t)V + w_m)_x.\end{aligned}$$

Treating each term separately (for $t \in [T_1, S_n]$)

$$\begin{aligned}&\Big|\int v^2(U(t)V + w_n)^{p-2}(U(t)V + w_n)_x\Big| \leq CK^{p-1}t^{-p/3}\|v\|^2_{L^2},\\ &\Big|\int v(U(t)V + w_n)^{p-1} - (U(t)V + w_m)^{p-1})(U(t)V + w_m)_x\Big|\\ &\qquad \leq \; C\int v^2(|U(t)V + w_n|^{p-2} + |U(t)V + w_m|^{p-2}|(U(t)V + w_m)_x|\\ &\qquad \leq \; CK^{p-1}t^{-p/3}\|v\|^2_{L^2},\end{aligned}$$

so that

$$\frac{d}{dt}\|v\|^2_{L^2} \leq CK^{p-1}t^{-p/3}\|v\|^2_{L^2}.$$

Now, using Gronwall's lemma on $[t, S_n]$

$$\|v_{n,m}(t)\|_{L^2} \leq C\|v_{n,m}(S_n)\|_{L^2} \leq CS_n^{-(p-3)/3}.$$

This proves that $(w_n)_{n\geq k}$ is a Cauchy sequence in the space $C([T_0, S_k], L^2)$ and so converges to a certain $w(t,x)$. Since this can be done for arbitrarily large n (and $S_n \to \infty$), $w \in C([T_0,\infty), L^2)$ is the only possible weak limit of $(w_n)_{n\in\mathbb{N}}$.

Given a fixed $t \geq T_0$, $M_0^t(w_n(t)) \leq CK^p t^{-\delta}$, so by weak limit

$$M_0^t(w(t)) \leq CK^p t^{-\delta} < \infty.$$

Now w_n satisfies (2.2), hence

$$w_n(t) = w_n(T_0) + \Big(\int_{T_0}^t U(t-s)(w_n(s))^p ds \Big)_x,$$

$w_n(T_0) \to w(T_0)$ in L^2 and $w_n(t) \to w(t)$ in L^2. Furthermore, $w_n \to w$ in $C_b([T_0,t],L^2)$ and w_n is a bounded sequence in $L^\infty([T_0,t],H^1)$ (for all $t \geq T_0$), so that by interpolation $w_n^p \to w_n^p$ in $C([T_0,t],L^2)$. Thus,

$$\Big(\int_{T_0}^t U(t-s)(w_n(s))^p ds \Big)_x \to \Big(\int_{T_0}^t U(t-s)(w(s))^p ds \Big)_x \quad \text{in } H^{-1}.$$

Hence, w satisfies the integral formulation of (2.1). As a consequence, when defining $u(t) = w(t) + U(t)V$, $u \in C_b([T_0,\infty),H^1)$ satisfies the conditions of Theorem 1.

Uniqueness of u. Let $\tilde{u}$ be another solution. We switch to $\tilde{w}(t,x) = \tilde{u}(t,x) - (U(t)V)(x)$. Then of course

$$\tilde{w}_t + \tilde{w}_{xxx} + (U(t)V + \tilde{w})_x^p = 0, \quad M_0^t(\tilde{w}(t)) \to 0 \quad \text{ as } t \to \infty.$$

Introduce T_1 such that for $t \geq T_1$, $M_0^t(\tilde{w}(t)) \leq 1$. Let us introduce again $v(t,x) = w(t,x) - \tilde{w}(t,x)$. Then

$$v_t + v_{xxx} + (U(t)V + w)_x^p - (U(t)V + \tilde{w})_x^p = 0,$$

which is basically identical to (3.15). So multiplying by v and integrating in x, one can do the same computations as previously, so that for $t \geq T_1$

$$\Big| \frac{d}{dt} \|v(t)\|_{L^2}^2 \Big| \leq C(V) t^{-p/3} \|v\|_{L^2}^2.$$

By Gronwall's lemma between t and τ

$$\|v(t)\|_{L^2} \leq C(V) \|v(\tau)\|_{L^2}.$$

By hypothesis, $M_0^\tau(v(\tau)) \to 0$ so that $\|v(\tau)\|_{L^2} \to 0$, and letting $\tau \to \infty$ gives for all $t \geq T_1$, $v(t,x) = 0$. So $w(t,x) = \tilde{w}(t,x)$, $u(t,x) = \tilde{u}(t,x)$ for $t \geq T_1$ and by H^1 uniqueness, for all t such that u and $\tilde{u}$ are defined, $u(t,x) = \tilde{u}(t,x)$. $\square$

4. PROOF OF THEOREM 2, THE CRITICAL CASE $p = 5$.

4.1. **Linear estimates.** Let us now set $p = 5$, and recall the linear estimates in [5].

Lemma 4. *Let ϕ be a function of space and ψ be a function of time and space. Then the following inequalities hold, assuming their right-hand side term is bounded*

(1) $\|\partial_x U(t)\phi\|_{L^\infty_x L^2_t([T,\infty))} \leq \|\phi\|_{L^2_x}$;

(2) $\sup_{t\geq T} \left\|\partial_x \int_t^\infty U(t-s)\psi(s)ds\right\|_{L^2_x} \leq \|\psi\|_{L^1_x L^2_t([T,\infty))}$;

(3) $\left\|\partial^2_{xx} \int_t^\infty U(t-s)\psi(s)ds\right\|_{L^\infty_x L^2_t([T,\infty))} \leq \|\psi\|_{L^1_x L^2_t([T,\infty))}$;

(4) $\|U(t)\phi\|_{L^5_x L^{10}_t([T,\infty))} \leq C\|\phi\|_{L^2_x}$;

(5) $\left\| \int_t^\infty U(t-s)\psi(s)ds\right\|_{L^5_x L^{10}_t([T,\infty))} \leq \|\psi\|_{L^{5/4}_x L^{10/9}_t([T,\infty))}$.

Proof. See [5]. In the proof, $T = -\infty$; for general T, the estimates (1) and (2) are clear, and for estimates (2), (3) and (5), replace ψ by $\psi\chi_{t\geq T}$.

4.2. **Construction of u. Proof of Theorem 2.** Let us do formal computations first: suppose u is the desired solution, and write

$$u(t) = U(t)V + w(t), \quad \text{i.e.} \quad w(t) = u(t) - U(t)V, \quad \lim_{t\to\infty} \|w\|_{L^2} = 0.$$

u satisfies

$$\begin{aligned} u(\tau) = U(\tau)u(0) + \partial_x \int_0^\tau U(\tau-s)u^5(s)ds \\ = U(\tau-t)u(t) + \partial_x \int_t^\tau U(\tau-s)u^5(s)ds. \end{aligned}$$

Thus, w satisfies

$$U(\tau)V+w(\tau) = U(\tau-t)(U(t)V+w(t))+\partial_x \int_t^\tau U(\tau-s)(U(s)V+w(s))^5(s)ds.$$

$U(\tau)V$ cancels and we obtain after composing by $U(t-\tau)$

$$U(t-\tau)w(\tau) = w(t) + \partial_x \int_t^\tau U(t-s)(U(s)V + w(s))^5(s)ds.$$

Now, let $\tau \to \infty$, $\|U(t-\tau)w(\tau)\|_{L^2} = \|w(\tau)\|_{L^2} \to 0$, so that we obtain w as a fixed point

$$w(t) = -\partial_x \int_t^\infty U(t-s)(U(s)V + w(s))^5(s)ds.$$

This explains the following scheme: we will show that such a fixed point exists by a contraction map argument. Let us introduce the function Φ.

$$\Phi : w \mapsto -\partial_x \int_t^\infty U(t-s)(U(s)V + w(s))^5 ds. \tag{4.1}$$

Our goal is to find a fixed point for Φ.

Let us introduce the following norm

$$\|\phi\|_{X_T} \quad = \quad \|\phi\|_{L_x^5 L_t^{10}([T,\infty))} + \|\partial_x \phi(t)\|_{L_x^\infty L_t^2([T,\infty))}.$$

From estimate (5) of Lemma 4, we get:

$$\begin{aligned}
&\|\Phi(w)\|_{L_x^5 L_t^{10}([T,\infty))} \le \|(U(t)V + w(t))^4 (U(t)V + w(t))_x\|_{L_x^{5/4} L_t^{10/9}([T,\infty))} \\
&\le \|(U(t)V + w(t))^4\|_{L_x^{5/4} L_t^{10/4}([T,\infty))} \|(U(t)V + w(t))_x\|_{L_x^\infty L_t^2([T,\infty))} \\
&\le C\|U(t)V + w(t)\|^4_{L_x^5 L_t^{10}([T,\infty))} \|\partial_x (U(t)V + w(t))\|_{L_x^\infty L_t^2([T,\infty))}.
\end{aligned}$$

By estimate (3)

$$\begin{aligned}
\|\partial_x \Phi(w)\|_{L_x^\infty L_t^2([T,\infty))} &\le \|(U(t)V + w(t))^5\|_{L_x^1 L_t^2([T,\infty))} \\
&\le \|U(t)V + w(t)\|^5_{L_x^5 L_t^{10}([T,\infty))}
\end{aligned}$$

Adding up the last two estimates (and using $|a+b|^k \le 2^{k-1}(|a|^k + |b|^k)$, $|a^4 b| \le |a|^5 + |b|^5$)

$$\begin{aligned}
&\|\Phi(w)\|_{X_T} \le C\|U(t)V + w(t)\|^4_{L_x^5 L_t^{10}([T,\infty))} \\
&\quad \times \Big(\|U(t)V + w(t)\|_{L_x^5 L_t^{10}([T,\infty))} + \|\partial_x (U(t)V + w(t))\|_{L_x^\infty L_t^2([T,\infty))} \Big) \\
&\qquad \le 2^3 (\|U(t)V\|^4_{L_x^5 L_t^{10}([T,\infty))} + \|w\|^4_{X_T})(\|U(t)V\|_{X_T} + \|w\|_{X_T}) \\
&\quad \le C(\|U(t)V\|^4_{L_x^5 L_t^{10}([T,\infty))} \|U(t)V\|_{X_T} + \|U(t)V\|_{X_T} \|w\|^4_{X_T} + \|w\|^5_{X_T}).
\end{aligned}$$

Let us prove the following simple lemma:

Lemma 5. *Let V be such that $U(t)V \in X_0$ (in particular, if $V \in L^2$). Then $\|U(t)V\|_{L_x^5 L_t^{10}([T,\infty))} \to 0$ as $T \to \infty$.*

Proof. Consider

$$f(\tau, x) = \|U(t)V\|^5_{L_t^{10}[\tau,\infty)}, \text{ and } h(x) = \|U(t)V\|^5_{L_t^{10}[0,\infty)} = f(0,x).$$

f and h are finite x-almost everywhere as $\int_x f < \infty$ and $\int_x h < \infty$. Then $h \in L_x^1$ and for all $\tau \ge 0$, $0 \le f(\tau, x) \le h(x)$. Now, since f is an exhausting integral, we have

$$x - \text{a.e.}, \lim_{\tau \to \infty} f(\tau, x) = 0.$$

Lebesgue's dominated convergence theorem implies $\lim_{\tau\to\infty}\|f(\tau,x)\|_{L^1_x}=0$, which is exactly $\lim_{\tau\to\infty}\|U(t)V\|_{L^5_xL^{10}_t[\tau,\infty)}=0$. □

As $V\in L^2$, by the previous lemma, $\|U(t)V\|_{L^5_xL^{10}_t([T,\infty))}\to 0$ as $T\to\infty$. Moreover, $\|U(t)V\|_{X_T}\le C\|V\|_{L^2}$. So our estimate may be written

$$\|\Phi(w)\|_{X_T}\le C(\|U(t)V\|^4_{L^5_xL^{10}_t([T,\infty))}\|V\|_{L^2}+\|V\|_{L^2}\|w\|^4_{X_T}+\|w\|^5_{X_T}). \quad (4.2)$$

This shows that for T_0 large enough, there exists $\delta>0$ so that Φ maps $B_{X_{T_0}}(0,\delta)$ to itself

$$\Phi: B_{X_{T_0}}(0,\delta)\to B_{X_{T_0}}(0,\delta).$$

The same computations show that for T_0 large enough, $\delta>0$ small enough, $\Phi: B_{X_{T_0}}(0,\delta)\to B_{X_{T_0}}(0,\delta)$ is a contraction. Thus, Φ has a unique fixed point, which we denote v.

$v=\Phi(v)$ may be written

$$v(t)=-\partial_x\int_t^\infty U(t-s)(U(s)V+v(s))^5ds,$$

and by construction $u(t)=U(t)V+v(t)$ is a solution to (1.4). Now by (4.2), if δ has been chosen small enough, we have

$$\|v\|_{X_T}\le C\|U(t)V\|^4_{L^5_xL^{10}_t([T,\infty))}\|V\|_{L^2}\to 0 \quad \text{as} \quad T\to\infty.$$

And by estimate (2) of Lemma 4

$$\begin{aligned}\|v\|_{L^\infty_t([T,\infty),L^2_x)} &\le \|U(t)V+v\|^5_{L^5_xL^{10}_t([T,\infty))}\\ &\le C(\|U(t)V\|^5_{L^5_xL^{10}_t([T,\infty))}+\|v\|^5_{X_T})\to 0 \text{ as } T\to\infty.\end{aligned}$$

To conclude, v satisfies the decay estimate $\|v(t)\|_{L^2}\to 0$ and moreover,

$$\|v\|_{L^\infty_t([T,\infty))L^2_x}+\|v\|_{L^5_xL^{10}_t([T,\infty))}+\|\partial_x v\|_{L^\infty_xL^2_t([T,\infty))}\to 0 \quad \text{as} \quad T\to\infty.$$

APPENDIX A.

For the sake of completeness, we present the proof of Lemma 1 of [4].

Proof of Lemma 1. Let us note $v(t,x)=U(-t)\phi$. We have the identity

$$\begin{aligned}\phi(x) &= U(t)v = \frac{1}{\pi}\Re\int_0^\infty e^{ipx+ip^3t/3}\hat{v}(t,p)dp\\ &= \frac{1}{\pi\sqrt[3]{t}}\Re\int_0^\infty e^{iq\eta+iq^3/3}\Big(\hat{v}(t,\chi)+\Big(\hat{v}\Big(t,\frac{q}{\sqrt[3]{t}}\Big)-\hat{v}(t,\chi)\Big)\Big)dq\\ &= \frac{1}{\sqrt[3]{t}}\Re\mathrm{Ai}\Big(\frac{x}{\sqrt[3]{t}}\Big)\hat{v}(t,\chi)+\mathcal{R}(t,x),\end{aligned}$$

where we made the change of variable $q = p\sqrt[3]{t}$ and $\eta = x/\sqrt[3]{t}$, and we introduce $\chi = \sqrt{-x/t}$ if $x \le 0$ and $\chi = 0$ if $x \ge 0$, and

$$\mathcal{R}(t,x) = \frac{1}{\pi\sqrt[3]{t}}\Re\int_0^\infty e^{iq\eta+iq^3/3}\Big(\hat{v}\Big(t,\frac{q}{\sqrt[3]{t}}\Big)-\hat{v}(t,\chi)\Big)dq.$$

We now estimate $\mathcal{R}(t,x)$, and consider two cases, $x \ge 0$ and $x \le 0$. For this, we will do appropriate integration by parts to use the decay related to the oscillatory integral.

Let us introduce a final notation $\mu = \sqrt{|\eta|}$. We have the identity

$$e^{iq\eta+iq^3/3} = \frac{1}{1+iq(q^2+\mu^2)}\frac{\partial}{\partial q}(qe^{iq\eta+iq^3/3}). \tag{4.3}$$

Consider the case $x \ge 0$; that is, $\chi = 0$. We can do an integration by parts using (4.3) in the remainder term $\mathcal{R}$

$$\begin{aligned}\mathcal{R}(t,x) \;&=\; \frac{1}{\pi\sqrt[3]{t}}\Re\int_0^\infty\Big(\frac{iq(q^2+\mu^2)(\hat{v}(t,q/\sqrt[3]{t})-\hat{v}(t,0))}{1+iq(q^2+\mu^2)}\\ &\quad -\frac{q}{\sqrt[3]{t}}\hat{v}_p\Big(t,\frac{q}{\sqrt[3]{t}}\Big)\Big)\frac{e^{iq\eta+iq^3/3}dq}{1+iq(q^2+\mu^2)}.\end{aligned}$$

Using the identity

$$\|D^\alpha J^t\phi\|_{L^2} = \|U(t)D^\alpha(xU(-t)\phi)\|_{L^2} = \|D^\alpha(xv)\|_{L^2} = \||p|^\alpha\hat{v}_p\|_{L^2},$$

and the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|\hat{v}(t,p)-\hat{v}(t,0)| \;&\le\; \int_0^p|\hat{v}_p(t,\rho)|d\rho \le \Big(\int_0^p|\rho|^{-2\alpha}d\rho\int_0^p|\hat{v}_p^2(t,\rho)\rho^{2\alpha}|d\rho\Big)^{\frac{1}{2}}\\ &\le\; C|p|^\gamma\||p|^\alpha\hat{v}_p(t,p)\|_{L^2} \le C|p|^\gamma\|D^\alpha J^t\phi\|_{L^2}.\end{aligned} \tag{4.4}$$

On the other hand, using the change of variable $z = q(1+\mu^2)$, a computation shows that, for any $a, b > 0$ such that $3b - a > 1$, we have

$$\begin{aligned}\int_0^\infty\frac{q^a dq}{(1+q(q^2+\mu^2))^b} &\le \int_0^{\mu+1}\frac{2q^a dq}{(1+q(1+\mu^2))^b}+\int_{\mu+1}^\infty\frac{q^a dq}{1+q^{3b}}\\ &\le \frac{C}{(1+\mu)^{3b-a-1}}.\end{aligned} \tag{4.5}$$

(We used the inequality $1+q(q^2+\mu^2) \ge \frac{1}{2}(1+q(1+\mu^2))$ for the first term and $q^2 \le q^2+\mu^2$ for the second.) Thus, we can estimate

$$|\mathcal{R}(t,x)| \;\le\; \frac{C}{\sqrt[3]{t}}\int_0^\infty\frac{(|\hat{v}(t,q/\sqrt[3]{t})-\hat{v}(t,0)|+q/\sqrt[3]{t}|\hat{v}_p(t,q/\sqrt[3]{t})|)dq}{|1+iq(q^2+\mu^2)|}$$

$$
\begin{aligned}
&\leq \frac{C}{t^{(1+\gamma)/3}}\|D^\alpha J^t\phi\|_{L^2}\int_0^\infty \frac{q^\gamma dq}{1+q(q^2+\mu^2)} \\
&\quad + \frac{C}{\sqrt[3]{t}^2}\Big(\int_0^\infty q^{2\alpha}\hat{v}_p^2\Big(t,\frac{q}{\sqrt[3]{t}}\Big)dq\int_0^\infty \frac{q^{1+2\gamma}dq}{(1+q(q^2+\mu^2))^2}\Big)^{1/2} \\
&\leq \frac{C\|D^\alpha J^t\phi\|_{L^2}}{t^{(1+\gamma)/3}(1+\mu)^{2-\gamma}}. \qquad (4.6)
\end{aligned}
$$

Let us now consider the case $x \leq 0$; that is, $\eta = -\mu^2 \leq 0$ (recall $\mu = \sqrt{|x|}/\sqrt[6]{t}$, $\chi = \mu/\sqrt[3]{t}$). We will now use the identity

$$
e^{iq\eta+iq^3/3} = \frac{1}{1+i(q^2-\mu^2)(q+\mu)}\frac{\partial}{\partial q}((q-\mu)e^{iq\eta+iq^3/3}). \qquad (4.7)
$$

We integrate by parts the remainder term $\mathcal{R}$

$$
\begin{aligned}
\mathcal{R}(t,x) &= \frac{1}{\pi\sqrt[3]{t}}\Re\int_0^\infty \Big(\frac{i(q-\mu)^2(3q+\mu)}{1+i(q^2-\mu^2)(q+\mu)}\Big(\hat{v}\Big(t,\frac{q}{\sqrt[3]{t}}\Big)-\hat{v}(t,\chi)\Big) \\
&\quad -\frac{q-\mu}{\sqrt[3]{t}}\hat{v}_p\Big(t,\frac{q}{\sqrt[3]{t}}\Big)\Big)\frac{e^{iq\eta+iq^3/3}dq}{1+i(q+\mu)(q-\mu)^2}.
\end{aligned}
$$

A computation similar to (4.5) gives (provided that $3c-a-b-1>0$)

$$
\begin{aligned}
&\int_0^\infty \frac{|q-\mu|^a q^b dq}{(1+(q-\mu)^2(q+\mu))^c} \leq \int_0^{2(1+\mu)} + \int_{2(1+\mu)}^\infty \\
&\leq C\int_{-1-\mu}^{1+\mu}\frac{|z|^a(1+\mu)^b dz}{1+(z^2(1+\mu))^c} + \int_{1+\mu}^\infty \frac{|z|^{a+b}dz}{1+z^{3c}} \leq C(1+\mu)^{a+b+1-3c},
\end{aligned}
$$

(with the change of variable $z = q-\mu$ and $z' = z\sqrt{1+\mu}$). Thus, by the Cauchy-Schwarz inequality, we obtain

$$
\begin{aligned}
|\mathcal{R}(t,x)| &\leq \frac{C}{\sqrt[3]{t}}\int_0^\infty \Big(\Big|\hat{v}\Big(t,\frac{q}{\sqrt[3]{t}}\Big)-\hat{v}(t,\chi)\Big| + \frac{|q-\mu|}{\sqrt[3]{t}}\Big|\hat{v}_p\Big(t,\frac{q}{\sqrt[3]{t}}\Big)\Big|\Big) \\
&\quad \times\frac{dq}{|1+i(q-\mu)^2(q+\mu)|} \\
&\leq \frac{C}{t^{(1+\gamma)/3}}\|D^\alpha J^t\phi\|_{L^2}\Big(\int_0^\infty \frac{|q-\mu|^\gamma dq}{1+(q-\mu)^2(q+\mu)} \\
&\quad +\Big(\int_0^\infty \frac{(q-\mu)^2 dq}{(1+(q-\mu)^2(q+\mu))^2 q^{2\alpha}}\Big)^{1/2}\Big) \\
&\leq \frac{C\|D^\alpha J^t\phi\|_{L^2}}{t^{(1+\gamma)/3}(1+\mu)^{2-\gamma}} \qquad (4.8)
\end{aligned}
$$

(recall $\alpha = 1/2 - \gamma$)). It remains to bound $\|\hat{v}(t,\cdot)\|_{L^\infty}$

$$\begin{aligned}
|\hat{v}(t,p)| &\leq \int_{\mathbb{R}} |\hat{v}_p(t,\rho)|d\rho \leq \int_{|\rho|\geq 1} |\hat{v}_p(t,\rho)|d\rho + \int_{|\rho|\leq 1} |\hat{v}_p(t,\rho)|d\rho \\
&\leq \Big(\int_{|\rho|\geq 1} |\hat{v}_p^2 \rho^2| d\rho \int_{|\rho|\geq 1} \frac{d\rho}{\rho^2}\Big)^{\frac{1}{2}} + \Big(\int_{|\rho|\leq 1} |\hat{v}_p^2 \rho^{2\alpha}| d\rho \int_{|\rho|\leq 1} \frac{d\rho}{\rho^{2\alpha}}\Big)^{\frac{1}{2}} \\
&\leq C(\|DJ^t\phi\|_{L^2} + \|D^\alpha J^t \phi\|_{L^2}). \qquad (4.9)
\end{aligned}$$

Thus, adding up (4.6), (4.8), along with the estimate $|\mathrm{Ai}(\eta)| \leq C(1+|\eta|)^{-1/4}$ and (4.9), we obtain (recall $\eta = x/\sqrt[3]{t}$)

$$|\phi(x)| \leq \frac{C}{t^{1/3}}(1+|\eta|)^{-1/4} M_0^t(\phi), \qquad (4.10)$$

which is the first pointwise estimate. The L^r-estimate follows

$$\begin{aligned}
\|\phi\|_{L^r} &\leq \frac{C}{t^{1/3}} M_0^t(\phi) \Big(\int \Big(1 + \frac{|x|}{\sqrt[3]{t}}\Big)^{r/4} dx\Big)^{1/r} \\
&\leq \frac{C}{t^{1/3-1/(3r)}} M_0^t(\phi) \Big(\int \Big(1+|\eta|\Big)^{r/4} d\eta \Big)^{1/r} \leq \frac{C}{t^{1/3-1/(3r)}} M_0^t(\phi).
\end{aligned}$$

Now we switch to the derivative ϕ_x

$$\phi_x(x) = \frac{i}{\pi} \Re \int_0^\infty e^{iq\eta + iq^3 t/3} \hat{v}\Big(t, \frac{q}{\sqrt[3]{t}}\Big) q dq.$$

Using identity (4.3), we obtain analogously to (4.6), in the domain $x \geq 0$ ($\eta = \mu^2 \geq 0$)

$$\begin{aligned}
|\phi_x(x)| &\leq \frac{C}{t^{2/3}} \int_0^\infty \frac{|\hat{v}(t, q/\sqrt[3]{t})| q dq}{|1 + iq(q^2+\mu^2)|} + \frac{C}{t} \int_0^\infty \frac{|\hat{v}_p(t, q/\sqrt[3]{t})| q^2 dq}{|1+iq(q^2+\mu^2)|} \\
&\leq \frac{C}{t^{2/3}} (\|\hat{v}(t)\|_{L^\infty} + \||p|^\alpha \hat{v}_p(t,p)\|_{L^2}) \\
&\leq \frac{C}{t^{2/3}} (\|DJ^t\phi\|_{L^2} + \|D^\alpha J^t \phi\|_{L^2}) \qquad (4.11)
\end{aligned}$$

(we used the Cauchy-Schwarz inequality as in (4.6), and in the last inequality, we used again (4.9)). In the domain $x \leq 0$ ($\eta = -\mu^2 \leq 0$), we use identity (4.7) to get analogously to (4.8)

$$\begin{aligned}
|\phi_x(x)| \leq \frac{C}{\sqrt[3]{t}^2} \int_0^\infty \Big(|\hat{v}\Big(t, \frac{q}{\sqrt[3]{t}}\Big) + \frac{|q-\mu|}{\sqrt[3]{t}} \Big|\hat{v}\Big(t, \frac{q}{\sqrt[3]{t}}\Big)\Big| \Big) \\
\times \frac{q dq}{|1 + i(q-\mu)^2(q+\mu)|}
\end{aligned}$$

$$\leq \frac{C}{\sqrt[3]{t}^2}\|\hat{v}\|_{L^\infty}\int_0^\infty \frac{qdq}{|1+i(q-\mu)^2(q+\mu)|}$$
$$+\frac{C}{\sqrt[3]{t}^2}\||p|^\alpha \hat{v}_p(t,p)\|_{L^2}\Big(\int_0^\infty \frac{(q-\mu)^2 q^{2\alpha}dq}{(1+(q-\mu)^2(q+\mu))^2}\Big)^{\frac{1}{2}}.$$

The integral of the second term can be estimated by our regular computations, but we have to be more careful with the first term

$$\int_0^\infty \frac{qdq}{1+(q-\mu)^2(q+\mu)} \leq C\int_0^{2(\mu+1)}\frac{(1+\mu)dq}{1+(q-\mu)^2\mu}+C\int_{2(\mu+1)}^\infty \frac{(q-\mu)dq}{1+(q-\mu)^3}$$
$$\leq C\sqrt{1+\mu}\int\frac{dz}{1+z^2}+\frac{C}{1+\mu} \quad \leq \quad C\sqrt{1+\mu}.$$

So that we obtain, for $x\leq 0$, the estimate

$$|\phi_x(x)| \leq \frac{C}{\sqrt[3]{t^2}}(\|DJ^t\phi\|_{L^2}+\|D^\alpha J^t\phi\|_{L^2})\sqrt{1+\mu}. \tag{4.12}$$

Finally, combining (4.11) and (4.12) gives the second pointwise estimate (recall $\eta = x/\sqrt[3]{t}$)

$$|\phi_x(x)| \leq \frac{CM_0^t(\phi)}{t^{2/3}}(1+\eta)^{1/4}. \tag{4.13}$$

APPENDIX B.

To conclude, we present a proof of a local existence theorem in M_0^t, which is needed in the discussion of Theorem 1. The proof is done for forward times, but of course, it is also true backwards.

Theorem 3. *Let $t_0\in\mathbb{R}$ and $u_0\in H^1(\mathbb{R})$ such that $M_0^{t_0}(u_0)<\infty$. Then there exist $T=T(M_0^{t_0}(u_0))$ and a unique solution $u:[t_0,t_0+T)\times\mathbb{R}\to\mathbb{R}$ to*

$$u_t+u_{xxx}+(u^p)_x=0,\quad u|_{t=t_0}=u_0.$$

Furthermore,

$$\forall t\in[t_0,T+t_0[,\quad M_0^t(u(t))<\infty,\ \text{and}\quad \limsup_{t\downarrow t_0} M_0^t(u(t))\leq M_0^{t_0}(u_0).$$

Proof. Uniqueness is straightforward as there is already uniqueness in H^1.

For the existence part, we proceed by regularization. Let $u_0^n\in H^3$ with compact support such that $M_0^{t_0}(u_0^n-u_0)\to 0$ (u_0^n exists by standard density arguments). In particular, we can suppose that

$$\forall n,\quad M_0^{t_0}(u_0^n)\leq 2M_0^{t_0}(u_0)=K.$$

For every u_0^n, the local existence theorem in H^3 ensures the existence of $u^n(t)$ on an interval $[t_0,t_0+T^n]$. Recall that existence in H^3 has the same time span as in H^1. Since the initial data sequence $(u_0^n)_n$ is uniformly bounded in H^1, $T_n = T_n(\|u_0^n\|_{H^1}) \geq T^* > 0$.

The point in working in H^3 is that $I^t(u^n(t)) = J^t(u^n(t)) - t(u^n)^p$ is well defined in H^1. Indeed, one computes

$$\left|\frac{d}{dt}\int x(u^n)^2\right| = \left| -3\int (u_x^n)^2 - \frac{2p}{p+1}\int (u^n)^{p+1}\right| \leq C\|u^n(t)\|_{H^1},$$

so that for all $t \in [t_0, t_0 + T^*)$, $\|\sqrt{x}u^n(t)\|_{L^2} < \infty$ (recall $u_0(t_0)$ has compact support, so that quantity is initially well defined at time t_0). In the same way, the derivatives in time of $\|xu^n\|_{L^2} + \|\sqrt{x}u_x^n(t)\|_{L^2}$ are controlled by $\|u^n(t)\|_{H^2}$, and that of $\|xu_x^n\|_{L^2}$ by $\|u^n(t)\|_{H^3}$. So finally, the H^3 bound on $u^n(t)$ ensures that for $t \in [t_0, t_0 + T^*]$

$$\|J^t u^n(t)\|_{H^1} \leq \|xu^n(t)\|_{H^1} + t\|u_{xx}^n(t)\|_{H^1} < \infty.$$

And for $I^t u_n(t)$

$$\|I^t(u^n(t))\|_{H^1} \leq \|J^t(u^n(t))\|_{H^1} + t\|(u^n(t))^p\|_{H^1} < \infty.$$

So now it is possible to do the a priori computations of Naumkin and Hayashi [4] for $u^n(t)$. Let $I = [t_0, t_0 + T_n^*)$ be such that for all $t \in I$, $M_0^t(u^n(t)) \leq 2K$ (and I maximal for this property); by H^3 continuity, $T_n^* > 0$. Their computations give (see equations (3.3), (3.8), and (3.9) of [4])

$$\frac{d}{dt}\|u^n\|_{H^1}^2 \leq C\frac{M_0^t(u^n(t))^{p-2}}{t^{2/3}(1+t)^{(p-2)/3}}\|u^n\|_{H^1}^2 \leq \frac{CK^p}{t^{2/3}(1+t)^{(p-2)/3}}, \tag{4.14}$$

and similar estimates for $\|D^\alpha I^t u^n\|_{L^2}$ and $\|DI^t u^n\|_{L^2}$.

Let T be such that

$$\int_{t_0}^{t_0+T} \frac{CK^p}{t^{2/3}(1+t)^{(p-2)/3}}dt \leq \frac{K}{3}$$

(there are 3 estimates). By a standard continuity argument, $T^* \geq T_n^* \geq T$ (independent of n), and

$$\forall n,\ \forall t \in [t_0, t_0 + T], \quad M_0^t(u_n(t)) \leq 2K.$$

In the same way, (4.14) gives that $t \mapsto M_0^t(u_n(t))$ is equicontinuous (in n).

Now, standard computations show that $u^n(t)$ is a Cauchy sequence in $C([t_0, t_0 + T], L^2)$, so it converges to $u(t)$. Indeed, with $v = u^n - u^m$

$$v_t + v_{xxx} + (u^n)_x^p - (u^m)_x^p = 0.$$

Multiply by v, and integrate in space

$$\frac{d}{dt}\|v\|_{L^2}^2(t) \leq \frac{(2K)^p}{t^{2/3}(1+t)^{(p-2)/3}}\|v\|_{L^2}^2(t).$$

And by Gronwall's lemma (taking the supremum)

$$\|v\|_{C^0([t_0,t_0+T],L^2)} \leq CK^p\|v(0)\|_{L^2}.$$

But $v(0) \to 0$ as $m, n \to \infty$ (in H^1, so in L^2).

$u(t)$ can also be seen as a weak limit in H^1, and in M_0^t, of $u^n(t)$ (for t fixed), therefore, we get the bounds in H^1 and M_0^t for $u(t)$. Finally, we take the H^{-1}-limit in the integral formulation

$$u^n(t) = S(t-t_0)u_0^n + \int_{t_0}^t S(t-\tau)(u^n)_x^p(\tau)d\tau.$$

There is no problem for $S(t-t_0)u_0^n$, and for the second term

$$\int_{t_0}^t S(t-\tau)\left((u^n)_x^p(\tau) - (u^m)_x^p\right)d\tau = \Big(\int_{t_0}^t S(t-\tau)\left((u^n)^p(\tau) - (u^m)^p\right)d\tau\Big)_x.$$

$(u^n)^p(\tau) - (u^m)^p \to 0$ in $C^0([t_0, t_0+T], L^2)$, so we can take the limit in L^2 in the integral, and then in the H^{-1} sense for the whole term.

$u(t)$ is a solution to (1.1) in the integral sense, $u(t_0) = u_0$, and $u \in L^\infty([t_0, t_0+T], H^1)$, so u is the unique $C^0([t_0, t_0+T], H^1)$ solution. By weak limit

$$\forall t \in [t_0, t_0+T[, \quad M_0^t(u(t)) \leq 2K, \quad \text{and} \quad \limsup_{t\downarrow t_0} M_0^t(u(t)) \leq M_0^{t_0}(u_0).$$

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