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EULER EQUATION IN A 3D CHANNEL WITH A NONCHARACTERISTIC BOUNDARY

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Abstract. In this paper we consider the Euler equations of an incompressible fluid in a $3D$ channel with permeable walls; a portion of the boundary is standing an inflow and another an outflow. We prove the existence, uniqueness and regularity of solutions, locally in time, in various function spaces of Hölder type.

1. Introduction

In this article, the Euler equations in a $3D$ channel are considered and we are interested in showing the well posedness of the problem for limited time. The flow in a $3D$ channel is a flow in an infinite domain limited by two parallel planes, with space periodicity in two orthogonal directions parallel to the planes. The study of the $2D$ and $3D$ Euler equations in a channel with the non-penetration boundary condition on the wall has been considered in a companion paper [10]. In this article we consider the case of a permeable boundary (noncharacteristic boundary).

The study of the Euler equations in different spaces and with homogeneous or non-homogeneous boundary conditions is a problem of interest in mathematical physics and we recall here the pioneering work in this field of Lichtenstein [9], Wolibner [15], Yudovich [5], and the more recent works of Beale, Kato and Majda [2], Ebin and Marsden [3], [4], Kato [6], [7], and Temam [12], [11], [13].

We consider the Euler equation in a three-dimensional channel, with a portion of the boundary standing an inflow and another an outflow; on the inflow portion of the boundary the full velocity is given and on the outflow portion we prescribe only the normal component of the velocity.

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The aim of this article is to prove the well posedness of the problem in the domain described above, for limited time; i.e. we want to prove the existence, uniqueness and regularity of solutions in various function spaces, of Hölder type. Beyond its intrinsic interest, the problem has applications in the study of boundary layers for the Navier-Stokes equations when the viscosity goes to zero; here we recall in particular the work of Temam and Wang [14] where the short time convergence of the solutions of the NSE to that of the Euler equations was proved, and the result proved here was announced and used. Our main result is Theorem 4.1.

As an overview of the methods used in this article, we first recall the work of Kato [6] where we find the proof of the existence and uniqueness of a smooth solution for all time of the Euler equations in space dimension two for a bounded domain. For the case of a bounded domain and in all dimensions, Temam gives a short proof of local in time existence of a solution, based on an appropriate estimate of the pressure p in terms of the velocity $\mathbf{u}$ (see [11], [13]); both Kato and Temam consider the case of non-penetration (homogeneous) boundary conditions ($\mathbf{u}\cdot\mathbf{n} = 0$ on the boundary). The case of permeable boundary in a $3D$ bounded domain was considered by Antontsev, Kazhikhov and Manakhov [1] using a method related to that used in [11]; in the present paper we extend the approach of [1] to our problem; more details will be given in the text.

2. The setting of the problem

We recall the Euler equations for incompressible fluids

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \operatorname{grad} p = \mathbf{f}, \tag{2.1a}$$

$$\operatorname{div}\mathbf{u} = 0, \tag{2.1b}$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the three-dimensional velocity of the flow and p is the pressure.

We study the flow in the infinite $3D$ channel $\Omega_\infty = \mathbb{R}^2 \times (0, h)$, and we set $\Omega = (0, L_1) \times (0, L_2) \times (0, h)$, where L_1, L_2 are respectively the periods in the $0x_1$ and the $0x_2$ directions. Equations (2.1) are supplemented with the initial and boundary conditions; the initial condition is

$$\mathbf{u}(x, 0) = \mathbf{u}_0(x), \quad \forall\, x \in \Omega_\infty, \tag{2.2}$$

and the boundary conditions for this system are

$$\begin{aligned} &\mathbf{u} = (0, 0, -U) \text{ on } \Gamma_1 = \{x_3 = h\},\ \mathbf{u}\cdot\mathbf{n} = U \text{ on } \Gamma_0 = \{x_3 = 0\},\\ &\text{and the functions are periodic in } x_1 \text{ and } x_2. \end{aligned} \tag{2.3}$$

Here and in all that follows, $\mathbf{n}$ is the external normal to the boundary and U is a positive scalar function. In order to simplify the notation, we denote by g a function on $\Gamma_0 \cup \Gamma_1$, which is equal to $-U$ on Γ_1 and to U on Γ_0. The part of the boundary where we impose periodic boundary conditions is globally referred to as Γ_l.

We assume the following regularity conditions on the data

$$\begin{aligned} &\mathbf{f} \in \mathcal{C}^{1+\alpha}(\Omega_\infty \times [0,T]),\ \mathbf{u}_0 \in \mathcal{C}^{1+\alpha}(\Omega_\infty \times [0,T]), \\ &U \in \mathcal{C}^{2+\alpha,\,1+\alpha}((\Gamma_0 \cup \Gamma_1) \times [0,T]),\ U \geq c > 0, \end{aligned} \tag{2.4}$$

for some α, with $0 < \alpha < 1$.

Other regularity conditions may be considered in what follows. In order to simplify the computations we will also suppose that $\operatorname{div} \mathbf{f} = 0$ and $\mathbf{f} \cdot \mathbf{n} = 0$ on $\Gamma_0 \cup \Gamma_1$ which amounts to changing the pressure. We assume that the following compatibility conditions between the initial and the boundary data hold

$$\mathbf{u}_0 = \mathbf{U}(t=0) \,\mathrm{on}\, \Gamma_1, \quad \mathbf{u}_0 \cdot \mathbf{n} = U(t=0) \,\mathrm{on}\, \Gamma_0, \tag{2.5a}$$

$$\frac{\partial \mathbf{U}}{\partial t}(t=0) + (\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 + \nabla p_0 = \mathbf{f}(t=0) \,\mathrm{on}\, \Gamma_1, \tag{2.5b}$$

where $\mathbf{U}$ denotes the vector $\mathbf{U} = (0, 0, -U)$.

Applying the divergence operator to equation (2.1a) and also multiplying the same equation by $\mathbf{n}$, we find an equation for the pressure at each instant of time t

$$\Delta p = -\sum_{i,j=1}^{3} \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i},\ x \in \Omega, \tag{2.6a}$$

$$\frac{\partial p}{\partial n} = -[(\mathbf{u} \cdot \nabla)\mathbf{u}] \cdot \mathbf{n} - \frac{\partial g}{\partial t},\ x \in \Gamma_1 \cup \Gamma_0, \tag{2.6b}$$

$$p \text{ is periodic in } x_1 \text{ and } x_2. \tag{2.6c}$$

Note that problem (2.6), which is of Neumann type, defines the pressure $p = p(t)$, for each $t \geq 0$, as a function of $\mathbf{u}(t)$ and $U(t)$. In order to be able to determine the pressure unambiguously, we add the condition

$$\int_\Omega p \,\mathrm{d}\Omega = 0. \tag{2.7}$$

In particular, considering (2.6) at $t = 0$, we obtain the initial pressure as a function of $\mathbf{u_0}$ and $U(0)$.

An Equivalent Formulation. We set $\boldsymbol{\omega} = \operatorname{curl} \mathbf{u}$ and, in view of eventually using a fixed-point procedure, we show how to successively determine some functions $\tilde{\mathbf{u}}$, $\tilde{p}$ and $\tilde{\boldsymbol{\omega}}$, when $\boldsymbol{\omega}$ is given. Here the presentation is formal; the actual proofs of existence of $\tilde{\mathbf{u}}$, $\tilde{p}$ and $\tilde{\boldsymbol{\omega}}$ and the estimates are given in Section 3. Later on, our goal is to define a mapping $\Lambda : \boldsymbol{\omega} \to \tilde{\boldsymbol{\omega}}$ on a suitable Hölder space, and to obtain the solution of (2.1) as a fixed point of Λ.

The equations determining $\tilde{\mathbf{u}}$ in terms of $\boldsymbol{\omega}$ and the data reads

$$\operatorname{curl} \tilde{\mathbf{u}} = \boldsymbol{\omega}, \tag{2.8a}$$

$$\operatorname{div} \tilde{\mathbf{u}} = 0, \tag{2.8b}$$

$$\tilde{u}_3 = -U \text{ on } \Gamma_1 \cup \Gamma_0, \tag{2.8c}$$

$$\tilde{\mathbf{u}} \text{ is periodic in } x_1,\, x_2, \tag{2.8d}$$

$$\int_{\tilde{\Gamma}_0} \tilde{u}_j(x_1, x_2, 0)\, \mathrm{d}x_1\, \mathrm{d}x_2 = 0, \quad \text{for } j = 1, 2, \tag{2.8e}$$

where $\tilde{\Gamma}_0 = (0, L_1) \times (0, L_2) \times (x_3 = 0)$.

Note that condition (2.8e) is added in order to insure the uniqueness of the solution for (2.8). The condition is coherent with the initial boundary condition for (2.1).

In order to simplify the computations, we set $\mathbf{u}_\tau = (u_1, u_2)$ on $\Gamma_1 \cup \Gamma_2$ and div_τ for the two-dimensional divergence operator. The equations for determining the pressure $\tilde{p}$ read

$$\Delta \tilde{p} = -\sum_{i,j=1}^{3} \frac{\partial \tilde{u}_i}{\partial x_j} \frac{\partial \tilde{u}_j}{\partial x_i} \text{ on } \Omega, \tag{2.9a}$$

$$\frac{\partial \tilde{p}}{\partial n} = \frac{\partial U}{\partial t} + 2 \tilde{\mathbf{u}}_\tau \cdot \nabla_\tau U \text{ on } \Gamma_1, \tag{2.9b}$$

$$\frac{\partial \tilde{p}}{\partial n} = -\frac{\partial U}{\partial t} - 2 \tilde{\mathbf{u}}_\tau \cdot \nabla_\tau U + \operatorname{div}_\tau (U \tilde{\mathbf{u}}_\tau) \text{ on } \Gamma_0, \tag{2.9c}$$

$$\tilde{p} \text{ is periodic in } x_1,\, x_2; \tag{2.9d}$$

system (2.9) is also supplemented with the condition $\int_\Omega \tilde{p}\, \mathrm{d}\Omega = 0$.

Remark 2.1. A more natural boundary condition for $\tilde{p}$ on Γ_1 would be $\partial \tilde{p}/\partial n = \partial U/\partial t$, since in the initial problem $\mathbf{u}_\tau = 0$ on Γ_1. The choice of (2.9b) is made to ensure the solvability of the Neumann problem (2.9) since the solvability condition is

$$\int_\Omega \Delta p\, \mathrm{d}\Omega = \int_{\partial\Omega} \frac{\partial p}{\partial n}\, \mathrm{d}(\partial\Omega). \tag{2.10}$$

Considering $\partial\tilde{p}/\partial n = \partial U/\partial t$ instead of (2.9b) as a boundary condition for (2.9), the solvability condition reduces to

$$\int_{\tilde{\Gamma}_1} \tilde{\mathbf{u}}_\tau \cdot \nabla_\tau U \, \mathrm{d}\Gamma = 0, \tag{2.11}$$

where $\tilde{\Gamma}_1 = (0, L_1) \times (0, L_2) \times (x_3 = h)$. Relation (2.11) is not known to be a priori true, since $\tilde{\mathbf{u}}$ is the solution of problem (2.8), so $\tilde{\mathbf{u}}_\tau$ is not necessarily zero on $\tilde{\Gamma}_1$.

For finding the equation for $\tilde{\boldsymbol{\omega}}$, we apply the curl operator to equation (2.1a) and since $\tilde{\mathbf{u}} \cdot \mathbf{n} < 0$ on Γ_1 and $\tilde{\mathbf{u}} \cdot \mathbf{n} > 0$ on Γ_0, we supplement the equation for $\tilde{\boldsymbol{\omega}}$ with a boundary condition on Γ_1 but not on Γ_0. The system of equations obtained is then

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\boldsymbol{\omega}} - (\tilde{\boldsymbol{\omega}} \cdot \nabla)\tilde{\mathbf{u}} = \operatorname{curl} \mathbf{f}, \tag{2.12a}$$

$$\tilde{\omega}_1 = \frac{1}{U}(f_2 - \frac{\partial \tilde{p}}{\partial x_2} - U\frac{\partial U}{\partial x_2}) \text{ on } \Gamma_1, \tag{2.12b}$$

$$\tilde{\omega}_2 = -\frac{1}{U}(f_1 - \frac{\partial \tilde{p}}{\partial x_1} - U\frac{\partial U}{\partial x_1}) \text{ on } \Gamma_1, \tag{2.12c}$$

$$\tilde{\omega}_3 = 0 \text{ on } \Gamma_1, \tag{2.12d}$$

$$\tilde{\boldsymbol{\omega}} \text{ is periodic in } x_1, x_2, \tag{2.12e}$$

$$\tilde{\boldsymbol{\omega}}(t = 0) = \operatorname{curl} \mathbf{u}_0. \tag{2.12f}$$

We now show the equivalence between the initial problem and the problem (2.8), (2.9) and (2.12). That is, supposing that we found a fixed point $\boldsymbol{\omega} = \tilde{\boldsymbol{\omega}}$ for the operator Λ, we need to show that $\tilde{\mathbf{u}}$ and $\tilde{p}$ respectively determined from (2.8) and (2.9) are actually solutions of the Euler problem (2.1).

Lemma 2.1. *Let us assume that $\tilde{\mathbf{u}}$ and $\tilde{p}$ belong to $\mathcal{C}^1(\Omega_\infty \times [0, T])$, that $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega}$ belongs to $\mathcal{C}^{0,1}(\Omega_\infty \times [0, T])$, and satisfies equations* (2.8), (2.9) *and* (2.12). *Then $(\tilde{\mathbf{u}}, \tilde{p})$ is a solution of the initial Euler problem* (2.1), *and by uniqueness $\tilde{\mathbf{u}} = \mathbf{u}$, $\tilde{p} = p$.*

Proof. Setting $\boldsymbol{\omega} = \tilde{\boldsymbol{\omega}}$ in (2.8), equation (2.12a) implies

$$\operatorname{curl} \Big(\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} - f\Big) = 0, \tag{2.13}$$

and thus, we can find a distribution π such that

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} - f = -\nabla\pi. \tag{2.14}$$

At this stage we first observe that, in the class of functions $\mathbf{u} \in \mathcal{C}^1(\Omega_\infty \times [0,T])$, $p \in \mathcal{C}^1(\Omega_\infty \times [0,T])$, the solution of the Euler problem is unique. Then in view of (2.14) and the conditions (2.8b)–(2.8d), we only need to prove that $\tilde{\mathbf{u}}_\tau = (\tilde{u}_1, \tilde{u}_2) = 0$ on Γ_1 and that $\pi = \tilde{p}$ in order to conclude that $(\tilde{\mathbf{u}}, \tilde{p})$ is the solution of the Euler problem.

The proof of uniqueness is standard and elementary at this level of regularity; if $(\mathbf{u}_i, p_i)$, $i = 1, 2$, are two solutions of (2.1), then the difference $(w, q) = (\mathbf{u}_1 - \mathbf{u}_2, p_1 - p_2)$ satisfies

$$\begin{aligned} &\frac{\partial w}{\partial t} + (\mathbf{u}_1 \cdot \nabla) w + (w \cdot \nabla)\mathbf{u}_2 + \nabla q = 0, \\ &\operatorname{div} w = 0. \end{aligned} \tag{2.15}$$

The initial condition for (2.15) is $w = 0$ at $t = 0$. The boundary condition on Γ_1 is $w = 0$ and on Γ_0 is $w \cdot \mathbf{n} = 0$.

We multiply equation (2.15) by w and integrate over Ω. We find, using the boundary conditions and $\operatorname{div} w = 0$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|^2 \le |\mathbf{u}_2|_{\mathcal{C}^1(\Omega)}\|w\|^2, \tag{2.16}$$

where by $\|\cdot\|$ we understand the norm in $L^2(\Omega)$.

Since the initial data is zero, we find $\|w(t)\| = 0$ for all $t > 0$, which implies that the solution is unique.

Showing that $\tilde{\mathbf{u}}_\tau = (\tilde{u}_1, \tilde{u}_2) = 0$ on Γ_1 is more delicate and it is done by showing at the same time that $\pi = \tilde{p} = p$. Hence we set $\psi = \pi - \tilde{p}$ and determine the Neumann problem of which ψ is a solution. An equation for π is obtained by applying the div operator to equation (2.14). We find that $\Delta\pi$ is the same as $\Delta\tilde{p}$ given by (2.9a) so that $\Delta\psi = 0$. Similarly the boundary conditions for π are obtained by considering the normal values of each side of (2.14) on Γ_0 and Γ_1. On Γ_0, $\partial\pi/\partial n$ is the same as $\partial\tilde{p}/\partial n$ since $\tilde{\mathbf{u}} \cdot \mathbf{n} = -U$ on Γ_0. On Γ_1 the boundary conditions for π and $\tilde{p}$ are not the same. Finally we find:

$$\Delta\psi = 0 \ \text{ for } x \in \Omega, \tag{2.17a}$$

$$\frac{\partial\psi}{\partial n} = -\operatorname{div}_\tau(U\tilde{\mathbf{u}}_\tau) \ \text{ on } \Gamma_1, \tag{2.17b}$$

$$\frac{\partial\psi}{\partial n} = 0 \ \text{ on } \Gamma_0, \tag{2.17c}$$

$$\psi \text{ is periodic in } x_1,\, x_2. \tag{2.17d}$$

Problem (2.17) is of Neumann type (similar to (2.6)) for which we know the existence and uniqueness of ψ up to a constant. In order to uniquely determine the function ψ, we impose its average over Ω to be zero.

We then multiply equation (2.17a) by ψ, integrate, and integrate by parts. Taking into account equation (2.17c) and the fact that we work with functions periodic in the $0x_1$ and $0x_2$ directions, we find

$$I = \|\nabla\psi\|^2 = \int_{\tilde{\Gamma}_1} \psi \frac{\partial \psi}{\partial n} \,\mathrm{d}\Gamma = -\int_{\tilde{\Gamma}_1} \psi \mathrm{div}_\tau (U \tilde{\mathbf{u}}_\tau) \,\mathrm{d}\Gamma = \int_{\tilde{\Gamma}_1} \nabla_\tau \psi \cdot \tilde{\mathbf{u}}_\tau U \,\mathrm{d}\Gamma, \tag{2.18}$$

where we took into account (2.17c).

In order to estimate this last integral, we need to compute $\partial\psi/\partial x_1$ and $\partial\psi/\partial x_2$ on $\tilde{\Gamma}_1$. Using (2.3), (2.9) and (2.14) we find

$$\nabla_\tau \psi = -\frac{1}{2}\nabla_\tau |\tilde{\mathbf{u}}_\tau|^2 - \frac{\partial}{\partial t}\tilde{\mathbf{u}}_\tau, \text{ on } \Gamma_1. \tag{2.19}$$

We can then compute

$$I = -\frac{1}{2}\int_{\tilde{\Gamma}_1} \nabla_\tau (|\tilde{\mathbf{u}}_\tau|^2) \cdot \tilde{\mathbf{u}}_\tau U \,\mathrm{d}\Gamma - \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\tilde{\Gamma}_1} |\tilde{\mathbf{u}}_\tau|^2 U \,\mathrm{d}\Gamma + \frac{1}{2}\int_{\tilde{\Gamma}_1} |\tilde{\mathbf{u}}_\tau|^2 \frac{\mathrm{d}U}{\mathrm{d}t} \,\mathrm{d}\Gamma. \tag{2.20}$$

From (2.20) we find the following energy estimate

$$\|\nabla\psi\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\tilde{\Gamma}_1} |\tilde{\mathbf{u}}_\tau|^2 U \,\mathrm{d}\Gamma \le K \int_{\tilde{\Gamma}_1} |\tilde{\mathbf{u}}_\tau|^2 U \,\mathrm{d}\Gamma, \tag{2.21}$$

where $K = K(\tilde{\mathbf{u}})$ depends on the $\mathcal{C}^1$ norm of $\tilde{\mathbf{u}}$.

Since $U \ge c_1$ for a constant $c_1 > 0$, we deduce from (2.21) that $\tilde{\mathbf{u}}_\tau = 0$, by using the Gronwall lemma and (2.5) (which gives us $\tilde{\mathbf{u}}_\tau(t = 0) = 0$). All these imply that $\tilde{\mathbf{u}}_\tau = 0$ at $x_3 = 0$. Then (2.21) implies that $\nabla\psi = 0$ which implies that $\psi = 0$ since we supposed the function ψ with zero average. □

We have thus shown that the initial problem (2.1) is equivalent to (2.8), (2.9) and (2.12), so we proceed from now on to solving these problems.

3. The Mapping Λ

In this section we rigorously define the operator Λ introduced in Section 2, and show that it possesses a fixed point in a suitable set, by using the Schauder fixed-point theorem. In particular, by proving the existence of the fixed point, we prove the existence of solutions for problems (2.8), (2.9) and (2.12). Supposing we know $(\mathbf{u}, p, \boldsymbol{\omega})$, we need to determine $(\tilde{\mathbf{u}}, \tilde{p}, \tilde{\boldsymbol{\omega}})$.

Setting $Q_\infty = \Omega_\infty \times (0,T)$, we also want to prove that the operator Λ, which associates $\tilde{\boldsymbol{\omega}}$ to $\boldsymbol{\omega}$, maps the Hölder space $\mathcal{C}^\alpha(Q_\infty)$ $(0<\alpha<1)$ into itself, and is continuous for T small enough. We then need to estimate $\tilde{\mathbf{u}}$, $\tilde{p}$ and $\tilde{\boldsymbol{\omega}}$ in the norm of the Hölder space $\mathcal{C}^\alpha(Q_\infty)$. We recall the definition of the norm on the Hölder space $\mathcal{C}^\alpha(Q_\infty)$

$$|f|_{\alpha,Q_\infty} = |f|_{0,Q_\infty} + H_x^\alpha(f) + H_t^\alpha(f), \tag{3.1}$$

where $|\cdot|_{0,Q_\infty}$ is the norm on $\mathcal{C}(Q_\infty)$ and

$$H_x^\alpha(f) = \sup_{\substack{x_1,x_2\in\Omega_\infty,\ x_1\neq x_2\\ t\in(0,T)}} \{|f(x_1,t)-f(x_2,t)|\cdot|x_1-x_2|^{-\alpha}\},$$

$$H_t^\alpha(f) = \sup_{\substack{x\in\Omega_\infty\\ t_1,t_2\in(0,T),\ t_1\neq t_2}} \{|f(x,t_1)-f(x,t_2)|\cdot|t_1-t_2|^{-\alpha}\}.$$

3.1. **Determination of $\tilde{\mathbf{u}}$.** We are now interested in solving problem (2.8)

$$\operatorname{curl}\tilde{\mathbf{u}} = \boldsymbol{\omega} \text{ on } \Omega, \tag{3.2a}$$

$$\operatorname{div}\tilde{\mathbf{u}} = 0 \text{ on } \Omega, \tag{3.2b}$$

$$\tilde{u}_3 = -U \text{ on } \Gamma_1\cup\Gamma_0, \tag{3.2c}$$

$$\tilde{\mathbf{u}} \text{ is periodic in } x_1,\ x_2, \tag{3.2d}$$

$$\int_{\tilde{\Gamma}_0} \tilde{u}_j(x_1,x_2,0)\,\mathrm{d}x_1\,\mathrm{d}x_2 = 0, \quad \text{for } j=1,2, \tag{3.2e}$$

assuming that

$$\begin{aligned} &\operatorname{div}\boldsymbol{\omega} = 0,\\ &\int_{\tilde{\Gamma}_0} \boldsymbol{\omega}\cdot\mathbf{n}\,\mathrm{d}\Gamma = \int_{\tilde{\Gamma}_1} \boldsymbol{\omega}\cdot\mathbf{n}\,\mathrm{d}\Gamma = 0. \end{aligned} \tag{3.3}$$

Equation (3.2) can be solved using Fourier series expansions. A function f of t, x_1, x_2, x_3, is written in Fourier modes as

$$f(t,x_1,x_2,x_3) = \sum_{k\in\mathbb{Z}^2} f_k(t,x_3)e^{i(k_1'x_1+k_2'x_2)}, \tag{3.4}$$

where $k_j' = 2\pi k_j/L_j$ for $j=1,2$.

Writing equation (3.2) in Fourier modes we find the following: For $|k|\neq 0$, we have the system,

$$ik_2'\tilde{u}_{3,k} - D_3\tilde{u}_{2,k} = \omega_{1,k}, \tag{3.5a}$$

$$D_3\tilde{u}_{1,k} - ik_1'\tilde{u}_{3,k} = \omega_{2,k}, \tag{3.5b}$$

$$ik_1'\tilde{u}_{2,k} - ik_2'\tilde{u}_{1,k} = \omega_{3,k}, \tag{3.5c}$$

$$ik_1'\tilde{u}_{1,k} + ik_2'\tilde{u}_{2,k} + D_3\tilde{u}_{3,k} = 0, \tag{3.5d}$$

$$\tilde{u}_{3,k}(t,0) = -U_k(t),\ \tilde{u}_{3,k}(t,h) = -U_k(t),, \tag{3.5e}$$

where D_3 is the differential operator $\partial/\partial x_3$.

For $|k| = 0$ we find the system

$$-D_3\tilde{u}_{2,0} = \omega_{1,0}, \tag{3.6a}$$

$$D_3\tilde{u}_{1,0} = \omega_{2,0}, \tag{3.6b}$$

$$\tilde{u}_{1,0}(t,0) = \tilde{u}_{2,0}(t,0) = 0, \tag{3.6c}$$

$$\tilde{u}_{3,0}(t,0) = \tilde{u}_{3,0}(t,h) = -U_0(t), \tag{3.6d}$$

$$D_3\tilde{u}_{3,0} = 0. \tag{3.6e}$$

We note here that equation (3.5c) for $|k| = 0$ reduces to $\omega_{3,0} = 0$ which is true because of the solvability conditions (3.3). Condition (3.2e), which reduces to (3.6c), was necessary in order to ensure the full determination of $\tilde{u}_{1,0}$ and $\tilde{u}_{2,0}$, not just up to a constant. The resolution of (3.6) is then easy.

For $|k| \neq 0$, from (3.5) we obtain a second-order differential equation for $\tilde{u}_{3,k}$:

$$\begin{aligned} &ik_2'\tilde{u}_{3,k} - \frac{ik_2'}{|k'|^2}D_3^2\tilde{u}_{3,k} = \omega_{1,k}, \\ &\tilde{u}_{3,k}(t,0) = \tilde{u}_{3,k}(t,h) = -U_k(t). \end{aligned} \tag{3.7}$$

We also determine $\tilde{u}_{1,k}$ and $\tilde{u}_{2,k}$ as

$$\tilde{u}_{1,k} = \frac{ik_2'\omega_{3,k} + ik_1'D_3\tilde{u}_{3,k}}{|k'|^2}, \qquad \tilde{u}_{1,k} = \frac{-ik_1'\omega_{3,k} + ik_2'D_3\tilde{u}_{3,k}}{|k'|^2}. \tag{3.8}$$

We then solved system (3.2); precise estimates will follow.

3.2. **Estimates for $\tilde{\mathbf{u}}$.** In all that follows, we are interested in finding a proper estimate for $\tilde{\mathbf{u}}$ in terms of $|\boldsymbol{\omega}|_\alpha$. We take M such that

$$|\boldsymbol{\omega}|_{\alpha,Q} \leq M < \infty.$$

In all that follows M and T are fixed. The precise choice of M and T is made in Section 4.

From the system of equations (3.2) and using classical imbedding results (see [1]), we find the following estimates

$$|\tilde{\mathbf{u}}(t)|_{1+\alpha,\Omega_\infty} \leq C[|\boldsymbol{\omega}(t)|_{\alpha,\Omega_\infty} + |U|_{1+\alpha,\Gamma_1\cup\Gamma_0}] \leq C(1+M), \tag{3.9}$$

and

$$\|\tilde{\mathbf{u}}(t)\|_{W^{1,q}(\Omega)} \le C[\|\boldsymbol{\omega}(t)\|_{L^q(\Omega)} + \|U\|_{W^{1,q}(\tilde{\Gamma}_1 \cup \tilde{\Gamma}_0)}] \text{ for } q \ge 1. \tag{3.10}$$

Here and in all that follows, we understand by C a constant depending only on the domain and on the initial data, which can vary at different instances.

We write equation (3.2) for two arbitrary values of t_1 and t_2. Subtracting the resulting equations one from another and writing (3.10) for $\tilde{\mathbf{u}}(t_1) - \tilde{\mathbf{u}}(t_2)$, we find

$$\begin{aligned} &\|\tilde{\mathbf{u}}(t_1) - \tilde{\mathbf{u}}(t_2)\|_{W^{1,q}(\Omega)} \\ &\le C[\|\boldsymbol{\omega}(t_1) - \boldsymbol{\omega}(t_2)\|_{L^q(\Omega)} + \|U(t_1) - U(t_2)\|_{W^{1,q}(\tilde{\Gamma}_1 \cup \tilde{\Gamma}_0)}]. \end{aligned} \tag{3.11}$$

Since we take U differentiable from $[0, T]$ into $W^{1,q}(\tilde{\Gamma}_1 \cup \tilde{\Gamma}_0)$, we have

$$\|U(t_1) - U(t_2)\|_{W^{1,q}(\tilde{\Gamma}_1 \cup \tilde{\Gamma}_0)} \le CT^{1-\alpha}|t_1 - t_2|^\alpha,$$

for all α with $0 < \alpha < 1$. A certain value of α will be chosen later.

Using this relation, (3.11) reads as

$$\|\tilde{\mathbf{u}}(t_1) - \tilde{\mathbf{u}}(t_2)\|_{W^{1,q}(\Omega)} \le C[H_t^\alpha(\boldsymbol{\omega}) + T^{1-\alpha}]|t_1 - t_2|^\alpha. \tag{3.12}$$

We remember that the equation for ω is

$$\frac{\partial \omega}{\partial t} + (\mathbf{u} \cdot \nabla)\omega - (\omega \cdot \nabla)\mathbf{u} = \text{curl } \mathbf{f}. \tag{3.13}$$

Since curl $\tilde{\mathbf{u}} = \boldsymbol{\omega}$, we find

$$\text{curl } \Big(\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\mathbf{u} \cdot \nabla)\tilde{\mathbf{u}} - \nabla \cdot (\mathbf{u} \otimes \tilde{\mathbf{u}}) - \mathbf{f}\Big) = 0, \tag{3.14}$$

where by $\nabla \cdot (\mathbf{u} \otimes \tilde{\mathbf{u}})$ we understand the vector having the components

$$(\nabla \cdot (\mathbf{u} \otimes \tilde{\mathbf{u}}))_j = \sum_{i=1}^{3} u_i \frac{\partial \tilde{u}_i}{\partial x_j}.$$

From (3.14) we find a distribution $\tilde{\pi}$ such that

$$\frac{\partial \tilde{\mathbf{u}}}{\partial t} + (\mathbf{u} \cdot \nabla)\tilde{\mathbf{u}} - \nabla \cdot (\mathbf{u} \otimes \tilde{\mathbf{u}}) + \nabla \tilde{\pi} = \mathbf{f}. \tag{3.15}$$

We introduce the function ϕ, a solution of the following problem:

$$\begin{aligned}&\Delta\phi = 0 \text{ for } x \in \Omega,\\&\frac{\partial\phi}{\partial n} = -U \text{ on } \Gamma_1,\\&\frac{\partial\phi}{\partial n} = U \text{ on } \Gamma_0,\\&\phi \text{ periodic in } x_1,\, x_2,\end{aligned} \tag{3.16}$$

and we ask ϕ to be of average zero on Ω (for ensuring the complete determination of the function). For ϕ the following estimates hold:

$$|\nabla\phi(t)|_{1+\alpha,\Omega_\infty} \le C, \quad |\nabla\phi(t_1) - \nabla\phi(t_2)|_{\alpha,\Omega_\infty} \le C|t_1 - t_2|. \tag{3.17}$$

We now take the scalar product of (3.15) with $\tilde{\mathbf{u}} - \nabla\phi$ in $L^2(\Omega)$. The advantage of multiplying with this function instead of $\tilde{\mathbf{u}}$ is that $\tilde{\mathbf{u}} - \nabla\phi$ has the normal velocity on $\Gamma_1 \cup \Gamma_0$ equal to zero besides being divergence free. We find

$$\Big(\frac{\partial\tilde{\mathbf{u}}}{\partial t}, \tilde{\mathbf{u}}\Big)_{L^2(\Omega)} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\mathbf{u}}\|^2_{L^2(\Omega)}.$$

Since the normal component of $\tilde{\mathbf{u}} - \nabla\phi$ on $\Gamma_1 \cup \Gamma_0$ is zero and its divergence vanishes, we have

$$(\nabla\pi, \tilde{\mathbf{u}} - \nabla\phi)_{L^2(\Omega)} = 0.$$

Using $\operatorname{div}\mathbf{f} = 0$ and $f_3 = 0$ on $\Gamma_0 \cup \Gamma_1$, we find

$$(\mathbf{f}, \nabla\phi)_{L^2(\Omega)} = 0.$$

We also compute

$$\begin{aligned}\Big(\frac{\partial\tilde{\mathbf{u}}}{\partial t}, \nabla\phi\Big)_{L^2(\Omega)} &= \int_{\tilde{\Gamma}_1\cup\tilde{\Gamma}_0} \frac{\partial(\tilde{\mathbf{u}}\cdot\mathbf{n})}{\partial t}\phi\,\mathrm{d}\Gamma = \int_{\tilde{\Gamma}_1\cup\tilde{\Gamma}_0} \frac{\partial}{\partial t}\Big(\frac{\partial\phi}{\partial n}\Big)\phi\,\mathrm{d}\Gamma\\&= \int_\Omega \Delta\frac{\partial\psi}{\partial t}\cdot\psi\,\mathrm{d}\Omega + \int_\Omega \nabla\psi_t\cdot\nabla\psi\,\mathrm{d}\Omega = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi\|^2_{L^2(\Omega)}.\end{aligned}$$

We find

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tilde{\mathbf{u}}\|^2_{L^2(\Omega)} &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla\phi\|^2_{L^2(\Omega)} + (\mathbf{f}, \tilde{\mathbf{u}} - \nabla\phi)_{L^2(\Omega)}\\&\quad + (\nabla\cdot(\mathbf{u}\otimes\tilde{\mathbf{u}}), \tilde{\mathbf{u}} - \nabla\phi)_{L^2(\Omega)} + ((\mathbf{u}\cdot\nabla)\tilde{\mathbf{u}}, \tilde{\mathbf{u}} - \nabla\phi)_{L^2(\Omega)}.\end{aligned} \tag{3.18}$$

It remains to estimate the last three terms from the right-hand side of (3.18). The last term can be estimated, using (3.9), as

$$((\mathbf{u}\cdot\nabla)\tilde{\mathbf{u}}, \tilde{\mathbf{u}})_{L^2(\Omega)} = \frac{1}{2}\int_{\tilde{\Gamma}_1\cup\tilde{\Gamma}_0} (\mathbf{u}\cdot\mathbf{n})|\tilde{\mathbf{u}}|^2\,\mathrm{d}\Gamma \le C(1 + M^2). \tag{3.19}$$

The other terms are treated in a similar manner. Introducing these estimates in (3.18), and using (3.17), the differential inequality obtained leads to

$$\|\tilde{\mathbf{u}}(t)\|_{L^2(\Omega)} \leq C(1 + M^2T)\,\forall\, t \in (0, T). \tag{3.20}$$

Repeating the same kind of reasoning for $\mathbf{u}(t_1) - \mathbf{u}(t_2)$, we deduce that

$$\|\tilde{\mathbf{u}}(t_1) - \tilde{\mathbf{u}}(t_2)\|_{L^2(\Omega)} \leq C|t_1 - t_2|(1 + M^2T) \leq NT^{1-\alpha}|t_1 - t_2|^{\alpha}, \tag{3.21}$$

where here, and in the sequel, N denotes a constant depending on T and M but the limit of this constant when $T \to 0$ is independent of M.

3.3. **Determination of the pressure $\tilde{p}$.** The next step is to find, from (2.9), the pressure $\tilde{p}$

$$\Delta \tilde{p} = -\sum_{i,j=1}^{3} \frac{\partial \tilde{u}_i}{\partial x_j}\frac{\partial \tilde{u}_j}{\partial x_i} \text{ on } \Omega, \tag{3.22a}$$

$$\frac{\partial \tilde{p}}{\partial n} = \frac{\partial U}{\partial t} + 2\tilde{\mathbf{u}}_\tau \cdot \nabla_\tau U \text{ on } \Gamma_1, \tag{3.22b}$$

$$\frac{\partial \tilde{p}}{\partial n} = -\frac{\partial U}{\partial t} - 2\tilde{\mathbf{u}}_\tau \cdot \nabla_\tau U + \mathrm{div}_\tau\,(U\tilde{\mathbf{u}}_\tau) \text{ on } \Gamma_0, \tag{3.22c}$$

$$\tilde{p} \text{ is periodic in } x_1, x_2, \tag{3.22d}$$

$$\int_\Omega \tilde{p}\ \mathrm{d}\Omega = 0. \tag{3.22e}$$

Equation (3.22) is a Neumann problem and all we need to prove is that the problem is solvable, meaning that

$$\int_\Omega \Delta \tilde{p}\,\mathrm{d}\Omega = \int_{\tilde{\Gamma}_1 \cup \tilde{\Gamma}_0} \frac{\partial \tilde{p}}{\partial \mathbf{n}}\,\mathrm{d}\Gamma. \tag{3.23}$$

From (3.22a) we deduce that

$$\int_\Omega \Delta \tilde{p}\,\mathrm{d}\Omega = -\int_\Omega \mathrm{div}\,((\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}})\,\mathrm{d}\Omega = -\int_{\tilde{\Gamma}_1 \cup \tilde{\Gamma}_0} [(\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}}] \cdot \mathbf{n}\,\mathrm{d}\Gamma. \tag{3.24}$$

We need to compute $(\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} \cdot \mathbf{n}$ on Γ_1 and Γ_0.

Using the fact that $\mathrm{div}\,\tilde{\mathbf{u}} = 0$ and $\tilde{\mathbf{u}} \cdot \mathbf{n} = U$ on $\Gamma_1 \cup \Gamma_0$, on Γ_1 we find

$$(\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} \cdot \mathbf{n} = -2\tilde{\mathbf{u}}_\tau \cdot \nabla_\tau U + \mathrm{div}_\tau(U\tilde{\mathbf{u}}_\tau) = \frac{\partial \tilde{p}}{\partial \mathbf{n}} + \frac{\partial U}{\partial t} + \mathrm{div}_\tau(U\tilde{\mathbf{u}}_\tau),$$

and on Γ_0 we find

$$(\tilde{\mathbf{u}} \cdot \nabla)\tilde{\mathbf{u}} \cdot \mathbf{n} = 2\tilde{\mathbf{u}}_\tau \cdot \nabla_\tau U - \mathrm{div}_\tau(U\tilde{\mathbf{u}}_\tau) = -\frac{\partial \tilde{p}}{\partial \mathbf{n}} - \frac{\partial U}{\partial t}.$$

Using the relations above, we obtain

$$\int_{\Omega} \Delta p \, \mathrm{d}\Omega = \int_{\tilde{\Gamma}_1 \cup \tilde{\Gamma}_0} \frac{\partial \tilde{p}}{\partial \mathbf{n}} \, \mathrm{d}\Gamma - \int_{\tilde{\Gamma}_1} \frac{\partial U}{\partial t} \, \mathrm{d}\Gamma + \int_{\tilde{\Gamma}_0} \frac{\partial U}{\partial t} \, \mathrm{d}\Gamma - \int_{\tilde{\Gamma}_1} \mathrm{div}_\tau (U \tilde{\mathbf{u}}_\tau) \, \mathrm{d}\Gamma$$
$$= \int_{\tilde{\Gamma}_1 \cup \tilde{\Gamma}_0} \frac{\partial \tilde{p}}{\partial \mathbf{n}} \, \mathrm{d}\Gamma, \tag{3.25}$$

which means that the solvability condition is satisfied.

3.4. **Estimates for the pressure $\tilde{p}$.** In this subsection we consider equation (3.22) and we need to estimate the pressure $\tilde{p}$. Note here that in the boundary condition (2.12b)–(2.12d) for $\tilde{\boldsymbol{\omega}}$ on Γ_1, the pressure $\tilde{p}$ appears, so we need to obtain an estimate for the pressure on Γ_1. The idea in what follows is to estimate $\tilde{p}$ on Γ_1 but not on Γ_0 because for the next step, when we find $\tilde{\boldsymbol{\omega}}$, we need only boundary conditions on Γ_1 and considering $\tilde{p}$ on Γ_0 we would demand more regularity on $\tilde{\mathbf{u}}$ than actually required. At this precise point the approaches in [1] and [11] diverge. Note that the difference between the boundary condition for $\tilde{p}$ on Γ_1 and on Γ_0 is that, on Γ_0, we need not only $\tilde{\mathbf{u}}_\tau$ but also the derivatives of $\tilde{\mathbf{u}}_\tau$.

Let Ω' be a subdomain of Ω containing Γ_1 as boundary such that $\bar{\Omega}' \cap \bar{\Gamma}_0 = \emptyset$. Then (see e.g. [1])

$$|\tilde{p}(t)|_{2+\alpha,\Omega'} \le C\Big[|\Delta \tilde{p}|_{\alpha,\Omega} + \Big|\frac{\partial \tilde{p}}{\partial \mathbf{n}}\Big|_{1+\alpha,\Gamma_1} + \|\tilde{p}(t)\|_{L^2(\Omega)}\Big], \tag{3.26a}$$

$$\|\tilde{p}(t)\|_{W^{2,q}(\Omega')} \le C\Big[\|\Delta \tilde{p}\|_{L^q(\Omega)} + \Big\|\frac{\partial \tilde{p}}{\partial \mathbf{n}}\Big\|_{W^{1,q}(\Gamma_1)} + \|\tilde{p}(t)\|_{L^2(\Omega)}\Big]. \tag{3.26b}$$

Taking into account (3.22a), we also have

$$\|\Delta \tilde{p}\|_{L^q(\Omega)} \le C|\nabla \tilde{\mathbf{u}}|_{0,\Omega} \|\nabla \tilde{\mathbf{u}}\|_{L^q(\Omega)}. \tag{3.27}$$

By the Sobolev imbeddings, we know that $W^{2,q}(\Omega') \subset \mathcal{C}^{1+\alpha}(\Omega')$ and the imbedding is continuous for $q > 3$ and $\alpha = (q-3)/q$. Using (3.9), (3.10) and (3.22b), we thus continue estimating $\tilde{p}$ as follows

$$|\nabla \tilde{p}(t)|_{\alpha,\Omega'} \le C|\tilde{p}|_{W^{2,q}(\Omega')} \le C\Big[|\nabla \tilde{\mathbf{u}}|_{0,\Omega} \|\nabla \tilde{\mathbf{u}}\|_{L^q(\Omega)} + \Big\|\frac{\partial \tilde{p}}{\partial \mathbf{n}}\Big\|_{W^{1,q}(\Gamma_1)} + \|\tilde{p}\|\Big]$$
$$\le C(1 + M^2 + \|\tilde{p}\|). \tag{3.28}$$

Taking t_1 and t_2 two arbitrary instants of time and repeating the reasoning for $\tilde{p}(t_1) - \tilde{p}(t_2)$, we find

$$|\nabla \tilde{p}(t_1) - \nabla \tilde{p}(t_2)|_{\alpha,\Omega'}$$

$$\leq C\Big\{\| \sum_{i,j=1}^{3} (D_i\tilde{u}_j\, D_j\tilde{u}_i)(t_1) - \sum_{i,j=1}^{3} (D_i\tilde{u}_j\, D_j\tilde{u}_i)(t_2)\|_{L^q(\Omega)} + \Big\|\frac{\partial\tilde{p}}{\partial\mathbf{n}}(t_1) - \frac{\partial\tilde{p}}{\partial\mathbf{n}}(t_2)\Big\|_{W^{1,q}(\Gamma_1)} + \|\tilde{p}(t_1) - \tilde{p}(t_2)\|\Big\}. \tag{3.29}$$

Using (3.9) and (3.12), we have

$$\begin{aligned} &\| \sum_{i,j=1}^{3} D_i\tilde{u}_j\, D_j\tilde{u}_i(t_1) - \sum_{i,j=1}^{3} D_i\tilde{u}_j\, D_j\tilde{u}_i(t_2)\|_{L^q(\Omega)} \\ &\quad\leq \text{(adding and substracting } \sum_{i,j=1}^{3} (D_i\tilde{u}_j)(t_1)\,(D_j\tilde{u}_i)(t_2)) \qquad (3.30)\\ &\leq C(1+M)\|\tilde{\mathbf{u}}(t_1) - \tilde{\mathbf{u}}(t_2)\|_{W^{1,q}(\Omega)} \leq C(1+M)\{H_t^\alpha(\tilde{\boldsymbol{\omega}}) + T^{1-\alpha}\}|t_1-t_2|^\alpha. \end{aligned}$$

Since we supposed $U \in \mathcal{C}^{2+\alpha,1+\alpha}((\Gamma_1\cup\Gamma_2)\times[0,T])$, using (3.22b) we also find

$$\Big\|\frac{\partial\tilde{p}}{\partial\mathbf{n}}(t_1) - \frac{\partial\tilde{p}}{\partial\mathbf{n}}(t_2)\Big\|_{W^{1,q}(\Gamma_1)} \leq C(1 + H_t^\alpha(\tilde{\boldsymbol{\omega}}) + T^{1-\alpha})|t_1-t_2|^\alpha. \tag{3.31}$$

Returning to (3.29), we find

$$\begin{aligned} |\nabla\tilde{p}(t_1) - \nabla\tilde{p}(t_2)|_{\alpha,\Omega'} \leq &C\{(1+M)(H_t^\alpha(\tilde{\boldsymbol{\omega}}) + T^{1-\alpha}) + 1\}|t_1-t_2|^\alpha \\ &+ C\|\tilde{p}(t_1) - \tilde{p}(t_2)\|. \end{aligned} \tag{3.32}$$

We need to estimate the $L^2(\Omega)$ norm of the pressure $\tilde{p}$. We introduce the function $\tilde{\psi}$, a solution of the Neumann-like problem

$$\Delta\tilde{\psi} = \tilde{p} \text{ on } \Omega, \tag{3.33a}$$

$$\frac{\partial\tilde{\psi}}{\partial\mathbf{n}} = 0 \text{ on } \Gamma_1\cup\Gamma_0, \tag{3.33b}$$

$$\tilde{\psi} \text{ is periodic in } x_1,\, x_2, \tag{3.33c}$$

and we also, suppose that the function $\tilde{\psi}$ has zero average on Ω. Since we required that $\int_\Omega \tilde{p}\,\mathrm{d}\Omega = 0$, the problem (3.33) is solvable and we estimate $\tilde{\psi}$ as

$$\|\tilde{\psi}(t)\|_{H^2(\Omega)} \leq C\|\tilde{p}(t)\|_{L^2(\Omega)}. \tag{3.34}$$

Using the fact that $\partial\tilde{\psi}/\partial\mathbf{n} = 0$ on $\Gamma_0 \cup \Gamma_1$, we deduce that

$$\begin{aligned}\|\tilde{p}(t)\|^2_{L^2(\Omega)} &= (\Delta\tilde{\psi}, \tilde{p})_{L^2(\Omega)} = -(\nabla\tilde{\psi}, \nabla\tilde{p})_{L^2(\Omega)} \\ &= (\tilde{\psi}, \Delta\tilde{p})_{L^2(\Omega)} - \int_{\tilde{\Gamma}_1\cup\tilde{\Gamma}_0} \frac{\partial\tilde{p}}{\partial\mathbf{n}}\tilde{\psi}\,\mathrm{d}\Gamma.\end{aligned} \tag{3.35}$$

Substituting Δp for the corresponding value from (3.22a), we find

$$\begin{aligned}(\tilde{\psi}, \Delta\tilde{p})_{L^2(\Omega)} &= -(\tilde{\psi}, \operatorname{div}[(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}}])_{L^2(\Omega)} \\ &= -\int_{\tilde{\Gamma}_1\cup\tilde{\Gamma}_0} \tilde{\psi}[(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}}]\cdot\mathbf{n}\,\mathrm{d}\Gamma + (\nabla\tilde{\psi}, (\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}})_{L^2(\Omega)}.\end{aligned} \tag{3.36}$$

Taking into account (2.9b)-(2.9c) we find, on Γ_1,

$$[(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}}]\cdot\mathbf{n} = -\frac{\partial\tilde{p}}{\partial n} + \frac{\partial U}{\partial t} + \operatorname{div}_\tau(U\tilde{\mathbf{u}}_\tau),$$

and on Γ_0

$$[(\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}}]\cdot\mathbf{n} = -\frac{\partial\tilde{p}}{\partial\mathbf{n}} - \frac{\partial U}{\partial t},$$

and we will use these relations in computing the first term from the right-hand side of (3.36).

We now treat the second term from the right-hand side of (3.36). Using integration by parts, we find

$$(\nabla\tilde{\psi}, (\tilde{\mathbf{u}}\cdot\nabla)\tilde{\mathbf{u}})_{L^2(\Omega)} = -\int_{\tilde{\Gamma}_1} U\tilde{\mathbf{u}}\cdot\nabla\tilde{\psi}\,\mathrm{d}\Gamma + \int_{\tilde{\Gamma}_0} U\tilde{\mathbf{u}}\cdot\nabla\tilde{\psi}\,\mathrm{d}\Gamma - (\tilde{\mathbf{u}}, (\tilde{\mathbf{u}}\cdot\nabla)\nabla\tilde{\psi})_{L^2(\Omega)}. \tag{3.37}$$

Returning to equation (3.35), we find

$$\begin{aligned}\|\tilde{p}(t)\|^2_{L^2(\Omega)} = &-\int_{\tilde{\Gamma}_1} U\tilde{\mathbf{u}}\cdot\nabla\tilde{\psi}\,\mathrm{d}\Gamma + \int_{\tilde{\Gamma}_0} U\tilde{\mathbf{u}}\cdot\nabla\tilde{\psi}\,\mathrm{d}\Gamma - \int_{\tilde{\Gamma}_1} \frac{\partial U}{\partial t}\tilde{\psi}\,\mathrm{d}\Gamma \\ &+ \int_{\tilde{\Gamma}_0} \frac{\partial U}{\partial t}\tilde{\psi}\,\mathrm{d}\Gamma - (\tilde{\mathbf{u}}, (\tilde{\mathbf{u}}\cdot\nabla)\nabla\tilde{\psi})_{L^2(\Omega)}.\end{aligned} \tag{3.38}$$

Taking into account (2.4), we have the following estimates

$$\Big|\int_{\tilde{\Gamma}_1} \frac{\partial U}{\partial t}\psi\,\mathrm{d}\Gamma\Big| \le C\|\tilde{\psi}\|_{L^2(\tilde{\Gamma}_1\cup\tilde{\Gamma}_0)}, \quad \Big|\int_{\tilde{\Gamma}_0} \frac{\partial U}{\partial t}\tilde{\psi}\,\mathrm{d}\Gamma\Big| \le C\|\tilde{\psi}\|_{L^2(\tilde{\Gamma}_1\cup\tilde{\Gamma}_0)}. \tag{3.39}$$

Using the Sobolev imbeddings (see e.g. [8]), more precisely using the fact that

$$\|f\|_{L^4(\tilde{\Gamma}_1\cup\tilde{\Gamma}_0)} \le C\|f\|_{H^{1/2}(\tilde{\Gamma}_0\cup\tilde{\Gamma}_1)} \le C\|f\|_{H^1(\Omega)},$$

we continue estimating

$$\Big|\int_{\tilde\Gamma_0} U\mathbf{u}\cdot\nabla\tilde\psi\,\mathrm{d}\Gamma\Big| \le C\|\tilde{\mathbf{u}}\|_{L^{4/3}(\tilde\Gamma_0)}\|\nabla\tilde\psi\|_{L^4(\tilde\Gamma_0)} \le C\|\tilde{\mathbf{u}}\|_{L^{4/3}(\tilde\Gamma_0)}\|\nabla\tilde\psi\|_{H^1(\Omega)}$$
$$\le C\|\tilde{\mathbf{u}}\|_{L^{4/3}(\tilde\Gamma_0)}\|\tilde\psi\|_{H^2(\Omega)} \le C\|\tilde{\mathbf{u}}\|_{L^{4/3}(\tilde\Gamma_0)}\|\tilde p\|_{L^2(\Omega)}. \tag{3.40}$$

We use similar arguments for the first term from (3.38).

It remains to estimate the last term from the right-hand side of (3.38). We find

$$|(\tilde{\mathbf{u}},(\tilde{\mathbf{u}}\cdot\nabla)\nabla\tilde\psi)_{L^2(\Omega)}| \le C\|\tilde{\mathbf{u}}\|^2_{L^4(\Omega)}\|\tilde p\|_{L^2(\Omega)}. \tag{3.41}$$

Returning to (3.38), we estimate $\tilde p$ as

$$\|\tilde p(t)\|_{L^2(\Omega)} \le C(1+\|\tilde{\mathbf{u}}(t)\|^2_{L^4(\Omega)} + \|\tilde{\mathbf{u}}(t)\|_{L^{4/3}(\tilde\Gamma_1\cup\tilde\Gamma_0)}). \tag{3.42}$$

Considering t_1 and t_2 two arbitrary instants of time and taking the difference $\tilde p(t_1)-\tilde p(t_2)$, we similarly have

$$\|\tilde p(t_1)-\tilde p(t_2)\|_{L^2(\Omega)} \le C(|t_1-t_2| + \sup_{0\le t\le T}\|\tilde{\mathbf{u}}(t)\|_{L^4(\Omega)}\|\tilde{\mathbf{u}}(t_1)-\tilde{\mathbf{u}}(t_2)\|_{L^4(\Omega)}$$
$$+\|\tilde{\mathbf{u}}(t_1)-\tilde{\mathbf{u}}(t_2)\|_{L^{4/3}(\tilde\Gamma_1\cup\tilde\Gamma_0)}). \tag{3.43}$$

In order to estimate the $L^4(\Omega)$ norm and the $L^{4/3}(\tilde\Gamma_1\cup\tilde\Gamma_0)$ norm of $\tilde{\mathbf{u}}$, we use the following imbedding and trace properties (see e.g. [1])

$$\|f\|_{L^4(\Omega)} \le C\|f\|^a_{W^{1,q}(\Omega)}\|f\|^{1-a}_{L^2(\Omega)},\ \text{where } q=3/(1-\alpha),\ a=3/(3+2\alpha), \tag{3.44a}$$

$$\|f\|_{L^{4/3}(\tilde\Gamma_1\cup\tilde\Gamma_0)} \le C\|f\|^b_{W^{1,q}(\Omega)}\|f\|^{1-b}_{L^2(\Omega)},\ \text{where } q=3/(1-\alpha),\ b=3/(3+\alpha). \tag{3.44b}$$

Applying these inequalities to $f=\tilde{\mathbf{u}}(t_1)-\tilde{\mathbf{u}}(t_2)$, we can estimate the right-hand side of (3.45) and we find

$$\|\tilde p(t_1)-\tilde p(t_2)\|_{L^2(\Omega)} \le NT^{\alpha(1-\alpha)/(3+\alpha)}|t_1-t_2|^\alpha. \tag{3.45}$$

Returning to (3.29), we conclude

$$|\nabla\tilde p(t_1)-\nabla\tilde p(t_2)|_{\alpha,\Omega'} \le C(1+M^2+NT^{\alpha(1-\alpha)/(3+\alpha)})|t_1-t_2|^\alpha. \tag{3.46}$$

3.5. **Determination of the vorticity $\tilde{\omega}$.** The last step of the method of approximation is finding the vorticity $\tilde{\boldsymbol{\omega}}$ knowing already $\tilde{\mathbf{u}}$ and $\tilde{p}$. The system of equations satisfied by $\boldsymbol{\omega}$ is

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\boldsymbol{\omega}} - (\tilde{\boldsymbol{\omega}} \cdot \nabla)\tilde{\mathbf{u}} = \text{curl } \mathbf{f}, \tag{3.47a}$$

$$\tilde{\omega}_1 = \frac{1}{U}(f_2 - \frac{\partial \tilde{p}}{\partial x_2}) - \frac{\partial U}{\partial x_2} \text{ on } \Gamma_1, \tag{3.47b}$$

$$\tilde{\omega}_2 = -\frac{1}{U}(f_1 - \frac{\partial \tilde{p}}{\partial x_1}) + \frac{\partial U}{\partial x_1}) \text{ on } \Gamma_1, \tag{3.47c}$$

$$\tilde{\omega}_3 = 0 \text{ on } \Gamma_1, \tag{3.47d}$$

$$\tilde{\boldsymbol{\omega}} \text{ is periodic in } x_1,\, x_2, \tag{3.47e}$$

$$\tilde{\boldsymbol{\omega}}(t=0) = \text{curl } \tilde{\mathbf{u}}_0. \tag{3.47f}$$

As we have seen at the first step of the iteration, the solvability condition for the problem in $\tilde{\mathbf{u}}$ is $\operatorname{div} \tilde{\boldsymbol{\omega}} = 0$, for every k. We start by proving that indeed $\operatorname{div} \tilde{\boldsymbol{\omega}} = 0$.

Proposition 3.1. *At every step k,* $\operatorname{div} \tilde{\boldsymbol{\omega}} = 0$ *for all* $(x,t) \in Q$.

Proof. We apply the operator div to equation (3.47a) and we find the following system for $\theta = \operatorname{div} \tilde{\boldsymbol{\omega}}$

$$\frac{\partial \theta}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\theta = 0, \tag{3.48a}$$

$$\theta(t=0) = \operatorname{div} \tilde{\boldsymbol{\omega}}(t=0) = \operatorname{div} \operatorname{curl} \tilde{\mathbf{u}}_0 = 0. \tag{3.48b}$$

Since we know that $\tilde{\mathbf{u}} \cdot \mathbf{n} < 0$ on Γ_1, we only need to prove that $\theta = \operatorname{div} \tilde{\boldsymbol{\omega}}$ is zero on Γ_1, and (3.48) will imply that $\theta = 0$ at all time.

On Γ_1 all the components of the vector $\tilde{\boldsymbol{\omega}}$ are given. Writing the third component of equation (3.47a), restricting it to Γ_1 and taking into account that $\tilde{\omega}_3 = 0$ on Γ_1, we find

$$\tilde{u}_3 \frac{\partial \tilde{\omega}_3}{\partial x_3} - \tilde{\omega}_1 \frac{\partial \tilde{u}_3}{\partial x_1} - \tilde{\omega}_2 \frac{\partial \tilde{u}_3}{\partial x_2} = (\text{curl } \mathbf{f})_3 \text{ on } \Gamma_1,$$

which implies

$$\tilde{u}_3 \operatorname{div} \tilde{\boldsymbol{\omega}} - \operatorname{div}_\tau(\tilde{u}_3 \tilde{\boldsymbol{\omega}}_\tau) = (\text{curl } \mathbf{f})_3 \text{ on } \Gamma_1. \tag{3.49}$$

Using relations (3.47b) and (3.47c), and substituting into $\operatorname{div}_\tau(\tilde{u}_3 \tilde{\boldsymbol{\omega}}_\tau)$, we find

$$\operatorname{div}_\tau(\tilde{u}_3 \tilde{\boldsymbol{\omega}}_\tau) = \operatorname{div}_\tau(U \tilde{\boldsymbol{\omega}}_\tau) = -(\text{curl } \mathbf{f})_3 \text{ on } \Gamma_1. \tag{3.50}$$

Returning to (3.49) we find that $\operatorname{div}\tilde{\boldsymbol{\omega}} = 0$ on Γ_1. With this, the proof of the proposition is completed. □

The method of characteristics. We can now start proving the existence of $\tilde{\boldsymbol{\omega}}$, a solution of (3.47). To solve the system (3.47), we use the method of characteristics. The characteristic trajectories of problem (3.47a) are found as the solutions of the following ordinary differential problem

$$\begin{aligned}&\frac{\mathrm{d}y_i}{\mathrm{d}s} = \tilde{u}_i(y_1, y_2, y_3, s),\\ &y_i|_{s=t} = x_i,\ i = 1, 2, 3,\ x = (x_1, x_2, x_3) \in \Omega,\ t \in [0, T],\ 0 \le s \le t.\end{aligned} \tag{3.51}$$

Since we know that the vector $\tilde{\mathbf{u}}$ is defined only on the domain Ω, we have to consider two cases for the problem; firstly the case where the trajectory $y(s; x, t)$ remains inside the domain Ω for all time $s \in [0, t]$ or secondly the case where there exists a time $s = s_0(x, t)$ when the trajectory reaches the surface Γ_1. We define the function $\tau(x, t)$ as the time when the particle (x, t) enters the domain Ω through Γ_1. Hence, the function $\tau(x, t)$ is equal to zero in the first case and to $s_0(x, t)$ in the second one. The function $\tau(x, t)$ can be found as the solution of the equation

$$\phi(y(\tau; x, t)) = 0, \tag{3.52}$$

where ϕ is the parametrization of the boundary Γ_1, that is, $\phi(x) = x_3 - h$.

Assuming enough regularity, we can formally compute, using (3.52), the derivatives of the function $\tau(x, t)$. Since $y_3(\tau; x, t) - h = 0$, we find

$$\Big(\frac{\partial y_3}{\partial t} + \frac{\partial y_3}{\partial s}\frac{\partial \tau}{\partial t}\Big)\Big|_{s=\tau} = 0, \tag{3.53}$$

which implies

$$\Big(\frac{\partial y_3}{\partial t} + \tilde{u}_3\frac{\partial \tau}{\partial t}\Big)\Big|_{s=\tau} = 0. \tag{3.54}$$

From the equation above, we find

$$\frac{\partial \tau}{\partial t} = U^{-1}\frac{\partial y_3}{\partial t}\Big|_{y=y(\tau)}. \tag{3.55}$$

Similarly, we find the derivatives in x_i of τ

$$\frac{\partial \tau}{\partial x_i} = U^{-1}\frac{\partial y_3}{\partial x_i}\Big|_{y=y(\tau)},\ i = 1, 2, 3. \tag{3.56}$$

For an arbitrary function $f = f(s, x, t)$, we set

$$[f](x, t) = f(\tau(x, t), x, t). \tag{3.57}$$

Using this definition, we see that the function $[y](x,t) = y(\tau(x,t), x, t)$ is the point where the trajectory touches Γ_1.

Considering a function f defined on $\Gamma_1 \times (0,T)$ and on Ω at $t=0$, we can extend it to the whole domain Ω, at each instant of time $t \in [0,T]$ as

$$[[f]](x,t) = f(y(0;x,t),0),\ \tau = 0,\ x \in \Omega, \tag{3.58a}$$

$$[[f]](x,t) = f(y(\tau(x,t);x,t),\tau(x,t)),\ \tau > 0,\ y \in \Gamma_1. \tag{3.58b}$$

We first suppose that curl $\mathbf{f} = 0$. Equation (3.47a) then reads

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\boldsymbol{\omega}} - (\tilde{\boldsymbol{\omega}} \cdot \nabla)\tilde{\mathbf{u}} = 0;\ (x,t) \in Q. \tag{3.59}$$

Taking into account (3.51), we can also write (3.59) as

$$\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + \sum_{j=1}^{3} \frac{\mathrm{d}y_j}{\mathrm{d}s}\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial x_j} = \sum_{j=1}^{3} \omega_j \frac{\partial \tilde{\mathbf{u}}}{\partial x_j},\ i = 1,2,3,\ (y(s,x,t),s) \in Q. \tag{3.60}$$

Equation (3.60) for the vortex $\tilde{\boldsymbol{\omega}}$ along the trajectories $y(s,x,t)$ can be written as

$$\frac{\mathrm{d}\tilde{\omega}_i}{\mathrm{d}s} = \sum_{j=1}^{3} \tilde{\omega}_j \frac{\partial \tilde{u}_i}{\partial x_j};\ i = 1,2,3. \tag{3.61}$$

On the other hand, differentiating (3.51) in x_m for $m = 1,2,3$, we find

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial y_i}{\partial x_m} = \sum_{j=1}^{3} \frac{\partial \tilde{u}_i}{\partial y_j}\frac{\partial y_j}{\partial x_m}, \\ &\frac{\partial y_i}{\partial x_m}\Big|_{s=t} = \delta_{im}. \end{aligned} \tag{3.62}$$

We notice the similitude between equations (3.61) and (3.62), meaning that $\tilde{\omega}_i$ and $\partial y_i/\partial x_m$ satisfy the same equations along the trajectories y but the initial data for $\tilde{\omega}_i$ is given at $s = \tau$ (the boundary conditions for $\tilde{\omega}$ are given on Γ_1) and for $\partial y_i/\partial x_m$ at $s = t$.

We also have, differentiating in t

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial y_i}{\partial t} = \sum_{j=1}^{3} \frac{\partial \tilde{u}_i}{\partial y_j}\frac{\partial y_j}{\partial t}, \\ &\frac{\partial y_i}{\partial t}\Big|_{s=t} = -\tilde{u}_i(x,t), \end{aligned} \tag{3.63}$$

where δ_{im} is the Kronecker symbol.

Combining equations (3.62) and (3.63), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\partial y_i}{\partial t}+\sum_{m=1}^{3}\frac{\partial y_i}{\partial x_m}\tilde{u}_m\Big)=\sum_{j=1}^{3}\frac{\partial \tilde{u}_i}{\partial y_j}\Big(\frac{\partial y_j}{\partial t}+\sum_{m=1}^{3}\frac{\partial y_j}{\partial x_m}\tilde{u}_m\Big). \tag{3.64}$$

Since the initial condition for equation (3.64) is

$$\frac{\partial y_i}{\partial t}\Big|_{s=t}+\sum_{m=1}^{3}\frac{\partial y_i}{\partial x_m}\tilde{u}_m\Big|_{s=t}=0,\ \text{for } i=1,2,3, \tag{3.65}$$

we write

$$\frac{\partial y_i}{\partial t}+\sum_{m=1}^{3}\frac{\partial y_i}{\partial x_m}\tilde{u}_m=0,\ \text{for } i=1,2,3. \tag{3.66}$$

Using (3.66), we obtain

$$\frac{\partial \tau}{\partial t}+\sum_{j=1}^{3}\tilde{u}_j(x,t)\frac{\partial \tau}{\partial x_j}=0. \tag{3.67}$$

In order to obtain an equation similar to (3.61) and having the initial condition at $s=\tau$ (the initial data are given on Γ_1 and at $t=0$), we need to change the coordinates. We consider the Lagrange coordinates (y,t), where $y=y(\tau(x,t),x,t)$. The Euler variables (x,t) are obtained as the solution of the Cauchy problem

$$\begin{aligned}&\frac{\mathrm{d}x_i}{\mathrm{d}s}=\tilde{u}_i(x_1,x_2,x_3,s),\\ &x_i|_{s=\tau}=y_i;\ i=1,2,3.\end{aligned} \tag{3.68}$$

We find the same equations for $\tilde{\omega}_j$ and $\partial x_j/\partial y_m$, the initial data being given at the same instant of time $s=\tau$. We then find $\tilde{\omega}_j$ as a combination of $\partial x_j/\partial y_m$,

$$\tilde{\omega}_i(s,y)=\sum_{j=1}^{3}\tilde{\omega}_j|_{s=\tau}\frac{\partial x_i}{\partial y_j},\ i=1,2,3. \tag{3.69}$$

Writing in (x,t) variables, we find the solution of problem (3.59) as

$$\tilde{\omega}_i^1(x,t)=\sum_{j=1}^{3}[[\tilde{\omega}_{0j}]](x,t)\big[J_{m,l}(y_{m+j-i},y_{l+j-i})\big](x,t), \tag{3.70}$$

where by $J_{m,l}(y_{m+j-i},y_{l+j-i})$ we understand

$$J_{m,l}(y_{m+j-i},y_{l+j-i})=\frac{\partial y_{m+j-i}}{\partial x_m}\frac{\partial y_{l+j-i}}{\partial x_l}-\frac{\partial y_{m+j-i}}{\partial x_l}\frac{\partial y_{l+j-i}}{\partial x_m},$$

and (i,m,l) is a cyclic permutation of $(1,2,3)$ and the indices $m+j-i$ and $l+j-i$ are taken modulo 3. Note also that in (3.70), by $\boldsymbol{\omega}_0$ we understand generically the initial data on Γ_1 (given by (3.47b)-(3.47d)) and the initial condition at $t=0$ (given by (3.47e)).

It now remains to find the solution of problem (3.47) for the general case when curl $\mathbf{f} \neq 0$. Since we know how to solve this problem when curl $\mathbf{f} = 0$ and the initial data are nonzero, we search for the general solution of problem (3.47) as the sum between the solution of the problem for curl $\mathbf{f} = 0$ and the solution of the following problem

$$\begin{aligned} &\frac{\partial \tilde{\boldsymbol{\omega}}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\tilde{\boldsymbol{\omega}} - (\tilde{\boldsymbol{\omega}} \cdot \nabla)\tilde{\mathbf{u}} = \text{curl } \mathbf{f}, \\ &\tilde{\boldsymbol{\omega}} = 0 \text{ on } \Gamma_1, \\ &\tilde{\boldsymbol{\omega}}|_{t=0} = 0. \end{aligned} \tag{3.71}$$

We solve this equation by Duhamel method (see for example [1]), which consists in solving the problem

$$\begin{aligned} &\frac{\partial \boldsymbol{\psi}}{\partial t} + (\tilde{\mathbf{u}} \cdot \nabla)\boldsymbol{\psi} - (\boldsymbol{\psi} \cdot \nabla)\tilde{\mathbf{u}} = 0, \text{ for } x \in \Omega,\ t > \xi \geq 0, \\ &\boldsymbol{\psi}|_{t=\xi} = \text{curl } \mathbf{f}(x,s),\ \psi|_{\Gamma_1} = 0. \end{aligned} \tag{3.72}$$

Problem (3.72) is solved using the reasoning made above, when we worked with curl $\mathbf{f} = 0$. The solution is

$$\psi_i(\xi,x,t) = \sum_{j=1}^{3} [[\psi_{0,j}]](\xi,x,t)[J_{m,l}(y_{m+j-i}, y_{l+j-i})](\xi,x,t), \tag{3.73}$$

where by ψ_0 we understand the initial condition for ψ, that is $\psi_0(x) =$ curl $\mathbf{f}(x,\xi)$ for $t=\xi$ and $x \in \Omega$ and $\psi_0(x) = 0$ for $x \in \Gamma_1$ and $t > \xi$.

The solution of problem (3.71) is

$$\tilde{\boldsymbol{\omega}}^2(x,t) = \int_0^t \boldsymbol{\psi}(\xi,x,t)\,\mathrm{d}\xi. \tag{3.74}$$

We find the solution of problem (3.47) as

$$\tilde{\boldsymbol{\omega}} = \tilde{\boldsymbol{\omega}}^1 + \tilde{\boldsymbol{\omega}}^2. \tag{3.75}$$

3.6. **Estimates for the trajectories and for the vorticity $\tilde{\boldsymbol{\omega}}$.** In all that follows, we want to estimate the vortex $\tilde{\boldsymbol{\omega}}$. We notice that we found the vortex $\tilde{\boldsymbol{\omega}}$ as the sum of two vectors $\tilde{\boldsymbol{\omega}}^1$ and $\tilde{\boldsymbol{\omega}}^2$, the difficult part being to determine $\tilde{\boldsymbol{\omega}}^1$ ($\tilde{\boldsymbol{\omega}}^2$ is obtained as the integral of a vector found by the same method as $\tilde{\boldsymbol{\omega}}^1$). Now we need to estimate only $\tilde{\boldsymbol{\omega}}^1$, because as long as we will be able to bound this vector, we can repeat the reasoning for $\tilde{\boldsymbol{\omega}}^2$ too.

Estimate of the $\mathcal{C}^0$ norm of the trajectories. We start estimating $\tilde{\boldsymbol{\omega}}^1$. For finding this vector we used the method of characteristics, so we need to estimate the trajectories. We remember that the equation for the trajectories is

$$\begin{aligned} &\frac{\mathrm{d}y_i}{\mathrm{d}s} = \tilde{u}_i(y_1, y_2, y_3, s), \\ &y_i|_{s=t} = x_i,\ i = 1,2,3,\ x = (x_1, x_2, x_3) \in \Omega,\ t \in [0,T],\ 0 \le s \le t. \end{aligned} \tag{3.76}$$

We find immediately, using (3.9), that

$$|y_i(s,x,t) - x_i| = |\int_t^s \tilde{u}_i(y_1, y_2, y_3, s')\,\mathrm{d}s'| \le C(1+M)T,\ \forall\, i = 1,2,3, \tag{3.77a}$$

$$|[y](x,t) - x| \le C(1+M)T, \tag{3.77b}$$

and we remember that $[y](x,t) = y(\tau(x,t),x,t)$. Note that we could write (3.77b) because in (3.77a) the bound is uniform in s.

From (3.62), we also estimate

$$\begin{aligned} &\Big|\frac{\partial y_i}{\partial x_m}\Big|_{0,Q_\infty} \le N,\ \Big|\frac{\partial y_i}{\partial x_m} - \delta_{im}\Big|_{0,Q_\infty} \le C(e^{MT}-1) \le NMT, \\ &\Big|\Big[\frac{\partial y_i}{\partial x_m}\Big]\Big|_{0,Q_\infty} \le N,\ \Big|\Big[\frac{\partial y_i}{\partial x_m}\Big] - \delta_{im}\Big|_{0,Q_\infty} \le NMT. \end{aligned} \tag{3.78}$$

From (3.66), we similarly find

$$\Big|\frac{\partial y_i}{\partial t}\Big|_{0,Q_\infty} \le NM,\ \Big|\Big[\frac{\partial y_i}{\partial t}\Big]\Big|_{0,Q_\infty} \le NM. \tag{3.79}$$

Using (3.56), (3.67) and (3.79), we also estimate the derivatives of τ

$$\Big|\frac{\partial \tau}{\partial t}\Big|_{0,Q_\infty} \le NM,\ \Big|\frac{\partial \tau}{\partial x_i}\Big|_{0,Q_\infty} \le N. \tag{3.80}$$

We need to study the Hölder continuity of the functions $\frac{\partial y_i}{\partial x_m}$ and $[\partial y_i/\partial x_m]$. System (3.62) can be written in a matrix form, as

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}s}\frac{\partial y}{\partial x} = \frac{\partial \tilde{u}}{\partial y} \cdot \frac{\partial y}{\partial x}, \\ &\frac{\partial y}{\partial x}\Big|_{s=t} = I, \end{aligned} \tag{3.81}$$

where by $\partial y/\partial x$ we understand the matrix with components $\partial y_i/\partial x_m$ (same rule for $\partial\tilde{u}/\partial y$); I is the unit matrix.

Hölder norm of the trajectories. Let us consider two arbitrary points x^1 and x^2 in Ω and the corresponding solutions $y^1 = y(s,x^1,t)$ and $y^2 = y(s,x^2,t)$ of problem (3.62). Setting $z = \partial y/\partial x(x^1) - \partial y/\partial x(x^2)$, we see that z satisfies the following system

$$\begin{aligned}&\frac{\mathrm{d}z}{\mathrm{d}s} = \frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^1)\cdot z + \Big\{\frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^1) - \frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^2)\Big\}\cdot\frac{\partial y}{\partial x}(x_2),\\ &z|_{s=t} = 0.\end{aligned} \tag{3.82}$$

We apply Gronwall's lemma to the following inequality

$$\frac{\mathrm{d}|z|}{\mathrm{d}s} \le \Big|\frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^1)\Big|\cdot|z| + \Big|\frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^1) - \frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^2)\Big|\cdot\Big|\frac{\partial y}{\partial x}(x_2)\Big|. \tag{3.83}$$

Using (3.78) and (3.9), we find

$$|z| \le N\Big|\frac{\partial y}{\partial x}\Big|_{0,Q_\infty}\int_s^t\Big|\frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^1) - \frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^2)\Big|\,\mathrm{d}s'. \tag{3.84}$$

Using (3.78), we continue estimating (3.84)

$$|z| \le N\int_s^t\Big|\frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^1) - \frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^2)\Big|\,\mathrm{d}s'. \tag{3.85}$$

It now remains to bound the integrand from the right-hand side of (3.85). Using estimate (3.9), we have

$$\Big|\frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^1) - \frac{\partial\tilde{\mathbf{u}}}{\partial y}(y^2)\Big| \le H_y^\alpha\Big(\frac{\partial\tilde{\mathbf{u}}}{\partial y}\Big)|y^1 - y^2|^\alpha \le CM\Big|\frac{\partial y}{\partial x}\Big|^\alpha_{0,Q_\infty}|x^1 - x^2|^\alpha. \tag{3.86}$$

So, returning to (3.85) we find

$$\Big|\frac{\partial y}{\partial x}(x^1) - \frac{\partial y}{\partial x}(x^2)\Big| \le NMT|x^1 - x^2|^\alpha, \tag{3.87}$$

which implies

$$H_x^\alpha\Big(\frac{\partial y}{\partial x}\Big) \le NMT. \tag{3.88}$$

We also have to study the Hölder continuity of $\partial y/\partial x$ with respect to t. Let t_1 and t_2 be two arbitrary instants of time from the interval $[0,T]$, such that $|\tilde{\mathbf{u}}|_{0,Q_\infty}|t_1 - t_2| < h = \operatorname{dist}(\Gamma_1,\Gamma_0)$. We can suppose, without loss of generality, that $t_1 < t_2$. Let $y^1 = y(s;x,t_1)$ and $y^2 = y(s;x,t_2)$ be the solutions of problem (3.51) having the same initial condition x respectively at $s = t_1$ and $s = t_2$. We also consider $\tau_i = \tau(x,t_i)$, for $i = 1,2$. There are two possible positions for τ_2 with respect to t_1 and t_2: when $\tau_2 \le t_1$ and when $t_1 \le \tau_2 \le t_2$. Note here that the case $\tau_2 > t_2$ is not possible, since by definition $\tau(x,t)$ is the time the particle (x,t) enters the domain.

For the first case, we notice that $y^2(t_1)$ makes sense and it is a point inside the domain. We define $x_0 = y^2(t_1; x, t_2)$. Then we look for y^2 as the solution of the Cauchy problem

$$\begin{aligned} &\frac{\mathrm{d}y}{\mathrm{d}s} = \tilde{\mathbf{u}}(y,s), \\ &y|_{s=t_1} = x_0. \end{aligned} \tag{3.89}$$

Taking into account the estimate (3.86) of $H_x^\alpha(\partial y/\partial x)$, we have

$$\Big|\frac{\partial y^1}{\partial x} - \frac{\partial y^2}{\partial x}\Big| = \Big|\frac{\partial y}{\partial x}(s; x_0, t_1) - \frac{\partial y}{\partial x}(s; x, t_1)\Big| \le NMT|x_0 - x|^\alpha. \tag{3.90}$$

Since y^2 is the solution of the Cauchy problem having x as initial data at t_2, we find

$$y(s; x, t_2) = x - \int_s^{t_2} \tilde{\mathbf{u}}(y(\xi; x, t_2), t_2)\,\mathrm{d}\xi, \tag{3.91}$$

and taking $s = t_1$ we have

$$|x_0 - x| = |\int_{t_1}^{t_2} \tilde{\mathbf{u}}(y(\xi; x, t_2), t_2)\,\mathrm{d}\xi| \le CM|t_1 - t_2|. \tag{3.92}$$

We conclude that

$$\Big|\frac{\partial y^1}{\partial x} - \frac{\partial y^2}{\partial x}\Big| \le NM^{1+\alpha}T|t_1 - t_2|^\alpha. \tag{3.93}$$

We now consider the second case: $\tau_1 < t_1 \le \tau_2 < t_2$. From the definition of the trajectory we know that y^1 remains inside the domain on the interval $[\tau_1, t_1]$. Moreover we want to prove that y^1 is also defined on $[t_1, \tau_2]$. We suppose that at an instant s_1, with $t_1 < s_1 < \tau_2$, the trajectory y^1 reaches Γ_0, meaning $y^1(s_1) \in \Gamma_0$. This means that the particle x reaches the boundary Γ_1 during the time $t_2 - \tau_2$ and the boundary Γ_0 during the time $s_1 - t_1$.

We obtain

$$\begin{aligned} h &\le |y^2(\tau_2; x, t_2) - y^1(s_1; x, t_1)| \le |\tilde{\mathbf{u}}|_{0,Q_\infty}(t_2 - \tau_2 + s_1 - t_1) \\ &\le |\tilde{\mathbf{u}}|_{0,Q_\infty}(t_2 - t_1) < h, \end{aligned} \tag{3.94}$$

so $s_1 < \tau_1$ is impossible.

The conclusion is that the trajectory remains inside the domain not only on the interval $[\tau_1, t_1]$ but also on the interval $[t_1, \tau_2]$.

It now makes sense to consider the following points from the domain

$$x^1 = y^1(\tau_2(x,t_2);x,t_1),\ x^2 = y^2(\tau_2(x,t_2);x,t_2).$$

We can look at y^1 and respectively y^2, as the solutions of the Cauchy problem (3.51) having as initial data $y^1|_{s=\tau_2} = x^1$ and $y^2|_{s=\tau_2} = x^2$. The problem is similar to the previous one, so we can estimate

$$\Big|\frac{\partial y^1}{\partial x} - \frac{\partial y^2}{\partial x}\Big| \le NMT|x^1 - x^2|^\alpha. \tag{3.95}$$

We also know, from the integral equations for y^1 and y^2, that

$$x^1 = x + \int_{t_1}^{\tau_2} \mathbf{u}(y^1(\xi;x,t_1),\xi)\,\mathrm{d}\xi,$$
$$x^2 = x - \int_{\tau_2}^{t_2} \mathbf{u}(y^2(\xi;x,t_2),\xi)\,\mathrm{d}\xi,$$

which leads to

$$|x^1 - x^2| = |\int_{t_1}^{\tau_2} \tilde{\mathbf{u}}(y^1(\xi;x,t_1),\xi)\,\mathrm{d}\xi + \int_{\tau_2}^{t_2} \tilde{\mathbf{u}}(y^2(\xi;x,t_2),\xi)\,\mathrm{d}\xi| \le M(t_2 - t_1). \tag{3.96}$$

For this case we obtained the same estimate for the Hölder constant with respect to t as for the previous one, so we conclude

$$H_t^\alpha\Big(\frac{\partial y}{\partial x}\Big) \le NM^{1+\alpha}T. \tag{3.97}$$

All that remains to do is to estimate the quantities

$$[\partial y/\partial x](x,t) = \partial y/\partial x(\tau(x,t);x,t).$$

Let us set $\tau_1 = \tau(x^1,t)$ and $\tau_2 = \tau(x^2,t)$. We start by estimating the Hölder constant with respect to x

$$\begin{aligned}\Big|\Big[\frac{\partial y}{\partial x}\Big](x^1,t) - \Big[\frac{\partial y}{\partial x}\Big](x^2,t)\Big| &= \Big|\frac{\partial y}{\partial x}(\tau_1;x^1,t) - \frac{\partial y}{\partial x}(\tau_2;x^2,t)\Big| \\ \le \Big|\frac{\partial y}{\partial x}(\tau_1;x^1,t) - \frac{\partial y}{\partial x}(\tau_1;x^2,t)\Big| &+ \Big|\frac{\partial y}{\partial x}(\tau_1;x^2,t) - \frac{\partial y}{\partial x}(\tau_2;x^2,t)\Big|.\end{aligned} \tag{3.98}$$

The first term from the right-hand side of (3.98) is estimated as

$$\Big|\frac{\partial y}{\partial x}(\tau_1;x^1,t) - \frac{\partial y}{\partial x}(\tau_1;x^2,t)\Big| \le NMT|x^1 - x^2|^\alpha. \tag{3.99}$$

For the second term from the right-hand side of (3.98), we use the fact that $\partial y/\partial x$ is the solution of problem (3.81), so we can write

$$\Big|\frac{\partial y}{\partial x}(\tau_1; x^2, t) - \frac{\partial y}{\partial x}(\tau_2; x^2, t)\Big| \le N|\tau_1 - \tau_2| \le NT^{1-\alpha}|\tau_1 - \tau_2|^{\alpha}. \tag{3.100}$$

Taking into account (3.80), we find

$$|\tau_1 - \tau_2| \le N|x^1 - x^2|,$$

so we conclude by

$$H_x^{\alpha}\Big(\Big[\frac{\partial y}{\partial x}\Big]\Big) \le NT^{1-\alpha}. \tag{3.101}$$

In order to find the Hölder continuity constant with respect to t, we repeat the same reasoning. Let us consider two arbitrary moments in time t_1 and t_2, and set $\tau_1 = \tau(x, t_1)$ and $\tau_2 = \tau(x, t_2)$. Then

$$\begin{aligned}\Big|\Big[\frac{\partial y}{\partial x}\Big](x, t_1) - \Big[\frac{\partial y}{\partial x}\Big](x, t_2)\Big| &= \Big|\frac{\partial y}{\partial x}(\tau_1; x, t_1) - \frac{\partial y}{\partial x}(\tau_2; x, t_2)\Big| \\ &\le \Big|\frac{\partial y}{\partial x}(\tau_1; x, t_1) - \frac{\partial y}{\partial x}(\tau_1; x, t_2)\Big| + \Big|\frac{\partial y}{\partial x}(\tau_1; x, t_2) - \frac{\partial y}{\partial x}(\tau_2; x, t_2)\Big|.\end{aligned} \tag{3.102}$$

Exactly as before, we find

$$H_t^{\alpha}\Big(\Big[\frac{\partial y}{\partial x}\Big]\Big) \le NT^{1-\alpha}. \tag{3.103}$$

Vortex estimates. Having estimated the trajectories, we can now estimate the norm $|\cdot|_{\alpha,Q_\infty}$ of the vortex $\tilde{\boldsymbol{\omega}}$, this being actually the task of this subsection. As we already mentioned at the beginning of this subsection, we need only to estimate $\tilde{\boldsymbol{\omega}}^1$, since the estimates for $\tilde{\boldsymbol{\omega}}^2$ are similar. The formula for $\tilde{\boldsymbol{\omega}}^1$ is

$$\tilde{\omega}_i^1(x, t) = \sum_{j=1}^{3} [[\tilde{\omega}_{0,j}]](x, t)\big[J_{m,l}(y_{m+j-i}, y_{l+j-i})\big](x, t), \tag{3.104}$$

where $J_{m,l}(y_{m+j-i}, y_{l+j-i})$ stands for

$$J_{m,l}(y_{m+j-i}, y_{l+j-i}) = \frac{\partial y_{m+j-i}}{\partial x_m}\frac{\partial y_{l+j-i}}{\partial x_l} - \frac{\partial y_{m+j-i}}{\partial x_l}\frac{\partial y_{l+j-i}}{\partial x_m},$$

and (i, m, l) is a cyclic recombination of $(1, 2, 3)$ and the indices $m + j - i$ and $l + j - i$ are taken modulo 3.

We can then estimate $\tilde{\boldsymbol{\omega}}^1$ as

$$|\tilde{\boldsymbol{\omega}}^1|_{\alpha,Q_\infty} \le |[[\tilde{\boldsymbol{\omega}}_0]]|_{\alpha,Q_\infty}\Big|\Big[\frac{\partial y}{\partial x}\Big]\Big|^2_{0,Q_\infty} + |[[\tilde{\boldsymbol{\omega}}_0]]|_{0,Q_\infty}\Big|\Big[\frac{\partial y}{\partial x}\Big]\Big|^2_{\alpha,Q_\infty}. \tag{3.105}$$

We add and subtract the matrix I in the terms containing $[\partial y/\partial x]$. Making use of relations (3.78), (3.101), (3.103), we find

$$|[[\tilde{\boldsymbol{\omega}}^1]]|_{\alpha,Q_\infty} \leq |[[\tilde{\boldsymbol{\omega}}_0]]|_{\alpha,Q_\infty}(C+N). \tag{3.106}$$

The problem reduces then to estimating $|\tilde{\boldsymbol{\omega}}_0|_{\alpha,Q_\infty}$. We recall the definition of $\tilde{\boldsymbol{\omega}}_0$: it is equal to curl $\tilde{\mathbf{u}}_0$ for $x \in \Omega$ at $t=0$, and with the boundary conditions on Γ_1, when $t>0$. From (3.47b)–(3.47d) and (3.47f), we find

$$|[[\tilde{\boldsymbol{\omega}}^1]]|_{\alpha,Q_\infty} \leq C(|[[\nabla\tilde{p}]]|_{\alpha,Q_\infty} + |[[\mathbf{f}]]|_{\alpha,Q_\infty} + |[[\nabla U]]|_{\alpha,Q_\infty} + |[[\text{curl }\tilde{\mathbf{u}}_0]]|_{\alpha,Q_\infty}). \tag{3.107}$$

We start by estimating the first term from (3.107), which is the most difficult. We deduce the Hölder constants for $[\nabla\tilde{p}]$, the norm in $\mathcal{C}(Q_\infty)$ being estimated in a similar way.

We consider x^1 and x^2 two arbitrary points in Ω and set $\tau_i = \tau(x^i,t)$ and $y^i = y(\tau_i;x^i,t)$, for $i=1,2$. Using the definition of $[[\nabla\tilde{p}]]$, we find

$$[[\nabla\tilde{p}]](x^2,t) - [[\nabla\tilde{p}]](x^1,t) = \nabla\tilde{p}(y^2,\tau_2) - \nabla\tilde{p}(y^1,\tau_1) = J_1 + J_2 + J_3, \tag{3.108}$$

where

$$\begin{aligned} J_1 &= \nabla\tilde{p}(y^2,0) - \nabla\tilde{p}(y^1,0),\\ J_2 &= \nabla\tilde{p}(y^2,\tau_2) - \nabla\tilde{p}(y^2,0) - \nabla\tilde{p}(y^1,\tau_2) + \nabla\tilde{p}(y^1,0),\\ J_3 &= \nabla\tilde{p}(y^1,\tau_2) - \nabla\tilde{p}(y^1,\tau_1). \end{aligned} \tag{3.109}$$

For the term J_1 we work with the initial pressure, so we have

$$\begin{aligned} |J_1| =& |\nabla p_0(y^2) - \nabla p_0(y^1)| \leq C|y^2-y^1|\\ &\leq C|y(\tau_1;x^1,t) - y(\tau_2;x^1,t)| + C|y(\tau_2;x^1,t) - y(\tau_2;x^2,t)|\\ &\leq C\Big\{\Big|\frac{\mathrm{d}y}{\mathrm{d}s}\Big|_{0,Q_\infty}|\tau_2-\tau_1| + \Big|\frac{\partial y}{\partial x}\Big|_{0,Q_\infty}|x^2-x^1|\Big\}. \end{aligned} \tag{3.110}$$

We continue estimating (3.110) using

$$|\tau(x^1,t)-\tau(x^2,t)| \leq CT^{1-\alpha}\Big|\frac{\partial\tau}{\partial x}\Big|^\alpha_{0,Q_\infty}|x^1-x^2|^\alpha \leq NT^{1-\alpha}|x^1-x^2|^\alpha,$$

which, by (3.78) leads to

$$|J_1| \leq N|x^1-x^2|^\alpha. \tag{3.111}$$

Since y^1 and y^2 are two points on Γ_1, the term J_2 is estimated as

$$|J_2| \leq |\nabla\tilde{p}(\tau_2) - \nabla\tilde{p}(0)|_{\alpha,\Gamma_1}|y^2-y^1|^\alpha. \tag{3.112}$$

Making the same reasoning as for (3.110), we find

$$|y^2 - y^1| \leq N|x^2 - x^1|, \tag{3.113}$$

and using (3.46), we find

$$|J_2| \leq CT^\alpha(1 + M^2 + NT^{\alpha(1-\alpha)/(3+\alpha)})N|x^1 - x^2|^\alpha. \tag{3.114}$$

It remains to estimate J_3

$$|J_3| = |\nabla\tilde{p}(y^1, \tau_2) - \nabla\tilde{p}(y^1, \tau_1)| \leq |\nabla\tilde{p}(\tau_2) - \nabla\tilde{p}(\tau_1)|_{0,\Gamma_1}. \tag{3.115}$$

We use the imbedding property

$$|\nabla\tilde{p}(\tau_2) - \nabla\tilde{p}(\tau_1)|_{0,\Gamma_1} \leq C|\nabla\tilde{p}(\tau_2) - \nabla\tilde{p}(\tau_1)|^\mu_{\alpha,\Omega'_\infty}\|\tilde{p}(\tau_2) - \tilde{p}(\tau_1)\|^{1-\mu}_{L^2(\Omega')}, \tag{3.116}$$

where, as mentioned before, Ω' is an arbitrary subdomain of Ω, having Γ_1 as a part of the boundary; $\mu = 5/(5 + 2\alpha)$. With (3.116), J_3 is estimated as

$$|J_3| \leq C|\nabla\tilde{p}(\tau_2) - \nabla\tilde{p}(\tau_1)|^\mu_{\alpha,\Omega'_\infty}\|\nabla\tilde{p}(\tau_2) - \nabla\tilde{p}(\tau_1)\|^{1-\mu}_{L^2(\Omega)}.$$

Using (3.45) and (3.113), we find

$$\|\tilde{p}(\tau_2) - \tilde{p}(\tau_1)\|_{L^2(\Omega)} \leq NT^{\alpha(1-\alpha)/(3+\alpha)}|\tau_1 - \tau_2|^\alpha \leq NT^{\alpha(1-\alpha)/(3+\alpha)}|x^1 - x^2|^\alpha. \tag{3.117}$$

Taking into account (3.29), we conclude

$$|\tilde{p}(\tau_2) - \tilde{p}(\tau_1)|_{0,\Gamma_1} \leq N(1 + M + NT^{\alpha(1-\alpha)/(3+\alpha)})^\mu T^{(1-\mu)\alpha(1-\alpha)/(3+\alpha)}|x^1 - x^2|^\alpha. \tag{3.118}$$

Gathering the estimates for J_1, J_2 and J_3, we find

$$H^\alpha_x([[\nabla\tilde{p}]]) \leq N. \tag{3.119}$$

Note that is was important to bound J_1 by a constant N which has a limit when $T \to 0$ independent of M.

For the estimates of the Hölder constant with respect to t, we can repeat the arguments and we obtain the same bound

$$H^\alpha_t([[\nabla\tilde{p}]]) \leq N. \tag{3.120}$$

It remains to estimate $|[[\text{curl }\tilde{\mathbf{u}}_0]]|_{\alpha,Q_\infty}$, $|[[\mathbf{f}]]|_{\alpha,Q_\infty}$ and $|[[\nabla U]]|_{\alpha,Q_\infty}$. We have

$$\begin{aligned} |[[\text{curl }\tilde{\mathbf{u}}_0]]|_{\alpha,Q_\infty} &= |\text{curl }\tilde{\mathbf{u}}_0(y(0,x,t))|_{\alpha,Q_\infty} \\ &\leq |\text{curl }\tilde{\mathbf{u}}_0|_{0,\Omega} + H^\alpha_x(\text{curl }\tilde{\mathbf{u}}_0)\Big(\Big|\frac{\partial y}{\partial x}\Big|^\alpha_{0,Q_\infty} + \Big|\frac{\partial y}{\partial t}\Big|^\alpha_{0,Q_\infty}\Big) \leq N, \end{aligned} \tag{3.121}$$

and similarly for the rest of the terms. We use here the fact that $\mathbf{f} \in \mathcal{C}^{1+\alpha}(Q_\infty)$, $\tilde{\mathbf{u}}_0 \in \mathcal{C}^{1+\alpha}(\Omega_\infty)$ and $U \in \mathcal{C}^{2+\alpha,1+\alpha}((\Gamma_1 \cup \Gamma_0) \times [0, T])$.

Conclusion 3.1. *We finally find the following estimate on* $\tilde{\boldsymbol{\omega}}$

$$|\tilde{\boldsymbol{\omega}}|_{\alpha,Q_\infty} \leq N, \tag{3.122}$$

where, as we mentioned before, N *is a constant depending on* M *and* T *but the limit of this constant when* T *goes to zero is independent of* M*.*

4. The main result

The task of this section is to deduce, based on the a priori estimates obtained above, the well posedness of the Euler problem considered.

Theorem 4.1. *We are given the functions* $\mathbf{f} \in \mathcal{C}^{1+\alpha}(Q_\infty)$, $\mathbf{u}_0 \in \mathcal{C}^{1+\alpha}(\Omega_\infty)$ *and* $U \in \mathcal{C}^{2+\alpha,1+\alpha}((\Gamma_1 \cup \Gamma_0) \times [0,T])$, $U \geq c > 0$ *where* c *is a constant. We also suppose that* U *is differentiable from* $[0,T]$ *to* $W^{1,q}(\tilde{\Gamma}_1 \cup \tilde{\Gamma}_0)$.

Then problem (2.1) *with initial condition* (2.2) *and boundary condition* (2.3) *has, locally in time, a unique solution* $(\mathbf{u}, p)$ *such that*

$$\mathbf{u} \in \mathcal{C}^{1+\alpha}([0,T_\star] \times \Omega_\infty),\ \nabla p \in \mathcal{C}^{\alpha}([0,T_\star] \times \Omega_\infty).$$

The choice of $T_\star$ *is explained below.*

Proof. The existence result is based on the Schauder fixed-point theorem. The operator considered in order to apply this theorem is

$$\Lambda : \boldsymbol{\omega} \to \tilde{\boldsymbol{\omega}}. \tag{4.1}$$

In the previous section we obtained $|\tilde{\boldsymbol{\omega}}|_{\alpha,Q_\infty} \leq N$. We observe that N is a constant of the type e^{CMT} so it depends on M and T, but the limit when T goes to zero is a constant N_0 independent of M. Hence if we chose $M = M_\star = 2N_0$, then for T small enough, with $T = T_\star$, we have $N(M_\star, T_\star) \leq M_\star$. This implies that Λ maps the ball of radius M from $\mathcal{C}^\alpha$ into itself. In order to be able to apply the Schauder theorem, we consider Λ as a function from $\mathcal{C}^\beta$ into itself, with $\beta < \alpha$. We proved that Λ is mapping the compact set $K = \{\boldsymbol{\omega} : |\boldsymbol{\omega}|_{\alpha,Q_\infty} \leq M\}$ of $\mathcal{C}^\beta$ into itself. The existence of a fixed point for Λ is ensured.

The uniqueness of the solution was already proved before. □

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