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ON EXTENSIONS OF HIGHER-ORDER HARDY'S INEQUALITIES

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Abstract. We describe some extensions of Hardy's inequalities induced by higher-order linear differential operators with singular potentials.

1. Introduction: Hardy's inequalities and extensions

1.1. **Hardy's inequalities.** Let $B = \{|x| < 1\}$ be the unit ball in $\mathbf{R}^N$ and $\langle \cdot, \cdot \rangle$ be the inner product in $L^2 = L^2(B)$ with the induced norm $\| \cdot \|$. Hardy's inequalities [17, 18] of higher order are well known in the mathematical literature and have various applications in PDE theory, see [10, 11, 18, 21, 22, 24, 26]. The $l^{\rm th}$-order Hardy inequality in dimension $N > 2l$ has the form

$$l \text{ is even:} \quad \int_B |\Delta^{l/2} w|^2 dx \geq \alpha_l \int_B \frac{w^2}{|x|^{2l}} \, {\rm d}x \quad \text{for } w \in H_0^l, \tag{1.1}$$

$$l \text{ is odd:} \quad \int_B |\nabla \Delta^{(l-1)/2} w|^2 {\rm d}x \geq \alpha_l \int_B \frac{w^2}{|x|^{2l}} \, {\rm d}x \quad \text{for } w \in H_0^l, \tag{1.2}$$

where $\alpha_l = \alpha_l(N) > 0$ is the best constant (for convenience, we present a short derivation of α_l in Section 2). Notice that, for $l = 2$, the inequality was derived by Rellich more than fifty years ago; see references in [16]. The spaces H_0^l consist of H^l functions for which $w = \partial w / \partial \nu = ... = \partial^k w / \partial \nu^k = 0$ on the boundary $S = \partial B$, where $k = (l-2)/2$ if l is even and $k = (l-1)/2$

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if l is odd. The equality in (1.1), (1.2) is asymptotically attained on H_0^l-approximations of the function

$$\psi(x) = |x|^{-(N-2l)/2}. \tag{1.3}$$

For any $l \geq 1$, the Hardy inequality is associated with the symmetric $2l$-th order linear elliptic operator with scaling invariant singular potential

$$\mathbf{B} = -(-\Delta)^l + \tfrac{\alpha_l}{|x|^{2l}}\, I. \tag{1.4}$$

There exist various approaches to improving and extending Hardy's inequalities. We refer to the first results established more than twenty years ago, [29], [23, page 98] (see also [27] about earlier related estimates), and more recent extensions in [2, 3, 5, 8, 14, 16, 19, 30]. Most of these papers are devoted to the second-order case $l = 1$. For the higher-order cases, some extensions of Hardy's inequalities admitting optimal characterization of the best constants are obtained in [16]. Other generalizations were constructed in [15] for $l = 1, 2, 3$ by using the classical theory [25] of self-adjoint extensions of singular ordinary differential operators (1.4) depending on the radial variable $r = |x| \in (0, 1)$.

In the present paper we show that a general extension scheme from [15, Section 1.3] based on using auxiliary positive multiplication operators can be applied for arbitrary l. The radial setting makes it possible to detect the origin of the *best constants* in such inequalities which lies in the theory of self-adjoint extensions of symmetric higher-order differential operators. Generalizations of some of the inequalities to the elliptic setting in B and for bounded smooth domains in $\mathbf{R}^N$ is then done by eigenfunction expansions and by Schwartz symmetrization techniques [21]; see details below.

2. Best constant in higher-order Hardy's inequality

Lemma 2.1. *Let $l \geq 2$ and $N > 2l$. The best constants in* (1.1) *and* (1.2) *are*

$$l \text{ is even: } \alpha_l = B_2 B_4 ... B_l, \quad \text{and} \quad l \text{ is odd: } \alpha_l = B_3 B_5 ... B_l \alpha_1, \tag{2.1}$$

where $B_k = [\frac{1}{4}(N-2k)(N+2k-4)]^2$ for $k = 1, 2, ..., l$ and $\alpha_1 = \frac{1}{4}(N-2)^2$.

Proof. It is enough to prove the result for functions $w \in C^{l,\alpha}(\bar{B})$ vanishing on ∂B with derivatives of order l which form a dense subset in H_0^l. Given a small $\varepsilon > 0$, denote $B_\varepsilon = \{\varepsilon < |x| < 1\}$ and $S_\varepsilon = \{|x| = \varepsilon\}$.

Noting that $r^{-l} \in L^2$ for $N > 2l$, since $w/r^{l-1} \in H_0^1$, by the first Hardy inequality

$$\alpha_1 \int \frac{w^2}{r^{2l}} \le \int |\nabla(\frac{w}{r^{l-1}})|^2. \tag{2.2}$$

Integrating by parts yields

$$\int_{B_\varepsilon} |\nabla(\frac{w}{r^{l-1}})|^2 \equiv \int_{B_\varepsilon} \nabla(\frac{w}{r^{l-1}}) \cdot \nabla(\frac{w}{r^{l-1}})$$
$$= -\int_{B_\varepsilon} \frac{w}{r^{l-1}} \Delta(\frac{w}{r^{l-1}}) - \int_{S_\varepsilon} \frac{w}{r^{l-1}} \frac{\partial}{\partial r}(\frac{w}{r^{l-1}})\, \mathrm{d}\sigma.$$

The last integral satisfies $|\int_{S_\varepsilon}| = O(\varepsilon^{N-2l}) \to 0$ as $\varepsilon \to 0$. Substituting $\Delta(\frac{1}{r^{l-1}} w) = \frac{1}{r^{l-1}} \Delta w + w \Delta \frac{1}{r^{l-1}} + 2\nabla w \cdot \nabla \frac{1}{r^{l-1}}$, integrating by parts in the last term by using the identity $\Delta r^\gamma = \gamma(\gamma + N - 2) r^{\gamma - 2}$ and passing to the limit $\varepsilon \to 0$ (cf. Appendix in [12]), we obtain

$$\alpha_1 \int \frac{w^2}{r^{2l}} \le -\int \frac{w \Delta w}{r^{2(l-2)}} + (l-1)^2 \int \frac{w^2}{r^{2l}}.$$

Since $\alpha_1 - (l-1)^2 = \sqrt{B_l}$, by the Hölder inequality in the equivalent estimate

$$\sqrt{B_l} \int \frac{w^2}{r^{2l}} \le -\int \frac{w\Delta w}{r^{2(l-2)}} \equiv -\int \frac{w}{r^l} \frac{\Delta w}{r^{l-2}} \le \left(\int \frac{w^2}{r^{2l}}\right)^{1/2} \left(\int \frac{|\Delta w|^2}{r^{2(l-2)}}\right)^{1/2},$$

we deduce that

$$\int \frac{|\Delta w|^2}{r^{2(l-2)}} \ge B_l \int \frac{w^2}{r^{2l}}. \tag{2.3}$$

This completes the proof for even l by iterating. If $l = 2m + 1$ is odd, then iterating gives

$$\int \frac{|\Delta^m w|^2}{r^2} \ge B_3 B_5 \cdots B_l \int \frac{w^2}{r^{2l}},$$

and using the first Hardy inequality

$$\int \frac{|\Delta^m w|^2}{r^2} \le \frac{1}{\alpha_1} \int |\nabla \Delta^m w|^2$$

completes the proof. □

3. EXTENSIONS OF HIGHER-ORDER OPERATOR INEQUALITIES

For the radial singular potential in (1.4), one can expect that the best constants of related operator inequalities admitting radial setting are induced by the radial part of the operator. We then begin with the analysis of singular ordinary differential operators.

3.1. **Preliminaries: Characteristic polynomial and deficiency indices.** Consider the $2l$-th order ordinary differential operator (1.4) with the single variable $r = |x| \in (0,1)$. For our main applications, in the case of deficiency indices (l,l) (see calculations below) and for Friedrichs extensions, we put regular zero Dirichlet conditions at $r = 1$

$$\psi(1) = \psi'(1) = \cdots = \psi^{(l-1)}(1) = 0. \tag{3.1}$$

For more general self-adjoint extensions, which also can be used for improved operator inequalities, the "boundary conditions" corresponding cannot be specified explicitly and independently of the differential expression $\mathbf{B}$; see [25, Section 18].

In the radial setting, the Laplacian $\Delta = \Delta_r \equiv \frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{N-1}{r}\frac{\mathrm{d}}{\mathrm{d}r}$ is a singular operator at $r = 0$, and, as usual, we consider it in the space $C_0^\infty = C_0^\infty([0,1))$ of radial infinitely many times differentiable functions $v(r)$ on $[0,1)$ vanishing in a neighbourhood of $r = 1$ and satisfying $v^{(k)}(0) = 0$ for $k = 1, ..., 2l-1$, e.g. assuming that $v(r) \equiv \text{const}$ for $r \approx 0$. It follows from Hardy's inequality that $\mathbf{B}$ is semibounded in C_0^∞,

$$\mathbf{B} \le 0. \tag{3.2}$$

The corresponding unique Friedrichs self-adjoint extension of $\mathbf{B}$ [7] will play a key role in extending Hardy's inequalities. We will also discuss generalizations of the operator inequality (3.2) by using other self-adjoint extensions. Our analysis is based on the classical theory of singular ordinary differential operators and their self-adjoint extensions [25].

Given a self-adjoint extension of $\mathbf{B}$, inequality (3.2) can be improved in $D(\mathbf{B})$ provided that $\sigma(\mathbf{B})$ is purely discrete (countable) and then $\mathbf{B} \le \lambda_1 I$ by completeness and closure of the eigenfunctions. Therefore, we need to study the corresponding eigenvalue problem

$$\mathbf{B}\psi = \lambda\psi \quad \text{on} \quad (0,1), \tag{3.3}$$

with the singular end-point at the origin $r = 0$. Let us estimate the deficiency indices of $\mathbf{B}$ at $r = 0$. In the radial setting, equation (3.3) written in the form

$$(-1)^{l+1} r^{2l} \Delta^l \psi + \alpha_l \psi = \lambda r^{2l} \psi \tag{3.4}$$

is a perturbed Euler equation and by the change $s = \ln r \to -\infty$ as $r \to 0$ reduces to an exponentially perturbed ODE with constant coefficients

$$(-1)^{l+1} \psi^{(2l)} + ... + (\alpha_l - \lambda e^{2ls})\psi = 0 \tag{3.5}$$

(see the characteristic polynomial below). The behaviour of all solutions as $s \to -\infty$ is given by exponentially small perturbations of solutions of the unperturbed equation, [9, Chapter III]. This gives the deficiency indices of the operator as the number of linearly independent L^2-solutions of equation (3.3) with $\operatorname{Im} \lambda \neq 0$. For convenience, instead of the space $L^2(0,1)$ for general self-adjoint differential expressions with real coefficients

$$l(y) = (-1)^l (p_0 y^{(l)})^{(l)} + (-1)^{l-1} (p_1 y^{(l-1)})^{(l-1)} + ... + p_l y, \tag{3.6}$$

$x \in (0,1]$, $l = 1, 2, ...$ we use the space $L^2(B)$, where operator (1.4) is symmetric in C_0^∞, so that we will include the extra weight r^{N-1} in the analysis of the singular end-point $r = 0$. The differential expression (1.4) reduces to the canonical form (3.6) in L^2 without weight by the change $x = r^N$.

Using (3.5) and looking for solutions close to the singular end-point in the form

$$\psi(r) = r^\gamma (1 + o(1)) \quad \text{as } \ r \to 0, \tag{3.7}$$

one obtains that $\gamma \in \mathbb{C}$ satisfies the algebraic equation

$$G(\gamma) \equiv G_*(\gamma) + \alpha_l = 0, \quad G_*(\gamma) = (-1)^{l+1} \textstyle\prod_{k=1}^{l} [\gamma - 2(k-1)](\gamma + N - 2k), \tag{3.8}$$

where $G(\gamma)$ is the characteristic polynomial of the operator with constant coefficients given in (3.5). It follows from Lemma 2.1 that

$$\alpha_l = -G_*(-\tfrac{1}{2}(N - 2l)) > 0, \tag{3.9}$$

so the best constant $\alpha_l > 0$ is such that (1.3) is the exact solution of the homogeneous equation $\mathbf{B}\psi = 0$ in $\mathbf{R}^N \setminus \{0\}$. Thus, the characteristic equation has the double root

$$\gamma_l = -\tfrac{1}{2}(N - 2l). \tag{3.10}$$

It generates two L^2-solutions denoted by

$$\bar{\psi}_l(r) = r^{-(N-2l)/2} \ln r (1 + o(1)) \quad \text{and} \quad \bar{\psi}_{l+1}(r) = r^{-(N-2l)/2}(1 + o(1)), \tag{3.11}$$

which are ordered relative to the growth rate as $r \to 0$. The other $2l - 2$ characteristic roots are real or complex, unlike the lower-order cases $l = 1$ and 2, where the roots are always real. Complex roots can occur for $l \geq 3$. For instance, one can see from the structure of the characteristic polynomial (3.8) that, for $N \gg 2l$ and l even, there exist precisely two more real roots $\hat{\gamma}_{l-1} > 2(l-1)$ and $\gamma_{l-1} < 2 - N$. On the other hand, for $N \gg 2l$ and l odd no more real roots exist. First, let us consider the real case.

Proposition 3.1. *There exists $N_* > 2l$ such that, for all $N \in (2l, N_*)$, the characteristic polynomial has precisely $2l-1$ distinct real roots. Besides the double root* (3.10)*, these are l negative roots satisfying*

$$\begin{aligned} \textit{odd } l\text{:}\ & 2-N < \gamma_1 < \gamma_2 < 4-N < ... < \\ & \gamma_{l-2} < \gamma_{l-1} < 2(l-2)-N < \gamma_l, \end{aligned} \tag{3.12}$$

$$\begin{aligned} \textit{even } l\text{:}\ & \gamma_1 < 2-N < \gamma_2 < \gamma_3 < 6-N < ... \\ & < \gamma_{l-2} < \gamma_{l-1} < 2(l-2)-N < \gamma_l, \end{aligned} \tag{3.13}$$

and $l-1$ positive roots,

$$\begin{aligned} \textit{odd } l\text{:}\ & 2 < \gamma_{l+1} < \gamma_{l+2} < 4 < ... \\ & < 2(l-2) < \gamma_{2l-2} < \gamma_{2l-1} < 2(l-1), \end{aligned} \tag{3.14}$$

$$\begin{aligned} \textit{even } l\text{:}\ & 2 < \gamma_{l+1} < \gamma_{l+2} < 4 < ... \\ & < \gamma_{2l-3} < \gamma_{2l-2} < 2(l-2) < 2(l-1) < \gamma_{2l-1}. \end{aligned} \tag{3.15}$$

If $N = N_*$, then two more double real roots occur generating a singularity with the extra $\ln r$-factor which does not affect the analysis. For $N > N_*$, there exist complex roots. Concentrating for simplicity on the real case $N < N_*$ of distinct roots, we will explain slight modifications that are necessary to extend the results to $N \geq N_*$.

3.2. **Domain of self-adjoint extensions.** Let us perform first estimates of the deficiency indices of operator $\mathbf{B}$ (more precisely, of $\mathbf{B}_0$ defined as the closure of the restriction of $\mathbf{B}$ to C_0^∞, [25, Section 17]. First, it is easy to see that, for the higher-order operators (1.4) with $l \geq 2$, in general, the deficiency indices cannot equal $(2l, 2l)$, which is the most convenient case to have compact resolvent and hence discrete spectra of any self-adjoint extension of $\mathbf{B}$, [25, Section 19]. Indeed, taking the maximal singularity $\psi_1(r) \sim r^{\gamma_1}$ as $r \to 0$, one can see that $\psi_1 \in L^2$ if and only if

$$\gamma_1 > -\tfrac{1}{2}\, N. \tag{3.16}$$

Since $\gamma_1 \approx 2 - N$ and $\gamma_1 < 2 - N$ for even l's by (3.13), it follows that (3.16) is not true for any $l = 2, 4, ...$ and, most probably, can be valid for $l = 1$ only.

On the other hand, $r^{\gamma_l} \ln r \in L^2$ so that the deficiency indices are, at least, $(l+1, l+1)$, and hence cannot be (l, l), which is another convenient case where self-adjoint extensions are defined by any l linearly independent self-adjoint boundary conditions at the regular end-point $r = 1$ including

(3.1), [25, Section 18]. It follows from (3.12), (3.13) that the

$$\text{deficiency indices are } (l+1, l+1) \quad \text{provided that } \; N > \max\{2l, 4(l-2)\}. \tag{3.17}$$

Indeed, we have that $r^{2(l-2)-N} \not\in L^2$ if and only if $N > 4(l-2)$. Thus, for $l \ge 2$, the deficiency indices can be equal to (m, m) with an integer $m \in [l+1, 2l]$ (and it seems $m < 2l$).

We now specify the growth condition at the singular end-point which will be shown to correspond to the Friedrichs self-adjoint extension,

$$\psi(r) = O(r^{-(N-2l)/2}) \quad \text{as } \; r \to 0, \tag{3.18}$$

excluding exactly l "maximal" singularities at the origin. and then forming the so-called principal solutions with the minimally possible growth at the singular point. Some of them are excluded by the functional setting (L^2 at this moment, but others will be introduced later on), so that, actually, (3.18) means that we impose $m-l$ extra "boundary conditions" at the singular end-point $r = 0$. Condition (3.18) is a convenient way to describe some important features of the domain of real self-adjoint extensions under consideration, and there holds:

Proposition 3.2. *For $N \in (2l, N_*)$, $0 \not\in \sigma_p(\mathbf{B})$ for operator* (1.4), (3.1), (3.18).

Proof. Assuming for the contrary that (3.3) has the eigenvalue $\lambda = 0$, we obtain

$$\psi(r) = C_1 r^{\gamma_l} + C_2 r^{\gamma_{l+1}} + ... + C_l r^{\gamma_{2l-1}},$$

where by the regular conditions (3.1) coefficients $\{C_1, ..., C_l\}$ satisfy a linear system of l equations. The corresponding determinant reduces to the Vandermonde one which does not vanish for mutually distinct exponents $\{\gamma_l, \gamma_{l+1}, ..., \gamma_{2l-1}\}$ and hence there exists the unique trivial solution. Indeed, the determinant is the Wronskian of a system of solutions of an ODE and does not vanish provided that the solutions are linearly independent. Obviously, this conclusion holds true for $N = N_*$ (then for a double root γ_k, function $r^{\gamma_k} \ln r$ becomes a new linearly independent solution) and also for any $N > N_*$. □

Let us comment on a general structure of the domain of self-adjoint extensions. Let the deficiency indices be (m, m). According to [25, Section 18], choosing a special orthonormal basis $\{\varphi_1, ..., \varphi_m\}$ in the deficiency space $\mathcal{N}_\lambda$, $\operatorname{Im} \lambda \ne 0$, of the operator $\mathbf{B}_0$, where the functions are ordered relative to their singularities as $r \to 0$ (φ_1 has the maximal one), any real self-adjoint extension of $\mathbf{B}$ is characterized by means of a unitary symmetric $m \times m$ matrix $u = [u_{\mu\nu}]$ ([25], page 74 and 81). Then we choose $u_{11} = ... = u_{m-l,m-l} = -1$ and remove those non-diagonal elements so that the first $m-l$ functions φ_ν not satisfying (3.18) are excluded from the domain, i.e., we set $u_{\mu\nu} = 0$ for $\mu = 1, ..., m-l$ and $\nu = 1, ..., m$ and extend these elements to complete a unitary symmetric matrix.

There exist infinitely many self-adjoint extensions which are minimal ones in the sense that condition (3.18) cancels the most singular behaviour at the origin. The functional setting for almost all such extensions and the corresponding operator inequalities cannot be described in an explicit way, [25]. Therefore, later on, we will concentrate on the unique, extremal Friedrichs self-adjoint extensions of semibounded symmetric operators corresponding to the regular conditions (3.1), [7, 20].

3.3. **Friedrichs extension and improved Hardy's inequality.** For our purposes, we will use the Friedrichs extension of semibounded symmetric operators on the basis of the space C_0^∞ by completing it via the positive definite quadratic form and intersecting the resulting Hilbert space with the domain of the maximal adjoint operator $D(\mathbf{B}^*)$, [7, 20]. In the present case of the scaling invariant operator $\mathbf{B}$ in (1.4), there exists an opportunity to describe it in more detail by using a different, "more regular" representation in a low-dimensional space.

Namely, setting $u = r^{-(N-2l)/2} v$ in (3.3), so that v is bounded by condition (3.18), we obtain the eigenvalue problem for the linear operator

$$L^{(l)} v = r^{(N-2l)/2} \mathbf{B}(r^{-(N-2l)/2} v) \equiv ((-1)^{l+1} \Delta_{2l}^l + P_{2l-1}) v, \tag{3.19}$$

where Δ_{2l} denotes the radial Laplacian in $\mathbf{R}^{2l}$ and P_{2l-1} is a lower-order non-positive operator. Therefore, it is natural to consider $L^{(l)}$ in the $L^2_{2l} = L^2(B)$-space, where the scalar product is defined in the topology of $\mathbf{R}^{2l}$ and the following property holds.

Proposition 3.3. *$L^{(l)}$ is symmetric in L^2_{2l}.*

Proof. For any $v, w, \in C_0^\infty$,

$$\langle L^{(l)} v, w \rangle_{L^2_{2l}} = N\omega_N \int_0^1 r^{2l-1} r^{(N-2l)/2} \mathbf{B}(r^{-(N-2l)/2} v)\, w \mathrm{d}r$$

$$\equiv N\omega_N \int_0^1 r^{N-1} \mathbf{B}(r^{-(N-2l)/2} v)\, r^{-(N-2l)/2} w \mathrm{d}r = \langle v, L^{(l)} w \rangle_{L^2_{2l}}$$

(ω_N is the volume of B), since $\mathbf{B}$ in (1.4) is composed from Δ's in $\mathbf{R}^N$. □

By representation (3.19) and Euler's structure in (3.4), we have that

$$L^{(l)} = (-1)^{l+1}\Delta_{2l}^l + \tfrac{1}{r^2}\, L_1 + \tfrac{1}{r^4}\, L_2 + ... + \tfrac{1}{r^{2l}}\, L_l, \tag{3.20}$$

where L_k for $k = 1, ..., l$ are linear operators which have to be symmetric in $L^2_{2(l-k)}$. Fixing the higher-order terms and putting others into the lower-order ones and bearing in mind the non-positivity of $L^{(l)}$, it is not difficult to see that $L_k = (-1)^{l+1-k} c_k \Delta_{2(l-k)}^{l-k}$, i.e., (3.20) reads

$$L^{(l)} = (-1)^{l+1}\Delta_{2l}^l + \tfrac{1}{r^2}\,(-1)^l c_1 \Delta_{2(l-1)}^{l-1} + \tfrac{1}{r^4}\,(-1)^{l-1} c_2 \Delta_{2(l-2)}^{l-2} + \cdots - \tfrac{1}{r^{2l}}\, c_l I, \tag{3.21}$$

where $c_k \le 0$ and $c_l = 0$ by (3.9). In particular, $L^{(1)} = \Delta_2$ (the Laplacian in $\mathbf{R}^2$) [6, 30] and $L^{(2)} = -\Delta_4^2 + \frac{N(N-4)}{2}\,\frac{1}{r^2}\,\Delta_2$ is symmetric in L^2_4 [15]. For $l = 2$ and 3, more general elliptic representations of $L^{(l)}$ were derived in [16].

Thus, by definition (3.9) of the best constant α_l, $L^{(l)}v$ does not contain the zero-order term with v and begins with v', i.e., its characteristic equation admits the root $\gamma = 0$. Consider its unique Friedrichs extension with regular conditions (3.1). As above, this extension is obtained by using the corresponding positive quadratic form in $C^\infty([0, 1))$ of radial functions whose derivatives up to order $2l - 1$ vanish at $r = 0$. The Hilbert space completed by using this quadratic form includes functions satisfying the Dirichlet conditions (3.1). Since the characteristic equations have the root $\gamma = 0$, it is not difficult to show that the domain of such a Friedrichs extension does not include functions generated by stronger singularities than (3.18) and the domain consists of bounded functions (the analysis does not differ from that for the radial Laplacian). The eigenfunctions of $L^{(l)}$ are bounded and in this sense, in view of scaling, the operator $L^{(l)}$ is more regular than $\mathbf{B}$.

The Hardy inequality implies that $L \le 0$ in C_0^∞ and the Friedrichs self-adjoint extension satisfies the same bound. Its discrete spectrum (the $L^{(l)-1}$'s are compact by embeddings of corresponding Sobolev spaces) has the first eigenvalue $\lambda_1 < 0$. This gives the extended inequality for operator (1.4), (3.1), (3.18)

$$\mathbf{B} = -(-\Delta)^l + \tfrac{\alpha_l}{|x|^{2l}}\, I \le \lambda_1 I \quad \text{in } D(\mathbf{B}), \tag{3.22}$$

which is *optimal* in the sense that the equality is attained at the first eigenfunction $\psi_1 = r^{-(N-2l)/2} v_1 \in D(\mathbf{B})$. It's H_{2l}-restriction

$$(-\Delta)^l \ge (\tfrac{\alpha_l}{|x|^{2l}} - \lambda_1) I \quad \text{in } D(\mathbf{B}_1) \cap H^{2l} \tag{3.23}$$

is not optimal since $\psi_1 \notin H^{2l}$. The corresponding improved Hardy's inequality (H_l-restriction) formulated using $|D^l w|$ as in (1.1), (1.2),

$$\int |D^l w|^2 \geq \alpha_l \int \frac{w^2}{|x|^{2l}} - \lambda_1 \int w^2 \quad \text{in } H_0^l, \tag{3.24}$$

is also not optimal since $\psi_1 \notin H^l$. Therefore, they admit further extensions.

3.4. **Generalization to elliptic setting.** It is easy to show that (3.24) holds for non-radial functions in H_0^l. In polar coordinates $x = (r, \sigma)$ in B,

$$\Delta = \Delta_r + \tfrac{1}{r^2} \Delta_\sigma, \tag{3.25}$$

where Δ_σ is the Laplace-Beltrami operator on the unit sphere S^{N-1} in $\mathbf{R}^N$, which is a regular operator with discrete spectrum in $L^2(S^{N-1})$ (each eigenvalue repeated as many times as its multiplicity)

$$\sigma(-\Delta_\sigma) = \{k(k+N-2) : k \geq 0\} \quad \Longrightarrow \quad \Delta_\sigma \leq 0 \text{ in } C_0^\infty, \tag{3.26}$$

and an orthonormal, complete, closed subset $\{f_k(\sigma)\}$ of eigenfunctions being k-th order homogeneous harmonic polynomials restricted to S^{N-1}. For convenience, we derive well-known bounds on the iterated Laplacian. Firstly,

$$\Delta \leq \Delta_r, \quad \text{and hence} \quad \Delta^2 \geq \Delta_r^2, \tag{3.27}$$

since by the first inequality in vector form,

$$\begin{aligned} \int \Delta^2 w\, w &= -\int \Delta \nabla w \cdot \nabla w \geq -\int \Delta_r \nabla w \cdot \nabla w \\ &= \int \nabla_r \nabla w \cdot \nabla_r \nabla w = \int \nabla \nabla_r w \cdot \nabla \nabla_r w = -\int \Delta \nabla_r w \cdot \nabla_r w \\ &\geq -N\omega_N \int r^{N-1} \Delta_r \nabla_r w \cdot \nabla_r w = \int \Delta_r^2 w\, w. \end{aligned}$$

Secondly and similarly, $\Delta^3 \leq \Delta_r^3$ by (3.27),

$$\begin{aligned} \int \Delta^3 w\, w &= \int \Delta \Delta w\, \Delta w \leq \int \Delta_r \Delta w\, \Delta w \\ &= -\int \nabla_r \Delta w \cdot \nabla_r \Delta w = -\int \Delta \nabla_r w \cdot \Delta \nabla_r w \\ &= -\int \Delta^2 \nabla_r w \cdot \nabla_r w \leq -N\omega_N \int r^{N-1} \Delta_r^2 \nabla_r w \cdot \nabla_r w = \int \Delta_r^3 w\, w, \end{aligned}$$

and finally, by induction,

$$(-1)^{l+1} \Delta^l \leq (-1)^{l+1} \Delta_r^l \quad \text{for any } l \geq 1. \tag{3.28}$$

Substituting converging eigenfunction expansions $w = \sum \varphi_k(r) f_k(\sigma)$ of functions in C_0^∞, where φ_k's are L^2_σ-projections of w onto f_k, into (3.24) and performing integration in $\mathrm{d}\sigma$ proves by density that the inequality remains true without the radial restriction.

Generalizations to arbitrary smooth bounded domains Ω of measure ω_N are performed by Schwartz symmetrization (rearrangement) which is a classical technique for $l = 1$ [21] and applies to functionals of arbitrary order $l \geq 1$; see details in [16, Section 3, 4].

3.5. **Extensions in weighted L^2-spaces.** This is a simple way to get a compact resolvent of self-similar extensions and discrete spectrum, though the approach can essentially change the functional space and the nature of inequalities. We characterize a couple of typical examples corresponding to different functional settings.

Deficiency indices $(2l, 2l)$. Consider the operator

$$\mathbf{B}_1 = \tfrac{1}{\rho}\,\mathbf{B} \quad \text{in the weighted space } L^2_\rho \tag{3.29}$$

with a positive continuous weight ρ on $(0, 1]$. Let

$$\rho(r) \to 0 \quad \text{as } r \to 0 \quad \text{and} \quad \int_0 r^{N-1} \rho(r) r^{2\gamma_1} \mathrm{d}r < \infty, \quad \text{e.g.,} \tag{3.30}$$

$$\rho(r) = r^{\mu_0} |\ln \tfrac{r}{2}|^\beta, \quad \beta < -1, \quad \text{where } \mu_0 = -N - 2\gamma_1 > 0. \tag{3.31}$$

For more general weights, the Osgood criterion occurs

$$\rho(r) = r^{\mu_0} \kappa(r), \quad \text{where} \quad \int^\infty \kappa(e^{-s}) ds < \infty. \tag{3.32}$$

In the case of the deficiency indices $(2l, 2l)$ in L^2_ρ, the construction of the minimal Friedrichs extension is the same as above and the spectrum is purely discrete. For general self-adjoint extensions for arbitrary deficiency indices not equal to $(2l, 2l)$, in order to get a discrete spectrum, one needs to check the L^2-integrability of the kernel of the inverse operator (or of the resolvent)

$$\mathbf{B}_\rho = \rho^{-1/2} \mathbf{B} \rho^{-1/2}, \quad \mathbf{B}_\rho^{-1} = \rho^{1/2} \mathbf{B}^{-1} \rho^{1/2}, \tag{3.33}$$

and to show that it is compact in L^2. If (3.30) holds, then such weights ρ improve the integrability properties of integral kernels and then such self-adjoint extensions have compact resolvent and the spectrum is discrete. The Friedrichs extension is non-positive, (3.1), (3.18) are valid and $\lambda_1 < 0$ by Proposition 3.2. This gives the optimal inequality

$$-(-\Delta)^l + \tfrac{\alpha_l}{|x|^{2l}}\, I \leq \lambda_1 \rho I \quad \text{in } D(\mathbf{B}_1). \tag{3.34}$$

Since $r^{-(N-2l)/2} \notin H^l$, both H_{2l} and H_l-restrictions are not optimal,

$$(-\Delta)^l \geq (\tfrac{\alpha_l}{|x|^{2l}} - \lambda_1 \rho) I \quad \text{in } D(\mathbf{B}_1) \cap H^{2l}, \tag{3.35}$$

$$\int |D^l w|^2 \geq \alpha_l \int \frac{w^2}{|x|^{2l}} - \lambda_1 \int \rho w^2 \quad \text{in } H_0^l. \tag{3.36}$$

Deficiency indices (l,l)**.** A harder functional setting occurs if we take a positive weight $\rho(r)$ satisfying (cf. (3.31)) $r^{2l}\rho(r) \to 0$ as $r \to 0$ and

$$\int_0 r^{2l-1}\rho(r)\mathrm{d}r < \infty, \ \int_0 r^{2l-1}\rho(r) \ln^2 r \mathrm{d}r = \infty, \tag{3.37}$$

$$\text{e.g.,} \quad \rho = r^{-2l} |\ln \tfrac{r}{2}|^\beta \quad \text{with } \beta \in [-3,1). \tag{3.38}$$

Then the deficiency indices in L^2_ρ are precisely (l,l) and hence any l regular self-adjoint boundary conditions at $r = 1$ including (3.1) for the Friedrichs extension define a self-adjoint extension of the operator (3.29) [25, page 78] and $0 \notin \sigma_p(\mathbf{B}_1)$. The main difficulty is then to establish conditions under which the resolvent is compact. For $l = 1$ this is easy to do using the standard representation of the kernel of $\mathbf{B}^{-1}$ composed from functions (3.11), i.e., of terms $\sim r^{-(N-2)/2} \ln r y^{-(N-2)/2}$ for $r < y$ and *vice versa* for $r > y$. Then, for the case (3.38), the kernel is $L^2(B \times B)$ if (it is curious that $\beta = -2$ is not included)

$$\beta < -2 \quad (l = 1).$$

For any $\beta \leq -2$, various improved Hardy inequalities in L^2 and in L^p were derived in [14, 4] in general elliptic settings including constuctions of infinite series inequalities with optimal coefficients.

For $l \geq 2$, the analysis of the L^2-integrability of the kernel becomes more tricky, and it seems, using optimal estimates of singularities (3.7), (3.11) in conjunction with Naimarks's analysis of kernels [25, page 59, 84] does not always imply the result. Therefore, a different technique for proving compactness is necessary for deficiency indices (l,l) and in other cases for $l \geq 2$. In [31], an inequality for $l = 2$ corresponding to the critical value $\beta = -2$ was proved directly for radial functions in B with further elliptic extension via symmetrization (we expect this approach be efficient for any $l \geq 2$).

FURTHER EXTENSIONS VIA MULTIPLIERS. We now extend the $(2l, 2l)$ case (3.30), (3.31). Using another multiplication operator $\mathbf{A}_1 = \frac{1}{\rho_1} I$ in $W_1 = L^2_{\rho_1}$, with a suitable positive weight ρ_1 satisfying (3.30), which is more degenerate at $r=0$ than $\rho(r)$, consider

$$\mathbf{B}_2 = \tfrac{1}{\rho_1}\left(\mathbf{B} - \lambda_1 \rho I\right) \tag{3.39}$$

that is symmetric in W_1. Then, determining suitable weights for indices $(2l, 2l)$ as above and denoting by $\lambda_1^{(1)} \le 0$ the first eigenvalue of $\mathbf{B}_2$, we obtain the optimal extended inequality

$$-(-\Delta)^l + \tfrac{\alpha_l}{|x|^{2l}}\, I \le \lambda_1 \rho I + \lambda_1^{(1)} \rho_1 I, \tag{3.40}$$

which is indeed (3.34) with the weight $\bar\rho = |\lambda_1|\rho + |\lambda_1^{(1)}|\rho_1$ and $\lambda_1 = -1$. One can obtain an arbitrary number of terms on the right-hand side. The restricted H_1-inequalities are still non-optimal and the extension can be continued until an infinite series inequalities occurs in the corresponding space

$$-(-\Delta)^l \le -\tfrac{\alpha_l}{|x|^{2l}}\, I + \textstyle\sum_{k=0}^{\infty} \lambda_1^{(k)} \rho_k I \quad (\lambda_1^{(0)} = \lambda_1). \tag{3.41}$$

Obviously, if the weight series converges to a suitable weight $\bar\rho$, then (3.41) is equivalent to (3.34) with such a weight. Anyway, by construction, all the coefficients $\{\alpha_1, \lambda_1, \lambda_1^{(1)}, ...\}$ in (3.41) are the best and cannot be improved.

Generalizations to elliptic settings are done by Schwartz symmetrization [21] which applies to higher-order operators in weighted spaces (see details and references in [16]) provided that all $\rho_k(|x|)$ are decreasing in $|x|$. For $l=1$, such series Hardy's inequalities with logarithmic terms were derived in [2, 3, 14, 4]. In the special case $\rho_k = |x|^{-2(l-k)}$, $k = 0, 1, ..., l$ for $l = 2, 3$, the best constants in the corresponding improved inequality admit a unified characterization via eigenvalues of iteration of the Laplacian; see [16] and Conjecture 1 in Section 4 therein (it is associated with representation (3.21)).

3.6. **Differential multipliers A.** Next, as a multiplier, we take the Laplacian

$$\mathbf{A}_1 = -\Delta > 0, \quad W_1 = H^{-1} \text{ with scalar product } \langle w, v\rangle_1 = \langle (-\Delta)^{-1} w, v\rangle, \tag{3.42}$$

where $(-\Delta)^{-1} : L^2 \to L^2$ is a positive compact integral operator and the product in H^{-1} is extended to $H^{-2} \times L^2$ by duality. Consider the operator

$$\mathbf{B}_1 = (-\Delta)(\mathbf{B} - \lambda_1 I) \tag{3.43}$$

and, as usual, fix regular Dirichlet conditions

$$\psi(1) = \psi'(1) = \cdots = \psi^{(l+1)}(1) = 0. \tag{3.44}$$

The deficiency indices in H^{-1} can be calculated by using the theory of extensions of symmetric operators, see [25, Chapter V] and [7]. It follows that the characteristic polynomial of $(-\Delta)\mathbf{B}$ has the additional multiplier $(\gamma-2l)[\gamma-2(l+1)+N]$ and extra roots $\gamma_{2l+1}=2(l+1)-N$ and $\gamma_{2l+2}=2l$. The deficiency indices are to be calculated in H^{-1} so that these are (m,m), where m is the number of exponents γ_k such that (cf. (3.16) for indices in L^2)

$$\gamma_k < \tfrac{1}{2}\,(N+2). \tag{3.45}$$

The same formula applies for complex roots for $N>N_*$, where $\gamma_k \mapsto \operatorname{Re}\gamma_k$.

Thus, $\mathbf{B}_1$ is symmetric in H^{-1}, semibounded in C_0^∞ and we will deal with its Friedrichs extension. Since by (3.42)

$$\langle \mathbf{B}_1 w, v\rangle_{-1} = \langle (\mathbf{B}-\lambda_1 I)w, v\rangle, \tag{3.46}$$

the extension and completing via the positive definite quadratic form leaves condition (3.18) unchanged. Consider the corresponding eigenvalue problem

$$\mathbf{B}_1\psi = \lambda^{(1)}\psi \quad \text{on } (0,1). \tag{3.47}$$

In order to show that the spectrum of $\mathbf{B}_1$ is discrete, we consider an inverse $(\mathbf{B}-\lambda_1 I-c(-\Delta)^{-1})^{-1}(-\Delta)^{-1}$ for some $c>0$ bearing in mind that $\mathbf{B}-\lambda_1 I \le 0$ in C_0^∞ so that the semi-axes $\{\lambda<0\}$ does not contain continuous spectrum. Hence for some $c>0$, the first multiplier is bounded and the inverse operator is compact as the product of bounded and a compact operator. In other words, the extra Laplacian multiplier in (3.43) improves the L^2-integrability of the kernel of the resolvent.

Thus, the spectrum of $\mathbf{B}_1$ is discrete and the subset of eigenfunctions is complete and closed in $D(\mathbf{B}_1)$. Since by construction of the Friedrichs extension $D(\mathbf{B}_1)\subset D(\mathbf{B})$, it follows from (3.22), that $\lambda_1^{(1)}\le 0$. One can expect that for most of the regular conditions (possibly, for "almost all"), the strict inequality holds,

$$\lambda_1^{(1)} < 0. \tag{3.48}$$

Indeed, if on the contrary $\lambda_1^{(1)}=0$, then (3.47) gives

$$(\mathbf{B}-\lambda_1)\psi = C_1 r^{2-N} + C_2 \equiv f. \tag{3.49}$$

In particular, it follows that $\psi \sim C_1 r^{2+2l-N} \notin H^{-1}$ provided that $N \ge 2(3+2l)$, so that in this case $C_1=0$. Furthermore, by the orthogonality condition, a solution in $D(\mathbf{B})$ exists provided that f satisfies $m_1<2l$ (m_1 being the geometric multiplicity of λ_1 for $\mathbf{B}$) orthogonality conditions, and then ψ depends on m_1+2 arbitrary constants. Then the regular conditions (3.44) give a linear system on the coefficients which is expected to admit a trivial solution only. This is obvious for sufficiently small $m_1<l$, where the

system is overdetermined. For m_1 large enough, the strict inequality (3.48) can be checked numerically though this is not straightforward and gets more difficult if l increases.

By completeness and closure of eigenfunctions (see [25, Section 21] and [7]), we get the optimal inequality $\mathbf{B}_1 \le \lambda_1^{(1)} I$ in $D(\mathbf{B}_1)$ and hence

$$-(-\Delta)^l + \tfrac{\alpha_l}{r^{2l}}\, I \le \lambda_1 I + \lambda_1^{(1)}(-\Delta)^{-1}, \tag{3.50}$$

where the constants α_l, λ_1 and $\lambda_1^{(1)}$ are the best. Of course, this is immediate from (3.24) by embedding but here and later on we concentrate on the "functional setting nature" of best constants in such inequalities. Differential operators having compact symmetric non-local perturbations as in (3.50) fall in the scope of the classical elliptic theory [1, pages 59-95] and, in the elliptic setting in bounded domains, are known to have discrete spectra and compact resolvents; see [28].

Since $r^{-(N-2l)/2} \notin H^{2l}$, the H_{2l}-restriction of (3.50) is non-optimal,

$$(-\Delta)^l \ge \tfrac{\alpha_l}{r^{2l}}\, I - \lambda_1 I - \lambda_1^{(1)}(-\Delta)^{-1} \quad \text{in } D(\mathbf{B}_1) \cap H^{2l}. \tag{3.51}$$

Moreover, $r^{-(N-2l)/2} \notin H^l$ and the H_l-restriction is also non-optimal,

$$\int |D^l w|^2 \ge \alpha_l \int \frac{w^2}{|x|^{2l}} dx - \lambda_1 \|w\|^2_{L^2} - \lambda_1^{(1)} \|w\|^2_{H^{-1}} \quad \text{in } D(\mathbf{B}_1) \cap H^l. \tag{3.52}$$

Taking next the same multiplier $\mathbf{A}_2 = \mathbf{A}_1 = -\Delta$, we get in H^{-2} the symmetric operator $\mathbf{B}_2 = (-\Delta)(\mathbf{B}_1 - \lambda_1^{(1)} I)$ with Dirichlet regular conditions at $r = 1$. The characteristic polynomial has extra roots $\gamma_{2l+1} = 2(l+2) - N$ and $\gamma_{2l+2} = 2(l+1)$. For the domain of the Friedrichs extension, (3.18) remains valid. By a similar analysis, the resolvent is compact and the spectrum is purely discrete. Then $\lambda_1^{(2)} \le 0$, and the strict inequality is expected. Then in $D(\mathbf{B}_2)$

$$-(-\Delta)^l + \tfrac{\alpha_l}{r^{2l}}\, I \le \lambda_1 I + \lambda_1^{(1)}(-\Delta)^{-1} + \lambda_1^{(2)}(-\Delta)^{-2}. \tag{3.53}$$

The corresponding non-optimal H_1-restriction in $D(\mathbf{B}_2) \cap H^l$ takes the form

$$\int |D^l w|^2 \ge \alpha_l \int \frac{w^2}{|x|^{2l}} - \lambda_1 \|w\|^2_{L^2} - \lambda_1^{(1)} \|w\|^2_{H^{-1}} - \lambda_1^{(2)} \|w\|^2_{H^{-2}}. \tag{3.54}$$

Applying the same differential multipliers leads to finite sums in the inequality

$$-(-\Delta)^l + \tfrac{\alpha_l}{|x|^{2l}}\, I \le \textstyle\sum_{k=0}^M \lambda_1^{(k)} (-\Delta)^{-k}, \tag{3.55}$$

which admits the corresponding H_{2l} and H_l restrictions.

3.7. **On extension via compact operator A.** For $l \ge 2$, it is possible to complete the Sobolev scale by using the multiplier $\mathbf{A}_1 = (-\Delta)^{-1}$, $W_1 = H^1$ with the scalar product in H_0^1 obtained as the extension to $H_0^2 \times L^2$ by duality, $\langle w, v\rangle_1 \equiv \langle(-\Delta)w, v\rangle$. Then we obtain a symmetric operator with integral (possibly, compact) perturbations $\mathbf{B}_1 = (-\Delta)^{-1}(\mathbf{B} - \lambda_1 I)$. One of the main difficulties here is to show that the resolvent of $\mathbf{B}_1$ is compact and the spectrum is discrete. We mention that Rellich's theorem (see [25, Section 24]) guarantees this if the eigenvalues of $\mathbf{B}$ satisfy $\sum \lambda_k^{-2} < \infty$. Actually, this implies that the resolvent is a compact operator $l^2 \to l^2$ in the Hilbert space of sequences l^2 bearing in mind the isomorphism $L^2 \to l^2$ via the eigenfunction expansion $v = \sum c_k \psi_k \in L^2$ with $\sum c_k^2 < \infty$ meaning that $\{c_k\} \in l^2$.

Therefore, assuming that a general spectral theory of non-local elliptic operators applies [28], the final inequalities take the form

$$-(-\Delta)^l + \tfrac{\alpha_l}{|x|^{2l}}\, I \le \lambda_1 I + \lambda_1^{(1)}(-\Delta) \quad \text{in } D(\mathbf{B}_1), \tag{3.56}$$

$$\int |D^l w|^2 \ge \alpha_l \int \frac{w^2}{|x|^{2l}} - \lambda_1 \|w\|_{L^2}^2 - \lambda_1^{(1)} \|w\|_{H^1}^2 \quad \text{in } D(\mathbf{B}_1) \cap H^l. \tag{3.57}$$

Note that unlike the previous cases, because of the last term, inequality (3.57) does not admit a standard elliptic setting, unless we define the norm in H^1 by using the radial part only (for the Dirichlet boundary data, it is a norm; see [15, Section 3]).

Finally, combining the restriction of (3.55) and (3.56) yields estimates

$$\int |D^l w|^2 \mathrm{d}x \ge \alpha_l \int \frac{w^2}{r^{2l}} \mathrm{d}x - \textstyle\sum_{k=-l+1}^{M} \lambda_1^{(k)} \|w\|_{H^{-k}}^2 \quad \text{in } D(\mathbf{B}_{M+l-1}) \cap H^l, \tag{3.58}$$

where all the coefficients α_l and $\{\lambda_1^{(k)}\}$ are the best possible. Passing to the limit $M \to \infty$ by using a suitable topology in the Sobolev scale originated by differential operators of infinite order (see [13]) implies a formal possibility to get an infinite series.

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