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HIGHER ORDER BOUNDARY ESTIMATES FOR BLOW-UP SOLUTIONS OF ELLIPTIC EQUATIONS

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Abstract. We investigate blow-up solutions of the equation $\Delta u = u^p + g(u)$ in a bounded smooth domain Ω. If $p > 1$ and if g satisfies appropriate growth conditions (compared with the growth of t^p) as t goes to infinity we find optimal asymptotic estimates of the solution $u(x)$ in terms of the distance of x from the boundary $\partial\Omega$.

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain, and let $f(t)$ be a smooth function, increasing for $t > 0$, which satisfies $f(0) = 0$ and the Keller-Osserman condition

$$\int_1^\infty \frac{dt}{\sqrt{2F(t)}} < \infty, \qquad F(t) = \int_0^t f(\tau)d\tau.$$

It is well known [10,13] that under these conditions the Dirichlet problem

$$\Delta u = f(u) \ \text{ in } \Omega, \qquad u(x) \to \infty \text{ as } x \to \partial\Omega, \tag{1.1}$$

has a classical solution called a boundary blow-up (explosive, large) solution. Moreover, the one-dimensional problem

$$\Phi'' = f(\Phi), \quad \Phi(s) > 0, \quad \lim_{s\to 0} \Phi(s) = \infty$$

has a solution satisfying

$$\int_{\Phi(s)}^\infty \frac{dt}{\sqrt{2F(t)}} = s, \qquad F(t) = \int_0^t f(\tau)d\tau. \tag{1.2}$$

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C. Bandle and M. Marcus [6], in convex domains under the additional assumption that $F(t)/t^2$ is increasing for large t, have proved that any solution u to problem (1.1) satisfies

$$\left|u(x) - \Phi(\delta(x))\right| \leq c\delta(x)\Phi(\delta(x)), \tag{1.3}$$

where $\delta(x)$ is the distance of x from the boundary $\partial\Omega$ and c is a suitable constant. In the special case $f(t) = t^p$, $p > 1$, one finds $\Phi(\delta) = \phi(\delta)$, with

$$\phi(s) = (a_p s)^{\frac{2}{1-p}}, \qquad a_p = \frac{p-1}{\sqrt{2(p+1)}}. \tag{1.4}$$

Recently, C. Bandle [2] has investigated solutions to problem (1.1) with $f(t) = t^p$ and proved the expansion

$$u(x) = \phi(\delta(x))\Big[1 + \frac{(N-1)K(\overline{x})}{p+3}\delta(x) + o(\delta(x))\Big],$$

where $K(\overline{x})$ denotes the mean curvature of $\partial\Omega$ at the point $\overline{x}$ nearest to x, and $o(\delta)$ has the usual meaning. In [4], C. Bandle and M. Marcus have investigated problem (1.1) for a special class of functions $f(t)$ and found interesting boundary estimates. Higher order estimates in special cases have been found in [1,7,12]. Nonlinear equations more general than (1.1) have been discussed in several papers; see for example [3] and references therein.

A special result of the present paper is the following

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded smooth convex domain, and let $p > 1$ be a real number. A boundary blow-up solution of the equation $\Delta u = u^p$ in Ω satisfies the estimate*

$$u(x) = \phi(\delta)\Big(1 + \sum_{k=1}^{m} A_k\delta^k + O(1)\delta^\sigma\Big),$$

where m is the maximum integer satisfying $m < \frac{2(p+1)}{p-1}$, $A_k = A_k(x)$, $k = 1, \cdots, m$ depend on the geometry of Ω and on p, $O(1)$ is a bounded quantity, and σ is any real number such that $m < \sigma < \frac{2(p+1)}{p-1}$.

This theorem will be proved below after Theorem 2.2. More generally, we consider equation (1.1) with $f(t) = t^p + g(t)$, where $g(t)$ is smooth and grows less than t^p. As a special result we find

Theorem 1.2. *Let $u(x)$ be a boundary blow-up solution of the equation $\Delta u = u^p + u^q$ in Ω, with $p \geq 5$ and $0 < q < p$.*

i) If $0 < q < 1$, *then* $u(x)$ *satisfies the estimate*

$$u(x) = \phi(\delta)\Big(1 + A_1\delta + A_2\delta^2 + O(1)\delta^{\frac{2(p-q)}{p-1}}\Big)$$

with

$$A_1 = \frac{(N-1)K}{p+3}, \quad A_2 = \frac{9-p-2p^2}{12}A_1^2 + \frac{p-3}{6}\nabla A_1 \cdot \nabla\delta,$$

where $K = K(x)$ *is the mean curvature of the surface* $\{x \in \Omega : \delta(x) = constant\}$.

ii) If $1 \le q < (p+1)/2$, *then*

$$u(x) = \phi(\delta)\Big(1 + A_1\delta + O(1)\delta^{\frac{2(p-q)}{p-1}}\Big).$$

iii) If $(p+1)/2 \le q < p$, *then*

$$u(x) = \phi(\delta)\Big(1 + O(1)\delta^{\frac{2(p-q)}{p-1}}\Big).$$

This theorem will be proved after Theorem 2.3.

2. Boundary estimates

Let $p > 1$ be a real number and let $g(t) : [0, \infty) \to \mathbb{R}$ be a smooth function such that

$$g(0) = 0, \quad pt^{p-1} + g'(t) > 0, \qquad g(t) = O(1)t^q, \quad -1 < q < p, \tag{2.1}$$

where $O(1)$ denotes a bounded quantity. Under these assumptions the Keller-Osserman condition for $f(t) = t^p + g(t)$ holds and the first integral in (1.2) defines a function $\Phi = \Phi(s)$ with $\Phi(s) \to \infty$ as $s \to 0$.

Lemma 2.1. *Let* p, q *be real numbers with* $p > 1$ *and* $-1 < q < p$. *Let* $f(t) = t^p + g(t)$, *where* $g(t)$ *satisfies* (2.1). *If* $u(x)$ *is a solution to problem* (1.1), *in* Ω *convex, then*

$$u(x) = \phi(\delta(x)) + O(1)\big(\delta(x)\big)^{\frac{p-3}{p-1}} + O(1)\big(\delta(x)\big)^{\frac{2(p-q-1)}{p-1}},$$

where ϕ *is the function defined in* (1.4) *and* $O(1)$ *denotes a bounded quantity.*

Proof. Let us show first that

$$\Phi(s) = \phi(s)(1 + O(1)s^{\beta}), \qquad \beta = \frac{2(p-q)}{p-1}, \tag{2.2}$$

where $\Phi = \Phi(s)$ is the function defined in (1.2). Let $\rho(s) = (\Phi(s))^{\frac{1-p}{2}}$. Since $\Phi(s) \to \infty$ as $s \to 0$ we have $\rho(0) = 0$. By (1.2) we find

$$\Phi'(s) = -\big(2F(\Phi)\big)^{\frac{1}{2}}. \tag{2.3}$$

On the other side,

$$\rho' = \frac{1-p}{2}\Phi^{-\frac{p+1}{2}}\Phi'. \tag{2.4}$$

We have

$$F(t) = \frac{t^{p+1}}{p+1} + G(t), \quad \text{with} \quad G(t) = \int_0^t g(\tau)d\tau.$$

Hence, by (2.4) and (2.3) we obtain

$$\begin{aligned} \rho' &= \frac{p-1}{\sqrt{2}}\Phi^{-\frac{p+1}{2}}\Big[\frac{\Phi^{p+1}}{p+1} + G(\Phi)\Big]^{\frac{1}{2}} \\ &= \frac{p-1}{\sqrt{2(p+1)}}\Big[1 + (p+1)\Phi^{-p-1}G(\Phi)\Big]^{\frac{1}{2}}. \end{aligned} \tag{2.5}$$

Since $-1 < q$, (2.1) implies $G(t) = O(1)t^{1+q}$. Therefore,

$$\Phi^{-p-1}G(\Phi) = O(1)\Phi^{q-p} = O(1)\rho^{\beta}.$$

Insertion of this estimate into (2.5) yields

$$\rho' = a_p\big[1 + O(1)\rho^{\beta}\big]. \tag{2.6}$$

By (2.6) we have $\rho'(0) = a_p$. As a consequence, $\rho(s) = O(1)s$. Hence, by (2.6) we find

$$\rho'(s) = a_p\big[1 + O(1)s^{\beta}\big].$$

Integrating the last equation on $(0, s)$ we find

$$\rho(s) = a_p s\big[1 + O(1)s^{\beta}\big],$$

whence

$$\big(\rho(s)\big)^{\frac{2}{1-p}} = \big(a_p s\big)^{\frac{2}{1-p}}\big[1 + O(1)s^{\beta}\big].$$

Estimate (2.2) follows.

Since under our assumptions the function $f(t)$ is increasing for $t > 0$ and $F(t)t^{-2}$ is increasing for large t, we can use the following result of Bandle-Marcus [6]

$$u(x) = \Phi(\delta(x)) + O(1)\delta(x)\Phi(\delta(x)).$$

Insertion of (2.2) into the last equation leads to the assertion of the lemma.

Recall that $\delta = \delta(x)$ is the distance of x from $\partial\Omega$. If Ω is smooth, then also $\delta(x)$ is smooth for x near to $\partial\Omega$. We have [8]

$$\sum_{i=1}^{N} \delta_{x_i}\delta_{x_i} = 1, \qquad \sum_{i=1}^{N} \delta_{x_i x_i} = -(N-1)K, \tag{2.7}$$

where $K = K(x)$ is the mean curvature of the surface $\{x \in \Omega : \delta(x) = \mathit{constant}\}$. Define

$$H = (N-1)K, \quad A_1 = \frac{H}{p+3}, \quad A_2 = \frac{9-p-2p^2}{12}A_1^2 + \frac{p-3}{6}\nabla A_1 \cdot \nabla\delta. \tag{2.8}$$

Let m be the maximum integer such that $m < 2(p+1)/(p-1)$. If $m > 2$ we define $A_3, \cdots, A_m$ as follows. Let $C_k = C_k(A_1, \cdots, A_k)$ such that the following identity holds for all t

$$\sum_{h=1}^{m} \binom{p}{h} (A_1 t + \cdots + A_m t^m)^h = \sum_{k=1}^{m} C_k t^k + O(1)t^{m+1}, \tag{2.9}$$

where

$$\binom{p}{h} = \frac{p(p-1)\cdots(p-h+1)}{h!}.$$

One finds

$$C_1 = pA_1, \quad C_k = pA_k + P_k(A_1, \cdots, A_{k-1}), \quad k = 2, \cdots, m,$$

where $P_k(A_1, \cdots, A_{k-1})$ are suitable polynomials. For example,

$$P_2(A_1) = \frac{p(p-1)}{2}A_1^2, \quad P_3(A_1, A_2) = p(p-1)A_1A_2 + \frac{p(p-1)(p-2)}{6}A_1^3.$$

Using these polynomials, for $3 \le k \le m$ we define A_k in terms of $A_1, \ldots, A_{k-1}$ according to the equation

$$\begin{aligned} &\frac{(p-1)^2}{2(p+1)}(k+1)\Big(k - 2\frac{p+1}{p-1}\Big)A_k + \Big(\frac{p-1}{p+1} - \frac{(p-1)^2}{2(p+1)}(k-1)\Big)HA_{k-1} \\ &+ \Big(2\frac{1-p}{p+1} + \frac{(p-1)^2}{p+1}(k-1)\Big)\nabla A_{k-1}\cdot\nabla\delta + \frac{(p-1)^2}{2(p+1)}\Delta A_{k-2} \\ &\quad = P_k(A_1, \cdots, A_{k-1}). \end{aligned} \tag{2.10}$$

Since for $k \le m$ we have $k < 2(p+1)/(p-1)$, (2.10) defines recursively A_k, $k = 3, \cdots, m$. Of course, we do not need (2.10) when $m = 2$.

In the case that $1 < p \le 3$ or $p - 1 \le q < p$ we use the additional assumption (cf. [5]):

$$\begin{aligned}&\text{there exist} \quad \theta_0 < 1,\ t_0 > 1 \quad \text{such that}\\ &\forall \theta \in (\theta_0, 1), \quad \forall\ t > t_0 \ \text{ we have } \ \theta f(t) > f(\theta t).\end{aligned} \tag{2.11}$$

This condition holds, for example, if $f(t) = t^p + A\, t^q$, with A constant and $q < p$.

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded smooth convex domain, let $p > 1$ be a real number and let m be the maximum integer such that $m < 2(p+1)/(p-1)$. Let $g(t)$ be a smooth function which satisfies* (2.1) *with $-1 < q < 1 + (m-2)(1-p)/2$. If $1 < p \leq 3$ or $p - 1 \leq q < p$, suppose $f(t) = t^p + g(t)$ satisfies* (2.11)*. If $u(x)$ is a solution of problem* (1.1) *with $f(u) = u^p + g(u)$, then*

$$u(x) = \phi(\delta)\Big(1 + \sum_{k=1}^{m} A_k \delta^k + O(1)\delta^\beta\Big), \qquad \beta = \frac{2(p-q)}{p-1},$$

where $\phi(\delta)$ is defined by (1.4) *and A_k are defined as in* (2.8) *and* (2.10)*.*

Proof. We look for a super-solution of the form

$$w(x) = \phi(\delta)\Big(1 + \sum_{k=1}^{m} A_k \delta^k + \alpha \delta^\sigma\Big),$$

where $m < \sigma < 2(p+1)/(p-1)$ and α is a positive constant to be determined. We have

$$\begin{aligned} w_{x_i} &= \phi' \delta_{x_i}\Big(1 + \sum_{k=1}^{m} A_k \delta^k + \alpha \delta^\sigma\Big) \\ &\quad + \phi\Big(\sum_{k=1}^{m} (A_k)_{x_i} \delta^k + \sum_{k=1}^{m} A_k k \delta^{k-1} \delta_{x_i} + \alpha \sigma \delta^{\sigma-1} \delta_{x_i}\Big). \end{aligned}$$

Recalling that $(N-1)K = H$ we have

$$\sum_{i=1}^{N} \delta_{x_i x_i} = -H, \qquad \sum_{i=1}^{N} \delta_{x_i} \delta_{x_i} = 1.$$

Hence, we find

$$\Delta w = \phi''\Big(1 + \sum_{k=1}^{m} A_k \delta^k + \alpha \delta^\sigma\Big) - \phi'\Big(H + \sum_{k=1}^{m} H A_k \delta^k + \alpha H \delta^\sigma\Big)$$

$$+2\phi'\Big(\sum_{k=1}^{m}\nabla A_k\cdot\nabla\delta\,\delta^k+\sum_{k=1}^{m}kA_k\delta^{k-1}+\alpha\sigma\delta^{\sigma-1}\Big)$$

$$+\phi\Big(\sum_{k=1}^{m}\Delta A_k\,\delta^k+\sum_{k=1}^{m}2k\nabla A_k\cdot\nabla\delta\,\delta^{k-1}+\sum_{k=1}^{m}k(k-1)A_k\delta^{k-2}-\sum_{k=1}^{m}kA_kH\delta^{k-1}\Big)$$

$$+\phi\Big(\alpha\sigma(\sigma-1)\delta^{\sigma-2}-\alpha\sigma H\delta^{\sigma-1}\Big). \tag{2.12}$$

By (1.4) we find

$$\phi''=\phi^p,\quad \phi'=\phi''\frac{1-p}{p+1}s,\quad \phi=\phi''\frac{(p-1)^2}{2(p+1)}s^2.$$

Using the last equations, by (2.12) we obtain

$$\Delta w=\phi^p\Big[1+\sum_{k=1}^{m}A_k\delta^k+\alpha\delta^\sigma+\frac{p-1}{p+1}\Big(H\delta+\sum_{k=1}^{m}HA_k\delta^{k+1}+\alpha H\delta^{\sigma+1}\Big)$$
$$+2\frac{1-p}{p+1}\Big(\sum_{k=1}^{m}\nabla A_k\cdot\nabla\delta\,\delta^{k+1}+\sum_{k=1}^{m}kA_k\delta^k+\alpha\sigma\delta^\sigma\Big)$$
$$+\frac{(p-1)^2}{2(p+1)}\Big(\sum_{k=1}^{m}\Delta A_k\,\delta^{k+2}+\sum_{k=1}^{m}2k\nabla A_k\cdot\nabla\delta\,\delta^{k+1}+\sum_{k=1}^{m}k(k-1)A_k\delta^k$$
$$-\sum_{k=1}^{m}kA_kH\delta^{k+1}\Big)+\frac{(p-1)^2}{2(p+1)}\Big(\alpha\sigma(\sigma-1)\delta^\sigma-\alpha\sigma H\delta^{\sigma+1}\Big)\Big].$$

Denoting by $M_i,\ i=1,2,\cdots$ nonnegative constants independent of α we find

$$\Delta w<\phi^p\Big\{1+\delta\Big[A_1+\frac{p-1}{p+1}H+2\frac{1-p}{p+1}A_1\Big]$$
$$+\delta^2\Big[\Big(1+4\frac{1-p}{p+1}+\frac{(p-1)^2}{p+1}\Big)A_2+\Big(\frac{p-1}{p+1}-\frac{(p-1)^2}{2(p+1)}\Big)HA_1$$
$$+\Big(2\frac{1-p}{p+1}+\frac{(p-1)^2}{p+1}\Big)\nabla A_1\cdot\nabla\delta\Big]$$
$$+\sum_{k=3}^{m}\delta^k\Big[\Big(1+2\frac{1-p}{p+1}k+\frac{(p-1)^2}{2(p+1)}k(k-1)\Big)A_k+\Big(\frac{p-1}{p+1}-\frac{(p-1)^2}{2(p+1)}(k-1)\Big)HA_{k-1}$$
$$+\Big(2\frac{1-p}{p+1}+\frac{(p-1)^2}{p+1}(k-1)\Big)\nabla A_{k-1}\cdot\nabla\delta+\frac{(p-1)^2}{2(p+1)}\Delta A_{k-2}\Big]$$
$$+\alpha\delta^\sigma\Big[1+2\frac{1-p}{p+1}\sigma+\frac{(p-1)^2}{2(p+1)}\sigma(\sigma-1)+M_1\delta\Big]+M_2\delta^{m+1}\Big\}. \tag{2.13}$$

We will take α and δ_0 so that, for $\{x \in \Omega : \delta(x) < \delta_0\}$,

$$-\frac{1}{2} < \sum_{k=1}^{m} A_k \delta^k + \alpha\delta^\sigma < 1, \quad \alpha\delta^\sigma < 1. \tag{2.14}$$

Then, using Taylor's expansion we have

$$w^p = \phi^p \Big(1 + \sum_{k=1}^{m} A_k \delta^k + \alpha\delta^\sigma\Big)^p$$

$$> \phi^p \Big\{1 + \sum_{h=1}^{m} \binom{p}{h} \Big(\sum_{k=1}^{m} A_k \delta^k + \alpha\delta^\sigma\Big)^h - M_3 \Big|\sum_{k=1}^{m} A_k \delta^k + \alpha\delta^\sigma\Big|^{m+1}\Big\}.$$

Using (2.1) we have $g(w)\phi^{-p} \geq -M_4\phi^{q-p} = -M_5 s^\beta$. Note that in the case of $g(t) \geq 0$ we can take $M_5 = 0$. Using this estimate and the previous inequality we find

$$w^p + g(w) > \phi^p \Big\{1 + p\Big(\sum_{k=1}^{m} A_k \delta^k + \alpha\delta^\sigma\Big) + \sum_{h=2}^{m} \binom{p}{h} \Big(\sum_{k=1}^{m} A_k \delta^k + \alpha\delta^\sigma\Big)^h$$

$$- M_3 \Big|\sum_{k=1}^{m} A_k \delta^k + \alpha\delta^\sigma\Big|^{m+1} - M_5 \delta^\beta\Big\}.$$

For $\alpha\delta^\sigma < 1$ and $h \geq 2$ we have

$$\Big(\sum_{k=1}^{m} A_k \delta^k + \alpha\delta^\sigma\Big)^h = \Big(\sum_{k=1}^{m} A_k \delta^k\Big)^h + O(1)\delta\alpha\delta^\sigma + O(1)\big(\alpha\delta^\sigma\big)^2,$$

and

$$\Big|\sum_{k=1}^{m} A_k \delta^k + \alpha\delta^\sigma\Big|^{m+1} = O(1)\delta^{m+1} + O(1)\delta\alpha\delta^\sigma + O(1)(\alpha\delta^\sigma)^2.$$

Hence,

$$w^p + g(w) > \phi^p \Big\{1 + \sum_{h=1}^{m} \binom{p}{h} \Big(\sum_{k=1}^{m} A_k \delta^k\Big)^h + p\alpha\delta^\sigma$$

$$- M_6 \delta\alpha\delta^\sigma - M_7 \big(\alpha\delta^\sigma\big)^2 - M_8 \delta^{m+1} - M_5 \delta^\beta\Big\}.$$

Using the quantities C_k defined according to (2.9) we get

$$w^p + g(w) > \phi^p \Big\{1 + \sum_{k=1}^{m} C_k \delta^k + p\alpha\delta^\sigma - M_6 \delta\alpha\delta^\sigma$$

$$-M_7\big(\alpha\delta^\sigma\big)^2 - M_9\delta^{m+1} - M_5\delta^\beta\Big\}. \tag{2.15}$$

Using (2.8) and the equations

$$C_1 = pA_1, \quad C_2 = pA_2 + \frac{p(p-1)}{2}A_1^2$$

we find

$$A_1 + \frac{p-1}{p+1}H + 2\frac{1-p}{p+1}A_1 = C_1, \tag{2.16}$$

and

$$\Big(1 + 4\frac{1-p}{p+1} + \frac{(p-1)^2}{p+1}\Big)A_2 + \Big(\frac{p-1}{p+1} - \frac{(p-1)^2}{2(p+1)}\Big)HA_1$$
$$+\Big(2\frac{1-p}{p+1} + \frac{(p-1)^2}{p+1}\Big)\nabla A_1 \cdot \nabla\delta = C_2. \tag{2.17}$$

Using (2.10) and recalling that $pA_k + P_k(A_1, \cdots, A_{k-1}) = C_k$, after some computation we get

$$\Big(1 + 2\frac{1-p}{p+1}k + \frac{(p-1)^2}{2(p+1)}k(k-1)\Big)A_k + \Big(\frac{p-1}{p+1} - \frac{(p-1)^2}{2(p+1)}(k-1)\Big)HA_{k-1}$$
$$+\Big(2\frac{1-p}{p+1} + \frac{(p-1)^2}{p+1}(k-1)\Big)\nabla A_{k-1}\cdot\nabla\delta + \frac{(p-1)^2}{2(p+1)}\Delta A_{k-2} = C_k,\ k = 3, \ldots, m. \tag{2.18}$$

By (2.13) and (2.15), using (2.16), (2.17) and (2.18) we find that

$$\Delta w < w^p + g(w) \tag{2.19}$$

when

$$\alpha\delta^\sigma\Big(1 + 2\frac{1-p}{p+1}\sigma + \frac{(p-1)^2}{2(p+1)}\sigma(\sigma-1) + M_1\delta\Big) + M_2\delta^{m+1}$$
$$< p\alpha\delta^\sigma - M_6\delta\alpha\delta^\sigma - M_7\big(\alpha\delta^\sigma\big)^2 - M_9\delta^{m+1} - M_5\delta^\beta.$$

Rearranging we find

$$(M_2 + M_9)\delta^{m+1} + M_5\delta^\beta$$
$$< \alpha\delta^\sigma\Big[\frac{(p-1)^2}{2(p+1)}(\sigma+1)\Big(\frac{2(p+1)}{p-1} - \sigma\Big) - (M_1 + M_6)\delta - M_7\alpha\delta^\sigma\Big]. \tag{2.20}$$

Since m is the maximum integer satisfying $m < 2(p+1)/(p-1)$, we have $m+1 \geq 2(p+1)/(p-1)$. Moreover, since $-1 < q$, we have $2(p+1)/(p-1) > \beta$. Hence, $m+1 > \beta$. Therefore, (2.20) holds with $\sigma = \beta$ when

$$M_2 + M_9 + M_5 < \alpha\Big[\frac{\big(2(p-q)+p-1\big)(q+1)}{p+1} - (M_1+M_6)\delta - M_7\alpha\delta^{\beta}\Big]. \tag{2.21}$$

We can take α and δ_0 so that (2.21) and (2.14) (with $\sigma = \beta$) hold for $\delta(x) < \delta_0$.

We show now that we can choose α and δ_0 so that $u(x) < w(x)$ for $\delta(x) = \delta_0$. By Lemma 2.1 we have

$$\frac{u(x)}{\phi(\delta(x))} = 1 + O(1)\delta + O(1)\delta^{\frac{2(p-q)}{p-1}}.$$

It follows that

$$\lim_{x\to\partial\Omega} \frac{\phi(\delta(x))}{u(x)} = 1.$$

Let $r = \alpha\delta_0^{\beta}$, where α and δ_0 are as in above. Decrease δ_0 (increasing α according to $\alpha\delta_0^{\beta} = r$) until

$$\frac{\phi(\delta(x))}{u(x)} > \frac{2}{2+r}$$

for $\delta(x) \leq \delta_0$. Multiplying by

$$1 + \sum_{k=1}^{m} A_k\delta^k + \alpha\delta^{\beta}$$

we have

$$\frac{w(x)}{u(x)} > \frac{2}{2+r}\Big(1 + \sum_{k=1}^{m} A_k\delta^k + \alpha\delta^{\beta}\Big).$$

Decrease δ_0 again (and increase α) in order to have $\sum_{k=1}^{m} A_k\delta_0^k > -r/2$ and $\alpha\delta_0^{\beta} = r$. Then $w(x) > u(x)$ for $\delta(x) = \delta_0$. If $p > 3$, since $q < 1$ we have $q < p-1$. Hence, by using Lemma 2.1 we find that $w(x) - u(x)$ approaches zero as x approaches $\partial\Omega$. By (2.19) it follows that $w(x) \geq u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$. If $1 < p \leq 3$ or $p-1 \leq q < p$ we use condition (2.11) and a trick taken by [5]. Let t_0 and θ_0 be the constants of condition (2.11). Decrease δ_0 (and increase α according to $\alpha\delta_0^{\beta} = r$) in order to have $u(x) > t_0$ for $\delta(x) < \delta_0$. For $\theta \in (\theta_0, 1)$ we have (trivially) $w(x) > \theta u(x)$ on

$\{x \in \Omega : \delta(x) = \delta_0\}$. On the other side, since $\phi(\delta(x))/u(x) \to 1$ as $x \to \partial\Omega$, we find that $w(x) > \theta u(x)$ near $\partial\Omega$. Moreover, using (2.11) we find

$$\Delta(\theta u) = \theta f(u) > f(\theta u). \tag{2.22}$$

By (2.19) and (2.22) it follows that $w(x) \geq \theta u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$. As $\theta \to 1$ we find that $w(x) \geq u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$.

We look for a sub-solution of the kind

$$v(x) = \phi(\delta)\Big(1 + \sum_{k=1}^{m} A_k \delta^k - \alpha\delta^\sigma\Big),$$

where m, A_k and σ are the same as before and α is a positive constant to be determined. Arguing as in the previous case we find the following inequality (similar to (2.13) with $-\alpha$ in place of α)

$$\begin{aligned}\Delta v > \phi^p\Big\{&1 + \delta\Big[A_1 + \frac{p-1}{p+1}H + 2\frac{1-p}{p+1}A_1\Big]\\
&+\delta^2\Big[\Big(1 + 4\frac{1-p}{p+1} + \frac{(p-1)^2}{p+1}\Big)A_2 + \Big(\frac{p-1}{p+1} - \frac{(p-1)^2}{2(p+1)}\Big)HA_1\\
&\qquad + \Big(2\frac{1-p}{p+1} + \frac{(p-1)^2}{p+1}\Big)\nabla A_1 \cdot \nabla\delta\Big]\\
&+\sum_{k=3}^{m}\delta^k\Big[\Big(1+2\frac{1-p}{p+1}k+\frac{(p-1)^2}{2(p+1)}k(k-1)\Big)A_k+\Big(\frac{p-1}{p+1}-\frac{(p-1)^2}{2(p+1)}(k-1)\Big)HA_{k-1}\\
&\qquad + \Big(2\frac{1-p}{p+1} + \frac{(p-1)^2}{p+1}(k-1)\Big)\nabla A_{k-1} \cdot \nabla\delta + \frac{(p-1)^2}{2(p+1)}\Delta A_{k-2}\Big]\\
&-\alpha\delta^\sigma\Big[1 + 2\frac{1-p}{p+1}\sigma + \frac{(p-1)^2}{2(p+1)}\sigma(\sigma-1) + M_1\delta\Big] - M_2\delta^{m+1}\Big\}.\end{aligned} \tag{2.23}$$

Now we take α and δ_0 so that, for $\{x \in \Omega : \delta(x) < \delta_0\}$,

$$-\frac{1}{2} < \sum_{k=1}^{m} A_k\delta^k - \alpha\delta^\sigma < 1, \quad \alpha\delta^\sigma < 1. \tag{2.24}$$

Using Taylor's expansion we find

$$\begin{aligned}v^p &= \phi^p\big(1 + \sum_{k=1}^{m} A_k\delta^k - \alpha\delta^\sigma\big)^p\\
&< \phi^p\Big\{1 + \sum_{h=1}^{m}\binom{p}{h}\Big(\sum_{k=1}^{m} A_k\delta^k - \alpha\delta^\sigma\Big)^h + M_3\Big|\sum_{k=1}^{m} A_k\delta^k + \alpha\delta^\sigma\Big|^{m+1}\Big\}.\end{aligned}$$

By (2.1) we have $g(w)\phi^{-p} \le M_4\phi^{q-p} = M_5 s^\beta$. Note that in the case of $g(t) \le 0$ we can take $M_5 = 0$. Using this estimate we find

$$v^p + g(v) < \phi^p \Big\{ 1 + \sum_{h=1}^m \binom{p}{h} \Big(\sum_{k=1}^m A_k \delta^k + \alpha\delta^\sigma \Big)^h + M_3 \Big| \sum_{k=1}^m A_k \delta^k + \alpha\delta^\sigma \Big|^{m+1} + M_5 \delta^\beta \Big\}.$$

Since

$$\Big| \sum_{k=1}^m A_k \delta^k - \alpha\delta^\sigma \Big|^{m+1} = O(1)\delta^{m+1} + O(1)\delta\alpha\delta^\sigma + O(1)\big(\alpha\delta^\sigma\big)^2,$$

we find an inequality similar to (2.15), that is,

$$v^p + g(v) < \phi^p \Big\{ 1 + \sum_{k=1}^m C_k \delta^k - p\alpha\delta^\sigma + M_6\delta\alpha\delta^\sigma + M_7\big(\alpha\delta^\sigma\big)^2 + M_9\delta^{m+1} + M_5\delta^\beta \Big\}, \tag{2.25}$$

where C_k are defined according to (2.9). Recalling (2.16), (2.17) and (2.18), from (2.23) and (2.25) we find that

$$\Delta v > v^p + g(v) \tag{2.26}$$

when

$$-\alpha\delta^\sigma \Big(1 + 2\frac{1-p}{p+1}\sigma + \frac{(p-1)^2}{2(p+1)}\sigma(\sigma-1) + M_1\delta \Big) - M_2\delta^{m+1} > -p\alpha\delta^\sigma + M_6\delta\alpha\delta^\sigma + M_7\big(\alpha\delta^\sigma\big)^2 + M_9\delta^{m+1} + M_5\delta^\beta.$$

Rearranging we find

$$(M_2 + M_9)\delta^{m+1} + M_5\delta^\beta < \alpha\delta^\sigma \Big[\frac{(p-1)^2}{2(p+1)}(\sigma+1)\Big(\frac{2(p+1)}{p-1} - \sigma \Big) - (M_1 + M_6)\delta - M_7\alpha\delta^\sigma \Big], \tag{2.27}$$

which is similar to (2.20). Note that the constants M_i of (2.27) may be different from the constants M_i used in (2.20). However, inequality (2.27) holds with $\sigma = \beta$ when

$$M_2 + M_9 + M_5 < \alpha \Big[\frac{\big(2(p-q) + p - 1\big)(q+1)}{p+1} - (M_1 + M_6)\delta - M_7\alpha\delta^\beta \Big]. \tag{2.28}$$

We can take α large and δ_0 small so that (2.28) and (2.24) (with $\sigma = \beta$) hold for $\delta(x) < \delta_0$.

Let $r = \alpha\delta_0^\beta$ with α and δ_0 as above. Since

$$\lim_{x\to\partial\Omega} \frac{\phi(\delta(x))}{u(x)} = 1,$$

we can decrease δ_0 (increasing α according to $\alpha\delta_0^\beta = r$) until

$$\frac{\phi(\delta(x))}{u(x)} < \frac{2}{2-r}$$

for $\delta(x) \le \delta_0$. Multiplying by

$$1 + \sum_{k=1}^m A_k\delta^k - \alpha\delta^\beta$$

we have

$$\frac{v(x)}{u(x)} < \frac{2}{2-r}\big(1 + \sum_{k=1}^m A_k\delta^k - \alpha\delta^\beta\big).$$

Decrease δ_0 again (and increase α) in order to have $\sum_1^m A_k\delta_0^k < r/2$ and $\alpha\delta_0^\beta = r$. Then $v(x) < u(x)$ for $\delta(x) = \delta_0$. If $p > 3$, using Lemma 2.1 we find that $v(x) - u(x)$ approaches zero as x approaches $\partial\Omega$. By (2.26) it follows that $v(x) \le u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$. If $1 < p \le 3$ or $p - 1 \le q < p$ we use condition (2.11). We may assume that $u(x) > t_0$ for $\delta(x) < \delta_0$. For $\theta \in (\theta_0, 1)$ we have (trivially) $v(x) < u(x)/\theta$ on $\{x \in \Omega : \delta(x) = \delta_0\}$. On the other side, since $\phi(\delta(x))/u(x) \to 1$ as $x \to \partial\Omega$, we find that $v(x) < u(x)/\theta$ near $\partial\Omega$. Moreover, using (2.11) we find

$$\Delta(u/\theta) = f(u)/\theta > f(u/\theta). \tag{2.29}$$

By (2.26) and (2.29) it follows that $v(x) \le u(x)/\theta$ on $\{x \in \Omega : \delta(x) < \delta_0\}$. As $\theta \to 1$ we find that $v(x) \le u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$.

Therefore, we have

$$u(x) = \phi(\delta(x))\Big(1 + \sum_{k=1}^m A_k\delta^k + O(1)\delta^\beta\Big),$$

which is the assertion of the theorem.

Proof of Theorem 1.1. We use notation and computations of the previous theorem. In the special case of $g \equiv 0$, we find that $\Delta w < w^p$ when (2.20) holds with $M_5 = 0$. Therefore, now we have

$$(M_2+M_9)\delta^{m+1-\sigma} < \alpha\Big[\frac{(p-1)^2}{2(p+1)}(\sigma+1)\Big(\frac{2(p+1)}{p-1}-\sigma\Big)-(M_1+M_6)\delta-M_7\alpha\delta^\sigma\Big]. \tag{2.30}$$

For $m < \sigma < 2(p+1)/(p-1)$, since $m+1 > \sigma$, this inequality holds with α fixed and δ small enough. We can increase α and decrease δ_0 so that (2.30) holds for $\delta(x) < \delta_0$, and $w(x) > u(x)$ for $\delta(x) = \delta_0$. On the other side, by Lemma 2.1 with $g = 0$ we find

$$u(x) - \phi(\delta(x)) = O(1)\big(\delta(x)\big)^{\frac{p-3}{p-1}}.$$

Hence, if $p > 3$, then $u(x) - \phi(\delta(x)) \to 0$ as $x \to \partial\Omega$. Therefore, we must have $w(x) \geq u(x)$ in $\{x \in \Omega : \delta(x) < \delta_0\}$. If $1 < p \leq 3$ we argue as in [5]. By (1.3) we have $\phi(\delta(x))/u(x) \to 1$ as $x \to \partial\Omega$. Hence, for $0 < \theta < 1$ we have $w(x) > \theta u(x)$ for x near $\partial\Omega$. Moreover, $\Delta(\theta u) = \theta u^p > (\theta u)^p$. This inequality together with $\Delta w < w^p$ implies that $w(x) \geq \theta u(x)$ in $\{x \in \Omega : \delta(x) < \delta_0\}$. As $\theta \to 1$ we find that $w(x) \geq u(x)$.

In proving $\Delta v > v^p$ we arrive at (2.27) with $M_5 = 0$. Arguing as in the previous case we find that $v(x) \leq u(x)$ in $\{x \in \Omega : \delta(x) < \delta_0\}$. The statement of the theorem follows.

Remark. Recall that $q < 1 + (m-2)(1-p)/2$ if and only if $m < \beta$. If the perturbation g satisfies (2.1) with $1 + (m-2)(1-p)/2 \leq q < p$, then some terms of the sum $\sum_{k=1}^{m} A_k \delta^k$ change. Indeed, we have

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be a bounded smooth convex domain, let $p > 1$ be a real number and let m be the maximum integer such that $m < 2(p+1)/(p-1)$. If $\beta = 2(p-q)/(p-1) \leq m$, let n be the integer such that*

$$n < \beta \leq n+1.$$

We have $0 \leq n \leq m-1$. Let $g(t)$ be a smooth function which satisfies (2.1)*. If $1 < p \leq 3$ or $p - 1 \leq q < p$, suppose $f(t) = t^p + g(t)$ satisfies* (2.11)*. If $u(x)$ is a solution to problem* (1.1) *with $f(u) = u^p + g(u)$, then if $n \geq 1$ we have*

$$u(x) = \phi(\delta)\Big(1 + \sum_{k=1}^{n} A_k \delta^k + O(1)\delta^\beta\Big), \qquad \beta = \frac{2(p-q)}{p-1},$$

where ϕ is defined by (1.4)*, and A_k, $k = 1, \cdots, n$, are defined as in* (2.8) *and* (2.10)*; if $n = 0$, then*

$$u(x) = \phi(\delta)\big(1 + O(1)\delta^\beta\big).$$

Proof. Consider first $n \geq 1$. Put

$$w(x) = \phi(\delta)\Big(1 + \sum_{k=1}^{n} A_k \delta^k + \alpha\delta^\sigma\Big).$$

By the same computation of the proof of Theorem 2.2 we find that (cf. (2.20) with n in place of m)

$$\Delta w < w^p + g(w) \tag{2.31}$$

when

$$(M_2 + M_9)\delta^{n+1} + M_5\delta^\beta$$
$$< \alpha\delta^\sigma\Big[\frac{(p-1)^2}{2(p+1)}(\sigma+1)\Big(\frac{2(p+1)}{p-1} - \sigma\Big) - (M_1 + M_6)\delta - M_7\alpha\delta^\sigma\Big].$$

Since $\beta \le n+1$, the last inequality holds with $\sigma = \beta$ when

$$M_2 + M_9 + M_5 < \alpha\Big[\frac{\big(2(p-q) + p - 1\big)(q+1)}{p+1} - (M_1 + M_6)\delta - M_7\alpha\delta^\beta\Big]. \tag{2.32}$$

Hence, we can take δ_0 small and α large such that, for $\delta < \delta_0$, inequality (2.32) holds. As in the proof of Theorem 2.2, we can decrease δ_0 and increase α so that $w(x) > u(x)$ for $\delta(x) = \delta_0$. If $p > 3$, since $1 < 2(p-q)/(p-1)$, we have $q < p-1$. Hence, by using Lemma 2.1 we find that $w(x) - u(x)$ approaches zero as x approaches $\partial\Omega$. By (2.31) it follows that $w(x) \ge u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$. If $1 < p \le 3$ or $p - 1 \le q < p$, using condition (2.11) we arrive at the same conclusion.

By the same argument one proves that

$$v(x) = \phi(\delta)\Big(1 + \sum_{k=1}^{n} A_k\delta^k - \alpha\delta^\sigma\Big)$$

with α and δ_0 suitable is a sub-solution on $\{x \in \Omega : \delta(x) < \delta_0\}$. The theorem follows when $n \ge 1$.

If $n = 0$ we look for a super-solution of the form

$$w(x) = \phi(\delta)\big(1 + \alpha\delta^\beta\big).$$

The proof is very similar to the previous case but easier, because it does not make use of (2.16), (2.17), (2.18). We find that

$$\Delta w < w^p + g(w)$$

whenever (2.32) holds. We can take δ_0 and α so that (2.32) holds for $\delta(x) < \delta_0$ and $w(x) > u(x)$ for $\delta(x) = \delta_0$. If $p > 3$ and $q < p-1$ we use Lemma 2.1 to prove that $w(x) - u(x) \to 0$ as $x \to \partial\Omega$. We conclude that $w(x) \ge u(x)$ on $\{x \in \Omega : \delta(x) < \delta_0\}$. If $1 < p \le 3$ or $q \ge p-1$, we use condition (2.11) to get the same conclusion.

By the same argument one proves that

$$v(x) = \phi(\delta)\big(1 - \alpha\delta^\beta\big)$$

is a sub-solution on $\{x \in \Omega : \delta(x) < \delta_0\}$. The theorem follows.

Proof of Theorem 1.2. The function $f(t) = t^p + t^q$ with $0 < q < p$ satisfies conditions (2.1) and (2.11), therefore we can use Theorem 2.2 and Theorem 2.3. Since now $p \geq 5$ we have $m = 2$, and assertion (i) follows by Theorem 2.1. Using the notation of Theorem 2.3, we have either $n = 1$ or $n = 0$, and assertions (ii) and (iii) follow from this theorem.

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