Differential and Integral Equations Volume 19, Number 3 (2006), 279–296

DEGREE THEORETIC METHODS IN THE STUDY OF NONLINEAR PERIODIC PROBLEMS WITH NONSMOOTH POTENTIALS

Ravi P. Agarwal
Department of Mathematical Sciences, Florida Institute of Technology
Melbourne 32901-6975, FL, USA

Michael E. Filippakis[1]
Department of Mathematics, National Technical University
Zografou Campus, Athens 15780, Greece

Donal O'Regan
Department of Mathematics, National University of Ireland, Galway, Ireland

Nikolaos S. Papageorgiou
Department of Mathematics, National Technical University
Zografou Campus, Athens 15780, Greece

(Submitted by: Roger Temam)

Abstract. In this paper we study periodic problems driven by the scalar ordinary p-Laplacian and with a nonsmooth potential. Using degree theoretic methods based on a fixed-point index for nonconvex-valued multifunctions, we prove two existence theorems. In the first we employ nonuniform nonresonance conditions between two successive eigenvalues of the negative p-Laplacian with periodic boundary conditions. In the second we use Landesman-Lazer conditions.

1. Introduction

In this paper, we study the following nonlinear periodic problem with nonsmooth potential

$$\left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' \in \partial j(t,x(t)) \text{ a.e. on } T=[0,b] \\ x(0)=x(b),\ x'(0)=x'(b),\ 1<p<\infty. \end{array} \right\} \tag{1.1}$$

Here $j(t,x)$ is a measurable function which is locally Lipschitz in $x \in \mathbb{R}$ and $\partial j(t,x)$ is the generalized subdifferential of $x \to j(t,x)$ (see Section 2).

Accepted for publication: September 2005.
AMS Subject Classifications: 34B15, 34B18.
[1]Supported by a grant of the National Scholarship Foundation of Greece (I.K.Y.).

Our approach is degree theoretic based on the topological fixed-point index for certain nonconvex valued multifunctions introduced recently by Bader [1] and our hypotheses permit partial interaction (resonance) with two successive eigenvalues of the negative scalar ordinary p-Laplacian with periodic boundary conditions (nonuniform nonresonance). So asymptotically at $\pm\infty$, we require that the "slope" $\frac{\partial j(t,x)}{|x|^{p-2}x}$ lies between two successive eigenvalues and partial interaction (nonuniform nonresonance) with them is allowed. In the past this problem was investigated in the context of semilinear (i.e. $p = 2$), smooth (i.e. $j(t, \cdot) \in C^1(\mathbb{R})$) problems. We mention the works of Iannacci-Nkashama [13], Habets-Metzen [12] and Fonda-Mawhin [9]. Recently Zhang [14] examined problems driven by the scalar ordinary p-Laplacian with Dirichlet boundary conditions, under conditions of nonuniform nonresonance. We should mention that it is not at all clear if variational methods can be used here, since for $p \neq 2$ we do not have an orthogonal direct sum decomposition of the state space $W^{1,p}_{per}((0, b))$ in terms of the eigenspaces.

In Section 4, we use Landesman-Lazer type conditions in place of the nonuniform nonresonance conditions of Section 3 and prove a second existence theorem for problem (1.1). Our result improves the works of Cesari-Kannan [3], Drabek [7] and Iannacci-Nkashama [13] where $p = 2$ (semilinear problem), the potential function $j(t, \cdot)$ is smooth (i.e. $j(t, \cdot) \in C^1(\mathbb{R})$) and the boundary conditions are Dirichlet. Finally, we would like to mention also the works of Del Pino-Elgueta-Manasevich [4], de Figueiredo-Miyagaki [8], Guo [11] which use similar types of arguments for semilinear (i.e. $p = 2$) and quasilinear boundary value problems.

2. Mathematical Background

In this section we will briefly recall some of the mathematical tools we will use in our analysis.

Let us start by recalling what we know about the spectrum of the negative scalar ordinary p-Laplacian with periodic boundary conditions. So consider the following nonlinear eigenvalue problem:

$$\left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' = \lambda |x(t)|^{p-2}x(t) \text{ a.e. on } T = [0, b] \\ x(0) = x(b),\ x'(0) = x'(b),\ 1 < p < \infty. \end{array} \right\} \tag{2.1}$$

As usual, an eigenvalue is a number $\lambda \in \mathbb{R}$ for which problem (2.1) has a nontrivial solution known as the eigenfunction corresponding to the eigenvalue $\lambda \in \mathbb{R}$. A simple integration argument reveals that a necessary condition for problem (2.1) to have a nontrivial solution is that $\lambda \geq 0$ and $\lambda_0 = 0$ is an eigenvalue with corresponding eigenspace $\mathbb{R}$ (the constant functions). Moreover, every nonconstant eigenfunction must change sign. It can be shown (see for example Gasinski-Papageorgiou [10]), that the eigenvalues of (2.1) are given by the sequence

$$\left\{\mu_{2n} = (\frac{2n\pi_p}{b})^p\right\}_{n\geq 0},$$

where $\pi_p = \frac{2\pi(p-1)^{1/p}}{p\sin\frac{\pi}{p}}$. If $p=2$, then $\pi_2=\pi$ and we recover the eigenvalues of the negative scalar ordinary Laplacian with periodic boundary conditions, namely $\left\{\mu_{2n} = \left(\frac{2n\pi}{b}\right)^2\right\}_{n\geq 0}$. Note that the corresponding eigenfunctions $u \in C^1_{per}(T)$ satisfy $u(t)\neq 0$ almost everywhere on T.

In our analysis of problem (1.1), we will also use the eigenvalues of the weighted periodic eigenvalue problem for the scalar ordinary p-Laplacian. Namely we consider the following nonlinear eigenvalue problem:

$$\left\{\begin{array}{l} -(|x'(t)|^{p-2}x'(t))' = (\lambda + g(t))\,|x(t)|^{p-2}x(t) \text{ a.e. on } T=[0,b] \\ x(0)=x(b),\ x'(0)=x'(b),\ 1<p<\infty,\ g\in L^1(T). \end{array}\right\} \qquad (2.2)$$

This problem was studied by Zhang [15], who proved that it has a double sequence of eigenavalues $\{\underline{\lambda}_{2n}(g)\}_{n\geq 1}$ and $\{\overline{\lambda}_{2n}(g)\}_{n\geq 0}$ such that

$$-\infty < \overline{\lambda}_0(g) < \underline{\lambda}_2(g) \leq \overline{\lambda}_2(g) < \cdots < \underline{\lambda}_{2n}(g) < \overline{\lambda}_{2n}(g) < \cdots$$

and $\underline{\lambda}_{2n}(g)\to +\infty$ as $n\to\infty$. If $p=2$, then $\{\underline{\lambda}_{2n}(g)\}_{n\geq 1}$ and $\{\overline{\lambda}_{2n}(g)\}_{n\geq 0}$ are all the eigenavalues of (2.2). If $p\neq 2$, then we do not know if this is the case.

If Y,Z are Hausdorff topological spaces and $G: Y\to 2^Z\backslash\{\emptyset\}$ is a multifunction, we say that G is upper semicontinuous (usc for short), if for every $C\subseteq Z$ nonempty, closed, the set $G^-(C)=\{y\in Y : G(y)\cap C\neq\emptyset\}$ is closed in Y. If Z is regular and G has closed values, then upper semicontinuity of G implies that its graph $GrG=\{(y,z)\in Y\times Z : z\in G(y)\}$ is closed in $Y\times Z$. The converse is true, if G is locally compact, i.e., for every $y\in Y$, we can find a neighborhood U of y such that $\overline{G(U)}$ is compact in Z. For details we refer to Denkowski-Migorski-Papageorgiou [5] (Chapter 4).

If X,V are Banach spaces and $K: D\subseteq X\to V$, then we say that K is completely continuous, if for every sequence $\{x_n\}_{n\geq 1}\subseteq D$ converging weakly to $x\in D$, we have $K(x_n)\to K(x)$ in Y. If X is reflexive and $D\subseteq X$ is nonempty closed convex, then complete continuity of K implies that K is compact, i.e., K is continuous and maps bounded sets onto relatively compacts sets.

Suppose that $C \subseteq V$ and $D \subseteq X$ are nonempty closed convex sets and $G : C \to 2^D \backslash \{\emptyset\}$ is a multifunction with weakly compact convex values which is usc from C with the relative norm topology into D with the relative weak topology. Also let $K : D \to C$ be completely continuous and set $S = K \circ G : C \to 2^C \backslash \{\emptyset\}$. Assume that S is compact, i.e., S maps bounded sets onto relatively compact ones (this is the case if for example G maps norm bounded sets to norm bounded sets and X is reflexive; also observe that in the present setting the compactness assumption on S implies that it is usc). We emphasize that S need not have convex values. Finally let U be a bounded (relatively) open subset of C such that $\mathrm{Fix}(S) \cap \partial U = \emptyset$, with $\mathrm{Fix} S = \{v \in C : v \in S(v)\}$ (the set of fixed points of S). For such triples (S, U, C) Bader [1] defined a fixed-point index denoted by $\mathrm{ind}_C(S, U)$, which exhibits all the usual properties and extends the classical Leray-Schauder fixed-point index. In this case if $S_0 = K_0 \circ G_0$ and $S_1 = K_1 \circ G_1$, we say that S_0 and S_1 are homotopic, if there exist an usc multifunction $\theta : [0,1] \times C \to 2^D \backslash \{\emptyset\}$ (D furnished with the relative weak topology) with weakly compact, convex values such that $\theta(0, \cdot) = G_0$ and $\theta(1, \cdot) = G_1$ and a sequentially continuous map $\xi : [0,1] \times D \to C$ (D with the relative weak topology) such that $\xi(0, \cdot) = K_0$ and $\xi(1, \cdot) = K_1$. We set $h(t,x) = \xi(\theta(t,x))$ and in the homotopy invariance property of the index we require that $x \neq h(t,x)$ for all $(t,x) \in [0,1] \times \partial U$ and h is compact.

If X is a Banach space, a function $j : X \to \mathbb{R}$ is said to be locally Lipschitz if for every $x \in X$, we can find a neighborhood U of x such that $|j(y) - j(z)| \le k_U \|y - z\|$ for some $k_U > 0$ (depending on U) and all $y, z \in U$. If j is continuous and convex, then it is locally Lipschitz. The same is true if $j \in C^1(X)$. For locally Lipschitz functions we have a subdifferential theory which extends the one for continuous convex functions. The generalized directional derivative of j at $x \in X$ in the direction $h \in X$, is defined by

$$j^0(x;h) = \limsup_{\substack{x' \to x \\ \lambda \downarrow 0}} \frac{j(x' + \lambda h) - j(x')}{\lambda}.$$

It is easy to check that $j^0(x; \cdot)$ is sublinear continuous, hence it is the support function of a nonempty, w^*-compact convex set $\partial j(x)$ defined by

$$\partial j(x) = \{x^* \in X^* : \langle x^*, h \rangle_{X^* X} \le j^0(x;h) \ \text{ for all } \ h \in X\}.$$

The multifunction $x \to \partial j(x)$ is the generalized subdifferential of j. If j is continuous convex, then the generalized subdifferential coincides with the subdifferential of convex analysis. If $j \in C^1(X)$, then $\partial j(x) = \{j'(x)\}$. For further details see Denkowski-Migorski-Papageorgiou [5] (Chapter 5).

3. Nonuniform nonresonance between higher eigenvalues

In this section we prove an existence theorem by assuming that we have nonuniform nonresonance of the "slope" $\frac{\partial j(t,x)}{|x|^{p-2}x}$ as $x \to \pm\infty$ between two successive eigenvalues of the negative scalar ordinary p-Laplacian with periodic boundary conditions.

Our hypotheses on $j(t,x)$ are the following:

$H(j)_1$: $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that

(i) for all $x \in \mathbb{R}$, $t \to j(t,x)$ is measurable;

(ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;

(iii) for every $r > 0$, there exists $\alpha_r \in L^1(T)_+$ such that for almost all $t \in T$, all $|x| \le r$ and all $u \in \partial j(t,x)$, we have $|u| \le \alpha_r(t)$;

(iv) there exist functions $\eta, \theta \in L^1(T)$ such that for some $n \ge 0$

$$\mu_{2n} \le \eta(t) \le \theta(t) \le \mu_{2n+2} \text{ a.e. on } T$$

and the first and third inequalities are strict on sets (in general different) of positive measure and

$$\eta(t) \le \liminf_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \le \limsup_{|x|\to\infty} \frac{u}{|x|^{p-2}x} \le \theta(t)$$

uniformly for almost all $t \in T$ and all $u \in \partial j(t,x)$.

Remark 3.1. Hypothesis $H(j)_1(iv)$ is the nonuniform nonresonance condition. Let $\widehat{\alpha} \in L^1(T)$ and $\widehat{c} > 0$ be such that $\mu_{2n} + \widehat{c} \le \alpha(t) \le \mu_{2n+2} - \widehat{c}$ almost everywhere on T for some $n \ge 1$ and the inequalities are strict on sets (not necessarily the same) of positive measure. Then if $j(t,x) = \frac{\widehat{\alpha}(t)}{p} \max\{|x|^p, |x|^r\} + \frac{\widehat{c}}{p} \sin|x|^p$ with $1 \le r < p$, then $j(t,\cdot)$ is nonsmooth, locally Lipschitz and satisfies hypotheses $H(j)_1$.

Proposition 3.2. *If $\eta, \theta \in L^1(T)$ are as in hypothesis $H(j)_1(iv)$ and $g \in L^1(T)$ satisfies $\eta(t) \le g(t) \le \theta(t)$ almost everywhere on T, then all eigenavalues of* (2.2) *are nonzero and do not have zero as a limit point.*

Proof. From the monotonicity of the eigenvalues

$$\{\underline{\lambda}_{2n}(g)\}_{n\ge1} \text{ and } \{\overline{\lambda}_{2n}(g)\}_{n\ge0}$$

with respect to the weight g (see Zhang [15]), we have

$$\overline{\lambda}_{2n}(g) \le \overline{\lambda}_{2n}(\eta) \le \overline{\lambda}_{2n}(\mu_{2n}) = 0, \tag{3.1}$$

$$0 = \underline{\lambda}_{2n+2}(\mu_{n+2}) < \underline{\lambda}_{2n+2}(\theta) \le \overline{\lambda}_{2n+2}(g). \tag{3.2}$$

Also from Zhang [15], we know that if $\lambda \in \mathbb{R}$ is an eigenvalue of (2.2), then

$$\lambda \in \bigcup_{k\geq 1} [\underline{\lambda}_{2k}(g), \overline{\lambda}_{2k}(g)] \cup (-\infty, \overline{\lambda}_0(g)]$$

and so from (3.1) and (3.2) it follows that $\lambda \neq 0$.

Let $\sigma(p)$ be the set of eigenvalues of (2.2). Suppose that we can find $\{\lambda_n\}_{n\geq 1} \subseteq \sigma(p)$ such that $\lambda_n \to 0$. We can find $u_n \in C^1(T)$, $u_n \neq 0$, such that

$$\left\{ \begin{array}{l} -(|u_n'(t)|^{p-2}u_n'(t))' = (\lambda_n + g(t))|u_n(t)|^{p-2}u_n(t) \text{ a.e. on } T \\ u_n(0) = u_n(b), \; u_n'(0) = u_n'(b). \end{array} \right\} \tag{3.3}$$

Due to the $(p-1)$-homogeneity of problem (3.3), we may assume that $\|u_n\| = 1$ for all $n \neq 1$ (hereafter by $\|\cdot\|$ we denote the norm of the Sobolev space $W^{1,p}_{per}((0,b)) = \{x \in W^{1,p}((0,b)) : x(0) = x(b)\}$). By passing to a suitable subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}_{per}((0,b)) \text{ and } u_n \to u \text{ in } C(T).$$

Let $\langle\cdot,\cdot\rangle$ denote the duality brackets for the pair $(W^{1,p}_{per}((0,b)), W^{1,p}_{per}((0,b))^*)$ and let $A : W^{1,p}_{per}((0,b)) \to W^{1,p}_{per}((0,b))^*$ be the nonlinear operator defined by

$$\langle A(x), y\rangle = \int_0^b |x'(t)|^{p-2}x'(t)y'(t)dt \text{ for all } x, y \in W^{1,p}_{per}((0,b)).$$

It is easy to see that A is monotone, demicontinuous, hence maximal monotone (see Brezis [2], page 33 (Hilbert space case) and Denkowski-Migorski-Papageorgiou [6], page 37). Also let $F : W^{1,p}_{per}((0,b)) \to L^q(T)$ $(\frac{1}{p} + \frac{1}{q} = 1)$ be the bounded continuous nonlinear operator defined by

$$F(x)(\cdot) = |x(\cdot)|^{p-2}x(\cdot).$$

Then we can equivalently rewrite (3.3) as the following abstract operator equation

$$A(u_n) = (\lambda_n + g)F(u_n), \tag{3.4}$$

$$\Rightarrow \langle A(u_n), u_n - u\rangle = \int_0^b (\lambda_n + g(t))|u_n(t)|^{p-2}u_n(t)(u_n - u)(t)dt.$$

Evidently

$$\int_0^b (\lambda_n + g(t))|u_n(t)|^{p-2}u_n(t)(u_n - u)(t)dt \to 0$$

as $n \to \infty$. So

$$\lim_{n\to\infty} \langle A(u_n), u_n - u\rangle = 0.$$

Because A is maximal monotone, it is generalized pseudomonotone (see Denkowski-Migorski-Papageorgiou [6], page 58) and so

$$\langle A(u_n), u_n\rangle \to \langle A(u), u\rangle \;\Rightarrow\; \|u_n'\|_p \to \|u'\|_p.$$

Recall that $u_n \xrightarrow{w} u$ in $L^p(T)$. The Lebesgue space $L^p(T)$ is uniformly convex and so it has the Kadec-Klee property. So it follows that $u_n' \to u'$ in $L^p(T)$, hence $u_n \to u$ in $W^{1,p}_{per}((0,b))$. Passing to the limit as $n \to \infty$ in (3.4), we obtain

$$A(u) = gF(u),\; u \in W^{1,p}_{per}((0,b)). \tag{3.5}$$

Let $\varphi \in C^1_c((0,b))$. Then since $(|u'|^{p-2}u')' \in W^{-1,q}((0,b)) = W_0^{1,p}((0,b))^*$ and if by $\langle\cdot,\cdot\rangle_0$ we denote the duality brackets for the pair $(W_0^{1,p}((0,b)), W^{-1,q}((0,b)))$, we have

$$\langle -(|u'|^{p-2}u')', \varphi\rangle_0 = \int_0^b g(t)|u(t)|^{p-2}u(t)\varphi(t)dt.$$

Since $C^1_c((0,b))$ is dense in $W_0^{1,p}((0,b))$, we infer that

$$-(|u'(t)|^{p-2}u'(t))' = g(t)|u(t)|^{p-2}u(t) \text{ a.e. on } T,\; u(0) = u(b). \tag{3.6}$$

From this it follows that $|u'|^{p-2}u' \in W^{1,1}((0,b))$, hence $u' \in C(T)$ and so $u \in C^1(T)$. Using as a test function $\psi \in W^{1,p}_{per}((0,b))$ on (3.5) and after integration by parts because of (3.6), we have

$$\begin{aligned} &|u'(0)|^{p-2}u'(0)\psi(0) = |u'(b)|^{p-2}u'(b)\psi(b) \text{ for all } \psi \in W^{1,p}_{per}((0,b)),\\ \Rightarrow &u'(0) = u'(b). \end{aligned}$$

So we conclude that

$$\left\{\begin{array}{l} -(|u'(t)|^{p-2}u'(t))' = g(t)|u(t)|^{p-2}u(t) \text{ a.e. on } T,\\ u(0) = u(b),\; u'(0) = u'(b). \end{array}\right\} \tag{3.7}$$

Note that $\|u\| = 1$, hence $u \neq 0$. So from (3.7) we infer that $0 \in \sigma(p)$, a contradiction. □

By virtue of Proposition 3.2, we can find $\varepsilon_0 \in (0,1)$ such that $(-\varepsilon_0, \varepsilon_0) \cap \sigma(p) = \emptyset$. Fix $0 < \varepsilon < \varepsilon_0$ and consider the following auxiliary periodic problem

$$\left\{\begin{array}{l} -(|x'(t)|^{p-2}x'(t))' + \varepsilon|x(t)|^{p-2}x(t) = h(t) \text{ a.e. on } T\\ x(0) = x(b),\; x'(0) = x'(b),\; h \in L^1(T). \end{array}\right\} \tag{3.8}$$

We have the following result concerning problem (3.8).

Proposition 3.3. *For every $h \in L^1(T)$ problem* (3.8) *has a unique solution $V_\varepsilon(h) \in C^1_{per}(T)$ and the map $V_\varepsilon : L^1(T) \to W^{1,p}_{per}((0,b))$ is completely continuous.*

Proof. Let $A, F : W^{1,p}_{per}((0,b)) \to W^{1,p}_{per}((0,b))^*$ be the operators defined in the proof of Proposition 3.2 (recall that $L^q(T) \subseteq W^{1,p}_{per}((0,b))^*$ and the embedding is compact). Consider the nonlinear operator $L_\varepsilon : W^{1,p}_{per}((0,b)) \to W^{1,p}_{per}((0,b))^*$ defined by

$$L_\varepsilon(x) = A(x) + \varepsilon F(x) \text{ for all } x \in W^{1,p}_{per}((0,b)).$$

Clearly, L_ε is monotone, demicontinuous, hence it is maximal monotone. Also it is strictly monotone. Moreover, for every $x \in W^{1,p}_{per}((0,b))$, we have

$$\langle L_\varepsilon(x), x\rangle = \langle A(x), x\rangle + \varepsilon \int_0^b |x(t)|^p dt = \|x'\|_p^p + \varepsilon\|x\|_p^p \geq \varepsilon\|x\|^p$$

(assuming $\varepsilon < 1$),

$\Rightarrow L_\varepsilon$ is coercive.

But a maximal monotone coercive operator is surjective (see Denkowski-Migorski-Papageorgiou [6], page 49). So given $h \in L^1(T) \subseteq W^{1,p}_{per}((0,b))^*$, we can find $x \in W^{1,p}_{per}((0,b))$ such that

$$L_\varepsilon(x) = h.$$

The strict monotonicity of L_ε, implies that this solution $x \in W^{1,p}_{per}((0,b))$ is unique. As in the proof of Proposition 3.2 we can check that $x \in C^1(T)$ and it solves the auxiliary problem (3.8). Therefore for every $h \in L^1(T)$ problem (3.8) has a unique solution $V_\varepsilon(h) \in C^1_{per}(T)$.

Next we show that the map $h \to V_\varepsilon(h)$ is completely continuous from $L^1(T)$ into $W^{1,p}_{per}((0,b))$. To this end suppose that $h_n \xrightarrow{w} h$ in $L^1(T)$ and let $x_n = V_\varepsilon(h_n)$, $n \geq 1$. We have

$$\left\{ \begin{array}{l} -(|x_n'(t)|^{p-2}x_n'(t))' + \varepsilon|x_n(t)|^{p-2}x_n(t) = h_n(t) \text{ a.e. on } T, \\ x_n(0) = x_n(b),\ x_n'(0) = x_n'(b). \end{array} \right\}$$

Multiplying with $x_n(t)$, integrating over T and using integration by parts, we obtain

$$\|x_n'\|_p^p + \varepsilon\|x_n\|_p^p = \int_0^b h_n(t)x_n(t)dt,$$

$$\Rightarrow \varepsilon\|x_n\|^p \leq \|h_n\|_1\|x_n\| \leq c_1\|x_n\| \text{ for some } c_1 > 0, \text{ all } n \geq 1,$$

$$\Rightarrow \{x_n\}_{n\geq 1} \subseteq W_0^{1,p}((0,b)) \text{ is bounded.}$$

We may assume that

$$x_n \xrightarrow{w} x \text{ in } W^{1,p}_{per}((0,b)) \text{ and } x_n \to x \text{ in } C(T).$$

For every $n \geq 1$, we have

$$A(x_n) + \varepsilon F(x_n) = h_n. \tag{3.9}$$

Using as a test function $x_n - x \in W^{1,p}_{per}((0,b))$ and since

$$\langle F(x_n), x_n - x\rangle = \int_0^b |x_n|^{p-2} x_n (x_n - x) dt$$

and

$$\int_0^b h_n(x_n - x)dt \to 0 \text{ as } n \to \infty,$$

we obtain

$$\lim \langle A(x_n), x_n - x\rangle = 0.$$

As in the proof of Proposition 3.2, from the above convergence it follows that $x_n \to x$ in $W^{1,p}_{per}((0,b))$. So if we pass to the limit as $n \to \infty$, in (3.9), we obtain

$$\begin{aligned} & A(x) + \varepsilon F(x) = h, \\ \Rightarrow & \left\{ \begin{array}{l} -(|x'(t)|^{p-2}x'(t))' + \varepsilon |x(t)|^{p-2}x(t) = h(t) \text{ a.e. on } T \\ x(0) = x(b),\ x'(0) = x'(b), \end{array} \right\} \\ \Rightarrow & x \in V_\varepsilon(h). \end{aligned}$$

By Urysohn's criterion for convergent sequences, for the original sequence we have $V_\varepsilon(h_n) \to V_\varepsilon(h)$ in $W^{1,p}_{per}((0,b))$. This proves that $h \to V_\varepsilon(h)$ is completely continuous from $L^1(T)$ into $W^{1,p}_{per}((0,b))$. □

Now consider the multifunction $G : W^{1,p}_{per}((0,b)) \to 2^{L^1(T)}$ defined by

$$G(x) = S^1_{\partial(\cdot, x(\cdot))} = \{h \in L^1(T) : h(t) \in \partial j(t, x(t)) \text{ a.e. on } T\}.$$

In the next proposition we establish the topological properties of G.

Proposition 3.4. *If hypotheses $H(j)_1(i) \to (iii)$ hold, then G has nonempty, weakly compact and convex values and it is usc from $W^{1,p}_{per}((0,b))$ with the strong topology into $L^1(T)$ with the weak topology.*

Proof. By definition, for every $y \in \mathbb{R}$, we have

$$j^0(t, x(t); y) = \limsup_{\substack{v \to x(t) \\ r \downarrow 0}} \frac{j(t, v + ry) - j(v, t)}{r}$$

$$= \inf_{\varepsilon,\delta>0} \sup_{\substack{|v-x(t)|<\delta \\ 0<r<\varepsilon}} \frac{j(t,v+ry)-j(t,v)}{r}$$

$$= \inf_{n,m\geq 1} \sup_{\substack{|v-x(t)|<\frac{1}{n} \\ 0<r<\frac{1}{m}, v,r\in Q}} \frac{j(t,v+ry)-j(v,t)}{r}. \tag{3.10}$$

By virtue of hypotheses $H(j)_1(i)$ and (ii), the function $(t,x) \to j(t,x)$ is $\mathcal{L}(T) \times B(\mathbb{R})$-measurable, with $\mathcal{L}(T)$ being the Lebesgue σ-field of T and $B(\mathbb{R})$ the Borel σ-field of $\mathbb{R}$ (see Denkowski-Migorski-Papageorgiou [5], page 189). From (3.10) it follows that $t \to j^0(t,x(t);y)$ is Lebesgue measurable. We know that $y \to j^0(t,x(t);y)$ is continuous. Moreover, from the definition of the generalized subdifferential, we have

$$Gr\partial j(\cdot,x(\cdot)) = \{(t,u) \in T\times\mathbb{R} : uy \leq j^0(t,x(t);y) \text{ for all } y\in\mathbb{R}\}.$$

Let $\{u_n\}_{n\geq 1}$ be an enumeration of the rationals in $\mathbb{R}$. Then due to the continuity of $j^0(t,x(t);\cdot)$, we have

$$\begin{aligned} Gr\partial j(\cdot,x(\cdot)) &= \{(t,u) \in T\times\mathbb{R} : uy_n \leq j^0(t,x(t);y_n) \text{ for all } n\geq 1\} \\ &\in \mathcal{L}(T)\times B(\mathbb{R}). \end{aligned}$$

So we can apply the Yankov-von Neumann-Aumann selection theorem (see Denkowski-Migorski-Papageorgiou [5], page 432) and obtain $h: T \to \mathbb{R}$ a Lebesgue measurable function such that $h(t) \in \partial j(t,x(t))$ almost everywhere on T. By virtue of hypothesis $H(j)_1(iii)$, if $r = \|x\|_\infty$, then

$$|h(t)| \leq \alpha_r(t) \text{ a.e. on } T, \text{ i.e. } h\in L^1(T).$$

So $G(x) \neq \emptyset$ for all $x \in W^{1,p}_{per}((0,b))$ and clearly it is closed and convex. Moreover, since it is pointwise bounded by $\alpha_r(\cdot) \in L^1(T)$, from the Dunford-Pettis theorem, we infer that $G(x)$ is also weakly compact in $L^1(T)$.

By virtue of hypothesis $H(j)_1(iii)$, $G(\cdot)$ is locally compact into $L^1(T)$ with the weak topology. Moreover, recall that weakly compact subsets of $L^1(T)$ with the relative weak topology are measurable. So to prove the desired upper semicontinuity of G, it suffices to show that GrG is sequentially closed in $W^{1,p}_{per}((0,b)) \times L^1(T)_w$ ($L^1(T)_w$ denotes the Lebesgue space $L^1(T)$ with the weak topology). Suppose that $(x_n,h_n) \in GrG$, $x_n \to x$ in $W^{1,p}_{per}((0,b))$ and $h_n \xrightarrow{w} h$ in $L^1(T)$. Since $h_n \xrightarrow{w} h$ in $L^1(T)$, from Denkowski-Migorski-Papageorgiou [5], page 484, we have

$$h(t) \subseteq \operatorname{conv}\limsup_{n\to\infty} \partial j(t,x_n(t)) \subseteq \partial j(t,x(t)) \text{ a.e. on } T$$

the last inclusion being a consequence of the fact that $x \to \partial j(t,x)$ has convex values and a closed graph. Therefore $(x,h) \in GrG$ and we are done. $\square$

Now we are ready for the first existence theorem.

Theorem 3.5. *If hypotheses $H(j)_1$ hold, then problem* (1.1) *has a solution $x \in C^1(T)$ with $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,1}((0,b))$.*

Proof. We will show that for every $g \in L^1(T)$ with $\eta(t) \le g(t) \le \theta(t)$ almost everywhere on T, such that if we consider the compact set-valued homotopy

$$H_\varepsilon(\beta,x) = V_\varepsilon\left(\varepsilon F(x) + \beta G(x) + (1-\beta)gF(x)\right)$$
$$\text{for all } (\beta,x) \in [0,1] \times W^{1,p}_{per}((0,b)),$$

then we can find $\rho > 0$ such that if $\beta \in [0,1]$ and $\|x\| = \rho$, then $x \ne H(\beta,x)$. We argue indirectly. So suppose that we can find $\{\beta_n\}_{n\ge1} \subseteq [0,1]$ and $\{x_n\}_{n\ge1} \subseteq W^{1,p}_{per}((0,b))$ such that $\|x_n\| \to \infty$ and $x_n \in H_\varepsilon(\beta_n,x_n)$ for all $n \ge 1$. We have

$$A(x_n) = \beta_n u_n + (1-\beta_n)gF(x_n) \text{ with } u_n \in G(x_n) \text{ for all } n \ge 1. \tag{3.11}$$

Let $y_n = \frac{x_n}{\|x_n\|}$, $n \ge 1$. We may assume that

$$\beta_n \to \beta \in [0,1],\ y_n \xrightarrow{w} y \text{ in } W^{1,p}_{per}((,b)) \text{ and } y_n \to y \text{ in } C(T) \text{ as } n \to \infty.$$

For $\delta > 0$ and $n \ge 1$, we introduce the sets

$$C^-_{\delta,n} = \{t \in T : x_n(t) < 0,\ \eta(t) - \delta \le \frac{u}{|x_n(t)|^{p-2}x_n(t)} \le \theta(t) + \delta$$
$$\text{for all } u \in \partial j(t,x_n(t))\}$$
$$\text{and } C^+_{\delta,n} = \{t \in T : x_n(t) > 0,\ \eta(t) - \delta \le \frac{u}{x_n(t)^{p-1}} \le \theta(t) + \delta$$
$$\text{for all } u \in \partial j(t,x_n(t))\}.$$

Note that for all $t \in \{y < 0\}$, $x_n(t) \to -\infty$ and for all $t \in \{y > 0\}$ $x_n(t) \to +\infty$ as $n \to \infty$. So by virtue of hypothesis $H(j)_1(iv)$ we have

$$\chi_{C^-_{\delta,n}}(t) \to 1 \text{ a.e. on } \{y<0\} \text{ and } \chi_{C^+_{\delta,n}}(t) \to 1 \text{ a.e. on } \{y>0\}.$$

Because of hypotheses $H(j)_1(iii)$ and (iv), we see that for almost all $t \in T$, all $x \in \mathbb{R}$ and all $u \in \partial j(t,x)$, we have

$$|u| \le \widehat{\alpha}(t) + \widehat{c}(t)|x|^{p-1} \text{ with } \widehat{\alpha},\widehat{c} \in L^1(T)_+,$$
$$\Rightarrow \frac{|u_n(t)|}{\|x_n\|^{p-1}} \le \frac{\widehat{\alpha}(t)}{\|x_n\|^{p-1}} + \widehat{c}(t)|y_n(t)|^{p-1} \text{ a.e. on } T, \tag{3.12}$$

$$\Rightarrow \left\{ \frac{u_n(\cdot)}{|x_n\|^{p-1}} \right\}_{n\geq 1} \subseteq L^1(T) \text{ is uniformly integrable.}$$

So by the Dunford-Pettis theorem, we may assume that

$$\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} w \text{ in } L^1(T) \text{ as } n \to \infty.$$

Note that

$$\left\|(1-\chi_{C^-_{\delta,n}})\frac{u_n}{\|x_n\|^{p-1}}\right\|_{L^1(\{y<0\})} \to 0 \quad\text{and}\quad \left\|(1-\chi_{C^+_{\delta,n}})\frac{u_n}{\|x_n\|^{p-1}}\right\|_{L^1(\{y>0\})} \to 0.$$

So we infer that

$$\chi_{C^-_{\delta,n}}\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} w \text{ in } L^1(\{y<0\}) \text{ and } \chi_{C^+_{\delta,n}}\frac{u_n}{\|x_n\|^{p-1}} \xrightarrow{w} w \text{ in } L^1(\{y>0\}).$$

We have

$$\begin{aligned}
&\chi_{C^-_{\delta,n}}(t)\,(\theta(t)+\delta)\,|y_n(t)|^{p-2}y_n(t) \leq \chi_{C^-_{\delta,n}}(t)\frac{u_n(t)}{\|x_n\|^{p-1}}\\
&= \chi_{C^-_{\delta,n}}(t)\frac{u_n(t)}{|x_n(t)|^{p-2}x_n(t)}|y_n(t)|^{p-2}y_n(t)\\
&\leq \chi_{C^-_{\delta,n}}(t)\,(\eta(t)-\delta)\,|y_n(t)|^{p-2}y_n(t) \text{ a.e. on } T\\
\text{and }&\chi_{C^+_{\delta,n}}(t)(\eta(t)-\delta)y_n(t)^{p-1} \leq \chi_{C^+_{\delta,n}}(t)\frac{u_n(t)}{\|x_n\|^{p-1}}\\
&= \chi_{C^+_{\delta,n}}(t)\frac{u_n(t)}{x_n(t)^{p-1}}y_n(t)^{p-1} \leq \chi_{C^+_{\delta,n}}(t)(\theta(t)+\delta)y_n(t)^{p-1} \text{ a.e. on } T.
\end{aligned}$$

Passing to the limit as $n \to \infty$ and from Denkowski-Migorski-Papageorgiou [5], page 484, we have

$$(\theta(t)+\delta)|y(t)|^{p-2}y(t) \leq w(t) \leq (\eta(t)-\delta)|y(t)|^{p-2}y(t) \text{ a.e. on } \{y<0\}$$
$$\text{and } (\eta(t)-\delta)y(t)^{p-1} \leq w(t) \leq (\theta(t)+\delta)y(t)^{p-1} \text{ a.e. on } \{y>0\}.$$

Letting $\delta \downarrow 0$, we obtain

$$\theta(t)|y(t)|^{p-2}y(t) \leq w(t) \leq \eta(t)|y(t)|^{p-2}y(t) \text{ a.e. on } \{y<0\} \tag{3.13}$$

$$\text{and } \eta(t)y(t)^{p-1} \leq w(t) \leq \theta(t)y(t)^{p-1} \text{ a.e. on } \{y>0\}. \tag{3.14}$$

Moreover, from (3.12) it is clear that

$$w(t) = 0 \text{ a.e. on } \{y=0\}. \tag{3.15}$$

From (3.13), (3.14) and (3.15) it follows that

$$w(t) = g_0(t)|y(t)|^{p-2}y(t) \ \text{ a.e. on } T,$$

with $g_0 \in L^1(T)_+$ such that $\eta(t) \leq g_0(t) \leq \theta(t)$ a.e. on T.

Now we return to (3.11) and divide with $\|x_n\|^{p-1}$. We obtain

$$A(y_n) = \beta_n \frac{u_n}{\|x_n\|^{p-1}} + (1-\beta_n) g F(y_n). \tag{3.16}$$

Using as a test function $x_n - x$ and exploiting the generalized pseudomonotonicity of A and the Kadec-Klee property of $L^p(T)$, as in the proof of Proposition 3.2, we can deduce that $y_n \to y$ in $W^{1,p}_{per}((0,b))$. So passing to the limit as $n \to \infty$ in (3.16), we obtain

$$A(y) = \beta g_0 F(y) + (1-\beta) g F(y).$$

Now set $\hat{g}_0 = \beta g_0 + (1-\beta) g$. Then

$$A(y) = \hat{g}_0 F(y)$$

$$\Rightarrow \left\{ \begin{array}{l} -(|y'(t)|^{p-2} y'(t))' = \hat{g}_0(t) |y(t)|^{p-2} y(t) \text{ a.e. on } T \\ y(0) = y(b),\ y'(0) = y'(b). \end{array} \right\}$$

Note that $\|y\| = 1$ and so $y \neq 0$. So it follows that $0 \in \sigma(p)$, a contradiction to Proposition 3.2.

This proves that we can find $\rho > 0$ such that

$$x \neq H_\varepsilon(\beta, x) \text{ for all } \beta \in [0,1] \text{ and all } \|x\| = \rho.$$

Because of the homotopy invariance of the fixed-point index of Bader [1] (see Proposition 6(ii) in [1]), we have

$$\mathrm{ind}_{W^{1,p}_{per}((0,b))} \left(V_\varepsilon \circ (g_0 + \varepsilon) F, B_\rho\right) = \mathrm{ind}_{W^{1,p}_{per}} \left(V_\varepsilon \circ (\varepsilon F + G), B_\rho\right),$$

where $B_\rho = \{x \in W^{1,p}_{per}((0,b)) : \|x\| < \rho\}$. From the choice of $\varepsilon > 0$ and Proposition 3.2, we see that $x \neq V_\varepsilon \circ (g_0 + \varepsilon) F(x)$ for all $\|x\| = \rho$. Also it is clear that $x \to V_\varepsilon \circ (g_0 + \varepsilon) F(x)$ is odd. So by Borsuk's theorem we have

$$\begin{aligned} &\mathrm{ind}_{W^{1,p}_{per}((0,b))} \left(V_\varepsilon \circ (g_0 + \varepsilon) F, B_\rho\right) \neq 0, \\ \Rightarrow &\mathrm{ind}_{W^{1,p}_{per}((0,b))} \left(V_\varepsilon \circ (\varepsilon F + G), B_\rho\right) \neq 0. \end{aligned}$$

This means that we can find $x \in W^{1,p}_{per}((0,b))$ such that

$$x \in \left(V_\varepsilon \circ (\varepsilon F + G)\right)(x) \Rightarrow A(x) = u \text{ for some } u \in G(x).$$

From this as before we conclude that $x \in C^1_{per}(T)$ solves problem (1.1). $\square$

For $g \in L^1(T)_+$ satisfying $\eta(t) \leq g(t) \leq \theta(t)$ almost everywhere on T, we consider the following second-order nonlinear periodic problem:

$$\left\{ \begin{array}{l} -(|x'(t)|^{p-2} x'(t))' = (g(t) + \lambda) |x(t)|^{p-2} x(t) \text{ a.e. on } T \\ x(0) = x(b),\ x'(0) = x'(b),\ f \in L^1(T),\ \lambda \in \mathbb{R}. \end{array} \right\} \tag{3.17}$$

As an interesting byproduct of the previous analysis, we deduce the following existence result of the problem (3.17).

Proposition 3.6. *There exists* $\delta > 0$ *such that for all* $|\lambda| < \delta$ *and all* $f \in L^1(T)$, *problem* (3.17) *has a solution* $x \in C^1(T)$ *with* $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,1}((0,b))$.

4. Landesman-Lazer Conditions

In this section we prove a second existence theorem for problem (1.1), using this time Landesman-Lazer conditions (LL-conditions for short).

To formulate the LL-conditions in the present multivalued setting, we introduce the following two functions:

$$m(x,t) = \min\{u \in \partial j(t,x)\} \text{ and } M(t,x) = \max\{u : u \in \partial j(t,x)\}.$$

Note that for all $\xi \in \mathbb{R}$, we have

$$\begin{aligned} &\{(t,x) \in T \times \mathbb{R} : M(t,x) \le \xi\} \\ =&\{(t,x) \times T \times \mathbb{R} : \max\left(j^0(t,x;-1), j^0(t,x;1)\right) \le \xi \end{aligned} \tag{4.1}$$

and from the proof of Proposition 3.3, we know that $(t,x) \to j^0(t,x;-1)$, $j^0(t,x;1)$ are both $\mathcal{L}(T) \times B(\mathbb{R})$ measurable. So from (4.1) we infer that $(t,x) \to M(t,x)$ is $\mathcal{L}(T) \times B(\mathbb{R})$-measurable. Also because

$$m(t,x) = -\max[-u : u \in \partial j(t,x)] = \max[\widehat{u} : \widehat{u} \in \partial j(t,x)],$$

we see that $(t,x) \to m(t,x)$ too is $\mathcal{L}(T) \times B(\mathbb{R})$-measurable.

Our hypotheses on the nonsmooth potential function $j(t,x)$ are the following:

$H(j)_2$: $j : T \times \mathbb{R} \to \mathbb{R}$ is a function such that
- (i) for all $x \in \mathbb{R}$, $t \to j(t,x)$ is measurable;
- (ii) for almost all $t \in T$, $x \to j(t,x)$ is locally Lipschitz;
- (iii) for every $r > 0$, there exists $\alpha_r \in L^1(T)_+$ such that for almost all $t \in T$, all $|x| \le r$ and all $u \in \partial j(t,x)$, we have $|u| \le \alpha_r(t)$;
- (iv) there exist functions $h_-, h_+ \in L^1(T)$ such that

$$h_-(t) = \liminf_{x \to -\infty} m(t,x), \quad h_+(t) = \limsup_{x \to +\infty} M(t,x)$$

uniformly for almost all $t \in T$

$$\text{and } \int_0^b h_+(t)dt < 0 < \int_0^b h_-(t)dt.$$

Remark 4.1. Hypothesis $H(j)_2(iv)$ is the multivalued analog of the well-known LL-conditions. Suppose $\alpha, \beta \in L^1(T)_+$, $\alpha, \beta \neq 0$. Let $j_+(t,x) = \frac{\alpha(t)}{2}(-x^2 - x)$ for all $t \in T$ and all $x \in [0,1]$. Extend by periodicity (with period 1) to all $\mathbb{R}_+$. Also let $j_-(t,x) = \beta(t)\tan^{-1}|x|$. Finally, set

$$j(t,x) = \begin{cases} j_-(t,x) & \text{if } t \in T,\ x \leq 0 \\ j_+(t,x) & \text{if } t \in T,\ \ x \geq 0 \end{cases}.$$

Then the nonsmooth function $j(t,x)$ satisfies hypotheses $H(j)_2$.

Theorem 4.2. *If hypotheses $H(j)_2$ hold, then problem* (1.1) *has a solution $x \in C^1(T)$ with $|x'(\cdot)|^{p-2}x'(\cdot) \in W^{1,1}((0,b))$.*

Proof. We keep the notation introduced in Section 3. So we consider the following compact multivalued homotopy

$$H(\beta,x) = V_\varepsilon\left(\beta(\varepsilon F + G)(x)\right) \text{ for all } (\beta,x) \in [0,1] \times W^{1,p}_{per}((0,b)).$$

We claim that there exists $\rho > 0$ such that $x \neq H(\beta,x)$ for all $\beta \in [0,1]$ and all $\|x\| = \rho$. Suppose that this is not true. Then we can find $\{\beta_n\}_{n\geq 1} \subseteq [0,1]$ and $\{x_n\}_{n\geq 1} \subseteq W^{1,p}_{per}((0,b))$ such that $\|x_n\| \to \infty$ and $x_n = V_\varepsilon\left(\beta_n(\varepsilon F(x_n) + u_n)\right)$ with $u_n \in G(x_n)$, $n \geq 1$. Set $y_n = \frac{x_n}{\|x_n\|}$, $n \geq 1$. We may assume that

$$\beta_n \to \beta \in [0,1],\ y_n \xrightarrow{w} y \text{ in } W^{1,p}_{per}((0,b)) \text{ and } y_n \to y \text{ in } C(T).$$

For every $n \geq 1$, we have

$$A(x_n) + \varepsilon F(x_n) = \beta_n u_n + \varepsilon\beta_n F(x_n).$$

Dividing with $\|x_n\|^{p-1}$, we obtain

$$A(y_n) + \varepsilon F(y_n) = \beta_n \frac{u_n}{\|x_n\|^{p-1}} + \varepsilon\beta_n F(y_n). \tag{4.2}$$

By virtue of hypothesis $H(j)_2(iv)$, we see that $\frac{u_n}{\|x_n\|^{p-1}} \to 0$ in $L^1(T)$. Moreover, since

$$\int_0^b |y_n|^{p-2} y_n(y_n - y)dt \to 0 \text{ as } n \to \infty,$$

we obtain that

$$\lim\langle A(y_n), y_n - y\rangle = 0$$

from which as before we infer that $y_n \to y$ in $W^{1,p}_{per}((0,b))$. Passing to the limit as $n \to \infty$ in (4.2), we obtain

$$A(y) = (\beta - 1)\varepsilon F(y) \text{ with } \|y\| = 1 \text{ (hence } y \neq 0). \tag{4.3}$$

Since $\beta \in [0,1]$ and $\varepsilon > 0$, from (4.3) we infer that $\beta = 1$. Therefore

$$A(y) = 0, \text{ i.e., } y = \xi \in \mathbb{R},\ \xi \neq 0.$$

First suppose that $\xi > 0$. Returning to (4.2) and acting with the test function $y \equiv \xi \in W^{1,p}_{per}((0,b))$, we obtain

$$(1-\beta_n)\|x_n\|^{p-1}\varepsilon\xi \int_0^b |y_n|^{p-2} y_n dt = \beta_n \xi \int_0^b u_n dt. \tag{4.4}$$

Because $\beta_n \to \beta$ and $y_n \to \xi$ in $C(T)$ and since we have assumed that $\xi > 0$, from (4.4) it follows that there exists $n_0 \geq 1$ such that

$$\int_0^b u_n(t)dt \geq 0 \text{ for all } n \geq 1. \tag{4.5}$$

Note that since $\xi > 0$, for all $t \in T$ we have $x_n(t) \to +\infty$ as $n \to \infty$. We claim that this convergence is uniform in $t \in T$. To this end let $\delta \in (0, \xi)$. Because $y_n \to \xi$ in $C(T)$, we can find $n_1 = n_1(\delta) \geq 1$ such that for all $n \geq n_1$ and all $t \in T$, we have

$$|y_n(t) - \xi| < \delta \quad \Rightarrow 0 < \xi - \delta < y_n(t) \text{ for all } n \geq n_1 \text{ and all } t \in T.$$

Since $\|x_n\| \to \infty$, given $M > 0$ we can find $n_2 = n_2(M) \geq n_1$ such that for all $n \geq n_2$ we have $\|x_n\| \geq M$. Then for all $n \geq n_2$ and all $t \in T$, we have

$$\frac{x_n(t)}{M} \geq \frac{x_n(t)}{\|x_n\|} = y_n(t) > \gamma = \xi - \delta > 0,$$
$$\Rightarrow x_n(t) > \gamma M \text{ for all } n \geq n_2 \text{ and all } t \in T.$$

Because $M > 0$ was arbitrary, we conclude that $x_n(t) \to +\infty$ uniformly in $t \in T$ as $n \to \infty$.

So by virtue of hypothesis $H(j)_2(iv)$ we have that $\limsup_{n\to\infty} M(t, x_n(t)) = h_+(t)$ for almost all $t \in T$. Note that

$$u_n(t) \leq M(t, x_n(t)) \text{ a.e. on } T,$$
$$\Rightarrow \limsup_{n\to\infty} u_n(t) \leq \limsup_{n\to\infty} M(t, x_n(t)) = h_+(t) \text{ a.e. on } T.$$

By Fatou's lemma, we have

$$0 \leq \limsup_{n\to\infty} \int_0^b u_n(t)dt \leq \int_0^b h_+(t)dt \text{ (see (4.5))}.$$

Similarly if $\xi < 0$, then

$$\int_0^b u_n(t)dt \leq 0 \text{ for all } n \geq n_0. \tag{4.6}$$

As before we have that $x_n(t) \to -\infty$ uniformly in $t \in T$ as $n \to \infty$ and so we have $\liminf_{n\to\infty} m(t, x_n(t)) = h_-(t)$ almost everywhere on T. Since $m(t, x_n(t)) \le u_n(t)$ almost everywhere on T, it follows that

$$h_-(t) \le \liminf_{n\to\infty} u_n(t) \text{ a.e. on } T.$$

Hence via Fatou's lemma, we deduce that

$$\int_0^b h_-(t)dt \le \liminf_{n\to\infty} \int_0^b u_n(t)dt \le 0 \quad \text{(see (4.6))},$$

which contradicts hypothesis $H(j)_2(iv)$.

Therefore, we can find $\rho > 0$ such that

$$x \ne H(\beta, x) \text{ for all } \beta \in [0,1] \text{ and all } \|x\| = \rho.$$

Due to the homotopy invariance of the fixed-point index, we have

$$\text{ind}_{W^{1,p}_{per}}(H(0,\cdot), B_\rho) = \text{ind}_{W^{1,p}_{per}}(H(1,\cdot), B_\rho).$$

Note that $H(0,x) = V_\varepsilon(0)$ and for $\varepsilon > 0$ small $V_\varepsilon(0) = 0$ (see Proposition 3.2). So

$$\text{ind}_{W^{1,p}_{per}}(H(0,\cdot), B_\rho) = 1 \quad \Rightarrow \text{ind}_{W^{1,p}_{per}}(H(1,\cdot), B_\rho) = 1.$$

This means that we can find $x \in W^{1,p}_{per}((0,b))$ such that

$$A(x) = u \text{ with } u \in G(x).$$

As before from this equation it follows that $x \in C^1(T)$ and $|x'|^{p-2}x' \in W^{1,1}((0,b))$ and it solves problem (1.1). □

Remark 4.3. A careful reading of the proofs reveals that the results are still valid if in (1.1) the term $\partial j(t,x)$ is replaced by the multifunction $F(t,x)$ with nonempty, closed convex values such that

(a) for all $x \in \mathbb{R}$, $t \to F(t,x)$ is graph measurable;
(b) for almost all $t \in T$, $x \to F(t,x)$ has a closed graph;
(c) for every $r > 0$, there exists $\alpha_r \in L^1(T)_+$ such that for almost all $t \in T$, all $|x| \le r$ and all $u \in F(t,x)$, we have $|u| \le \alpha_r(t)$.

Note that in this setting $F(t,x) = [\psi(t,x), \varphi(t,x)]$ with $t \to \psi(t,x), \varphi(t,x)$ measurable and $x \to \psi(t,x), -\varphi(t,x)$ lower semicontinuous. Finally we should mention that our setting incorporates problems with discontinuous nonlinearities. Indeed if the nonlinearity of the problem is a measurable function $f : T \times \mathbb{R} \to \mathbb{R}$ such that for every $r > 0$ we can find $\alpha_r \in L^1(T)_+$ such that for almost all $t \in T$ and all $|x| \le r$, we have $|f(t,x)| \le \alpha_r(t)$, for the setting $j(t,x) = \int_0^x f(t,v)dv$, we have a locally Lipschitz potential

$j(t,x)$ and we can reasonably approximate the original problem with (1.1) where $j(t,x)$ is as defined above. This way we will fill in the gaps at the discontinuity points of $f(t,\cdot)$ and we can have an existence theory for the problem.

Acknowledgement. The authors wish to thank the referee for his corrections and remarks which helped clarify and improve the content of this paper.

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