EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS TO A P-LAPLACIAN EQUATION IN $\mathbb{R}^N$

Claudianor O. Alves [1]
Universidade Federal de Campina Grande
Departamento de Matemática e Estatística
CEP: 58109-970, Campina Grande - Pb, Brazil

Giovany M. Figueiredo
Universidade Federal do Pará, Campus do Baixo Tocantins-Abaetetuba
CEP: 68440-000 Abatetuba - Pa, Brazil

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Abstract. In this work we study the existence, multiplicity and concentration of positive solutions for the following class of problem

$$-\epsilon^p \Delta_p u + V(z)|u|^{p-2}u = f(u), \;\; u(z) > 0, \forall\, z \in \mathbb{R}^N, \qquad (P_\epsilon)$$

where $\Delta_p u$ is the p-Laplacian operator, ϵ is a positive parameter, $2 \leq p < N, V : \mathbb{R}^N \to \mathbb{R}$ is a continuous functions and $f : \mathbb{R} \to \mathbb{R}$ is a function of C^1 class.

1. Introduction

In this paper we are concerned with the existence, multiplicity and concentration of positive solutions for the following class of quasilinear problem

$$\begin{cases} -\epsilon^p \Delta_p u + V(z)|u|^{p-2}u = f(u) \;\text{ in }\; \mathbb{R}^N \\ u \in W^{1,p}(\mathbb{R}^N) \;\text{ with }\; 2 \leq p < N, \\ u(z) > 0, \forall\, z \in \mathbb{R}^N, \end{cases} \qquad (P_\epsilon)$$

where $\epsilon > 0$, $\Delta_p u$ is the p-Laplacian operator, that is,

$$\Delta_p u = div(|\nabla u|^{p-2}\nabla u),$$

V is a continuous function satisfying:

$$V_\infty = \liminf_{|x|\to\infty} V(x) > V_0 = \inf_{\mathbb{R}^N} V(x) > 0 \qquad (V)$$

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where, in this work, we will consider the cases $V_\infty < \infty$ or $V_\infty = \infty$. This kind of hypothesis was introduced by Rabinowitz in [22]. The nonlinearity $f : \mathbb{R} \to \mathbb{R}$ is a function of C^1 class and since we are looking for positive solutions we suppose that

$$f(s) = 0 \text{ for all } s < 0. \tag{f_1}$$

Moreover, we assume the following growth conditions at the origin and at infinity:

$$\lim_{|s|\to 0} \frac{|f(s)|}{|s|^{p-1}} = 0, \tag{f_2}$$

and there exists $q \in (p, p^* - 1)$ satisfying

$$\lim_{|s|\to \infty} \frac{|f(s)|}{|s|^{q}} = 0. \tag{f_3}$$

In this article, we use the classical Palais-Smale condition. Related to this condition, we suppose that f verifies the well-known Ambrosetti-Rabinowitz superlinear condition, that is, there exists $\theta > p$ such that

$$0 < \theta F(s) = \int_0^s f(t)dt \leq sf(s) \text{ for all } s > 0. \tag{f_4}$$

We assume the following monotonicity condition for f:

$$\text{The function } s \to \frac{f(s)}{s^{p-1}} \text{ is increasing in } (0, +\infty), \tag{f_5}$$

and the existence of constants $C > 0$ and $\sigma \in (p, p^*)$ satisfying

$$f'(s)s^2 - (p-1)f(s)s \geq Cs^\sigma \text{ for all } s \geq 0. \tag{f_6}$$

For the case $p = 2$, equations of the kind

$$-\epsilon^2 \Delta u + V(z)u = f(u) \text{ in } \mathbb{R}^N \tag{P_*}$$

arise in different models, for example, they are related with the existence of standing waves of the nonlinear Schrödinger equation

$$i\epsilon \frac{\partial \Psi}{\partial t} = -\epsilon^2 \Delta \Psi + (V(x) + E)\Psi - f(\Psi) \text{ for all } x \in \mathbb{R}^N, \tag{NLS}$$

when $f(s) = |s|^{q-2}u$, $2 < q < 2^* = 2N/(N-2)$. A standing wave of (NLS) is a solution of the form $\Psi(x,t) = exp(-iEt/\epsilon)u(x)$. In this case, u is a solution of (P_*).

Existence and concentration of positive solutions for the problem (P_*) have been extensively studied in recent years, see for example, Ambrosetti, Badiale, and Cingolani [4], Ambrosetti, Malchiodi, and Secchi [5], Cingolani and Lazzo [7, 8], Del Pino and Felmer [10], Floer [13], Lazzo [16], Oh [19, 20, 21], Rabinowitz [22], Serrin and Tang [23], Wang [26] and their references.

In this work, motivated by [8], we prove the existence, multiplicity and concentration of positive solutions to (P_ϵ). Our main result completes the study made in [8] in the following sense: we are working with p-Laplacian operators with $p \geq 2$ and the hypotheses on f are satisfied by a large class of nonlinearities which includes u^q for $q > p-1$. Here, in the proof of some lemmas and propositions, we use different arguments from those found in [8], because the nonlinearity is not necessarily homogeneous and the p-Laplacian operator is not linear. To obtain our main result, we prove a compactness result on Nehari manifolds and use the Moser iteration method to overcome some technical difficulties to show some estimates because, for example, for problems involving p-Laplacians we can not use in general convergence in the C^2 sense, and also, we do not know if the problem below has a unique positive solution

$$\begin{cases} -\Delta_p u + \mu|u|^{p-2}u = f(u) \text{ in } \mathbb{R}^N \\ u \in W^{1,p}(\mathbb{R}^N) \text{ with } 2 \leq p < N, \\ u(x) > 0, \forall\, x \in \mathbb{R}^N, \ \ \mu > 0. \end{cases} \qquad (P_\mu)$$

The above properties are used in a lot of papers when $p = 2$.

Our main result, Theorem 1.1 below, establishes the existence of multiplicity of solutions to (P_ϵ) involving the Lusternik-Schnirelman category of the sets M and M_δ given by

$$M = \{x \in \mathbb{R}^N : V(x) = V_0\}$$

and

$$M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \leq \delta\}, \quad \text{for} \quad \delta > 0.$$

Theorem 1.1. *Suppose that f satisfies $(f_1) - (f_6)$ and the function V satisfies (V). Then, for any $\delta > 0$, there exist $\epsilon_\delta > 0$ such that (P_ϵ) has at least $cat_{M_\delta}(M)$ positive solutions, for any $0 < \epsilon < \epsilon_\delta$. Moreover, if u_ϵ denotes one of these positive solutions and $\eta_\epsilon \in \mathbb{R}^N$ its global maximum, then*

$$\lim_{\epsilon \to 0} V(\eta_\epsilon) = V_0.$$

Remark 1.1. We recall that, if Y is a closed subset of a topological space X, the Lusternik-Schnirelman category $cat_X(Y)$ is the least number of closed and contractible sets in X which cover Y.

2. Variational framework and notation

Hereafter, we will work with the following problem equivalent to (P_ϵ), which is obtained under the change of variable $\epsilon z = x$

$$\begin{cases} -\Delta_p u + V(\epsilon x)|u|^{p-2}u = f(u) \ \text{ in } \ \mathbb{R}^N \\ u \in W^{1,p}(\mathbb{R}^N) \ \text{ with } \ 2 \le p < N, \\ u(x) > 0, \forall \ x \in \mathbb{R}^N. \end{cases} \tag{P_ϵ^*}$$

In this section, we fix some notation involving the functionals used to get the solutions to (P_ϵ^*); this way, we divide this section into some subsections.

2.1. **Energy functionals and their spaces.**

In this paper, the main tool used to prove Theorem 1.1 is the variational method, where the solutions to (P_ϵ^*) are the critical points of the functional given by

$$I_\epsilon(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon x)|u|^p - \int_{\mathbb{R}^N} F(u),$$

which is well defined on the Banach space W_ϵ defined by

$$W_\epsilon = \Big\{ u \in W^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(\epsilon x)|u|^p < \infty \Big\}$$

endowed with the norm

$$\|u\|_\epsilon^p = \int_{\mathbb{R}^N} |\nabla u|^p + \int_{\mathbb{R}^N} V(\epsilon x)|u|^p.$$

Next, let us denote the Nehari manifold associated to I_ϵ by

$$\mathcal{N}_\epsilon = \Big\{ u \in W_\epsilon \setminus \{0\} : \int_{\mathbb{R}^N} |\nabla u|^p + \int_{\mathbb{R}^N} V(\epsilon x)|u|^p = \int_{\mathbb{R}^N} f(u)u \Big\}.$$

It is easy to check that there exists $r > 0$ such that

$$\|u\|_\epsilon \ge r > 0, \ \text{ for all } \ u \in \mathcal{N}_\epsilon. \tag{2.1}$$

Moreover, let us also denote by E_μ the energy functional associated to the problem (P_μ) and by $\mathcal{M}_\mu$ its Nehari manifold, that is,

$$E_\mu(u) = \frac{1}{p}\int_{\mathbb{R}^N} |\nabla u|^p + \frac{1}{p}\int_{\mathbb{R}^N} \mu|u|^p - \int_{\mathbb{R}^N} F(u)$$

and

$$\mathcal{M}_\mu = \Big\{ u \in X_\mu \setminus \{0\} : \int_{\mathbb{R}^N} |\nabla u|^p + \int_{\mathbb{R}^N} \mu|u|^p = \int_{\mathbb{R}^N} f(u)u \Big\},$$

where $X_\mu = W^{1,p}(\mathbb{R}^N)$ endowed with the norm

$$\|u\|_\mu^p = \int_{\mathbb{R}^N} |\nabla u|^p + \int_{\mathbb{R}^N} \mu |u|^p.$$

In what follows, $m(\mu)$ denotes the minimax level related to the mountain pass theorem applied to E_μ.

2.2. **Technical results.** In this subsection, we will show some lemmas related to the functional I_ϵ. We start recalling that the functional I_ϵ verifies the mountain pass geometry and its proof follows by well-known arguments.

Lemma 2.1. *The functional I_ϵ satisfies the following conditions:*
(i) There exists α, $\rho > 0$ such that $I_\epsilon(u) \geq \alpha$ with $\|u\|_\epsilon = \rho$.
(ii) There exist $e \in B_\rho^c(0)$ with $I_\epsilon(e) < 0$.

By the mountain pass theorem without the (PS) condition (see Willem [27]), it follows that there exists a $(PS)_{c_\epsilon}$ sequence $(u_n) \subset W_\epsilon$, that is, a sequence satisfying $I_\epsilon(u_n) \to c_\epsilon$ and $I'_\epsilon(u_n) \to 0$, where c_ϵ is the minimax level of the mountain pass theorem applied to I_ϵ. By (f_1), we can suppose that for each n, u_n is nonnegative. Moreover, by standard arguments, we will assume that (u_n) is bounded and thus there exist a subsequence, still denoted by (u_n), and $u \in W_\epsilon$ such that

$$u_n \rightharpoonup u \text{ in } W_\epsilon \text{ and } u_n(x) \to u(x) \text{ a.e in } \mathbb{R}^N.$$

As in [22](Proposition 3.11), we shall use the following equivalent characterization of c_ϵ, which is more adequate to our purpose:

$$c_\epsilon = \inf_{u \in W_\epsilon \setminus \{0\}} \sup_{t \geq 0} I_\epsilon(tu) = \inf_{u \in \mathcal{N}_\epsilon} I_\epsilon(u).$$

Remark 2.1. It is easy to check that for each nonzero nonnegative $u \in W_\epsilon$, there exists a unique $t_0 = t_0(u)$ such that

$$I_\epsilon(t_0 u) = \max_{t \geq 0} I_\epsilon(tu).$$

Lemma 2.2. *Let $(u_n) \subset W_\epsilon$ be a $(PS)_d$ sequence for I_ϵ such that $u_n \rightharpoonup 0$ in W_ϵ. Then we have either:*
a) $u_n \to 0$ in W_ϵ or
b) there exist a sequence $(y_n) \subset \mathbb{R}^N$ and constants R , $\beta > 0$ such that

$$\liminf_{n \to +\infty} \int_{B_R(y_n)} |u_n|^p \geq \beta > 0.$$

Proof. Suppose that b) does not occur. Using Lemma 1.1 in [17], it follows:

$$u_n \to 0 \ \text{ in } \ L^s(\mathbb{R}^N) \ \text{ for } \ s \in (p\,,\, p^*).$$

Given $\xi > 0$, from $(f_2) - (f_3)$

$$0 \le \int_{\mathbb{R}^N} f(u_n)u_n \le \xi \int_{\mathbb{R}^N} |u_n|^p + C_\xi \int_{\mathbb{R}^N} |u_n|^{q+1}.$$

Using the fact that (u_n) is bounded in $L^p(\mathbb{R}^N)$, $u_n \to 0$ in $L^{q+1}(\mathbb{R}^N)$, and that ξ is arbitrary, we can conclude that

$$\int_{\mathbb{R}^N} f(u_n)u_n \to 0.$$

Recalling that $I'_\epsilon(u_n)u_n = o_n(1)$, it follows that $u_n \to 0$ in W_ϵ, thus, the proof is finished. □

Lemma 2.3. *Assume that $V_\infty < \infty$ and let (v_n) be a $(PS)_d$ sequence for the I_ϵ in W_ϵ with $v_n \rightharpoonup 0$ in W_ϵ. If $v_n \not\to 0$ in W_ϵ, then $d \ge m(V_\infty)$, where $m(V_\infty)$ is the minimax level of E_{V_∞}.*

Proof. Let $(t_n) \subset (0, +\infty)$ be a sequence such that $(t_n v_n) \subset \mathcal{M}_{V_\infty}$. We start by showing the following claim

Claim 1. *The sequence $\{t_n\}$ satisfies* $\limsup_{n\to\infty} t_n \le 1$.

In fact, supposing by contradiction that the above claim does not hold, there exist $\delta > 0$ and a subsequence still denoted by (t_n), such that

$$t_n \ge 1 + \delta \ \text{ for all } \ n \in \mathbb{N}. \tag{2.2}$$

Since (v_n) is bounded in W_ϵ, $I'_\epsilon(v_n)v_n = o_n(1)$, that is,

$$\int_{\mathbb{R}^N} \big[\, |\, \nabla v_n \,|^p + V(\epsilon x) \,|\, v_n \,|^p \,\big] = \int_{\mathbb{R}^N} f(v_n)v_n + o_n(1).$$

Moreover, recalling that $(t_n v_n) \subset \mathcal{M}_{V_\infty}$, we get

$$t_n^p \Big(\int_{\mathbb{R}^N} \big[\, |\, \nabla v_n \,|^p + V_\infty \,|\, v_n \,|^p \,\big] \Big) = \int_{\mathbb{R}^N} f(t_n v_n) t_n v_n.$$

The last two equalities imply that

$$\int_{\mathbb{R}^N} \Big[\frac{f(t_n v_n)(v_n)^p}{(t_n v_n)^{p-1}} - \frac{f(v_n)(v_n)^p}{(v_n)^{p-1}} \Big] = \int_{\mathbb{R}^N} [V_\infty - V(\epsilon x)] \,|\, v_n \,|^p + o_n(1).$$

Given $\xi > 0$, from condition (V) there exists $R = R(\xi) > 0$ such that

$$V(\epsilon x) \ge V_\infty - \xi \ \text{ for any } \ |x| \ge R.$$

Let $C > 0$ be such that $\|v_n\|_\epsilon \leq C$. Since $v_n \to 0$ in $L^p(B_R(0))$, we conclude that

$$\int_{\mathbb{R}^N} \Big[\frac{f(t_n v_n)}{(t_n v_n)^{p-1}} - \frac{f(v_n)}{(v_n)^{p-1}}\Big](v_n)^p \leq \xi C + o_n(1). \tag{2.3}$$

Since $v_n \not\to 0$ in W_ϵ, we may invoke Lemma 2.2 to obtain $(y_n) \subset \mathbb{R}^N$ and $\check{R}, \beta > 0$ such that

$$\int_{|y_n| \leq \check{R}} | v_n |^p \geq \beta. \tag{2.4}$$

If we define $\check{v}_n(x) = v_n(x + y_n)$, we may suppose that, up to a subsequence, $\check{v}_n \rightharpoonup \check{v}$ in $W^{1,p}(\mathbb{R}^N)$. Moreover, in view of (2.4), there exists a subset $\Omega \subset \mathbb{R}^N$ with positive measure such that $\check{v} > 0$ in Ω. From (f_5), we can use (2.2) to rewrite (2.3) as

$$0 < \int_\Omega \Big[\frac{f((1+\delta)\check{v}_n)}{((1+\delta)\check{v}_n)^{p-1}} - \frac{f(\check{v}_n)}{\check{v}_n^{p-1}}\Big]\check{v}_n^p \leq \xi C + o_n(1), \quad \text{for any} \quad \xi > 0.$$

Letting $n \to \infty$ in the last inequality and applying Fatou's lemma, it follows that

$$0 < \int_\Omega \Big[\frac{f((1+\delta)\check{v})}{((1+\delta)\check{v})^{p-1}} - \frac{f(\check{v})}{\check{v}^{p-1}}\Big]\check{v}^p \leq \xi\, C, \quad \text{for any} \quad \xi > 0,$$

which is absurd, and the claim is proved.

Now, we will consider the following cases:

Case 1: $\limsup_{n\to\infty} t_n = 1$.

In this case, there exists a subsequence, still denoted by $\{t_n\}$, such that $t_n \to 1$. Thus,

$$d + o_n(1) = I_\epsilon(v_n) \geq m(V_\infty) + I_\epsilon(v_n) - E_{V_\infty}(t_n v_n). \tag{2.5}$$

Recalling that

$$I_\epsilon(v_n) - E_{V_\infty}(t_n v_n) = \int_{\mathbb{R}^N} \frac{(1 - t_n^p)}{p}|\nabla v_n|^p + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon x)|v_n|^p - \frac{t_n^p}{p}\int_{\mathbb{R}^N} V_\infty |v_n|^p + \int_{\mathbb{R}^N} [F(t_n v_n) - F(v_n)],$$

and using the fact that (v_n) is bounded in W_ϵ together with the condition (V), we get

$$I_\epsilon(v_n) - E_{V_\infty}(t_n v_n) \geq \; o_n(1) - C\xi + \int_{\mathbb{R}^N} [F(t_n v_n) - F(v_n)].$$

Moreover, from the mean value theorem,

$$\int_{\mathbb{R}^N}[F(t_nv_n)-F(v_n)]=o_n(1),$$

therefore, $d+o_n(1)\geq m(V_\infty)-C\xi+o_n(1)$, and taking the limit, we have $d\geq m(V_\infty)$.

Case 2: $\limsup_{n\to\infty} t_n=t_0<1$.

In this case, we may suppose that there exists a subsequence, still denoted by $\{t_n\}$, satisfying $t_n\to t_0$ and $t_n<1$ for all $n\in\mathbb{N}$. Since $\frac{1}{p}f(s)s-F(s)$ is increasing

$$m(V_\infty)\leq\int_{\mathbb{R}^N}\big[\frac{1}{p}f(t_nv_n)(t_nv_n)-F(t_nv_n)\big]\leq\int_{\mathbb{R}^N}\big[\frac{1}{p}f(v_n)(v_n)-F(v_n)\big]$$

hence,

$$m(V_\infty)\leq I_\epsilon(v_n)-\frac{1}{p}I'_\epsilon(v_n)(v_n)=d+o_n(1).$$

Taking the limit as $n\to\infty$ in the last inequality, it follows that $d\geq m(V_\infty)$. □

Proposition 2.1. *The functional I_ϵ satisfies the $(PS)_c$ condition at any level $c<m(V_\infty)$ if $V_\infty<\infty$ and at any level $c\in\mathbb{R}$ if $V_\infty=\infty$.*

Proof. Let $(u_n)\subset W_\epsilon$ be such that $I_\epsilon(u_n)\to c$ and $I'_\epsilon(u_n)\to 0$. By standard calculations, we can see that (u_n) is bounded in W_ϵ. Thus there exists $u\in W_\epsilon$ such that, up to a subsequence, $u_n\rightharpoonup u$ in W_ϵ. Moreover, arguing as in [12], we conclude that $I'_\epsilon(u)=0$.

Let $v_n=u_n-u$, reasoning as in [1] (see also [12]) , we can show that $I'_\epsilon(v_n)\to 0$ and

$$I_\epsilon(v_n)=I_\epsilon(u_n)-I_\epsilon(u)+o_n(1)=c-I_\epsilon(u)+o_n(1)=d+o_n(1).$$

By (f_4),

$$I_\epsilon(u)=I_\epsilon(u)-\frac{1}{p}I'_\epsilon(u)(u)=\int_{\mathbb{R}^N}[\frac{1}{p}f(u)u-F(u)]\geq 0,$$

thus, if $V_\infty<\infty$, we obtain $d\leq c<m(V_\infty)$. It follows from Lemma 2.3 that $v_n\to 0$, consequently $u_n\to u$ in W_ϵ.

If $V_\infty=\infty$, it follows that V is coercive and by [9], the continuous Sobolev imbedding $W_\epsilon\hookrightarrow L^s(\mathbb{R}^N)$ is compact for $p\leq s<p^*$. Hence, up to a subsequence, $v_n\to 0$ in $L^s(\mathbb{R}^N)$ and by $(f_2)-(f_3)$

$$\| v_n \|^p=\int_{\mathbb{R}^N}f(v_n)v_n=o_n(1).$$

The last equality implies that $u_n \to u$ in W_ϵ. □

Proposition 2.2. *The functional I_ϵ restricted to $\mathcal{N}_\epsilon$ satisfies the $(PS)_c$ condition at any level $c < m(V_\infty)$ if $V_\infty < \infty$ and at any level $c \in \mathbb{R}$ if $V_\infty = \infty$.*

Proof. Let $(u_n) \subset \mathcal{N}_\epsilon$ be such that $I_\epsilon(u_n) \to c$ and $\|I'_\epsilon(u_n)\|_* = o_n(1)$. Then there exists $(\lambda_n) \subset \mathbb{R}$ such that

$$I'_\epsilon(u_n) = \lambda_n J'_\epsilon(u_n) + o_n(1), \tag{2.6}$$

where $J_\epsilon : W_\epsilon \to \mathbb{R}$ is given by

$$J_\epsilon(u) = \int_{\mathbb{R}^N} |\nabla u|^p + \int_{\mathbb{R}^N} V(\epsilon x)|u|^p - \int_{\mathbb{R}^N} f(u)u.$$

Note that by (f_6)

$$\begin{aligned} J'_\epsilon(u_n)u_n &= p\int_{\mathbb{R}^N} |\nabla u_n|^p + p\int_{\mathbb{R}^N} V(\epsilon x)|u_n|^p - \int_{\mathbb{R}^N} f(u_n)u_n - \int_{\mathbb{R}^N} f'(u_n)(u_n)^2 \\ &= \int_{\mathbb{R}^N} (p-1)f(u_n)u_n - \int_{\mathbb{R}^N} f'(u_n)(u_n)^2 \le -C\int_{\mathbb{R}^N} |u_n|^\sigma < 0. \end{aligned}$$

We may suppose that $J'_\epsilon(u_n)u_n \to l \le 0$. If $l = 0$, it follows from

$$|J'_\epsilon(u_n)u_n| \ge C\int_{\mathbb{R}^N} |u_n|^\sigma$$

that $u_n \to 0$ in $L^\sigma(\mathbb{R}^N)$, consequently by interpolation, it is possible to check that $u_n \to 0$ in W_ϵ, which contradicts (2.1). Thus, $l \ne 0$ and $\lambda_n = o_n(1)$. From (2.6), $I'_\epsilon(u_n) = o_n(1)$, and so, (u_n) is a $(PS)_c$ sequence for I_ϵ and the result follows from Proposition 2.1. □

Corollary 2.1. *The critical points of the functional I_ϵ on $\mathcal{N}_\epsilon$ are critical points of I_ϵ in W_ϵ*

Proof. The proof follows by using similar arguments explored in the last proposition. □

3. Existence of ground state solution

In this section we prove the existence of a ground state solution to (P^*_ϵ), that is, a positive solution u_ϵ of (P^*_ϵ) satisfying $I(u_\epsilon) = c_\epsilon$. We adapt some ideas found in [3]. The main result in this section is the following

Theorem 3.1. *Suppose that f satisfies $(f_1)-(f_5)$ and V satisfies (V), then there exist $\bar\epsilon > 0$ such that (P^*_ϵ) has a ground state solution $u_\epsilon \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ for all $0 < \epsilon < \bar\epsilon$.*

Proof. By Lemma 2.1, the functional I_ϵ satisfies the mountain pass geometry. Then using a version of the mountain pass theorem without Palais-Smale [27], there exists $(u_n) \subset W^{1,p}(\mathbb{R}^N)$ satisfying $I_\epsilon(u_n) \to c_\epsilon$ and $I'_\epsilon(u_n) \to 0$. If $V_\infty = \infty$, it follows that $I_\epsilon(u) = c_\epsilon$ and $I'_\epsilon(u) = 0$, where $u \in W^{1,p}(\mathbb{R}^N)$ is the weak limit of $\{u_n\}$ in $W^{1,p}(\mathbb{R}^N)$. Using the Proposition 2.1, [14], [11] and [25], we have $u \in L^\infty(\mathbb{R}^N)$ and $u \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$ with $0 < \alpha < 1$. By Harnack's inequality [24], $u(x) > 0$ for all $x \in \mathbb{R}^N$.

If $V_\infty < \infty$, consider without loss of generality that

$$V(0) = V_0 = \inf_{x \in \mathbb{R}^N} V(x).$$

Let $\mu \in \mathbb{R}$ such that $V_0 < \mu < V_\infty$. Since $m(V_0) < m(\mu) < m(V_\infty)$, there exists a non-negative function $w \in W^{1,p}(\mathbb{R}^N)$ with compact support such that $E_\mu(w) = \max_{t \geq 0} E_\mu(tw)$ and $E_\mu(w) < m(V_\infty)$. Condition (V) implies that for some $\overline{\epsilon} > 0$ $V(\epsilon x) \leq \mu$, for all $x \in \operatorname{supp} w$ and $\epsilon \leq \overline{\epsilon}$, so

$$\int_{\mathbb{R}^N} V(\epsilon x) \mid w \mid^p \leq \int_{\mathbb{R}^N} \mu \mid w \mid^p \quad \text{for all } \epsilon \leq \overline{\epsilon}$$

and consequently

$$I_\epsilon(tw) \leq E_\mu(tw) \leq E_\mu(w) \text{ for all } t > 0.$$

Therefore, $\max_{t>0} I_\epsilon(tw) \leq E_\mu(w)$, and $c_\epsilon < m(V_\infty)$. So, the theorem follows from Proposition 2.1. □

4. MULTIPLICITY OF SOLUTIONS TO (P_{ϵ^*})

In this section, our main goal is to show the existence of multiple solutions and study the behavior of their maximum points in relation to the set M. The main result in this section has the following statement

Theorem 4.1. *Suppose that f satisfies $(f_1)-(f_6)$ and the function V satisfies (V). Then, for any $\delta > 0$, there exist $\epsilon_\delta > 0$ such that (P^*_ϵ) has at least $cat_{M_\delta}(M)$ positive solutions, for any $0 < \epsilon < \epsilon_\delta$. Moreover, if u_ϵ denotes one of these positive solutions and $z_\epsilon \in \mathbb{R}^N$ its global maximum, then*

$$\lim_{\epsilon \to 0} V(\epsilon z_\epsilon) = V_0.$$

In order to prove the above theorem, in the next subsection we fix some notation and prove some preliminary lemmas.

4.1. **Preliminary results.** Let $\delta > 0$ be fixed and w be a ground state solution of problem (P_{V_0}). Let η be a smooth nonincreasing cut-off function defined in $[0, \infty)$ such that $\eta(s) = 1$ if $0 \leq s \leq \frac{\delta}{2}$ and $\eta(s) = 0$ if $s \geq \delta$.

For any $y \in M$, let us define

$$\Psi_{\epsilon,y}(x) = \eta(| \epsilon x - y |)w\big(\frac{\epsilon x - y}{\epsilon}\big)$$

and $t_\epsilon > 0$ satisfying

$$\max_{t \geq 0} I_\epsilon(t\Psi_{\epsilon,y}) = I_\epsilon(t_\epsilon \Psi_{\epsilon,y})$$

and $\Phi_\epsilon : M \to \mathcal{N}_\epsilon$ by $\Phi_\epsilon(y) = t_\epsilon \Psi_{\epsilon,y}$. By construction, $\Phi_\epsilon(y)$ has compact support for any $y \in M$.

Lemma 4.1. *The function Φ_ϵ has the following limit*

$$\lim_{\epsilon \to 0} I_\epsilon(\Phi_\epsilon(y)) = m(V_0), \ \textit{uniformly in } y \in M.$$

Proof. Suppose by contradiction that the lemma is false. Then there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\epsilon_n \to 0$ such that

$$| I_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) - m(V_0) | \geq \delta_0. \tag{4.1}$$

Repeating the same arguments explored in [12] (see also [2]) , it is possible to check that $t_{\epsilon_n} \to 1$. From Lebesgue's theorem, we can check that

$$\lim_{n \to \infty} \| \Psi_{\epsilon_n, y_n} \|_{\epsilon_n}^p = \| w \|_{V_0}^p$$

and

$$\lim_{n \to \infty} \int_{\mathbb{R}^N} F(\Psi_{\epsilon_n, y_n}) = \int_{\mathbb{R}^N} F(w).$$

Now, note that

$$\begin{aligned} I_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) = \frac{t_{\epsilon_n}^p}{p} \int_{\mathbb{R}^N} |\nabla(\eta(|\epsilon_n z|)w(z))|^p \\ + \frac{t_{\epsilon_n}^p}{p} \int_{\mathbb{R}^N} V(\epsilon_n z + y_n)|\eta(| \epsilon_n z|)w(z)|^p - \int_{\mathbb{R}^N} F(t_{\epsilon_n}\eta(|\epsilon_n z|)w(z)). \end{aligned}$$

Letting $n \to \infty$, we get $\lim_{n \to \infty} I_{\epsilon_n}(\Phi_{\epsilon_n}(y_n)) = E_{V_0}(w) = m(V_0)$, which contradicts (4.1). Thus, the lemma holds. □

For any $\delta > 0$, let $\rho = \rho(\delta) > 0$ be such that $M_\delta \subset B_\rho(0)$. Let $\chi : \mathbb{R}^N \to \mathbb{R}^N$ be defined as $\chi(x) = x$ for $| x | \leq \rho$ and $\chi(x) = \rho x/|x|$ for $|x| \geq \rho$. Finally, let us consider $\beta : \mathcal{N}_\epsilon \to \mathbb{R}^N$ given by

$$\beta(u) = \frac{\int_{\mathbb{R}^N} \chi(\epsilon x) \mid u(x) \mid^p}{\int_{\mathbb{R}^N} \mid u(x) \mid^p}.$$

Lemma 4.2. *The function Φ_ϵ has the following limit*

$$\lim_{\epsilon\to 0}\beta(\Phi_\epsilon(y)) = y, \quad \textit{uniformly in} \quad y \in M.$$

Proof. Suppose, by contradiction, that the lemma is false. Then, there exist $\delta_0 > 0$, $(y_n) \subset M$ and $\epsilon_n \to 0$ such that

$$\mid \beta(\Phi_{\epsilon_n}(y_n)) - y_n \mid \geq \delta_0. \tag{4.2}$$

Using the definition of $\Phi_{\epsilon_n}(y_n)$ and β, we have the below equality

$$\beta(\Phi_{\epsilon_n}(y_n)) = y_n + \frac{\int_{\mathbb{R}^N}[\chi(\epsilon_n z + y_n) - y_n] \mid \eta(\mid \epsilon_n z \mid) w(z) \mid^p}{\int_{\mathbb{R}^N} \mid \eta(\mid \epsilon_n z \mid) w(z) \mid^p}.$$

Using the fact that $(y_n) \subset M \subset B_\rho(0)$ and Lebesgue's theorem, it follows that

$$\mid \beta(\Phi_{\epsilon_n}(y_n)) - y_n \mid = o_n(1),$$

which contradicts (4.2) and the lemma is proved. □

Lemma 4.3. (*A compactness lemma*) *Let $(u_n) \subset \mathcal{M}_\mu$ be a sequence satisfying $E_\mu(u_n) \to m(\mu)$. Then,*
a) (u_n) has a subsequence strongly convergent in $W^{1,p}(\mathbb{R}^N)$ or
b) there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that, up to a subsequence, $v_n(x) = u_n(x + \tilde{y}_n)$ converges strongly in $W^{1,p}(\mathbb{R}^N)$.

In particular, there exists a minimizer for $m(\mu)$.

Proof. Applying Ekeland's variational principle (see Theorem 8.5 in [27]), we may suppose that (u_n) is a $(PS)_{m(\mu)}$ for E_μ. Thus going to a subsequence if necessary, we have that $u_n \rightharpoonup u$ weakly in $W^{1,p}(\mathbb{R}^N)$ and $E'_\mu(u) = 0$.

If $u \neq 0$, it is easy to check that u is a ground state solution of the autonomous problem (P_μ), that is, $E_\mu(u) = m(\mu)$.

If $u \equiv 0$, applying the same arguments employed in the proof of Lemma 2.2, there exists a sequence $(y_n) \subset \mathbb{R}^N$ such that $v_n \rightharpoonup v$ in $W^{1,p}(\mathbb{R}^N)$, where $v_n = u_n(x + y_n)$. Therefore, v_n is also a $(PS)_{m(\mu)}$ sequence of E_μ and $v \not\equiv 0$. It follows from the above arguments that, up to a subsequence, (v_n) converges strongly in $W^{1,p}(\mathbb{R}^N)$ and the proof of the lemma is over. □

Proposition 4.1. *Let $\epsilon_n \to 0$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$ be such that $I_{\epsilon_n}(u_n) \to m(V_0)$. Then there exists a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ such that $v_n(x) = u_n(x + \tilde{y}_n)$ has a convergent subsequence in $W^{1,p}(\mathbb{R}^N)$. Moreover, up to a subsequence, $y_n \to y \in M$, where $y_n = \epsilon_n \tilde{y}_n$.*

Proof. Arguing as in the proof of Lemma 2.2, we obtain a sequence $(\tilde{y}_n) \subset \mathbb{R}^N$ and constants R and β such that

$$\liminf_{n\to\infty} \int_{B_R(\tilde{y}_n)} | u_n |^p \geq \beta > 0.$$

Thus, if $v_n(x) = u_n(x + \tilde{y}_n)$, up to a subsequence, $v_n \rightharpoonup v \not\equiv 0$ in $W^{1,p}(\mathbb{R}^N)$. Let $t_n > 0$ be such that $\tilde{v}_n = t_n v_n \in \mathcal{M}_{V_0}$. Then,

$$E_{V_0}(\tilde{v}_n) \to m(V_0) \text{ and } (\tilde{v}_n) \subset \mathcal{M}_{V_0}.$$

Since $\{t_n\}$ is bounded, the sequence $(\tilde{v}_n)$ also is bounded, thus for some subsequence, $\tilde{v}_n \rightharpoonup \tilde{v}$ weakly in $W^{1,p}(\mathbb{R}^N)$. Moreover, reasoning as in [12], for some subsequence still denoted by $\{t_n\}$, we can assume that $t_n \to t_0 > 0$, and this limit implies that $\tilde{v} \not\equiv 0$. From Lemma 4.3, $\tilde{v_n} \to \tilde{v}$ in $W^{1,p}(\mathbb{R}^N)$, and so, $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$.

Now, we will show that $(y_n) = (\epsilon_n \tilde{y}_n)$ has a subsequence satisfying $y_n \to y \in M$.

Claim 2. *The sequence (y_n) is bounded in $\mathbb{R}^N$.*

Indeed, suppose by contradiction that $\{y_n\}$ is not bounded, then there exists a subsequence, still denoted by $\{y_n\}$, such that $|y_n| \to \infty$. Considering firstly the case $V_\infty = \infty$, the below inequality

$$\int_{\mathbb{R}^N} V(\epsilon_n x + y_n)|v_n|^p \leq \int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} V(\epsilon_n x + y_n)|v_n|^p = \int_{\mathbb{R}^N} f(v_n)v_n,$$

together with Fatou's lemma imply

$$\infty = \liminf_{n\to\infty} \int_{\mathbb{R}^N} f(v_n)v_n$$

which is absurd, because the sequence $\{f(v_n)v_n\}$ is bounded in $L^1(\mathbb{R}^N)$.

Now, let us consider the case $V_\infty < \infty$. Since $\tilde{v_n} \to \tilde{v}$ in $W^{1,p}(\mathbb{R}^N)$ and $V_0 < V_\infty$, we have

$$\begin{aligned} m(V_0) &= \frac{1}{p}\int_{\mathbb{R}^N} | \nabla \tilde{v} |^p + \frac{1}{p}\int_{\mathbb{R}^N} V_0 | \tilde{v} |^p - \int_{\mathbb{R}^N} F(\tilde{v}) \\ &< \frac{1}{p}\int_{\mathbb{R}^N} | \nabla \tilde{v} |^p + \frac{1}{p}\int_{\mathbb{R}^N} V_\infty | \tilde{v} |^p - \int_{\mathbb{R}^N} F(\tilde{v}) \\ &\leq \liminf_{n\to\infty} \Big[\frac{1}{p}\int_{\mathbb{R}^N} | \nabla \tilde{v}_n |^p + \frac{1}{p}\int_{\mathbb{R}^N} V(\epsilon_n x + y_n) | \tilde{v}_n |^p - \int_{\mathbb{R}^N} F(\tilde{v}_n)\Big], \end{aligned}$$

or equivalently

$$m(V_0) < \liminf_{n\to\infty} \Big[\frac{t_n^p}{p}\int_{\mathbb{R}^N} | \nabla u_n |^p + \frac{t_n^p}{p}\int_{\mathbb{R}^N} V(\epsilon_n z) | u_n |^p - \int_{\mathbb{R}^N} F(t_n u_n)\Big].$$

The last inequality implies,

$$m(V_0) < \liminf_{n\to\infty} I_{\epsilon_n}(t_n u_n) \le \liminf_{n\to\infty} I_{\epsilon_n}(u_n) = m(V_0),$$

which does not make sense. Hence, (y_n) is bounded and, up to a subsequence, $y_n \to y \in \mathbb{R}^N$. If $y \notin M$, then $V(y) > V_0$ and we obtain a contradiction arguing as above. Thus, $y \in M$ and the lemma is proved. □

Let $h : \mathbb{R}^+ \to \mathbb{R}^+$ be a positive function tending to 0 such that $h(\epsilon) \to 0$ as $\epsilon \to 0$ and let $\widetilde{\mathcal{N}}_\epsilon = \{u \in \mathcal{N}_\epsilon : I_\epsilon(u) \le m(V_0) + h(\epsilon)\}$.

Lemma 4.4. *Let $\delta > 0$ and $M_\delta = \{x \in \mathbb{R}^N : dist(x, M) \le \delta\}$. Then*

$$\lim_{\epsilon\to 0} \sup_{u\in\widetilde{\mathcal{N}}_\epsilon} \inf_{y\in M_\delta} |\beta(u) - y| = 0.$$

Proof. Let $(\epsilon_n) \subset \mathbb{R}$ be such that $\epsilon_n \to 0$. For each $n \in I\!N$, there exists $(u_n) \subset \widetilde{\mathcal{N}}_\epsilon$ such that

$$\inf_{y\in M_\delta} \big|\beta(u_n) - y\big| = \sup_{u\in\widetilde{\mathcal{N}}_{\epsilon_n}} \inf_{y\in M_\delta} \big|\beta(u) - y\big| + o_n(1).$$

Thus, it suffices to find a sequence $(y_n) \subset M_\delta$ such that

$$\lim_{n\to\infty} \big|\beta(u_n) - y_n\big| = 0. \tag{4.3}$$

In order to obtain this sequence, we note that $(u_n) \subset \widetilde{\mathcal{N}}_{\epsilon_n} \subset \mathcal{N}_{\epsilon_n}$ and thus,

$$m(V_0) \le c_{\epsilon_n} \le I_{\epsilon_n}(u_n) \le m(V_0) + h(\epsilon_n),$$

and so $I_{\epsilon_n}(u_n) \to m(V_0)$ and $(u_n) \subset \mathcal{N}_{\epsilon_n}$. From Proposition 4.1, we get a sequence $(\tilde{y_n}) \subset \mathbb{R}^N$ such that $(y_n) = (\epsilon_n \tilde{y_n}) \subset M_\delta$, for n sufficiently large. Thus,

$$\beta(u_n) = y_n + \frac{\int_{\mathbb{R}^N} [\chi(\epsilon_n z + y_n) - y_n] \mid v_n(z) \mid^p}{\int_{\mathbb{R}^N} \mid v_n(z) \mid^p},$$

where $v_n(x) = u_n(x + \tilde{y}_n)$. Recalling that $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$, it is easy to check that the sequence (y_n) satisfies (4.3). □

The next two lemmas play a role in the study of the behavior of the maximum points of the solutions. In the proof of the next lemma, we adapted some arguments found in [14], which are related to the Moser iteration method [18].

Lemma 4.5. *Let v_n be a solution of the following problem*

$$\begin{cases} -\Delta_p v_n + V_n(x)|v_n|^{p-2} v_n = f(v_n) \quad in \quad \mathbb{R}^N \\ v_n \in W^{1,p}(\mathbb{R}^N) \ \ with \ \ 2 \le p < N, \\ v_n(x) > 0, \forall\ x \in \mathbb{R}^N, \end{cases}$$

where $V_n(x) = V(\epsilon_n x + \epsilon_n \tilde{y_n})$. Assuming that the conditions (V) and $(f_1)-(f_6)$ hold and that $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$ with $v \not\equiv 0$, then $v_n \in L^\infty(\mathbb{R}^N)$ and there exists $C > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \leq C$ for all $n \in \mathbb{N}$. Furthermore

$$\lim_{|x|\to\infty} v_n(x) = 0 \text{ uniformly in } n.$$

Proof. For any $R > 0$, $0 < r \leq \frac{R}{2}$, let $\eta \in C^\infty(\mathbb{R}^N)$, $0 \leq \eta \leq 1$ with $\eta(x) = 1$ if $|x| \geq R$ and $\eta(x) = 0$ if $|x| \leq R - r$ and $|\nabla\eta| \leq \frac{2}{r}$. Note that by (f_3), we obtain the following growth condition for f:

$$f(s) \leq \xi|s|^{p-1} + C_\xi|s|^{p^*-1}. \tag{4.4}$$

For each $n \in \mathbb{N}$ and for $L > 0$, let

$$v_{L,n}(x) = \begin{cases} v_n(x), & v_n(x) \leq L \\ L, & v_n(x) \geq L, \end{cases}$$

$$z_{L,n} = \eta^p v_{L,n}^{p(\beta-1)} v_n \quad \text{and} \quad w_{L,n} = \eta v_n v_{L,n}^{\beta-1}$$

with $\beta > 1$ to be determined later.

Taking $z_{L,n}$ as a test function, we obtain

$$\begin{aligned}\int_{\mathbb{R}^N} \eta^p v_{L,n}^{p(\beta-1)}|\nabla v_n|^p &= -p(\beta-1)\int_{\mathbb{R}^N} v_{L,n}^{p\beta-p-1}\eta^p v_n|\nabla v_n|^{p-2}\nabla v_n \nabla v_{L,n} \\ &+ \int_{\mathbb{R}^N} f(v_n)\eta^p v_n v_{L,n}^{p(\beta-1)} - \int_{\mathbb{R}^N} V_n|v_n|^p\eta^p v_{L,n}^{p(\beta-1)} \\ &- p\int_{\mathbb{R}^N} \eta^{p-1}v_{L,n}^{p(\beta-1)} v_n|\nabla v_n|^{p-2}\nabla v_n \nabla\eta.\end{aligned}$$

By (4.4) and for a ξ sufficiently small, we have the following inequality:

$$\begin{aligned}&\int_{\mathbb{R}^N} \eta^p v_{L,n}^{p(\beta-1)}|\nabla v_n|^p \\ &\leq C_\xi \int_{\mathbb{R}^N} v_n^{p^*}\eta^p v_{L,n}^{p(\beta-1)} - p\int_{\mathbb{R}^N} \eta^{p-1}v_{L,n}^{p(\beta-1)} v_n|\nabla v_n|^{p-2}\nabla v_n\nabla\eta.\end{aligned}$$

For each $\epsilon > 0$, using Young's inequality we get

$$\begin{aligned}\int_{\mathbb{R}^N} \eta^p v_{L,n}^{p(\beta-1)}|\nabla v_n|^p &\leq C_\xi \int_{\mathbb{R}^N} v_n^{p^*}\eta^p v_{L,n}^{p(\beta-1)} + p\epsilon\int_{\mathbb{R}^N} \eta^p v_{L,n}^{p(\beta-1)}|\nabla v_n|^p \\ &+ pC_\epsilon \int_{\mathbb{R}^N} v_n^p v_{L,n}^{p(\beta-1)}|\nabla\eta|^p.\end{aligned}$$

Choosing $\epsilon > 0$ sufficiently small,

$$\int_{\mathbb{R}^N} \eta^p v_{L,n}^{p(\beta-1)} |\nabla v_n|^p \le C \int_{\mathbb{R}^N} v_n^{p^*} \eta^p v_{L,n}^{p(\beta-1)} + C \int_{\mathbb{R}^N} v_n^p v_{L,n}^{p(\beta-1)} |\nabla \eta|^p. \quad (4.5)$$

Now, from Sobolev imbedding and Hölder inequalities

$$|w_{L,n}|_{p^*}^p \le C\beta^p \Big[\int_{\mathbb{R}^N} v_n^p v_{L,n}^{p(\beta-1)} |\nabla \eta|^p + \int_{\mathbb{R}^N} \eta^p v_{L,n}^{p(\beta-1)} |\nabla v_n|^p \Big]. \quad (4.6)$$

Using (4.5) in (4.6), we have

$$|w_{L,n}|_{p^*}^p \le C\beta^p \Big[\int_{\mathbb{R}^N} v_n^p v_{L,n}^{p(\beta-1)} |\nabla \eta|^p + \int_{\mathbb{R}^N} v_n^{p^*} \eta^p v_{L,n}^{p(\beta-1)} \Big]. \quad (4.7)$$

We claim that $v_n \in L^{\frac{{p^*}^2}{p}}(|x| \ge R)$ for R large enough and uniformly in n. In fact, let $\beta = \frac{p^*}{p}$. From (4.7), we have

$$|w_{L,n}|_{p^*}^p \le C\beta^p \Big[\int_{\mathbb{R}^N} v_n^p v_{L,n}^{(p^*-p)} |\nabla \eta|^p + \int_{\mathbb{R}^N} v_n^{p^*} \eta^p v_{L,n}^{(p^*-p)} \Big]$$

or equivalently

$$|w_{L,n}|_{p^*}^p \le C\beta^p \Big[\int_{\mathbb{R}^N} v_n^p v_{L,n}^{(p^*-p)} |\nabla \eta|^p + \int_{\mathbb{R}^N} v_n^p \eta^p v_{L,n}^{(p^*-p)} v_n^{(p^*-p)} \Big].$$

Using the Hölder inequality with exponent $\frac{p^*}{p}$ and $\frac{p^*}{p^*-p}$

$$\begin{aligned} |w_{L,n}|_{p^*}^p \le\ & C\beta^p \int_{\mathbb{R}^N} v_n^p v_{L,n}^{(p^*-p)} |\nabla \eta|^p \\ & + C\beta^p \Big(\int_{\mathbb{R}^N} \Big[v_n \eta v_{L,n}^{\frac{(p^*-p)}{p}} \Big]^{p^*} \Big)^{\frac{p}{p^*}} \Big(\int_{|x| \ge R/2} v_n^{p^*} \Big)^{\frac{p^*-p}{p^*}}. \end{aligned}$$

From the definition of $w_{L,n}$, we have

$$\begin{aligned} \Big(\int_{\mathbb{R}^N} \Big[v_n \eta v_{L,n}^{\frac{(p^*-p)}{p}} \Big]^{p^*} \Big)^{\frac{p}{p^*}} \le\ & C\beta^p \int_{\mathbb{R}^N} v_n^p v_{L,n}^{(p^*-p)} |\nabla \eta|^p \\ & + C\beta^p \Big(\int_{\mathbb{R}^N} \Big[v_n \eta v_{L,n}^{\frac{(p^*-p)}{p}} \Big]^{p^*} \Big)^{\frac{p}{p^*}} \Big(\int_{|x| \ge R/2} v_n^{p^*} \Big)^{\frac{p^*-p}{p^*}}. \end{aligned}$$

Since $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$, for R sufficiently large, we conclude

$$\int_{|x| \ge R/2} v_n^{p^*} \le \epsilon \ \text{ uniformly in } n.$$

Hence,

$$\Big(\int_{|x|\geq R}\Big[v_n v_{L,n}^{\frac{(p^*-p)}{p}}\Big]^{p^*}\Big)^{\frac{p}{p^*}} \leq C\beta^p \int_{\mathbb{R}^N} v_n^p v_{L,n}^{(p^*-p)}$$

or equivalently

$$\Big(\int_{|x|\geq R}\Big[v_n v_{L,n}^{\frac{(p^*-p)}{p}}\Big]^{p^*}\Big)^{\frac{p}{p^*}} \leq C\beta^p \int_{\mathbb{R}^N} v_n^{p^*} \leq K < \infty.$$

Using Fatou's lemma in the variable L, we have

$$\int_{|x|\geq R} v_n^{\frac{{p^*}^2}{p}} < \infty$$

and therefore the claim holds.

Next, we note that if $\beta = p^*\frac{(t-1)}{pt}$ with $t = \frac{{p^*}^2}{p(p^*-p)}$, then $\beta > 1$, $\frac{pt}{t-1} < p^*$ and $v_n \in L^{(\beta pt)/t-1}(|x| \geq R - r)$.

Returning to inequality (4.7), we obtain

$$|w_{L,n}|_{p^*}^p \leq C\beta^p\Big[\int_{R\geq|x|\geq R-r} v_n^p v_{L,n}^{p(\beta-1)} + \int_{|x|\geq R-r} v_n^{p^*} v_{L,n}^{p(\beta-1)}\Big]$$

or equivalently

$$|w_{L,n}|_{p^*}^p \leq C\beta^p\Big[\int_{R\geq|x|\geq R-r} v_n^{p\beta} + \int_{|x|\geq R-r} v_n^{p^*-p} v_n^{p\beta}\Big].$$

Using Hölder's inequality with exponent $t/(t-1)$ and t, we get

$$\begin{aligned} |w_{L,n}|_{p^*}^p \;\; &\leq \;\; C\beta^p\Big\{\Big[\int_{R\geq|x|\geq R-r} v_n^{p\beta t/(t-1)}\Big]^{(t-1)/t}\Big[\int_{R\geq|x|\geq R-r} 1\Big]^{1/t} \\ &+ \;\; \Big[\int_{|x|\geq R-r} v_n^{(p^*-p)t}\Big]^{1/t}\Big[\int_{|x|\geq R-r} v_n^{p\beta t/(t-1)}\Big]^{t/(t-1)}\Big\}. \end{aligned}$$

Since $(p^*-p)t = {p^*}^2$, we conclude

$$|w_{L,n}|_{p^*}^p \leq C\beta^p\Big(\int_{|x|\geq R-r} v_n^{p\beta t/(t-1)}\Big)^{(t-1)/t}.$$

Note that

$$|v_{L,n}|_{p^*\beta(|x|\geq R)}^{p\beta} \leq \Big(\int_{|x|\geq R-r} v_{L,n}^{p^*\beta}\Big)^{p/p^*} \leq \Big(\int_{\mathbb{R}^N} \eta^p v_n^{p^*} v_{L,n}^{p^*(\beta-1)}\Big)^{p/p^*}$$

$$= |w_{L,n}|_{p^*}^p \leq C\beta^p\Big(\int_{|x|\geq R-r} v_n^{p\beta t/(t-1)}\Big)^{(t-1)/t} = C\beta^p |v_n|_{p\beta t/(t-1)(|x|\geq R-r)}^{p\beta}.$$

Applying Fatou's lemma

$$|v_n|^{p\beta}_{p^*\beta(|x|\geq R)} \leq C\beta^p |v_n|^{p\beta}_{p\beta t/(t-1)(|x|\geq R-r)}.$$

Considering $\chi = \frac{p^*(t-1)}{pt}$, $s = \frac{pt}{t-1}$ and the last inequality, we can prove that

$$|v_n|_{\chi^{m+1}s(|x|\geq R)} \leq C^{\sum_{i=1}^{m}\chi^{-i}}\chi^{\sum_{i=1}^{m} i\chi^{-i}}|v_n|_{p^*(|x|\geq R-r)},$$

which implies $\|v_n\|_{L^\infty(|x|\geq R)} \leq C|v_n|_{p^*(|x|\geq R-r)}$. Using again the convergence of $\{v_n\}$ to v in $W^{1,p}(\mathbb{R}^N)$, for $\epsilon > 0$ fixed there exists $R > 0$ such that

$$\|v_n\|_{L^\infty(|x|\geq R)} < \epsilon \;\; \forall\, n \in \mathbb{N}.$$

Thus,

$$\lim_{|x|\to\infty} v_n(x) = 0 \;\text{ uniformly in n}$$

and the proof of the lemma is finished. □

Lemma 4.6. *There exists $\delta > 0$ such that $\|v_n\|_\infty \geq \delta$ for all $n \in \mathbb{N}$.*

Proof. Suppose that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \to 0$. For fixed $\epsilon_0 = \frac{V_0}{2}$, it follows from (f_5) that there exists $n_0 \in \mathbb{N}$ such that,

$$\frac{f(\|v_n\|_{L^\infty(\mathbb{R}^N)})}{\|v_n\|^{p-1}_{L^\infty(\mathbb{R}^N)}} < \epsilon_0, \;\text{ for }\; n \geq n_0.$$

Hence,

$$\int_{\mathbb{R}^N} |\nabla v_n|^p + \int_{\mathbb{R}^N} V_0|v_n|^p \leq \int_{\mathbb{R}^N} \frac{f(\|v_n\|_{L^\infty(\mathbb{R}^N)})}{\|v_n\|^{p-1}_{L^\infty(\mathbb{R}^N)}}|v_n|^p \leq \epsilon_0 \int_{\mathbb{R}^N} |v_n|^p,$$

thus, $\|v_n\|_{W^{1,p}(\mathbb{R}^N)} = 0$ for $n \geq n_0$, which is absurd, because $v_n \neq 0$ for all $n \in \mathbb{N}$. Then there exists $\delta > 0$ such that $\|v_n\|_{L^\infty(\mathbb{R}^N)} \geq \delta$ for all $n \in \mathbb{N}$. □

Using the lemmas and propositions proved up to now, we can prove Theorem 4.1.

4.2. **Proof of Theorem 4.1.** We will divide the proof into two parts:

Part I: Multiplicity of solutions. We fix a small $\epsilon > 0$. Then, by Lemmas 4.1 and 4.4, we have $\beta \circ \Phi_\epsilon$ is homotopic to the inclusion map $id : M \to M_\delta$; this fact implies

$$cat_{\widetilde{\mathcal{N}}_\epsilon}(\widetilde{\mathcal{N}}_\epsilon) \geq cat_{M_\delta}(M).$$

Since that functional I_ϵ satisfies the $(PS)_c$ condition for $c \in (m(V_0), m(V_0)+h(\epsilon))$, by the Lusternik-Schnirelman theory of critical points (see [15], [27]) we can conclude that I_ϵ has at least $cat_{M_\delta}(M)$ critical points on $\mathcal{N}_\epsilon$. Consequently by Corollary 2.1, I_ϵ has at least $cat_{M_\delta}(M)$ critical points in W_ϵ.

Part II: The behavior of maximum points. If u_{ϵ_n} is a solution of problem (P_{ϵ_n}), then $v_n(x) = u_{\epsilon_n}(x + \tilde{y_n})$ is a solution of problem

$$\begin{cases} -\Delta_p v_n + V_n(x)|v_n|^{p-2}v_n = f(v_n) \ \ in \ \ \mathbb{R}^N \\ v_n \in W^{1,p}(\mathbb{R}^N) \ \text{with} \ 2 \le p < N, \\ v_n(x) > 0, \forall\ x \in \mathbb{R}^N, \end{cases}$$

with $V_n(x) = V(\epsilon_n x + \epsilon_n \tilde{y_n})$ and $(\tilde{y_n}) \subset \mathbb{R}^N$ given in Proposition 4.1. Moreover, up to a subsequence, $v_n \to v$ in $W^{1,p}(\mathbb{R}^N)$ and $y_n \to y$ in M, where $y_n = \epsilon_n \tilde{y_n}$. Considering p_n the global maximum of v_n, by Lemmas 4.5 and 4.6, we have that $p_n \in B_R(0)$ for some $R > 0$. Thus, the global maximum of u_{ϵ_n} is $z_\epsilon = p_n + \tilde{y_n}$ and therefore

$$\epsilon_n z_{\epsilon_n} = \epsilon_n p_n + \epsilon_n \tilde{y_n} = \epsilon_n p_n + y_n.$$

Since (p_n) is bounded, we have

$$\lim_{n\to\infty} V(\epsilon_n z_{\epsilon_n}) = V_0.$$

4.3. **Final comments.** If u_ϵ is a positive solution of (P_ϵ^*), the function $w_\epsilon(x) = u_\epsilon(x/\epsilon)$ is a positive solution of (P_ϵ). Thus, the maximum points η_ϵ and z_ϵ of w_ϵ and u_ϵ respectively, satisfy the equality $\eta_\epsilon = \epsilon z_\epsilon$, consequently,

$$\lim_{\epsilon\to 0} V(\eta_\epsilon) = V_0.$$

Thus, Theorem 1.1 follows from Theorem 4.1 and the last limit.

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