

ON THE EXISTENCE OF SOLUTIONS OF THE HELFRICH FLOW AND ITS CENTER MANIFOLD NEAR SPHERES

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Abstract. The Helfrich variational problem is the minimizing problem of the bending energy among closed surface with prescribed area and enclosed volume. This is one of the models for shape transformation theory of human red blood cells. Here the associated gradient flow, called the *Helfrich flow*, is studied. The existence of this geometric flow is proved locally for arbitrary initial data, and globally near spheres. Furthermore its center manifold near spheres is investigated.

1. INTRODUCTION

Let Σ be a smooth compact surface embedded in \mathbb{R}^3 . Consider the minimizing problem of the functional

$$\mathcal{W}_{c_0}(\Sigma) = \int_{\Sigma} (H - c_0)^2 dS$$

with prescribed area $\mathcal{A}(\Sigma) = A_0$ and enclosed volume $\mathcal{V}(\Sigma) = V_0$. Here H is the mean curvature of Σ with respect to the *inner* normal vector, and c_0 is a constant. This is a model of shapes of human red blood cells, where c_0 is called the *spontaneous curvature* (see [2, 5, 6]). If Σ is a critical point of this variational problem with constraints, then the surface must satisfy the equation

$$\Delta_{\Sigma} H + 2H(H^2 - K) + 2c_0 K - 2(c_0^2 + \lambda_1)H - \lambda_2 = 0.$$

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Here Δ_Σ is the Laplace-Beltrami operator of Σ , K is the Gaussian curvature, λ_1 and λ_2 are Lagrange multipliers. This is the first variation of the *Helfrich* functional

$$\mathcal{H}(\Sigma) = \mathcal{W}_{c_0}(\Sigma) + \lambda_1 \mathcal{A}(\Sigma) + \lambda_2 \mathcal{V}(\Sigma)$$

without constraints.

Now consider the L^2 -gradient flow (the *Helfrich flow*) for the Helfrich functional. Let $\Sigma(t)$ be a one-parameter family of surfaces, and let V be the velocity in the direction of the *inner* normal vector. The equation of flow is

$$V = -\Delta_{\Sigma(t)}H - 2H(H^2 - K) - 2c_0K + 2(c_0^2 + \lambda_1)H + \lambda_2. \quad (1.1)$$

We should state related results on geometrical gradient flows, that is, the surface diffusion flow, which is the H^{-1} -gradient flow for the area functional,

$$V = -\Delta_{\Sigma(t)}H$$

by Escher-Mayer-Simonett [4]; the Willmore flow, which corresponds to the Helfrich flow with $c_0 = \lambda_1 = \lambda_2 = 0$,

$$V = -\Delta_{\Sigma(t)}H - 2H(H^2 - K)$$

by Simonett [13], Kuwert-Schätzle [8, 9]. Roughly speaking they showed the local existence of a flow, the global existence near spheres, and the stability of spheres.

For the Helfrich flow, spheres are equilibria under suitable choices of λ_i , however, they are not necessarily stable. Furthermore finite-time blow-up occurs even if the initial surface is a sphere.

The first aim of ours is to show the local existence of a flow for arbitrary λ_i , and the existence of a global flow. In addition, we will derive the existence of a parameter-dependent center manifold for λ_i as unknown parameters. This is worthwhile since λ_i are Lagrange multipliers for the Helfrich problem and are determined together with critical points.

In the previous paper [10] we showed the existence of stationary solutions with rotational symmetry bifurcating from the spheres. We can show the existence of bifurcating solutions without symmetry by Krasnoselski's theorem [3, Theorem 7.3], however, it does not give us any information about the structure of all bifurcating branches.

The second aim is to show that branches are embedded in a finite dimensional smooth manifold by the center manifold analysis.

2. FORMULATION OF PROBLEM

We can formulate our problem as in Simonett [13], however, we should pay attention to the signature of mean curvature and normal velocity. We would like to follow the definition of mean curvature of our previous paper [10], which has the opposite signature to that of [13]. Since these differences of signature confuse us sometimes, we reformulate our problem under our choice of signature.

Let Σ_0 be an orientable compact closed surface in \mathbb{R}^3 , and let $\bigcup_{\ell=1}^m U_\ell$ be its open covering. We denote the inner unit normal vector fields of Σ_0 by ν . The mapping $X_\ell : U_\ell \times (-a, a) \ni (s, r) \rightarrow s + r\nu(s) \in \mathbb{R}^n$ is a C^∞ -diffeomorphism from $U_\ell \times (-a, a)$ to $\mathcal{R}_\ell = \text{Im}(X_\ell)$ provided $a > 0$ is sufficiently small. Let us denote the inverse mapping X_ℓ^{-1} by (S_ℓ, Λ_ℓ) , where $S_\ell(X_\ell(s, r)) = s \in U_\ell$, and $\Lambda_\ell(X_\ell(s, r)) = r \in (-a, a)$.

Assuming that $\Sigma(t)$ is sufficiently close to Σ_0 , we can represent it as a graph of a function on Σ_0 as

$$\Sigma_{\rho(t)} = \Sigma(t) = \bigcup_{\ell=1}^m \text{Im}(X_\ell : U_\ell \rightarrow \mathbb{R}^n, [s \mapsto X_\ell(s, \rho(s, t))]).$$

Conversely, for a given function $\rho : \Sigma_0 \times [0, T) \rightarrow (-a, a)$ we define the mapping $\Phi_{\ell, \rho}$ from $\mathcal{R}_\ell \times [0, T)$ to \mathbb{R} by

$$\Phi_{\ell, \rho}(x, t) = \Lambda_\ell(x) - \rho(S_\ell(x), t).$$

Then $(\Phi_{\ell, \rho}(\cdot, t))^{-1}$ gives the surface $\Sigma_{\rho(t)}$.

The velocity in the direction of the inner normal vector field of $\Sigma = \{\Sigma_{\rho(t)} : t \in [0, T)\}$ at $(x, t) = (X_\ell(s, \rho(s, t)), t)$ is given by

$$V(s, t) = - \frac{\partial_t \Phi_{\ell, \rho}(x, t)}{\|\nabla_x \Phi_{\ell, \rho}(x, t)\|} \Big|_{x=X_\ell(s, \rho(s, t))} = \frac{\partial_t \rho(s, t)}{\|\nabla_x \Phi_{\ell, \rho}(x, t)\|} \Big|_{x=X_\ell(s, \rho(s, t))}.$$

Equation (1.1) is represented as

$$\begin{aligned} \partial_t \rho = & L_\rho \{ -\Delta_\rho H(\rho) - 2H(\rho) (H^2(\rho) - K(\rho)) \\ & - 2c_0 K(\rho) + 2(c_0^2 + \lambda_1) H(\rho) + \lambda_2 \}, \end{aligned} \quad (2.1)$$

where $L_\rho = \|\nabla_x \Phi_{\ell, \rho}(x, t)\| \Big|_{x=X_\ell(s, \rho(s, t))}$.

We would like to write down the mean curvature and the Gaussian curvature in terms of the function ρ and its derivatives. As far as the authors know, there are no references for the expression of $K(\rho)$. We can refer to [13] for the expression of $H(\rho)$, but its definition has the opposite signature.

Therefore we state here these expressions for the sake of completeness. Since their proof is lengthy, we will give it in the last section. The surface is defined by $\{x \in \mathbb{R}^3 : \Phi_{\ell,\rho}(x) = 0\}$ locally, where the interior near the surface is located in the side $\{x \in \mathbb{R}^3 : \Phi_{\ell,\rho}(x) > 0\}$. Assume that $\nabla_x \Phi_{\ell,\rho} \neq 0$ everywhere near the surface.

Define the diffeomorphism θ_ρ between Σ_0 and Σ_ρ by

$$\theta_\rho(s) = X_\ell(s, \rho(s)) \quad \text{for } s \in U_\ell.$$

We denote its pull back by θ_ρ^* . Let $g_{\mathbb{R}^3}$ be the Euclidean metric on \mathbb{R}^3 , and put $\eta = g_{\mathbb{R}^3}|_{\mathcal{R}_\ell}$, $g_\ell = X_\ell^* \eta$, which are metrics on $T(\mathcal{R}_\ell)$ and $T(U_\ell \times (-a, a))$ respectively. The metric g_ℓ can be written as

$$g_\ell = w_\ell(r) + dr \otimes dr,$$

where $w_\ell(r)$ is the metric on $T(U_\ell \times \{r\})$. Put

$$g_\rho = w_\rho + dr \otimes dr = g_\ell|_{(s, \rho(s))},$$

where $w_\rho = (w_{jk}(\rho))$ is a metric on $T(\Sigma_0)$ defined by

$$w_\rho|_{T(U_\ell)} = w_\ell(\rho).$$

We denote the metric $(w_{jk}(\rho))^{-1} = (w^{jk}(\rho))$ on $T^*(\Sigma_0)$ simply by w_ρ^* .

Remark 2.1. w_ρ is not $\theta_\rho^* \eta$.

Put $U_{\ell,\rho} = (U_\ell, w(\rho))$ and $\Xi_\ell = (U_\ell \times (-a, a), g_\ell)$, and define the function $\tilde{\Phi}_{\ell,\rho}$ on $U_\ell \times (-a, a)$ by

$$\tilde{\Phi}_{\ell,\rho}(s, r) = \Phi_{\ell,\rho}(X_\ell(s, r)) = r - \rho(s).$$

In what follows, Latin indices except n range from 1 to 2, and Greek ones from 1 to 3. And the Einstein convention is used. Furthermore we denote the coordinate system by $y = (y^\alpha)$ defined by

$$y^\alpha = \begin{cases} s^i & (\alpha = i), \\ r & (\alpha = 3). \end{cases}$$

For a while we fix ℓ , ρ , and the time variable t , and omit them.

The entries of the metric g are

$$g_{\alpha\beta} = \begin{cases} w_{ij} & (\alpha = i, \beta = j), \\ 0 & (\alpha = i, \beta = 3), \\ 1 & (\alpha = 3, \beta = 3). \end{cases}$$

Therefore, we have

$$\begin{aligned} \left(\nabla_y \tilde{\Phi}\right)^\alpha &= \begin{cases} -w^{ij}\partial_i\rho & (\alpha = j), \\ 1 & (\alpha = 3), \end{cases} \\ \left(\nabla_y \tilde{\Phi}\right)_\beta &= \begin{cases} -\partial_i\rho & (\beta = i), \\ 1 & (\beta = 3), \end{cases} \\ \left(\text{Hess}_y \tilde{\Phi}\right)_{\alpha\beta} &= \begin{cases} -\partial_i\partial_j\rho + \Gamma_{ij}^k\partial_k\rho - \Gamma_{ij}^3 & (\alpha = i, \beta = j), \\ \Gamma_{i3}^k\partial_k\rho & (\alpha = i, \beta = 3), \\ 0 & (\alpha = \beta = 3), \end{cases} \\ \Delta_y \tilde{\Phi} &= -w^{ij}\partial_i\partial_j\rho + w^{ik}\Gamma_{ik}^j\partial_j\rho - w^{ik}\Gamma_{ik}^3. \end{aligned}$$

Here

$$\left(\nabla_y \tilde{\Phi}\right)^\alpha = g^{\alpha\beta} \left(\nabla_y \tilde{\Phi}\right)_\beta, \quad \left(\nabla \tilde{\Phi}\right)_\beta = \frac{\partial \tilde{\Phi}}{\partial y_\beta},$$

and Γ_{ij}^k is the Christoffel symbol of Ξ .

Using this fact we get:

Lemma 2.1 ([13]). *The mean curvature $H(\rho)$ can be written as*

$$H(\rho) = P_1(\rho)\rho + F_1(\rho),$$

where

$$\begin{aligned} P_1(\rho) &= \frac{1}{2L_\rho^3} \left[\left\{ L_\rho^2 w^{jk}(\rho) - w^{j\ell}(\rho)w^{km}(\rho)\partial_\ell\rho\partial_m\rho \right\} \partial_j\partial_k \right. \\ &\quad \left. - \left\{ L_\rho^2 w^{jk}(\rho)\Gamma_{jk}^i(\rho) + w^{j\ell}(\rho)w^{ki}(\rho)\Gamma_{jk}^3(\rho)\partial_\ell\rho \right. \right. \\ &\quad \left. \left. + 2w^{km}(\rho)\Gamma_{3k}^i(\rho)\partial_m\rho - w^{j\ell}(\rho)w^{km}(\rho)\Gamma_{jk}^i(\rho)\partial_\ell\rho\partial_m\rho \right\} \partial_i \right], \\ F_1(\rho) &= \frac{1}{2L_\rho} w^{jk}(\rho)\Gamma_{jk}^3(\rho), \end{aligned}$$

$$\Gamma_{jk}^i(\rho) = \Gamma_{jk}^i|_{(s,\rho(s))} \quad \text{on } T_{(s,\rho(s))}(\Xi_\ell).$$

Proof. Let x be the standard Euclidean coordinate system, then we have

$$\begin{aligned}
2H(\rho) &= -\operatorname{div}_x \left(\frac{\nabla_x \Phi}{\|\nabla_x \Phi\|} \right) \Big|_{x=X_\ell(s,\rho(s))} \\
&= - \left(\frac{\Delta_x \Phi}{\|\nabla_x \Phi\|} - \frac{\nabla_x \Phi \cdot \nabla_x \|\Phi\|}{\|\nabla_x \Phi\|^2} \right) \Big|_{x=X_\ell(s,\rho(s))} \\
&= - \left(\frac{\Delta_x \Phi}{\|\nabla_x \Phi\|} - \frac{\nabla_x \Phi \cdot \nabla_x \|\Phi\|^2}{2\|\nabla_x \Phi\|^3} \right) \Big|_{x=X_\ell(s,\rho(s))} \\
&= - \left(\frac{\Delta_x \Phi}{\|\nabla_x \Phi\|} - \frac{\operatorname{Hess}_x \Phi (\nabla_x \Phi, \nabla_x \Phi)}{\|\nabla_x \Phi\|^3} \right) \Big|_{x=X_\ell(s,\rho(s))}.
\end{aligned}$$

Since the differential operators ∇_x , Hess_x , Δ_x , and the length $\|\cdot\|$ are geometric, *i. e.*, independent of the choice of coordinate system, it holds that

$$2H(\rho) = - \left(\frac{\Delta_y \tilde{\Phi}}{\|\nabla_y \tilde{\Phi}\|} - \frac{(\operatorname{Hess}_y \tilde{\Phi})_{\alpha\beta} (\nabla_y \tilde{\Phi})^\alpha (\nabla_y \tilde{\Phi})^\beta}{\|\nabla_y \tilde{\Phi}\|^3} \right) \Big|_{(s,\rho(s))}.$$

Using the previous fact and $\|\nabla_y \tilde{\Phi}\| = L_\rho$, we get the assertion. \square

Lemma 2.2. *The Gaussian curvature $K(\rho)$ can be written as*

$$K(\rho) = w(\rho)^{-1} \det_3 \left\{ L_\rho^{-\frac{1}{2}} H_{\alpha\beta}(\rho) (\delta_\mu^\alpha - k_\mu^\alpha(\rho)) (\delta_\nu^\beta - k_\nu^\beta(\rho)) + \ell_{\mu\nu}(\rho) \right\},$$

where

$$\begin{aligned}
w(\rho) &= \det_2 (w_{ij}(\rho)), \\
H_{\alpha\beta}(\rho) &= \begin{cases} -\partial_i \partial_j \rho + \Gamma_{ij}^k(\rho) \partial_k \rho - \Gamma_{ij}^3(\rho) & (\alpha = i, \beta = j), \\ \Gamma_{i3}^k(\rho) \partial_k \rho & (\alpha = i, \beta = 3), \\ 0 & (\alpha = \beta = 3), \end{cases} \\
k_\mu^\alpha(\rho) &= \begin{cases} L_\rho^{-2} w^{ij}(\rho) \partial_i \rho \partial_j \rho & (\alpha = i, \mu = j), \\ -L_\rho^{-2} w^{ij}(\rho) \partial_j \rho & (\alpha = i, \mu = 3), \\ -L_\rho^{-2} \partial_j \rho & (\alpha = 3, \mu = j), \\ L_\rho^{-2} & (\alpha = \mu = 3), \end{cases} \\
\ell_{\mu\nu}(\rho) &= \begin{cases} L_\rho^{-2} \partial_i \rho \partial_j \rho & (\mu = i, \nu = j), \\ -L_\rho^{-2} \partial_i \rho & (\mu = i, \nu = 3; \text{ or } \mu = 3, \nu = i), \\ L_\rho^{-2} & (\mu = \nu = 3). \end{cases}
\end{aligned}$$

\det_k is the determinant for $k \times k$ matrices.

Proof. Noticing that

$$\begin{aligned} g|_{(s,\rho(s))} &= \det_3(g_{\alpha\beta})|_{(s,\rho(s))} = \det_2(w_{ij})|_{(s,\rho(s))} = w(\rho), \\ \left(\text{Hess}_y \tilde{\Phi}\right)_{\alpha\beta} \Big|_{(s,\rho(s))} &= H_{\alpha\beta}(\rho), \\ \|\nabla_y \tilde{\Phi}\|^{-2} \left(\nabla_y \tilde{\Phi}\right)^\alpha \left(\nabla_y \tilde{\Phi}\right)_\mu \Big|_{(s,\rho(s))} &= k_\nu^\alpha(\rho), \\ \|\nabla_y \tilde{\Phi}\|^{-2} \left(\nabla_y \tilde{\Phi}\right)_\mu \left(\nabla_y \tilde{\Phi}\right)_\nu \Big|_{(s,\rho(s))} &= \ell_{\mu\nu}(\rho), \end{aligned}$$

we obtain the assertion from Lemma 5.2 in the last section. \square

Now we go back to the equation (2.1). Define the metric σ_ρ by

$$\theta_\rho^* \eta = \sigma(\rho) = (\sigma_{jk}(\rho)).$$

Writing the Christoffel symbols with respect to this metric by $\gamma_{jk}^i(\rho)$, we have

$$\Delta_\rho = \sigma^{jk}(\rho) (\partial_j \partial_k - \gamma_{jk}^i(\rho) \partial_i).$$

Let $h^\gamma(\Sigma_0)$ be the little Hölder space on Σ_0 of order γ . We fix $0 < \alpha < \beta < 1$. Then, for $\beta_0 \in (\alpha, \beta)$ and $a > 0$, put

$$\mathcal{U} = \{\rho \in h^{2+\beta_0}(\Sigma_0) : \|\rho\|_\infty < a\}.$$

For two Banach spaces E_0 and E_1 satisfying $E_1 \hookrightarrow E_0$, the set $\mathcal{H}(E_1, E_0)$ is the class of $A \in \mathcal{L}(E_1, E_0)$ such that $-A$, considered as an unbounded operator in E_0 , generates a strongly continuous analytic semigroup on E_0 .

Proposition 2.1. *There exist $P \in C^\infty(\mathcal{U}, \mathcal{H}(h^{4+\alpha}(\Sigma_0), h^\alpha(\Sigma_0)))$, and $F \in C^\infty(\mathcal{U}, h^{\beta_0}(\Sigma_0))$ such that the equation (2.1) is in the form*

$$\rho_t + P(\rho)\rho + F(\rho) = 0.$$

Proof. The principal term $-L_\rho \Delta_\rho H(\rho)$ in (2.1) is the same as those of the surface diffusion flow [4] and of the Willmore flow [13] (see Remark below). Taking Lemmata 2.1 – 2.2 into consideration, we can obtain the assertion in a similar manner to the proof of [4, Lemma 2.1] and [13, Lemma 2.1]. \square

Remark 2.2. Our mean curvature $H(\rho)$ corresponds to $-H_\rho$ in [4] and $-H(\rho)$ in [13], having the opposite signature. Therefore our $-L_\rho \Delta_\rho H(\rho)$ is $L_\rho \Delta_\rho H_\rho$ in [4] and $L_\rho \Delta_\rho H(\rho)$ in [13], and the principal terms of equations for ρ are the same as each other.

Applying [1, Theorem 6.3], we get an existence result for the Helfrich flow.

Theorem 2.1. *For any $\rho_0 \in h^{2+\beta}(\Sigma_0) \cap \mathcal{U}$, there exists $t^+ = t^+(\rho_0) \in (0, \infty]$ such that (3.2) has a unique maximal solution $\rho \in C([0, t^+); h^{2+\beta}(\Sigma_0) \cap \mathcal{U}) \cap C^\infty((0, t^+); C^\infty(\Sigma_0))$ satisfying $\rho(0) = \rho_0$.*

3. THE LINEARIZED OPERATOR AROUND THE UNIT SPHERE

In this section we consider the case where the reference manifold Σ_0 is the unit sphere S^2 , for which $H(0) = K(0) = L_0 = 1$ holds. Therefore, S^2 is a stationary solution to (2.1), when

$$\lambda_2 = 2c_0 - 2(c_0^2 + \lambda_1). \quad (3.1)$$

Here we study the stability of S^2 under (3.1). Put $\gamma = \lambda_2 + 2c_0$, then the equation with which we concern ourselves is

$$\rho_t = L_\rho \{ -\Delta_\rho H(\rho) - 2H(\rho)(H^2(\rho) - K(\rho)) - 2c_0(K(\rho) - 1) + (4c_0 - \gamma)(H(\rho) - 1) \}. \quad (3.2)$$

We have already shown in [10] the following fact. Let us consider the case where γ is close to $n(n+1)$. Here n is an integer greater than 1. Then there exists an axially symmetric critical surface near $\gamma = n(n+1)$, which bifurcates from the unit sphere, to the variational problem of the Helfrich functional. Such phenomenon does not occur for $\gamma < 6$. That is, $\gamma = 6$ is the first bifurcation point. The existence of such critical surfaces near S^2 suggests that S^2 (*i.e.*, $\rho \equiv 0$) is not unconditional stable for $\gamma > 6$.

Furthermore if $\gamma < 0$, then S^2 is unstable. To show this, it is enough to analyze the ordinary differential equation associated with (2.1). Assume that $\rho(0)$ is a constant function on S^2 , and that $\rho(t)$ is also. Then we have

$$L_\rho = 1, \quad H(\rho) = \frac{1}{1-\rho}, \quad K(\rho) = \frac{1}{(1-\rho)^2}.$$

The first relation follows from $\Phi(x, t) = 1 - \|x\| - \rho$. Thus (2.1) is reduced to an ordinary differential equation

$$\rho_t = -2c_0 \left\{ \frac{1}{(1-\rho)^2} - 1 \right\} + (4c_0 - \gamma) \left(\frac{1}{1-\rho} - 1 \right).$$

Put $\frac{1}{1-\rho} = 1 + k$. Since

$$k_t = \frac{\rho_t}{(1-\rho)^2} = (1+k)^2 \rho_t,$$

we obtain the equation which k must satisfy:

$$k_t = -k(1+k)^2(2c_0k + \gamma).$$

The stationary solution $k \equiv 0$ (*i.e.*, the unit sphere) is unstable if

$$\left. \frac{d}{dk} \{-k(1+k)^2(2c_0k + \gamma)\} \right|_{k=0} = -\gamma > 0.$$

Therefore, $\gamma \geq 0$ is a necessary condition for the stability of S^2 as a stationary solution to (2.1).

Of course the same conclusion follows from the functional \mathcal{H} . If ρ is constant on S^2 , and if $\lambda_2 = 2c_0 - 2(c_0^2 + \lambda_1)$, then

$$\begin{aligned} \mathcal{H}(\Sigma_\rho) &= \int_{\Sigma_\rho} (H^2(\rho) - 2c_0H(\rho)) dS + (c_0^2 + \lambda_1) \mathcal{A}(\Sigma_\rho) + \lambda_2 \mathcal{V}(\Sigma_\rho) \\ &= 4\pi \left(1 - c_0 - \frac{\lambda_2}{6} + \frac{\gamma}{2} \rho^2 - \frac{\lambda_2}{3} \rho \right). \end{aligned}$$

Therefore, we have

$$\left. \frac{d}{d\rho} \mathcal{H}(\Sigma_\rho) \right|_{\rho=0} = 0, \quad \left. \frac{d^2}{d\rho^2} \mathcal{H}(\Sigma_\rho) \right|_{\rho=0} = 4\pi\gamma,$$

and the unit sphere S^2 is unstable as a critical point of the Helfrich functional when $\gamma < 0$.

We now investigate the stability of the unit sphere for $\gamma \in [0, 6]$. In fact we can construct a center manifold for $\gamma \in [0, 6]$, which attracts for all solutions to (2.1) with sufficiently small initial data if either $\gamma \in (0, 6)$ or $\gamma = c_0 = 0$.

Lemma 3.1. *It holds that*

$$\left. \frac{d}{d\varepsilon} L_{\varepsilon h} \right|_{\varepsilon=0} = 0.$$

Proof. The assertion follows from $L_{\varepsilon h} = \sqrt{1 + \varepsilon^2 \|\nabla_{\varepsilon h} h\|^2}$. □

Lemma 3.2. *It holds that*

$$\left. \frac{d}{d\varepsilon} H(\varepsilon h) \right|_{\varepsilon=0} = \frac{1}{2}(\Delta_0 + 2)h.$$

Proof. Since

$$H(\varepsilon h) = P_1(\varepsilon h)\varepsilon h + F_1(\varepsilon h),$$

we have

$$\left. \frac{d}{d\varepsilon} H(\varepsilon h) \right|_{\varepsilon=0} = P_1(0)h + \left. \frac{d}{d\varepsilon} F_1(\varepsilon h) \right|_{\varepsilon=0}.$$

Lemma 2.1 implies

$$P_1(0)h = \frac{1}{2}w^{jk}(0) (\partial_j \partial_k - \Gamma_{jk}^i(0) \partial_i) h = \frac{1}{2} \Delta_0 h.$$

By Lemma 3.1, it holds that

$$\frac{d}{d\varepsilon} F_1(\varepsilon h) \Big|_{\varepsilon=0} = -\frac{1}{2L_0} \frac{d}{d\varepsilon} w^{jk}(\varepsilon h) \Gamma_{jk}^3(\varepsilon h) \Big|_{\varepsilon=0}.$$

Since $\frac{1}{L_0} w^{jk}(\varepsilon h) \Gamma_{jk}^3(\varepsilon h)$ does not contain derivatives of h , its value at $s = s_0$ can be calculated by use of the constant function $f(s) = \varepsilon h(s_0)$. Noticing $L_0 = L_f$ holds for constant functions, we get

$$-\frac{1}{2L_0} w^{jk}(\varepsilon h(s_0)) \Gamma_{jk}^3(\varepsilon h(s_0)) = -\frac{1}{2L_f} w^{jk}(f) \Gamma_{jk}^3(f) = F_1(f).$$

Furthermore, because of $P_1(f) = 0$ for constant functions, we have

$$F_1(f) = H(f) = \frac{1}{1-f} = \frac{1}{1-\varepsilon h(s_0)}.$$

Therefore, we find

$$\begin{aligned} \frac{d}{d\varepsilon} F_1(\varepsilon h) \Big|_{\varepsilon=0} &= -\frac{1}{2L_f} \frac{d}{d\varepsilon} w^{jk}(f) \Gamma_{jk}^3(f) \Big|_{\varepsilon=0} = -\frac{d}{d\varepsilon} \frac{1}{2L_f} w^{jk}(f) \Gamma_{jk}^3(f) \Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon} \frac{1}{1-\varepsilon h(s_0)} \Big|_{\varepsilon=0} = \frac{h(s_0)}{(1-\varepsilon h(s_0))^2} \Big|_{\varepsilon=0} = h(s_0). \end{aligned}$$

Since s_0 is arbitrary, we obtain the assertion. \square

Lemma 3.3. *It holds that*

$$\frac{d}{d\varepsilon} (L_{\varepsilon h} \Delta_{\varepsilon h} H(\varepsilon h)) \Big|_{\varepsilon=0} = \frac{1}{2} \Delta_0 (\Delta_0 + 2) h.$$

Proof. We have

$$\frac{d}{d\varepsilon} (L_{\varepsilon h} \Delta_{\varepsilon h} H(\varepsilon h)) \Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} (L_{\varepsilon h} \Delta_{\varepsilon h} H(0)) \Big|_{\varepsilon=0} + L_0 \Delta_0 \frac{d}{d\varepsilon} H(\varepsilon h) \Big|_{\varepsilon=0}.$$

Taking $H(0) = L_0 \equiv 1$ into account, the previous lemma gives the assertion. \square

Lemma 3.4. *It holds that*

$$\frac{d}{d\varepsilon} \{L_{\varepsilon h} H(\varepsilon h) (H^2(\varepsilon h) - K(\varepsilon h))\} \Big|_{\varepsilon=0} = 0.$$

Proof. Let $\kappa_1(\varepsilon h)$ and $\kappa_2(\varepsilon h)$ be the principal curvatures. Since

$$H^2(\varepsilon h) - K(\varepsilon h) = \left(\frac{\kappa_1(\varepsilon h) - \kappa_2(\varepsilon h)}{2} \right)^2$$

and since $\kappa_1(0) = \kappa_2(0) = 1$, we get

$$(H^2(\varepsilon h) - K(\varepsilon h)) \Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} (H^2(\varepsilon h) - K(\varepsilon h)) \Big|_{\varepsilon=0} = 0.$$

Lemma 3.5. *It holds that*

$$\frac{d}{d\varepsilon} (L_{\varepsilon h} H(\varepsilon h)) \Big|_{\varepsilon=0} = \frac{1}{2}(\Delta_0 + 2)h, \quad \frac{d}{d\varepsilon} (L_{\varepsilon h} K(\varepsilon h)) \Big|_{\varepsilon=0} = (\Delta_0 + 2)h.$$

Proof. It follows from the lemmata above that

$$\frac{d}{d\varepsilon} (L_{\varepsilon h} H(\varepsilon h)) \Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} H(\varepsilon h) \Big|_{\varepsilon=0} = \frac{1}{2}(\Delta_0 + 2)h.$$

Similarly, we have

$$\begin{aligned} \frac{d}{d\varepsilon} (L_{\varepsilon h} K(\varepsilon h)) \Big|_{\varepsilon=0} &= \frac{d}{d\varepsilon} K(\varepsilon h) \Big|_{\varepsilon=0} \\ &= \frac{d}{d\varepsilon} \{H^2(\varepsilon h) - (H^2(\varepsilon) - K(\varepsilon h))\} \Big|_{\varepsilon=0} \\ &= 2H(0) \frac{d}{d\varepsilon} (H(\varepsilon h)) \Big|_{\varepsilon=0} = (\Delta_0 + 2)h. \end{aligned}$$

□

Combining these lemmata, we obtain

Proposition 3.1. *The linearized equation around the unit sphere of (2.1) is*

$$h_t + Ah = 0,$$

where $A = \frac{1}{2}(\Delta_0 + 2)(\Delta_0 + \gamma)$ and $\gamma = 4c_0 - 2(c_0^2 + \lambda_1)$.

We now investigate the spectrum of $-A$.

Proposition 3.2. *The spectrum of $-A$ consists of eigenvalues*

$$\text{spec}(-A) = \left\{ -\frac{1}{2}(\mu_i - 2)(\mu_i - \gamma) : \mu_i \in \text{spec}(-\Delta_0) \right\}.$$

Proof. The linearized operator A , defined on $h^{4+\alpha}(S^2)$ into $h^\alpha(S^2)$, is strongly elliptic with smooth coefficients. Furthermore, the imbedding $h^{4+\alpha}(S^2) \hookrightarrow h^\alpha(S^2)$ is compact. Hence the spectrum of A consists only of eigenvalues, and the eigenfunctions are smooth. In particular since the eigenfunctions belong to $L^2(S^2)$, they are expanded into Fourier series using the orthonormalized system of L^2 -eigenfunctions of $-\Delta_0$. Let $\text{spec}(-\Delta_0) = \{\mu_i\}$ and ϕ_i be an eigenfunction belonging to the eigenvalue μ_i , satisfying $\langle \phi_i, \phi_j \rangle = \delta_{ij}$. Consider the equation

$$(A + \nu)\phi = 0$$

and look for ϕ in the form

$$\phi = \sum_{i=1}^{\infty} c_i \phi_i.$$

Then $\{c_i\}$ must satisfy

$$\sum_{i=1}^{\infty} c_i \left\{ \frac{1}{2} (-\mu_i + 2) (-\mu_i + \gamma) + \nu \right\} \phi_i = 0.$$

Therefore, ν is an eigenvalue of $-A$ if and only if

$$\nu = -\frac{1}{2} (-\mu_i + 2) (-\mu_i + \gamma)$$

for some i . □

Corollary 3.1. *The spectrum $\text{spec}(-A)$ is contained in $(-\infty, 0]$ if and only if $0 \leq \gamma \leq 6$.*

Proof. The assertion follows from the fact that $\mu_i = k(k+1)$ for $k \in \mathbb{N} \cup \{0\}$. □

Let W_k be the eigenspace of $-\Delta_0$ belonging to the eigenvalue $k(k+1)$. In particular,

$$W_0 = \text{span}\{1\}, \quad W_1 = \text{span}\{x_1|_{S^2}, x_2|_{S^2}, x_3|_{S^2}\},$$

$$W_2 = \text{span}\{x_1x_2|_{S^2}, x_2x_3|_{S^2}, x_1x_3|_{S^2}, x_1^2 - x_2^2|_{S^2}, x_2^2 - x_3^2|_{S^2}\},$$

where x_i is the Euclidean coordinate function in \mathbb{R}^3 .

Remark 3.1. By use of the standard polar coordinate system

$$x_1|_{S^2} = \sin \phi \cos \theta, \quad x_2|_{S^2} = \sin \phi \sin \theta, \quad x_3|_{S^2} = \cos \phi,$$

it holds that

$$W_k = \text{span}\{P_k(\cos \phi), P_k^m(\cos \phi) \cos m\theta, P_k^m(\cos \phi) \sin m\theta \mid 1 \leq m \leq k\}.$$

Here P_k^m is the associated Legendre function, and $P_k = P_k^0$. In particular

$$x_1|_{S^2} = P_1^1(\cos \phi) \cos \theta, \quad x_2|_{S^2} = P_1^1(\cos \phi) \sin \theta, \quad x_3|_{S^2} = P_1(\cos \phi),$$

$$x_1x_2|_{S^2} = \frac{1}{6} P_2^2(\cos \phi) \sin 2\theta, \quad x_2x_3|_{S^2} = \frac{1}{3} P_2^1(\cos \phi) \sin \theta,$$

$$x_1x_3|_{S^2} = \frac{1}{3} P_2^1(\cos \phi) \cos \theta, \quad x_1^2 - x_2^2|_{S^2} = \frac{1}{3} P_2^2(\cos \phi) \cos 2\theta,$$

$$x_2^2 - x_3^2|_{S^2} = -P_2(\cos \phi) - \frac{1}{6} P_2^2(\cos \phi) \cos 2\theta,$$

$$P_2(\cos \phi) = -\frac{1}{2} (x_1^2 - x_2^2) - (x_2^2 - x_3^2) \Big|_{S^2},$$

$$P_2^1(\cos \phi) \cos \theta = 3x_3x_1|_{S^2}, \quad P_2^1(\cos \phi) \sin \theta = 3x_2x_3|_{S^2},$$

$$P_2^2(\cos \phi) \cos 2\theta = 3(x_1^2 - x_2^2) \Big|_{S^2}, \quad P_2^2(\cos \phi) \sin 2\theta = 6x_1x_2|_{S^2}.$$

In what follows we assume $0 \leq \gamma \leq 6$. We arrange eigenvalues

$$\nu = -\frac{1}{2}(k-1)(k+2)(k^2+k-\gamma)$$

of $-A$ in descending order as

$$0 = \nu_0 > \nu_1 > \cdots > \nu_m > \cdots.$$

Denoting the eigenspace belonging to ν_m by V_m , we have

Proposition 3.3. (1) When $\gamma = 0$,

$$\nu_m = -\frac{1}{2}m(m+1)(m+2)(m+3), \quad V_0 = W_0 \oplus W_1,$$

the geometric and algebraic multiplicity of $\nu_0 = \dim V_0 = 4$.

(2) When $0 < \gamma < 4$,

$$\nu_0 = 0, \quad \nu_1 = -\gamma,$$

$$\nu_m = -\frac{1}{2}(m-1)(m+2)(m^2+m-\gamma) \quad \text{for } m \geq 2,$$

$$V_0 = W_1,$$

the geometric and algebraic multiplicity of $\nu_0 = \dim V_0 = 3$.

(3) When $\gamma = 4$,

$$\nu_0 = 0, \quad \nu_1 = -\gamma,$$

$$\nu_m = -\frac{1}{2}m(m+3)(m^2+3m+2-\gamma) \quad \text{for } m \geq 2,$$

$$V_0 = W_1,$$

the geometric and algebraic multiplicity of $\nu_0 = \dim V_0 = 3$.

(4) When $4 < \gamma < 6$,

$$\nu_0 = 0, \quad \nu_1 = -2(6-\gamma), \quad \nu_2 = -\gamma,$$

$$\nu_m = -\frac{1}{2}(m-1)(m+2)(m^2+m-\gamma) \quad \text{for } m \geq 3,$$

$$V_0 = W_1,$$

the geometric and algebraic multiplicity of $\nu_0 = \dim V_0 = 3$.

(5) When $\gamma = 6$,

$$\nu_0 = 0, \quad \nu_1 = -\gamma,$$

$$\nu_m = -\frac{1}{2}m(m+3)(m^2+3m+2-\gamma) \quad \text{for } m \geq 2,$$

$$V_0 = W_1 \oplus W_2,$$

the geometric and algebraic multiplicity of $\nu_0 = \dim V_0 = 8$.

In a manner similar to [4, 13], we can obtain the following result.

Theorem 3.1. *For the Helfrich flow obtained in Theorem 2.1 with $\Sigma_0 = S^2$, the following statements hold.*

- (1) *For $\gamma \in [0, 6]$ there exists uniquely a local center manifold \mathcal{M} for (3.2) with*

$$\dim \mathcal{M} = \begin{cases} 4 & \gamma = 0, \\ 3 & \gamma \in (0, 6) \\ 8 & \gamma = 6. \end{cases}$$

- (2) *For $\gamma \in (0, 6)$ we have*

$$\mathcal{M} = \{ \text{all spheres with center close to the original } S^2 \text{ and with radius } 1 \}.$$

That is, the center manifold \mathcal{M} is generated by small translation.

- (3) *For $\gamma = c_0 = 0$ we have*

$$\mathcal{M} = \{ \text{all spheres with center close to the original } S^2 \text{ and with radius close to } 1 \}.$$

That is, the center manifold \mathcal{M} is generated by small translation and dilation with ratio near 1.

- (4) *Unit spheres obtained by small translation are included in \mathcal{M} even for cases “ $\gamma = 0$ and $c_0 \neq 0$ ”, and $\gamma = 6$.*
- (5) *For $\gamma \in (0, 6)$ or $\gamma = c_0 = 0$ the center manifold \mathcal{M} attracts the solutions of (3.2) that start in a small $h^{2+\beta}(S^2)$ neighborhood of 0, where β is that of Theorem 2.1.*

Proof. The first assertion follows from the results of Simonett [11, 12] (see also [4, 13]).

A non-zero constant function on S^2 is a generator of dilation, and $x_i|_{S^2}$ does that of translation. Furthermore unit spheres are stationary solutions for (3.2), and so are spheres for (3.2) with $c_0 = 0$. These facts yield the assertions (2)–(5) (see [4, 13]). Note that the assertions for $c_0 = 0$ were obtained by Simonett [13]. \square

Proposition 3.4. *When $\gamma = 0$ and $c_0 \neq 0$, the unit sphere is unstable.*

Proof. Let $\Sigma(0)$ be a sphere. By the uniqueness of the flow, $\Sigma(t)$ is also a sphere for every t . Denoting the curvature of $\Sigma(t)$ by $1 + k(t)$, we find that k is governed by

$$k_t = -2c_0k^2(1 + k)^2. \quad (3.3)$$

We consider the initial-value problem for (3.3) with sufficiently small $|k(0)|$. It is easily shown that:

- (1) The case $c_0 < 0$:
- (a) If $k(0) > 0$, then the solution k exists on $(-\infty, t_*)$ for some $t_* \in (0, \infty)$, and $k(t) \uparrow \infty$ as $t \uparrow t_*$. That is, the sphere $\Sigma(t)$ shrinks in finite time.
 - (b) If $-1 < k(0) < 0$, then k exists for all $t \in \mathbb{R}$, and satisfies $-ct^{-1} < k(t) < 0$ for sufficiently large t .
- (2) The case $c_0 > 0$:
- (a) If $k(0) > 0$, then the solution k exists on (t_*, ∞) for some $t_* \in (-\infty, 0)$, and satisfies $0 < k(t) < ct^{-1}$.
 - (b) If $-1 < k(0) < 0$, then k exists for all $t \in \mathbb{R}$, and satisfies $-1 < k(t) < -1 + ct^{-1}$ for sufficiently large t . That is, the sphere $\Sigma(t)$ grows up infinitely taking infinite time.

These suggest the assertion. \square

We do not know the more precise structure of \mathcal{M} for $\gamma = 6$. The bifurcating surface Σ_ε from the unit sphere around $\gamma = 6$ obtained in [10] satisfies

$$H(\rho) = 1 + \varepsilon P_2(x_3|_{S^2}) + \mathcal{O}(\varepsilon^2).$$

The axis of rotation for this surface is the x_3 -axis. The addition theorem for P_k^m implies that there also exists the bifurcating surface satisfying

$$H(\rho) = 1 + \varepsilon \left\{ P_2(\cos \phi_0) P_2(x_3|_{S^2}) + \frac{1}{3} P_2^1(\cos \phi_0) P_2^1(x_3|_{S^2}) \cos(\theta - \theta_0) + \frac{1}{12} P_2^2(\cos \phi_0) P_2^2(x_3|_{S^2}) \cos 2(\theta - \theta_0) \right\} + \mathcal{O}(\varepsilon^2).$$

Here $\theta \in [0, 2\pi)$ is the angle around the x_3 -axis. The surface is also rotationally symmetric around the axis passing through the origin and $(\sin \phi_0 \cos \theta_0, \sin \phi_0 \sin \theta_0, \cos \phi_0) \in S^2$.

4. A PARAMETER-DEPENDENT CENTER MANIFOLD

The Helfrich problem is a variational problem with two constraints, and λ_1 and λ_2 are Lagrange multipliers. Therefore they are determined together with critical points, and need not satisfy (3.1) for critical points other than the unit sphere. We should deal with λ_1 and λ_2 as unknown parameters rather than given constants in studying the center manifolds. More precisely we introduce a new parameter λ by

$$\lambda_2 = 2c_0 - 2(c_0^2 + \lambda_1) + 2\lambda$$

(cf. (3.1)), and study the parameter-dependent center manifolds (see [11]) as follows. Put $\gamma = \lambda_2 + 2c_0$ as before. We consider the problem described by

$$\begin{cases} \partial_t \rho = L_\rho \{ -\Delta_\rho H(\rho) - 2H(\rho) (H^2(\rho) - K(\rho)) \\ \quad - 2c_0 (K(\rho) - 1) + (4c_0 - \gamma) (H(\rho) - 1) + 2\lambda H(\rho) \}, \\ \partial_t \lambda = 0, \\ \partial_t \gamma = 0. \end{cases} \quad (4.1)$$

The unit sphere corresponds to $(\rho, \lambda, \gamma) \equiv (0, 0, \gamma)$, which is an equilibrium for any γ .

For any fixed γ_0 , we consider $\gamma = \gamma_0 + \hat{\gamma}$ as the small perturbation. We write the first equation in (4.1) as

$$\partial_t \rho + P(\rho)\rho + F(\lambda, \gamma_0 + \hat{\gamma}, \rho) = 0.$$

Then it holds that

$$P(0) - \partial_\rho F(0, \gamma_0, 0) = -\frac{1}{2}(\Delta_0 + 2)(\Delta_0 + \gamma_0) = A(\gamma_0).$$

Denote the operator $(\lambda, \hat{\gamma}, \rho) \mapsto A(\gamma_0)\rho$ by $\mathbf{A}(\gamma_0)$. Then we have

$$\text{spec}(-\mathbf{A}(\gamma_0)) = \text{spec}(-A(\gamma_0)) \cup \{0\} = \text{spec}(-A(\gamma_0)).$$

The eigenspace $\mathbf{V}_\nu(\gamma_0)$ belonging to $\nu \in \text{spec}(-\mathbf{A}(\gamma_0))$ is

$$\begin{aligned} \mathbf{V}_\nu(\gamma_0) &= \{(0, 0, \rho) : (A(\gamma_0) + \nu)\rho = 0\} \quad \text{if } \nu \neq 0, \\ \mathbf{V}_0(\gamma_0) &= \{(\lambda, \hat{\gamma}, \rho) : \lambda \in \mathbb{R}, \hat{\gamma} \in \mathbb{R}, A(\gamma_0)\rho = 0\}, \end{aligned}$$

In particular, $\text{spec}(-\mathbf{A}(\gamma_0)) \subset (-\infty, 0]$ if and only if $\gamma_0 \in [0, 6]$. Furthermore we have

Proposition 4.1. (1) *When $\gamma_0 = 0$,*

$$\mathbf{V}_0(0) = \{(\lambda, \hat{\gamma}, \rho) : \lambda \in \mathbb{R}, \hat{\gamma} \in \mathbb{R}, \rho \in W_0 \oplus W_1\}, \quad \dim \mathbf{V}_0(0) = 6.$$

(2) *When $\gamma_0 \in (0, 6)$,*

$$\mathbf{V}_0(\gamma_0) = \{(\lambda, \hat{\gamma}, \rho) : \lambda \in \mathbb{R}, \hat{\gamma} \in \mathbb{R}, \rho \in W_1\}, \quad \dim \mathbf{V}_0(\gamma_0) = 5.$$

(3) *When $\gamma_0 = 6$,*

$$\mathbf{V}_0(\gamma_0) = \{(\lambda, \hat{\gamma}, \rho) : \lambda \in \mathbb{R}, \hat{\gamma} \in \mathbb{R}, \rho \in W_1 \oplus W_2\}, \quad \dim \mathbf{V}_0(\gamma_0) = 10.$$

According to [11], we obtain

Theorem 4.1. *For $\gamma_0 \in [0, 6]$ there exists a parameter-dependent center manifold $\mathcal{M}(\gamma_0)$ uniquely with $\dim \mathcal{M}(\gamma_0) = \dim \mathbf{V}_0(\gamma_0)$.*

5. APPENDIX

In this section we shall give expressions of mean curvature and Gaussian curvature in terms of ρ . Lemmata in this section are valid not only for surfaces in \mathbb{R}^3 but also for hypersurfaces in \mathbb{R}^{n+1} . As in section 2, we assume that a closed hypersurface is defined by $\{x \in \mathbb{R}^{n+1} : \Phi(x) = 0\}$ locally, where the interior near the hypersurface is located in the side $\{x \in \mathbb{R}^{n+1} : \Phi(x) > 0\}$. Assume also that $\nabla_x \Phi \neq 0$ everywhere near the hypersurface.

Lemma 5.1. *The mean curvature H and Gaussian curvature K are given by*

$$H = -\frac{1}{n} \operatorname{div}_x \left(\frac{\nabla_x \Phi}{\|\nabla_x \Phi\|} \right) \Big|_{\{x: \Phi(x)=0\}},$$

$$K = \mathcal{G}(\nabla_x \Phi, \operatorname{Hess}_x \Phi) \Big|_{\{x: \Phi(x)=0\}}$$

where

$$\mathcal{G}(\mathbf{p}, X) = \det_{n+1} (\|\mathbf{p}\|^{-1} (I_{n+1} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}) X (I_{n+1} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}) + \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}),$$

$$\bar{\mathbf{p}} = \|\mathbf{p}\|^{-1} \mathbf{p}$$

for $\mathbf{p} \in \mathbb{R}^{n+1}$ and $X \in \mathcal{S}^{n+1}$.

Proof. The assertion for the mean curvature is well known. We give here the proof for the Gaussian curvature, which is an adapted version of [7, Lemma A]. We may assume $x = 0$, and

$$\mathbf{e}_{n+1} = -\|\nabla_x \Phi(0)\|^{-1} \nabla_x \Phi(0).$$

Introduce an orthonormal system $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ so that $\{\mathbf{e}_1, \dots, \mathbf{e}_n, \mathbf{e}_{n+1}\}$ is an orthogonal basis in \mathbb{R}^{n+1} , and the surface $\Phi(x) = 0$ is represented locally as a graph of a function $f : U \rightarrow \mathbb{R}$. Here U is a neighborhood of $0 \in \mathbb{R}^n$. Put $\tilde{x} = (x_1, \dots, x_n)$. Then eigenvalues of $\operatorname{Hess}_{\tilde{x}} f(0)$ are the principal curvatures of the surface at $x = 0$. Differentiating the relation

$$\Phi(\tilde{x}, f(\tilde{x})) = 0,$$

with respect to x_i ($i = 1, \dots, n$), we have

$$\Phi_{x_i}(\tilde{x}, f(\tilde{x})) + \Phi_{x_{n+1}}(\tilde{x}, f(\tilde{x})) f_{x_i}(\tilde{x}) = 0.$$

Differentiate with respect to x_j ($j = 1, \dots, n$), and we get

$$\Phi_{x_i x_j}(\tilde{x}, f(\tilde{x})) + \Phi_{x_i x_{n+1}}(\tilde{x}, f(\tilde{x})) f_{x_j}(\tilde{x}) + \Phi_{x_j x_{n+1}}(\tilde{x}, f(\tilde{x})) f_{x_i}(\tilde{x})$$

$$+ \Phi_{x_{n+1} x_{n+1}}(\tilde{x}, f(\tilde{x})) f_{x_i}(\tilde{x}) f_{x_j}(\tilde{x}) + \Phi_{x_{n+1}}(\tilde{x}, f(\tilde{x})) f_{x_i x_j}(\tilde{x}) = 0.$$

Since

$$f_{x_i}(0) = 0 \quad (i = 1, \dots, n), \quad \Phi_{x_{n+1}}(0) = -\|\nabla_x \Phi(0)\|,$$

we obtain

$$f_{x_i x_j}(0) = \|\nabla_x \Phi(0)\|^{-1} \Phi_{x_i x_j}(0).$$

We denote the set of all real-valued $j \times k$ matrices by $M_{j,k}(\mathbb{R})$, and $M_{jj}(\mathbb{R}) = M_j(\mathbb{R})$. Put

$$\mathbf{p} = \nabla_x \Phi(0), \quad X = \text{Hess}_x \Phi(0) \in M_{n+1}(\mathbb{R}), \quad E = (\mathbf{e}_1, \dots, \mathbf{e}_n) \in M_{n+1,n}(\mathbb{R}).$$

Then we have

$$\text{Hess}_{\bar{x}} f(0) = \|\nabla_x \Phi(0)\|^{-1} {}^t E \text{Hess}_x \Phi(0) E = \|\mathbf{p}\|^{-1} {}^t E X E,$$

$$\mathbf{e}_{n+1} = -\|\mathbf{p}\|^{-1} \mathbf{p} = -\bar{\mathbf{p}}.$$

Since $(E \ \mathbf{e}_{n+1}) = I_{n+1} \in M_{n+1}(\mathbb{R})$,

$$E {}^t E + \mathbf{e}_{n+1} {}^t \mathbf{e}_{n+1} = (E \ \mathbf{e}_{n+1}) \begin{pmatrix} {}^t E \\ {}^t \mathbf{e}_{n+1} \end{pmatrix} = I_{n+1}, \quad \mathbf{e}_{n+1} {}^t \mathbf{e}_{n+1} = \bar{\mathbf{p}} {}^t \bar{\mathbf{p}} = \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}$$

hold. Therefore,

$$\begin{aligned} \begin{pmatrix} \|\mathbf{p}\|^{-1} {}^t E X E & 0 \\ 0 & 1 \end{pmatrix} &= (E \ \mathbf{e}_{n+1}) \begin{pmatrix} \|\mathbf{p}\|^{-1} {}^t E X E & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} {}^t E \\ {}^t \mathbf{e}_{n+1} \end{pmatrix} \\ &= \|\mathbf{p}\|^{-1} E {}^t E X E {}^t E + \mathbf{e}_{n+1} {}^t \mathbf{e}_{n+1} \\ &= \|\mathbf{p}\|^{-1} (I_{n+1} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}) X (I_{n+1} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}) + \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}, \end{aligned}$$

from which it follows that

$$K = \det_n \text{Hess}_{\bar{x}} f(0) = \det_{n+1} \begin{pmatrix} \|\mathbf{p}\|^{-1} {}^t E X E & 0 \\ 0 & 1 \end{pmatrix} = \mathcal{G}(\mathbf{p}, X).$$

□

Let $y = (y_1, \dots, y_{n+1})$ be an arbitrary (local) coordinate system on \mathbb{R}^{n+1} . We would like to know the expression of Gaussian curvature in this system. Choose the standard coordinate system $x = (x_1, \dots, x_{n+1})$ so that

$$\Phi(0) = 0, \quad \partial_{x_{n+1}} = -\|\nabla_x \Phi(0)\|^{-1} \nabla_x \Phi(0),$$

$\text{span} \{\partial_{x_1}, \dots, \partial_{x_n}\} =$ the tangent space of the surface at $x = 0$.

Put

$$\tilde{\Phi}(y) = \Phi(x(y)) \quad (\text{or } \Phi(x) = \tilde{\Phi}(y(x))), \quad g_{ij} = (\partial_{y_i}, \partial_{y_j})_{\mathbb{R}^{n+1}}.$$

We have

$$\frac{\partial \Phi}{\partial x_i} = \frac{\partial \tilde{\Phi}}{\partial y_k} \frac{\partial y_k}{\partial x_i}.$$

Let $\Gamma_{x_{ij}}^s$ and $\Gamma_{y_{k\ell}}^m$ be the Christoffel symbol with respect to the coordinate systems x and y respectively. Since

$$\left(\frac{\partial^2 y_m}{\partial x_i \partial x_j} + \frac{\partial y_k}{\partial x_i} \frac{\partial y_\ell}{\partial x_j} \Gamma_{y_{k\ell}}^m \right) \frac{\partial x_s}{\partial y_m} = \Gamma_{x_{ij}}^s = 0,$$

it holds that

$$\frac{\partial^2 \Phi}{\partial x_i \partial x_j} = \frac{\partial^2 \tilde{\Phi}}{\partial y_k \partial y_\ell} \frac{\partial y_k}{\partial x_i} \frac{\partial y_\ell}{\partial x_j} + \frac{\partial \tilde{\Phi}}{\partial y_m} \frac{\partial^2 y_m}{\partial x_i \partial x_j} = \frac{\partial y_k}{\partial x_i} \frac{\partial y_\ell}{\partial x_j} \left(\frac{\partial^2 \tilde{\Phi}}{\partial y_k \partial y_\ell} - \Gamma_{y_{k\ell}}^m \frac{\partial \tilde{\Phi}}{\partial y_m} \right).$$

Therefore, we have, with $\mathbf{p} = \nabla_x \Phi$,

$$(\bar{\mathbf{p}} \otimes \bar{\mathbf{p}})_{ij} = \left(\|\nabla_x \Phi\|^{-2} \frac{\partial \Phi}{\partial x_i} \frac{\partial \Phi}{\partial x_j} \right) = \|\nabla_y \tilde{\Phi}\|_g^{-2} \frac{\partial y_k}{\partial x_i} \frac{\partial y_\ell}{\partial x_j} \frac{\partial \tilde{\Phi}}{\partial y_k} \frac{\partial \tilde{\Phi}}{\partial y_\ell}.$$

We multiply both sides by $\delta^{\lambda i} \frac{\partial y_\mu}{\partial x_\lambda} \delta^{j\rho} \frac{\partial y_\nu}{\partial x_\rho}$. Because of $g^{k\ell} = \delta^{ij} \frac{\partial y_k}{\partial x_i} \frac{\partial y_\ell}{\partial x_j}$, we get

$$\begin{aligned} \delta^{\lambda i} \frac{\partial y_\mu}{\partial x_\lambda} (\bar{\mathbf{p}} \otimes \bar{\mathbf{p}})_{ij} \delta^{j\rho} \frac{\partial y_\nu}{\partial x_\rho} &= g^{k\mu} g^{\ell\nu} \|\nabla_y \tilde{\Phi}\|_g^{-2} \frac{\partial \tilde{\Phi}}{\partial y_k} \frac{\partial \tilde{\Phi}}{\partial y_\ell} \\ &= \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\mu \left(\nabla_y \tilde{\Phi} \right)^\nu. \end{aligned}$$

Furthermore, since $\delta_{ij} = \frac{\partial y_k}{\partial x_i} \frac{\partial y_\ell}{\partial x_j} g_{k\ell}$, it holds that

$$(I_{n+1} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}})_{ij} = \left(g_{k\ell} - \|\nabla_y \tilde{\Phi}\|_g^{-2} \frac{\partial \tilde{\Phi}}{\partial y_k} \frac{\partial \tilde{\Phi}}{\partial y_\ell} \right) \frac{\partial y_k}{\partial x_i} \frac{\partial y_\ell}{\partial x_j}.$$

Using $g^{k\ell} = \delta^{ij} \frac{\partial y_k}{\partial x_i} \frac{\partial y_\ell}{\partial x_j}$ again, we get

$$\begin{aligned} &((I_{n+1} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}) \text{Hess}_x \Phi (I_{n+1} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}))_{ij} \\ &= \left(g_{k\ell} - \|\nabla_y \tilde{\Phi}\|_g^{-2} \frac{\partial \tilde{\Phi}}{\partial y_k} \frac{\partial \tilde{\Phi}}{\partial y_\ell} \right) \left(g_{st} - \|\nabla_y \tilde{\Phi}\|_g^{-2} \frac{\partial \tilde{\Phi}}{\partial y_s} \frac{\partial \tilde{\Phi}}{\partial y_t} \right) \\ &\quad \times \frac{\partial y_k}{\partial x_i} \frac{\partial y_\ell}{\partial x_p} \frac{\partial y_\alpha}{\partial x_u} \frac{\partial y_\beta}{\partial x_q} \frac{\partial y_s}{\partial x_v} \frac{\partial y_t}{\partial x_j} \delta^{pu} \delta^{qv} \left(\frac{\partial^2 \tilde{\Phi}}{\partial y_\alpha \partial y_\beta} - \Gamma_{y_{\alpha\beta}}^\gamma \frac{\partial \tilde{\Phi}}{\partial y_\gamma} \right) \end{aligned}$$

$$\begin{aligned}
&= \left(g_{kl} - \|\nabla_y \tilde{\Phi}\|_g^{-2} \frac{\partial \tilde{\Phi}}{\partial y_k} \frac{\partial \tilde{\Phi}}{\partial y_l} \right) \left(g_{st} - \|\nabla_y \tilde{\Phi}\|_g^{-2} \frac{\partial \tilde{\Phi}}{\partial y_s} \frac{\partial \tilde{\Phi}}{\partial y_t} \right) \\
&\quad \times g^{\ell\alpha} g^{\beta s} \frac{\partial y_k}{\partial x_i} \frac{\partial y_t}{\partial x_j} \left(\frac{\partial^2 \tilde{\Phi}}{\partial y_\alpha \partial y_\beta} - \Gamma_{y_{\alpha\beta}}^\gamma \frac{\partial \tilde{\Phi}}{\partial y_\gamma} \right) \\
&= \left(\delta_k^\alpha - \|\nabla_y \tilde{\Phi}\|_g^{-2} g^{\ell\alpha} \frac{\partial \tilde{\Phi}}{\partial y_k} \frac{\partial \tilde{\Phi}}{\partial y_\ell} \right) \left(\delta_t^\beta - \|\nabla_y \tilde{\Phi}\|_g^{-2} g^{\beta s} \frac{\partial \tilde{\Phi}}{\partial y_s} \frac{\partial \tilde{\Phi}}{\partial y_t} \right) \\
&\quad \times \frac{\partial y_k}{\partial x_i} \frac{\partial y_t}{\partial x_j} \left(\frac{\partial^2 \tilde{\Phi}}{\partial y_\alpha \partial y_\beta} - \Gamma_{y_{\alpha\beta}}^\gamma \frac{\partial \tilde{\Phi}}{\partial y_\gamma} \right) \\
&= \left\{ \delta_k^\alpha - \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\alpha \frac{\partial \tilde{\Phi}}{\partial y_k} \right\} \left\{ \delta_t^\beta - \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\beta \frac{\partial \tilde{\Phi}}{\partial y_t} \right\} \\
&\quad \times \frac{\partial y_k}{\partial x_i} \frac{\partial y_t}{\partial x_j} \left(\text{Hess}_y \tilde{\Phi} \right)_{\alpha\beta}.
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
&\delta^{\lambda i} \frac{\partial y_\mu}{\partial x_\lambda} \left((I_{n+1} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}) \text{Hess}_x \Phi (I_{n+1} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}) \right)_{ij} \delta^{j\rho} \frac{\partial y_\nu}{\partial x_\rho} \\
&= \left\{ \delta_k^\alpha - \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\alpha \frac{\partial \tilde{\Phi}}{\partial y_k} \right\} \\
&\quad \times \left\{ \delta_t^\beta - \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\beta \frac{\partial \tilde{\Phi}}{\partial y_t} \right\} g^{\mu k} g^{\nu t} \left(\text{Hess}_y \tilde{\Phi} \right)_{\alpha\beta} \\
&= \left\{ g^{\alpha\mu} - \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\alpha \left(\nabla_y \tilde{\Phi} \right)^\mu \right\} \\
&\quad \times \left\{ g^{\beta\nu} - \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\beta \left(\nabla_y \tilde{\Phi} \right)^\nu \right\} \left(\text{Hess}_y \tilde{\Phi} \right)_{\alpha\beta}.
\end{aligned}$$

Since

$$\begin{aligned}
\det_{n+1} \left(\delta^{\lambda i} \frac{\partial y_\mu}{\partial x_\lambda} \right) &= \det_{n+1} \left(\frac{\partial y_\mu}{\partial x_\lambda} \right) = \det_{n+1} \left(\frac{\partial x_\lambda}{\partial y_\mu} \right)^{-1} \\
&= \text{sgn} \left\{ \det_{n+1} \left(\frac{\partial x_\lambda}{\partial y_\mu} \right) \right\} g^{-\frac{1}{2}},
\end{aligned}$$

we obtain

$$\begin{aligned}
g^{-1} \Big|_{\{y|\tilde{\Phi}(y)=0\}} &K = \det_{n+1} \left\{ \left(\delta^{\lambda i} \frac{\partial y_\mu}{\partial x_\lambda} \right) \right. \\
&\quad \times \left(\|\mathbf{p}\|^{-1} (I_{n+1} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}) \text{Hess}_x \Phi (I_{n+1} - \bar{\mathbf{p}} \otimes \bar{\mathbf{p}}) + \bar{\mathbf{p}} \otimes \bar{\mathbf{p}} \right)
\end{aligned}$$

$$\begin{aligned}
& \times \left(\delta^{j\nu} \frac{\partial y_\rho}{\partial x_\nu} \right) \Big|_{\{y:\tilde{\Phi}(y)=0\}} \\
& = \det_{n+1} \left[\|\nabla_y \tilde{\Phi}\|_g^{-1} \left\{ g^{\alpha\mu} - \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\alpha \left(\nabla_y \tilde{\Phi} \right)^\mu \right\} \right. \\
& \quad \times \left\{ g^{\beta\nu} - \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\beta \left(\nabla_y \tilde{\Phi} \right)^\nu \right\} \left(\text{Hess}_y \tilde{\Phi} \right)_{\alpha\beta} \\
& \quad \left. + \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\mu \left(\nabla_y \tilde{\Phi} \right)^\nu \right] \Big|_{\{y:\tilde{\Phi}(y)=0\}}.
\end{aligned}$$

Lowering the indices μ and ν , we obtain

Lemma 5.2. *The Gaussian curvature is given by*

$$\begin{aligned}
K & = g^{-1} \det_{n+1} \left[\|\nabla_y \tilde{\Phi}\|_g^{-1} \left\{ \delta_\mu^\alpha - \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\alpha \left(\nabla_y \tilde{\Phi} \right)_\mu \right\} \right. \\
& \quad \times \left\{ \delta_\nu^\beta - \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)^\beta \left(\nabla_y \tilde{\Phi} \right)_\nu \right\} \left(\text{Hess}_y \tilde{\Phi} \right)_{\alpha\beta} \\
& \quad \left. + \|\nabla_y \tilde{\Phi}\|_g^{-2} \left(\nabla_y \tilde{\Phi} \right)_\mu \left(\nabla_y \tilde{\Phi} \right)_\nu \right] \Big|_{\{y:\tilde{\Phi}(y)=0\}}.
\end{aligned}$$

Lemmata 2.1 and 2.2 also hold for hypersurfaces with the replacement of 3 by $n + 1$.

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