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TRAVELING WAVES IN TIME DEPENDENT BISTABLE EQUATIONS

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Abstract. The current paper is to explore traveling waves in general time dependent bistable equations. In order to do so, it first introduces a notion of traveling wave solutions in general time dependent equations, which is a natural extension of classical traveling wave solutions. The basic point of view in the paper is that traveling wave solutions are certain limits of wave-like solutions. It then introduces a notion of wave-like solutions and shows in terms of certain backward-forward limits that the existence of wave-like solutions in general time dependent equations implies the existence of traveling wave solutions. It is shown that wave-like solutions exist in time dependent bistable equations and hence traveling wave solutions exist in such equations. Moreover, it is shown that traveling wave solutions in a time dependent bistable equation are stable and unique. The results obtained in the paper extend many of the results on traveling wave solutions of time independent (periodic, almost periodic) bistable equations.

1. Introduction

The current paper is to explore traveling wave solutions in general time dependent bistable equations.

Traveling wave solutions play an important role in the understanding of nonlinear dynamics in numerous applied problems. Many of such problems are modelled by reaction diffusion equations. For example, it was derived by S.M. Allen and J.W. Cahn in 1979 ([2]) that (ignoring time dependence and spatial inhomogeneouity) antiphase grain boundary motion (after proper rescaling) is modelled by

$$u_t = \Delta u - W_u(u), \quad x \in \mathbb{R}^n, \tag{1.1}$$

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where $W(u) = (1-u^2)^2$. In this case, $u(t,x)$ is an order parameter representing the state of the material at time t and position x, Δu represents the internal interaction energy, W is a double well potential function representing the entropy. The minima $u = \pm 1$ of W correspond to the pure phases. The antiphase grain boundary is the interface between two regions, one with order parameter 1, and the other -1. Therefore, among others, one is interested in the solutions $u(t,x)$ with $\lim_{x\cdot\sigma\to-\infty} u(t,x) = -1$ and $\lim_{x\cdot\sigma\to\infty} u(t,x) = 1$, where $\sigma \in \mathbb{R}^n$ is a unit vector, in particular, traveling wave solutions propagating along the direction of σ $u(t,x) = \phi(x\cdot\sigma - ct)$ with $\lim_{x\to\pm\infty} \phi(x) = \pm 1$. Equation (1.1) is often referred as the Allen-Cahn equation.

Notice that (1.1) has exactly three constant solutions $u = \pm 1$ and $u = 0$, and $u = \pm 1$ are stable, $u = 0$ is unstable. In general, a time independent equation

$$u_t = \Delta u + f(u), \quad x \in \mathbb{R}^n \tag{1.2}$$

is called *bistable* if there are three constant solutions $u = u^{\pm}$ and $u = u^0$ with $u^- < u^0 < u^+$, $u = u^{\pm}$ being stable and $u = u^0$ being unstable, and there is no other constant solution lying between u^- and u^+. For example, when $f(u) = u(u-a)(1-u)$ for some $0 < a < 1$, (1.2) is bistable. Bistable equations appear in modelling phase transition, nerve propagation and many other problems in science and engineering (see [4], [8], [16], [31], etc.).

When $f(u) = u(1-u)$, (1.2) serves as a model for the spatial spread of an allele in a migrating diploid species with two type alleles (ignoring time dependence and spatial inhomogeneouity) (see [4], [15], [18], etc.). Traveling wave solutions propagating along the direction of some given unit vector σ $u(t,x) = \phi(x\cdot\sigma - ct)$ with $\lim_{x\to-\infty}\phi(x) = 0$ and $\lim_{x\to\infty}\phi(x) = 1$ is also of particular interest in this case. Equation (1.2) with $f(u) = u(1-u)$ is often referred to as Fisher or KPP equation due to the pioneer works [18], [25].

In nature, many systems are subject to certain time dependence. When time dependence is taken into account, the above mentioned problems are modelled by the time dependent equation

$$u_t = \Delta u + f(t,u), \quad x \in \mathbb{R}^n \tag{1.3}$$

with appropriate nonlinearity $f(t,u)$. Due to the practical relevance, it is important to study front propagating dynamics in general time dependent evolution equations. However, though front propagating dynamics (i.e. traveling wave solutions) in time independent equations have been studied for a long time (see [4], [16], [18], [23], [25], [30], [31], [38], [40], [43], [44], etc.) and recently there is also some study on front propagating dynamics in time periodic and almost periodic equations (see [1], [32], [33], [34], [41], etc.), the understanding of front propagating dynamics in general time dependent equations is very little. To explore front propagating dynamics in general time dependent equations, one first needs to introduce a proper notion of traveling wave solutions in such equations. As is known, many traditional approaches developed for time independent equations are difficult to apply to time dependent equations. One also needs to discover new techniques to investigate front propagating dynamics in general time dependent equations. The objective of the current paper is to extend the notion of classical traveling wave solutions in time independent equations to general time dependent equations and to establish approaches for the study of traveling wave solutions in such equations, in particular, the existence, uniqueness, and stability of traveling waves in time dependent equations with bistable nonlinearity.

To be more specific, we first note that traveling wave solutions of (1.2) propagating along the direction of some unit vector are characterized by their corresponding equation in one space dimension

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}.$$

Therefore, throughout the paper, we consider the equation in one space dimension corresponding to (1.3),

$$u_t = u_{xx} + f(t, u), \quad x \in \mathbb{R}, \tag{1.4}$$

where f is a smooth function and f, f_t, f_u are bounded in $\mathbb{R} \times B$ for any bounded set $B \subset \mathbb{R}$. Recall that in the time independent case ($f(t,u) = f(u)$), a traveling wave solution of (1.4) is a solution $u(t,x)$ with a fixed profile $\phi(\cdot)$ and a constant speed c, that is, $u(t,x) = \phi(x - ct)$ (it is referred to as *classical traveling wave solution* in the following). A traveling wave solution of (1.4) in the time periodic case ($f(t,u)$ is periodic in t) is defined to be a solution $u(t,x)$ with a periodically varing profile $\phi : \mathbb{R} \to \mathbb{R}$ and a constant speed c, that is, $u(t,x) = \phi(x - ct, t)$ and $\phi(x,t)$ is periodic in t (see [1], [41])(it is referred to as *periodic traveling wave solution* in the following). In [32], [33], [34], the author of the current paper initiated a study of traveling waves of (1.4) in the time almost periodic case ($f(t,u)$ is almost periodic in t). A notion of almost periodic (almost automorphic) traveling wave solution is introduced in [32] ([34]) for the first time. Roughly speaking, an almost periodic (almost automorphic) traveling wave solution in the time almost periodic case is a solution with an almost periodic (almost automorphic) wave profile and an almost periodic (almost automorphic) wave speed. It should be mentioned that an almost periodic equation may not have almost periodic traveling wave solutions but almost automorphic ones.

In order to extend the notion of classical (periodic, almost periodic/almost automorphic) traveling wave solutions in time independent (periodic, almost periodic) equations to the general time dependent equation (1.4), we consider all the equations in the hull of (1.4),

$$u_t = u_{xx} + g(t,x), \tag{1.4$_g$}$$

where $g \in H(f) = \mathrm{cl}\{\sigma_t f : t \in \mathbb{R}\}$, $\sigma_t f(\cdot,\cdot) = f(t+\cdot,\cdot)$ and the closure is taken in the open compact topology.

In the following, $\sigma_t g$ denotes $\sigma_t g(\cdot,\cdot) = g(t+\cdot,\cdot)$. $H(f)$ is endowed with the open compact topology. Hence $(H(f), \{\sigma_t\}_{t\in\mathbb{R}})$ is a compact flow. f is said to be *recurrent in* t (recurrent for short) if $(H(f), \{\sigma_t\}_{t\in\mathbb{R}})$ is a compact minimal flow. f is said to be *almost periodic in* t *uniformly in* u (almost periodic for short) if for any given sequences $\{\alpha_n^{'}\}$, $\{\beta_n^{'}\} \subset \mathbb{R}$, there are subsequences $\{\alpha_n\} \subset \{\alpha_n^{'}\}$, $\{\beta_n\} \subset \{\beta_n^{'}\}$ such that

$$\lim_{n\to\infty}\lim_{m\to\infty} f(t+\alpha_n+\beta_m,u) = \lim_{n\to\infty} f(t+\alpha_n+\beta_n,u)$$

uniformly for t, u in compact sets. f is said to be *almost automorphic in* t *uniformly in* u (almost automorphic for short) if for any given sequence $\{\alpha_n^{'}\} \subset \mathbb{R}$, there is a subsequence $\{\alpha_n\} \subset \{\alpha_n^{'}\}$ such that

$$\lim_{n\to\infty}\lim_{m\to\infty} f(t+\alpha_n-\alpha_m,u) = f(t,u)$$

uniformly for t, u in compact sets. Note that if f is almost periodic (almost automorphic), then it is almost automorphic (recurrent). We refer the reader to [17] for definitions and properties of almost periodic functions and to [39], [46] for definitions and properties of almost automorphic functions. For given $f(t,x)$, $\tilde{f}(t,x)$ which are recurrent in t, $\tilde{f}$ is said to have *at least the same recurrence as* f if when $\{\alpha_n\} \subset \mathbb{R}$ is such that $\lim_{n\to\infty} f(t+\alpha_n,u)$ exists uniformly for t,u in compact sets, then $\lim_{n\to\infty} \tilde{f}(t+\alpha_n,u)$ also exists uniformly for t,u in compact sets (this implies that if $\lim_{n\to\infty} f(t+\alpha_n,u) = f(t,u)$, then $\lim_{n\to\infty} \tilde{f}(t+\alpha_n,u) = \tilde{f}(t,u)$ by considering $\lim_{n\to\infty} f(t+\beta_n,u)$ and $\lim_{n\to\infty} \tilde{f}(t+\beta_n,u)$ with $\beta_{2n-1} = \alpha_n$ and $\beta_{2n} = 0$ for $n = 1,2,\cdots$). $\tilde{f}$ is said to have *at least almost the same recurrence as* f if when $\{\alpha_n\} \subset \mathbb{R}^n$ is such that $\lim_{n\to\infty} f(t+\alpha_n,u) = f(t,u)$ uniformly for t,u in compact sets, then $\lim_{n\to\infty} \tilde{f}(t+\alpha_n,u) = \tilde{f}(t,u)$ uniformly for t,u in compact sets. Note that if $f(t,x)$ is almost periodic in t and $\tilde{f}$ is recurrent in t with at least (almost) the same recurrence as f, then $\tilde{f}$ is almost periodic (almost automorphic) in t with $\mathcal{M}(\tilde{f}) \subset \mathcal{M}(f)$, where $\mathcal{M}(\cdot)$ denotes the frequency module of an almost periodic or almost automorphic function (see [46]).

Observe that $\{(1.4)_g\}_{g\in H(f)}$ generates a skew-product semiflow $\Pi_t : X \times H(f) \to X \times H(f)$ (X is an appropriate function space, see (2.1)-(2.3) for detail),

$$\Pi_t(u_0, g) = (u(t, \cdot; u_0, g), \sigma_t g),$$

where $u(t, x; u_0, g)$ is the solution of $(1.4)_g$ with $u(0, x; u_0, g) = u_0(x)$.

Throughout this paper, we assume

(H1) *There are $u^\pm : H(f) \to \mathbb{R}$ satisfying that $u^\pm$ are continuous at any $g \in H(g)$, $u^-(g) < u^+(g)$, and $u^\pm(t, g) := u^\pm(\sigma_t g)$ are solutions of $(1.4)_g$ for any $g \in H(f)$.*

Note that $u^\pm(t, f)$ are among the simplest solutions of (1.4). More precisely, if f is time independent (periodic, almost periodic in t), then $u^\pm(t, f)$ are also time independent (periodic with the same period as that of f, almost periodic with $\mathcal{M}(u^\pm) \subset \mathcal{M}(f)$). If f is recurrent in t , then $u^\pm(t, f)$ are recurrent with at least the same recurrence as f.

A solution $u(t, x; f)$ of (1.4) is called a *traveling wave solution connecting $u^\pm$* (traveling wave solution for short) if there are $U : \mathbb{R} \times H(f) \to \mathbb{R}$ and $c : \mathbb{R} \times H(f) \to \mathbb{R}$ such that $\lim_{x\to\pm\infty} U(x, g) = u^\pm(g)$, $u(t, x; f) = U(x - c(t, f), \sigma_t f)$, $u(t, x; U(\cdot, g), g) = U(x - c(t, g), \sigma_t g)$, and $U(\cdot, g)$ is continuous at residually many $g \in H(f)$ (see Definition 3.1 for details) ($U(\cdot, g)$ is usually referred to as *wave-front function* and $c_t(t, g)$ is referred to as *propagating speed*). It is called a *critical traveling wave solution* if it is steeper than any other traveling wave solution (see Definition 3.1 for detail). In many cases, the critical traveling wave solution is the one with relatively minimal propagating speed among all the traveling wave solutions (see [34]). A traveling wave solution with wave-front function $U(\cdot, g)$ is called *(strictly) monotone* if $U_x(x, g) \geq (>)0$ for $x \in \mathbb{R}$ and $g \in H(f)$.

Note that when f is independent of t (periodic, almost periodic in t), a traveling wave solution defined in the above sense is a classical (periodic, almost automorphic) traveling wave solution (see Theorem 3.1). Hence the notion of traveling wave solutions introduced in the current paper is a natural extension of classical traveling wave solutions. As almost periodic equations may not have almost periodic but almost automorphic solutions, in general, we may not expect the wave front function $U(\cdot, g)$ to be continuous at any $g \in H(f)$.

We shall adopt the same point of view as in [32], [33], [34] (that is, traveling wave solutions are the limit of wave-like solutions in certain sense) to study the existence of traveling wave solutions. Roughly, a solution of $(1.4)_g$ is called a *wave-like solution connecting* $u^{\pm}$ (wave-like solution for short) if it connects $u^{\pm}$ and does not become flat as time increases (see Definition 3.2 for details). However, the approaches developed in [32], [33], [34] to show that the existence of a wave-like solution implies the existence of traveling wave solutions are difficult to apply to the general time dependent case. In the current paper, we employ certain backward-forward limit arguments to show that the existence of a wave-like solution implies the existence of traveling wave solutions. The following is a general existence theorem about traveling wave solutions established in the paper.

Theorem A. (Theorem 3.1) *Consider* (1.4). *The existence of a wave-like solution implies the existence of a strictly monotone critical traveling wave solution. Moreover, if f is almost periodic (recurrent) in t, then for residually many $g \in H(f)$, $U^g(t,x) = U(x,\sigma_t g)$ is almost automorphic (recurrent) in t with $\mathcal{M}(U^g(\cdot,x)) \subset \mathcal{M}(g) \equiv \mathcal{M}(f)$ (with at least almost the same recurrence as g), where $U(\cdot,g)$ is the wave-front function of the critical traveling wave solution. If f is periodic of period T in t, then so is $U^g(t,x) = U(x,\sigma_t g)$ for any $g \in H(f)$.*

We say that (1.4) or f is bistable if it satisfies

(H2) *There are $u^0 : H(f) \to \mathbb{R}$ and $\delta^0, T^0 > 0$ satisfying that*

i) *u^0 is continuous at any $g \in H(g)$, $u^-(g) < u^0(g) < u^+(g)$, and $u^0(t,g) = u^0(\sigma_t g)$ is a solution of $(1.4)_g$ for any $g \in H(f)$;*

ii) *$u^{\pm}$ in* **(H1)** *are stable in the sense that*

$$\frac{1}{t}\int_0^t g_u(s, u^{\pm}(\sigma_s g))ds \le -\delta^0$$

for $t \ge T^0$, $g \in H(f)$, and for any $\delta > 0$ and $M > 0$,

$$\lim_{t\to\infty} u(t,\cdot;u_0,g) - u^{+(-)}(\sigma_t g) = 0$$

uniformly in $g \in H(f)$, where $u^0(g) + \delta < u_0 < u^+(g) + M$ ($u^-(g) - M < u_0 < u^0(g) - \delta$).

iii) *u^0 is unstable in the sense that*

$$\frac{1}{t}\int_0^t g_u(s, u^0(\sigma_s g))ds \ge \delta^0$$

for $t \ge T^0$, $g \in H(f)$.

Clearly, if $f(t,u) = u(u-a)(1-u)$, then it is bistable. We have the following theorem about traveling wave solutions in time dependent bistable equations.

Theorem B. *Consider* (1.4).

1) (*Theorem* 4.1 *and Corollary* 4.2) *If f is of bistable type, then there is a wave-like solution of* $(1.4)_g$ *for any $g \in H(f)$. Moreover, it is uniformly in $g \in H(f)$ (see Definition 3.2 2)). Hence* (1.4) *has a strictly monotone critical traveling wave solution.*

2) (*Theorem* 5.1) *If f is bistable and recurrent, then* (1.4) *has a strictly monotone traveling wave solution whose wave front function is continuous. In particular, if f is periodic* (*quasi-periodic, almost periodic*), *then so is the traveling wave solution.*

3) (*Theorem* 5.1) *If f is bistable and recurrent, then traveling wave solutions are unique in the sense that for any two traveling wave solutions with continuous wave-front functions $U(\cdot,g)$ and $V(\cdot,g)$, there is a continuous function $\xi^* : H(f) \to \mathbb{R}$ such that $V(\cdot,g) = U(\cdot + \xi^*(g), g)$.*

4) (*Theorem* 5.1) *If f is bistable and recurrent, the unique traveling wave solution is globally and asymptotically stable in the sense that for any $g \in H(f)$ and $u_0 \in X$ with $u^-(g) \le u_0(x) \le u^+(g)$ for $x \in \mathbb{R}$ and $\lim_{x\to\infty} u_0(x) = u^{\pm}(g)$, there is $\xi(u_0,g)$ such that*

$$u(t,x;u_0,g) - u(t,x+\xi(u_0,g);U(\cdot,g),g) \to 0 \quad \textit{as } t \to \infty.$$

We make the following remarks. First, the above results generalize most existing ones on traveling wave solutions in bistable equations. Second, traveling wave solutions have also been widely studied for nonlocal evolution equations (see [5], [7], [10], [11], etc.), delayed reaction diffusion equations (see [22], [26], [37], [42], [49], etc.), and spatial discrete evolution equations (see [6], [9], [12], [13], [14], [20], [24], [27], [28], [35], [41], [47], [48], etc.). Finally, it should be pointed out that there are some recent works on traveling wave solutions in time independent but spatially inhomogeneous media (see [29], [45], etc.). In [36], the author of the current paper initiated a study on traveling waves in (both space and time) random media.

The paper is organized as follows. In section 2, we present preliminary lemmas to be used in later sections. We introduce notions of traveling wave solutions and wave-like solutions in general time dependent equations and prove Theorem A in section 3. Section 4 is devoted to the study of wave-like solutions in time dependent bistable equations and Theorem B 1) is proved in this section. Theorem B 2), 3), and 4) are proved in section 5.

2. Preliminary Lemmas

Let

$$X = C^b_{unif}(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R} : u \text{ is bounded and uniformly continuous}\} \tag{2.1}$$

with the uniform convergence topology. For given $u_0 \in X$, denote $u(t, \cdot; u_0, g)$ as the solution of $(1.4)_g$ with $u(0, \cdot; u_0, g) = u_0(\cdot)$. It is not difficult to see that $u(t, x; u_0, g)$ is continuous in t, x, u_0, and g (see [21]). Therefore, (1.4) generates a (local) skew-product semiflow,

$$\Pi_t : X \times H(f) \to X \times H(f), \tag{2.2}$$

$$\Pi_t(u_0, g) = (\pi_t(u_0, g), \sigma_t g), \tag{2.3}$$

where $t \in \mathbb{R}^+$, $x \in \mathbb{R}$, and $\pi_t(u_0, g) = u(t, \cdot; u_0, g)$. Let

$$BPC(\mathbb{R}) = \{u : \mathbb{R} \to \mathbb{R} : u \text{ is bounded, piecewise continuous, and has finitely many discontinuous points}\}. \tag{2.4}$$

Observe that for any $u_0 \in BPC(\mathbb{R})$ and $g \in H(f)$, a solution of $(1.4)_g$ with initial data $u_0(\cdot)$ exists and we may also write it as $u(t, x; u_0, g)$.

Throughout the paper, $u_0^*(g)$ is defined as follows,

$$u_0^*(g)(x) = \begin{cases} u^+(g) & \text{for} \quad x \geq 0 \\ u^-(g) & \text{for} \quad x < 0. \end{cases} \tag{2.5}$$

Clearly, $u_0^* \in BPC(\mathbb{R})$.

For any $u_0 \in X$ or $BPC(\mathbb{R})$, $u_0(\pm\infty) := \lim_{x \to \pm\infty} u_0(x)$ if the limits exist.

Lemma 2.1. (1) *Given $u_0 \in X$ or $BPC(\mathbb{R})$ and $g \in H(f)$, if $u_0(\pm\infty) = u^\pm(g)$, then $u(t, \pm\infty; u_0, g) = u^\pm(\sigma_t g)$ for any $t > 0$ at which $u(t, x; u_0, g)$ exists.*

(2) *Given $u_0 \in X$ or $BPC(\mathbb{R})$, if $u_0(x)$ is non-decreasing (non-increasing) in x and $u_0 \not\equiv$ constant, then $u(t, x; u_0, g)$ is strictly increasing (decreasing) in x for any $g \in H(f)$ and $t > 0$ at which $u(t, x; u_0, g)$ exists.*

Proof. If follows from standard theory for parabolic equations (see for example, [19], [21]). □

Lemma 2.2. *Given $g_n \in H(f)$ and u_n, $u_0 \in X$ with u_n, u_0 being bounded, if $g_n(t, u) \to f^*(t, u)$ and $u_n(x) \to u_0(x)$ in the open compact topology, then*

$$u(t, x; u_n, g_n) \to u^*(t, x; u_0)$$

in the open compact topology, where $u^*(t,x;u_0)$ *is the solution of*

$$u_t = u_{xx} + f^*(t,u)$$

with $u^*(0,x;u_0) = u_0(x)$*.*

Proof. Let $v_n(t,x) = u(t,x;u_n,g_n) - u^*(t,x;u_0)$. Then $v_n(t,x)$ satisfies

$$\begin{aligned}(v_n)_t = (v_n)_{xx} + (g_n)_u(t,u_n^*(t,x))v_n \\ + g_n(t,u^*(t,x;u_0)) - f^*(t,u^*(t,x;u_0)),\end{aligned}$$

where $u_n^*(t,x)$ lies between $u(t,x;u_n,g_n)$ and $u^*(t,x;u_0)$. Let

$$\tilde{v}_n(t,x) = \frac{v_n(t,x)}{1+x^2}.$$

Then $\tilde{v}_n$ satisfies

$$\begin{aligned}(\tilde{v}_n)_t = (\tilde{v}_n)_{xx} + \frac{4x}{1+x^2}(\tilde{v}_n)_x + \Big[(g_n)_u(t,u_n^*(t,x)) + \frac{2}{1+x^2}\Big]\tilde{v}_n \\ + \frac{g_n(t,u^*(t,x;u_0)) - f^*(t,u^*(t,x;u_0))}{1+x^2}.\end{aligned}$$

Note that

$$\tilde{v}_n(0,x) = \frac{v_n(0,x)}{1+x^2} \to 0$$

and

$$\frac{g_n(s,u^*(s,x;u_0)) - f^*(s,u^*(s,x;u_0))}{1+x^2} \to 0$$

as $n \to \infty$ uniformly in $0 \le s \le t$ and $x \in \mathbb{R}$, and $(g_n)_u(s,u_n^*(s,x)) + 2/(1+x^2)$ is uniformly bounded for $0 \le s \le t$ and $x \in \mathbb{R}$. It then follows that $\tilde{v}_n(t,x) \to 0$ as $n \to \infty$ uniformly in $x \in \mathbb{R}$. Hence $v_n(t,x) \to 0$ and then $u(t,x;u_0,g_n) \to u^*(t,x;u_0)$ as $n \to \infty$ in the open compact topology. □

Lemma 2.3. *Let* $u_\epsilon(\cdot)$*,* $u_0(\cdot) \in BPC(\mathbb{R})$ *be such that*

$$\int_{-\infty}^{\infty} |u_\epsilon(x) - u_0(x)|dx \to 0$$

as $\epsilon \to 0$*. Then for any* $t > 0$ *at which* $u(t,x;u_\epsilon,g)$ *and* $u(t,x;u_0,g)$ *exist,*

$$\lim_{\epsilon \to 0} u(t,x;u_\epsilon,g) = u(t,x;u_0,\mathbb{R})$$

uniformly in $x \in \mathbb{R}$*.*

Proof. See [25]. □

Lemma 2.4. *For any* y_1, $y_2 \in \mathbb{R}$ *with* $y_1 > (<) y_2$ *and any* $t > 0$, *there holds*

$$u(t,\cdot;u_0^*(\sigma_{-t}g)(\cdot + y_1),\sigma_{-t}g) > (<) u(t,\cdot;u_0^*(\sigma_{-t}g)(\cdot + y_2),\sigma_{-t}g).$$

Proof. Note that for any $y \in \mathbb{R}$,

$$u_0^*(\sigma_{-t}g)(x+y) = \begin{cases} u^+(\sigma_{-t}g) & \text{for} \quad x \geq -y \\ u^-(\sigma_{-t}g) & \text{for} \quad x < -y. \end{cases}$$

Hence, for any y_1, $y_2 \in \mathbb{R}$ with $y_1 > (<) y_2$,

$$u_0^*(\sigma_{-t}g)(x+y_1) \geq (\leq) u_0^*(\sigma_{-t}g)(x+y_2)$$

for $x \in \mathbb{R}$, but $u_0^*(\sigma_{-t}g)(x+y_1) \not\equiv u_0^*(\sigma_{-t}g)(x+y_2)$. The lemma then follows from the comparison principle for parabolic equations. □

Lemma 2.5. *Consider*

$$u_t = u_{xx} + q(t,x)u, \quad x \in \mathbb{R}, \quad t > 0, \tag{2.6}$$

where q *is a bounded and continuous function. Let* $u(t,x)$ *be a nonzero solution of* (2.6).

1) *For each* $t > 0$, *the zero set of* $u(t,x)$, $z_t = \{x \in \mathbb{R} : u(t,x) = 0\}$ *is a discrete subset of* $\mathbb{R}$.
2) *If at* (t_0, x_0) *both* u *and* u_x *vanish, then there is a neighborhood* $N = [t_0 - \delta, t_0 + \delta] \times [x_0 - \epsilon, x_0 + \epsilon]$ *of* (t_0, x_0) *such that*

(i) $u(t, x_0 \pm \epsilon) \neq 0$ *for* $|t - t_0| \leq \delta$;
(ii) $u(t_0 + \delta, \cdot)$ *has at most one zero in the interval* $[x_0 - \epsilon, x_0 + \epsilon]$;
(iii) $u(t_0 - \delta, \cdot)$ *has at least two zeros in the interval* $[x_0 - \epsilon, x_0 + \epsilon]$.

Proof. See [3]. □

Lemma 2.6. *Given* $g \in H(f)$ *and* $u_0 \in X$, *if* $u_0(\pm\infty) = u^\pm(g)$,

$$u^-(g) < u_0(x) < u^+(g) \quad \text{for} \quad -\infty < x < \infty,$$

and $u_0(x)$ *is differentiable and* $u_0'(x)$ *is bounded for* $x \in \mathbb{R}$, *then for any* $t > 0$, *there is a unique* $\xi(t,g)$ *such that*

$$u(t,x;u_0^*(g),g) \begin{cases} > u(t,x;u_0,g) & \text{for} \quad x > \xi(t,g) \\ < u(t,x;u_0,g) & \text{for} \quad x < \xi(t,g). \end{cases}$$

Proof. First, it is not difficult to see that there is $u_\epsilon^*(g) \in X$ such that $u_\epsilon^*(g)(x) = u_0^*(g)(x)$ for $|x| \gg 1$,

$$\int_{-\infty}^{\infty} |u_\epsilon^*(g)(x) - u_0^*(g)(x)| dx \to 0$$

as $\epsilon \to 0$ and for $0 < \epsilon \ll 1$, $u_\epsilon^*(g)(\cdot) - u_0(\cdot)$ has exactly one simple zero. Then by Lemma 2.5, for any $t > 0$, there is $\xi_\epsilon(t,g)$ such that

$$u(t,x;u_\epsilon^*(g),g)\begin{cases} > u(t,x;u_0,g) & \text{for} \quad x > \xi_\epsilon(t,g) \\ < u(t,x;u_0,g) & \text{for} \quad x < \xi_\epsilon(t,g). \end{cases}$$

Take a sequence $\epsilon_n \to 0$. Without loss of generality, we may assume that $\xi_{\epsilon_n}(t,g) \to \xi(t,g) \in [-\infty,\infty]$. By Lemma 2.3, we have

$$u(t,x;u_0^*(g),g)\begin{cases} \geq u(t,x;u_0,g) & \text{for} \quad x > \xi(t,g) \\ \leq u(t,x;u_0,g) & \text{for} \quad x < \xi(t,g). \end{cases}$$

Note that the above holds for any $t > 0$. It then follows from Lemma 2.5 and the comparison principle for parabolic equations that

$$u(t,x;u_0^*(g),g)\begin{cases} > u(t,x;u_0,g) & \text{for} \quad x > \xi(t,g) \\ < u(t,x;u_0,g) & \text{for} \quad x < \xi(t,g). \end{cases}$$

□

Let Y and Z be two compact metric spaces and $P : Z \to Y$ be a homomorphism.

Lemma 2.7. *There is a residual subset $Y_0 \subset Y$ such that for any $y_0 \in Y_0$ and $y_n \in Y$ with $y_n \to y_0$ as $n \to \infty$, for any $z_0 \in P^{-1}(y_0)$, there are $z_n \in P^{-1}(y_n)$ such that $z_n \to z_0$ as $n \to \infty$.*

Proof. See [39] or [46]. □

3. Traveling Wave and Wave-Like Solutions in General Equations

Consider (1.4). In this section, we first introduce notions of traveling wave solutions and wave-like solutions in general time dependent case. Then we show Theorem A, that is, that the existence of wave-like solutions implies the existence of traveling wave solutions.

Definition 3.1. 1) A global solution $u^*(t,x)$ of (1.4) is called a traveling wave solution connecting $u^\pm$ (traveling wave solution for short) if there are $U : H(f) \to X$, which is continuous on a residual subset $H_0(f) \subset H(f)$ with $\sigma_t H_0(f) = H_0(f)$ and $\lim_{x\to\pm\infty} U(g)(x) = u^\pm(g)$ for $g \in H(f)$, and $c : \mathbb{R} \times H(f) \to \mathbb{R}$ such that $u(t,x;U(\cdot,g),g) = U(x - c(t,g), \sigma_t g)$ ($U(x,g) \equiv U(g)(x)$ is called a wave front function) and $u^*(t,x) = u(t,x;U(f),f)(= U(x - c(t,f), \sigma_t f))$. We say that $U : H(f) \to X$ generates a traveling wave solution of (1.4) if U is continuous on a residual subset $H_0(f) \subset H(f)$ with $\sigma_t H_0(f) = H_0(f)$, $\lim_{x\to\pm\infty} U(g)(x) = u^{\pm}(g)$, and there is $c : \mathbb{R}\times H(f) \to \mathbb{R}$ such that $u(t, x; U(\cdot, g), g) = U(x - c(t, g), \sigma_t g)$.

2) A traveling wave solution $u^*(t, x)$ of (1.4) generated by $U : H(f) \to X$ is said to be critical if for any $V : H(f) \to X$ which generates a traveling wave solution of (1.4), there is $\xi : H(f) \to \mathbb{R}$ such that

$$U(x, g) \begin{cases} \geq V(x, g) & \text{for} \quad x > \xi(g) \\ \leq V(x, g) & \text{for} \quad x < \xi(g). \end{cases}$$

3) A traveling wave solution generated by $U : H(f) \to \mathbb{R}$ is (strictly) monotone if $U_x(x, g) \geq (>) 0$ for $x \in \mathbb{R}$ and $g \in H(f)$ $(U(x, g) = U(g)(x))$.

Definition 3.2. 1) Given $g \in H(f)$ and $u_0 \in X$ with bounded derivative $u_{0x}(x)$ for $x \in \mathbb{R}$ and

$$u^-(g) < u_0(x) < u^+(g) \quad \text{for} \quad -\infty < x < \infty, \quad u_0(\pm\infty) = u^{\pm}(g),$$

$u(t, \cdot; u_0, g)$ is called a wave-like solution of $(1.4)_g$ connecting $u^{\pm}$ (wave-like solution for short) if for any $0 < \delta \ll 1$ there is $m(\delta, u_0, g) > 0$ such that

$$x_+(t, g) - x_-(t, g) \leq m(\delta, u_0, g) \quad \text{for} \quad t \geq 0,$$

where

$$x_-(t, g) = \sup\{\xi : u(t, x; u_0, g) \leq u^-(\sigma_t g) + \delta \quad \text{for} \quad x \leq \xi\}$$

and

$$x_+(t, g) = \inf\{\xi : u(t, x, u_0, g) \geq u^+(\sigma_t g) - \delta \quad \text{for} \quad x \geq \xi\}.$$

2) Given $u_0(\cdot, g) \in X$ with bounded derivative $u_{0x}(x, g)$ for $x \in \mathbb{R}$ and

$$u^-(g) < u_0(x, g) < u^+(g) \quad \text{for} \quad -\infty < x < \infty, \quad u_0(\pm\infty, g) = u^{\pm}(g),$$

$u(t, \cdot; u_0(\cdot, g), g)$ is said to be wave-like uniformly in $g \in H(f)$ if it is a wave-like solution of $(1.4)_g$ and $m(\delta, u_0(\cdot, g), g) \leq m(\delta)$ for some $m(\delta) > 0$ and any $g \in H(f)$.

Theorem 3.1. *Suppose that* $(1.4)_g$ *has a wave-like solution that is uniformly in* $g \in H(f)$*. Then*

1) *There is a strictly monotone critical traveling wave solution of* (1.4)*. More precisely, there are* $U(\cdot, g) \in X$ *and* $c(\cdot, g) \in \mathbb{R}$ *such that*

$$u(t, x; U(\cdot, g), g) = U(x - c(t, g), \sigma_t g), \quad U(\pm\infty, g) = u^{\pm}(g), \quad U_x(x, g) > 0$$

for $x \in \mathbb{R}$ *and* $g \in H(f)$, *and* $U(\cdot, g)$ *is continuous on a residual subset* $H_0(g)$ *of* $H(f)$ *with* $\sigma_t H_0(f) = H_0(f)$. *Moreover, for any* $V : H(f) \to X$ *which generates a traveling wave solution of* (1.4), *there is* $\xi : H(f) \to \mathbb{R}$ *such that*

$$U(x,g) \begin{cases} \geq V(x,g) & \text{for} \quad x > \xi(g) \\ \leq V(x,g) & \text{for} \quad x < \xi(g), \end{cases}$$

where $V(x,g) = V(g)(x)$.

2) *If* f *is recurrent, then for any* $g_0 \in H_0(f)$, $h(t, g_0) \equiv U(\cdot, \sigma_t g_0) \in X$ *is a recurrent function from* $\mathbb{R}$ *to* X *with at least almost the same recurrence as* g_0. *If* f *is almost periodic, then* $h(t, g_0)$ *is an almost automorphic function from* $\mathbb{R}$ *to* X *with* $\mathcal{M}(h(\cdot, g_0)) \subset \mathcal{M}(g_0) \equiv \mathcal{M}(f)$ *for any* $g_0 \in H_0(f)$. *If* f *is periodic of period* T, *then* $H_0(f) = H(f)$ *and* $U(\cdot, t) = U(\cdot, \sigma_t g_0)$ *is periodic in* t *with period* T. *In this case, let*

$$c_0 = \frac{c(T, g_0) - C(0, g_0)}{T} \quad \textit{and} \quad V(x,t) = U(x + c_0 t - c(t, g_0), \sigma_t g_0).$$

Then

$$u(t, x; U(\cdot, g_0), g_0) = V(x - c_0 t, t) \quad \textit{and} \quad V(x, t+T) = V(x,t).$$

Proof. 1) We first prove the existence of U and c such that $U(\pm\infty, g) = u^{\pm}(g)$, $U_x(x,g) > 0$ for $x \in \mathbb{R}$ and $g \in H(f)$, and $u(t, x; U(\cdot, g), g) = U(x - c(t,g), \sigma_t g)$ for $g \in H(f)$, $t \in \mathbb{R}$.

Let $u_0^*(g)$ be as in (2.5). Take a function $\tilde{u}^0 : H(f) \to \mathbb{R}$ such that it is continuous on $H(f)$, $\inf\{u^+(g) - \tilde{u}^0(g), \tilde{u}^0(g) - u^-(g), g \in H(f)\} > 0$, and $\tilde{u}^0(\sigma_t g)$ is differentiable in t with bounded derivative (for example, we may take $\tilde{u}^0(g) = \frac{u^+(g) + u^-(g)}{2}$). Then by Lemma 2.1, for any $g \in H(f)$ and $s > 0$, there is a unique $y(s,g)$ such that

$$u(s, y(s,g); u_0^*(\sigma_{-s} g), \sigma_{-s} g) = \tilde{u}^0(g),$$

and for any $0 < \delta \ll 1$, there are $y^{\pm}(s, g, \delta)$ such that

$$u(s, y^{\pm}(s, g, \delta); u_0^*(\sigma_{-s} g), \sigma_{-s} g) = u^{\pm}(g) \mp \delta.$$

By Lemma 2.6, for any $0 < \delta \ll 1$, there is $m_0^*(\delta)$ such that

$$y^+(s, g, \delta) - y^-(s, g, \delta) \leq m_0^*(\delta)$$

for $g \in H(f)$ and $s > 0$. Hence

$$\lim_{x \to \pm\infty} u(s, x + y(s,g); u_0^*(\sigma_{-s} g), \sigma_{-s} g) = u^{\pm}(g)$$

uniformly for $g \in H(f)$. For given $g \in H(f)$ and $s \geq 0$, let

$$U(s, x; g) = u(s, x + y(s,g); u_0^*(\sigma_{-s} g), \sigma_{-s} g).$$

Then by Lemma 2.6 again, for any $0 < s_1 < s_2$, we have

$$U(s_1, x; g) \begin{cases} > U(s_2, x; g) & \text{for} \quad x > 0 \\ < U(s_2, x; g) & \text{for} \quad x < 0. \end{cases}$$

Hence $\lim_{s\to\infty} U(s, x; g)$ exists uniformly for $x \in \mathbb{R}$. Denote it as

$$U(x, g) = \lim_{s\to\infty} U(s, x; g).$$

Then

$$U(0, g) = \tilde{u}^0(g), \qquad \lim_{x\to\pm\infty} U(x; g) = u^{\pm}(g)$$

uniformly for $g \in H(f)$, and $u(t, x; U(\cdot, g), g)$ exists for $t \in \mathbb{R}$. Clearly $u(t, x; U(\cdot, g), g)$ is nondecreasing in x for any $t \in \mathbb{R}$. It then follows that $U_x(x, g) > 0$ for $x \in \mathbb{R}$ and $g \in H(f)$.

We claim that for any $g \in H(f)$ and $t \in \mathbb{R}$, there is $c(t, g)$ such that

$$u(t, x; U(\cdot, g), g) = U(x - c(t, g), \sigma_t g).$$

In fact, by Lemma 2.1, there is unique $c(t, g)$ such that

$$u(t, c(t, g); U(\cdot, g), g) = \tilde{u}^0(\sigma_t g).$$

Note that

$$u(t, x + c(t, g); U(\cdot, g), g) = \lim_{s\to\infty} u(t + s, x + y(s, g) + c(t, g); u_0^*(\sigma_{-s} g), \sigma_{-s} g)$$

and

$$\begin{aligned} U(x, \sigma_t g) &= \lim_{s\to\infty} u(s, x + y(s, \sigma_t g); u_0^*(\sigma_{-s+t} g), \sigma_{-s+t} g) \\ &= \lim_{s\to\infty} u(t + s, x + y(t + s, \sigma_t g); u_0^*(\sigma_{-s} g), \sigma_{-s} g). \end{aligned}$$

Take $s_n \to \infty$. Without loss of generality, we may assume that

$$y(s_n, g) + c(t, g) \geq y(t + s_n, \sigma_t g)$$

and

$$u^*(x) = \lim_{n\to\infty} u(t + s_n - 1, x + y(s_n, g) + c(t, g); u_0^*(\sigma_{-s_n} g), \sigma_{-s_n} g),$$

$$U^*(x) = \lim_{n\to\infty} u(t + s_n - 1, x + y(t + s_n, \sigma_t g); u_0^*(\sigma_{-s_n} g), \sigma_{-s_n} g)$$

in the open compact topology. Then by Lemma 2.2,

$$u(t, x + c(t, g); U(\cdot, g), g) = u(1, x; u^*, \sigma_{t-1} g)$$

and $U(x,\sigma_t g) = u(1,x;U^*,\sigma_{t-1}g)$. By Lemma 2.4, $u^*(x) \geq U^*(x)$ for $x \in \mathbb{R}$. It then follows from the comparison principle for parabolic equations and the fact that $u(t,c(t,g);U(\cdot,g),g) = U(0,\sigma_t g) = \tilde{u}^0(\sigma_t)$ that

$$u(t,x+c(t,g);U(\cdot,g),g) \equiv U(x,\sigma_t g).$$

Next, we prove that $U(\cdot,g)$ is continuous on a residual subset $H_0(f)$ of $H(f)$. Let

$$Y = cl\{(U(\cdot,g),g) : g \in H(f)\} \subset X \times H(f),$$

where the closure is taken in the $X \times H(f)$-topology. By the continuity of $u^{\pm}(g)$, for any $(V,g) \in Y$,

$$V(\pm\infty) = u^{\pm}(g),$$

$$u^-(g) \leq V(x) \leq u^+(g) \quad \text{for} \quad x \in \mathbb{R},$$

and

$$V(0) = \tilde{u}^0(g).$$

Let $P : Y \to H(f)$, $P(V(\cdot),g) = g$. Then Y is a compact subset of $X \times H(f)$ and P is continuous. Let

$$\bar{h} : H(f) \to 2^Y, \quad \bar{h}(g) = P^{-1}(g).$$

By Lemma 2.7, there is a residual subset $H_0(f) \subset H(f)$ such that $\bar{h}$ is continuous on $H_0(f)$.

We claim that for any $g \in H(f)$ and $(V(\cdot),g) \in Y$, there holds

$$U(x,g) \begin{cases} \geq V(x) & \text{for} \quad x \geq 0 \\ \leq V(x) & \text{for} \quad x \leq 0. \end{cases} \tag{3.1}$$

In fact, for any $g \in H(f)$ and $(V(\cdot),g) \in Y$, suppose that

$$(V(\cdot),g) = \lim_{n\to\infty} (U(\cdot,g_n),g_n).$$

Note that

$$U(x,g_n) = \lim_{s\to\infty} u(s,x+y(s,g_n);u_0^*(\sigma_{-s}g_n),\sigma_{-s}g_n)$$

in the uniform convergence topology. Without loss of generality, we may assume that for some $s_n \to \infty$,

$$V(x) = \lim_{n\to\infty} u(s_n,x+y(s_n,g_n);u_0^*(\sigma_{-s_n}g_n),\sigma_{-s_n}g_n)$$

in the uniform convergence topology. Then

$$u(t,x;V(\cdot),g) = \lim_{n\to\infty} u(t+s_n,x+y(s_n,g_n);u_0^*(\sigma_{-s_n}g_n),\sigma_{-s_n}g_n)$$

exists for any $t \in \mathbb{R}$. Note also that

$$U(x,g) = \lim_{t\to\infty} u(t,x;u_0^*(\sigma_{-t}g)(\cdot + y(t,g)),\sigma_{-t}g)$$

and

$$V(x) = u(t,x;u(-t,\cdot;V(\cdot),g),\sigma_{-t}g).$$

By the comparison principle for parabolic equations,

$$u^-(g) < V(x) < u^+(g) \quad \text{for} \quad x \in \mathbb{R}.$$

Following from Lemma 2.6 and the fact that $V(0) = u(t,0;u_0^*(\sigma_{-t}g)(\cdot + y(t,g)),\sigma_{-t}g) = \tilde{u}^0(g)$, we must have (3.1) holding.

We also claim that for any $g_0 \in H_0(f)$, $\bar{h}(g_0) = \{(U(\cdot,g_0),g_0)\}$ is a singleton. In fact, if $(V(\cdot),g_0) \in \bar{h}(g_0)$, then there are $(U(\cdot,g_n),g_n)$ such that $(U(\cdot,g_n),g_n) \to (V(\cdot),g_0)$ as $n \to \infty$. By the continuity of $\bar{h}$ at g_0, there are $(V_n,g_n) \in Y$ such that $(V_n,g_n) \to (U(\cdot,g_0),g_0)$. By (3.1),

$$U(x,g_n) \begin{cases} \ge V_n(x) & \text{for} \quad x \ge 0 \\ \le V_n(x) & \text{for} \quad x \le 0. \end{cases}$$

Hence we must have

$$V(x) \begin{cases} \ge U(x,g_0) & \text{for} \quad x \ge 0 \\ \le U(x,g_0) & \text{for} \quad x \le 0. \end{cases}$$

By (3.1) again,

$$U(x,g_0) \begin{cases} \ge V(x) & \text{for} \quad x \ge 0 \\ \le V(x) & \text{for} \quad x \le 0. \end{cases}$$

Hence $V(x) = U(x,g_0)$ for $x \in \mathbb{R}$.

It then follows that $h : H(f) \to X$, $h(g) = U(\cdot,g)$ is continuous at $g \in H_0(f)$.

Finally, we prove that $\sigma_t H_0(f) = H_0(f)$ for any $t \in \mathbb{R}$. Note that $g \in H_0(f)$ if and only if $P^{-1}(g)$ is a singleton.

Assume $g_0 \in H_0(f)$. For any $t > 0$, suppose that $(V(\cdot),\sigma_t g_0) \in Y$. Then there is $g_n \to g_0$ such that $U(\cdot,\sigma_t g_n) \to V(\cdot)$. Note that $\lim_{x\to\pm\infty} U(x,g) = u^\pm(g)$ uniformly for $g \in H(f)$. Hence,

$$u(-t,x;U(\cdot,\sigma_t g_n),\sigma_t g_n) - u^\pm(g_n) \to 0$$

as $x \to \pm\infty$ uniformly in $n \ge 1$ and therefore $c(-t,\sigma_t g_n)$ is bounded with respect to n. Without loss of generality, suppose that $c(-t,\sigma_t g_n) \to c^*$ as $n \to \infty$. Then

$$U(x,g_n) = u(-t,x + c(-t,\sigma_t g_n);U(\cdot,\sigma_t g_n),\sigma_t g_n)$$

$$
\begin{aligned}
&= u(-t, x + c^*; U(\cdot, \sigma_t g_n), \sigma_t g_n) \\
&\quad + u(-t, x + c(-t, \sigma_t g_n); U(\cdot, \sigma_t g_n), \sigma_t g_n) \\
&\quad - u(-t, x + c^*; U(\cdot, \sigma_t g_n), \sigma_t g_n) \\
&\to u(-t, x + c^*; V(\cdot), \sigma_t g_0) \\
&= u(-t, x; V(\cdot + c^*), \sigma_t g_0).
\end{aligned}
$$

Note that

$$U(x, g_n) \to U(x, g_0) = u(-t, x; U(\cdot + c(-t, \sigma_t g_0), \sigma_t g_0), \sigma_t g_0).$$

We then must have

$$U(\cdot + c(-t, \sigma_t g_0), \sigma_t g_0) = V(\cdot + c^*).$$

Since $U(0, \sigma_t g_0) = V(0) = \tilde{u}^0(\sigma_t g_0)$ and $U(x, g)$,$V(x)$ are strictly monotone in x, we must have $c^* = c(-t, \sigma_t g_0)$ and then $V(\cdot) = U(\cdot, \sigma_t g_0)$. Therefore $P^{-1}(\sigma_t g)$ is a singleton and then $\sigma_t g \in H_0(f)$. This implies that $\sigma_t H_0(f) = H_0(f)$ for any $t \in \mathbb{R}$.

2) First, if $(H(f), \mathbb{R})$ is minimal, then for any $g_0 \in H_0(f)$, by the continuity of $\bar{h}$ and h at g_0, $cl\{(U(\cdot, \sigma_t g_0), \sigma_t g_0) : t \in \mathbb{R}\}$ is minimal and $F(t, g_0) = U(\cdot, \sigma_t g_0)$ is a recurrent function from $\mathbb{R}$ to X with at least almost the same recurrence as that of g_0.

Now suppose that $(H(f), \mathbb{R})$ is almost periodic. Then for any $g_0 \in H_0(f)$, for any $\{\alpha_n^{'}\} \subset \mathbb{R}$, there is $\{\alpha_n\} \subset \{\alpha_n^{'}\}$ such that

$$\lim_{n\to\infty} \lim_{m\to\infty} \sigma_{-\alpha_n + \alpha_m} g_0 = g_0.$$

By the continuity of $\bar{h}$ at g_0 and the continuity of $u^{\pm}(g)$, we have

$$\lim_{n\to\infty} \lim_{m\to\infty} U(\cdot, \sigma_{-\alpha_n + \alpha_m} g_0) = U(\cdot, g_0)$$

in the uniform convergence topology. Therefore, $U(\cdot, \sigma_t g_0)$ is an almost automorphic function from $\mathbb{R}$ to X.

Finally suppose that $\sigma_T g_0 = g_0$ and $H(f) = \{\sigma_t g_0 : 0 \le t \le T\}$. By $\sigma_t H_0(f) = H_0(f)$, we have $H_0(f) = H(f)$ and hence $U(\cdot, t) \equiv U(\cdot, \sigma_t g_0)$ is periodic in t with periodic T. Let

$$c_0 = \frac{c(T, g_0) - c(0, g_0)}{T}$$

and

$$V(x, t) = U(x + c_0 t - c(t, g_0), \sigma_t g_0).$$

Note that $c(t + T, g_0) = c(t, g_0) + c(T, g_0)$. Then

$$c_0 \cdot (t + T) - c(t + T, g_0) = c_0 \cdot t - c(t, g_0).$$

Hence, $V(x, t+T) = V(x, T)$ and $u(t, x; U(\cdot, g_0), g_0) = V(x - c_0 t, t)$. □

Corollary 3.2. 1) *For any $M > 0$, there is $\alpha(M) > 0$ such that*

$$U_x(x, g) \geq \alpha(M)$$

for any $|x| \leq M$ and $g \in H(f)$.

2) *$c_t(t, g)$ exists and is uniformly bounded in $g \in H(f)$.*

Proof. 1) By the arguments in Theorem 4.1, for any $(V, g) \in Y$, $V_x(x) > 0$. Now assume that there is $M_0 > 0$ and $|x_n| \leq M_0$, $g_n \in H(f)$ such that $U_x(x_n, g_n) \to 0$ as $n \to \infty$. Without loss of generality, we may assume that $x_n \to x^*$, $g_n \to g^*$, $U(\cdot, g_n) \to V^*$, and $U_x(\cdot, g_n) \to V_x^*(\cdot)$ in the open compact topology. It then follows that $V_x^*(x^*, g^*) = 0$, a contradiction.

2) We first show that $c(t, g)$ is continuous in t. In fact, for any $t, t_0 \in \mathbb{R}$ and $g \in H(f)$,

$$\begin{aligned}
&|u(t, c(t, g); U(\cdot, g), g) - u(t, c(t_0, g); U(\cdot, g), g)| \\
&\leq |u(t, c(t, g); U(\cdot, g), g) - u(t_0, c(t_0, g); U(\cdot, g), g)| \\
&\quad + |u(t_0, c(t_0, g); U(\cdot, g), g) - u(t, c(t_0, g); U(\cdot, g), g)| \\
&= |\tilde{u}^0(\sigma_t g) - \tilde{u}^0(\sigma_{t_0} g)| + |u(t_0, c(t_0, g); U(\cdot, g), g) - u(t, c(t_0, g); U(\cdot, g), g)|.
\end{aligned}$$

Hence,

$$u(t, c(t, g); U(\cdot, g), g) - u(t, c(t_0, g); U(\cdot, g), g) \to 0$$

as $t \to t_0$. This together with 1) and $u(t, c(t, g); U(\cdot, g), g) = \tilde{u}^0(\sigma_t g)$ implies that $c(t, g) \to c(t_0, g)$ as $t \to t_0$.

Now note that $u(t, c(t, g); U(\cdot, g), g) = \tilde{u}^0(\sigma_t g)$. Then

$$\begin{aligned}
\frac{\tilde{u}^0(\sigma_t g) - \tilde{u}^0(\sigma_{t_0} g)}{t - t_0} &= \frac{u(t, c(t, g); U(\cdot, g), g) - u(t_0, c(t_0, g); U(\cdot, g), g)}{t - t_0} \\
&= \frac{u(t, c(t, g); U(\cdot, g), g) - u(t_0, c(t, g); U(\cdot, g), g)}{t - t_0} \\
&\quad + \frac{u(t_0, c(t, g); U(\cdot, g), g) - u(t_0, c(t_0, g); U(\cdot, g), g)}{t - t_0}.
\end{aligned}$$

This together with the continuity of $c(t, g)$ implies that $c_t(t, g)$ exists and

$$c_t(t, g) = \frac{(\tilde{u}^0(\sigma_t g))_t - u_t(t, c(t, g); U(\cdot, g), g)}{u_x(t, c(t, g); U(\cdot, g), g)}.$$

□

4. Existence of Wave-like and Traveling Wave Solutions in Bistable Equations

In this section, we consider the existence of wave-like solutions in time dependent bistable equations and prove Theorem B 1). Throughout this section, $u(t,\cdot;u_0,g)$ denotes the solution of $(1.4)_g$ with $u(0,x;u_0,g)=u_0(x)$. Let $\zeta(\cdot)$, $\eta(\cdot)$, and $H(\cdot)$ be functions with the following properties:

$$\zeta(s)=\frac{1}{2}\Big(1+\tanh\frac{s}{2}\Big),\quad s\in\mathbb{R},$$

$$\eta(s)=\begin{cases}0 & \text{if} \quad s<0\\ 1 & \text{if} \quad s\geq 4,\end{cases}$$

$$\eta^{'}(s)\geq 0 \quad \text{and} \quad |\eta^{''}(s)|\leq 2 \quad \text{for} \quad s\in\mathbb{R},$$

$$H(s)=\begin{cases}1 & \text{for} \quad s\geq 0\\ 0 & \text{for} \quad s<0.\end{cases}$$

Let $u_0^g(x)=u^-(g)(1-\zeta(x))+u^+(g)\zeta(x)$.

Theorem 4.1. *Consider* (1.4). *Suppose that f is of bistable type. Then for any $g\in H(f)$, $u(t,x;u_0^g(\cdot),g)$ is a wave-like solution of* $(1.4)_g$. *Moreover, $u(t,x;u_0^g(\cdot),g)$ is wave-like uniformly in $g\in H(f)$.*

The following corollary follows directly from Theorems 3.1 and 4.1.

Corollary 4.2. *Consider* (1.4). *Suppose that f is of bistable type. Then it has a critical monotone traveling wave solution.*

Theorem 4.1 is a generalization of [33, Theorem 3.1]. It can be proved by arguments similar to (but not the same as) those in proving [33, Theorem 3.1]. We provide a proof in the following. To do so, we first show the following lemmas.

Lemma 4.3. *There is $\tau>0$ such that the following hold.*

1) *Let $v^1(t,x;g)$, $v^2(t,x;g)$ be the solutions of*

$$v_t=v_{xx}+g_u(t,u^0(\sigma_t g))v(t,x) \tag{4.1}$$

with $v^1(t,0;g)=H(x)$, $v^2(0,x;g)=-1+2H(x)$. Then there is $\chi(g)\in\mathbb{R}$ such that

$$v^1(\tau,\chi(g);g)\geq 3,\quad v^2(\tau,\chi(g);g)\leq -3.$$

Moreover, $\chi(g)$ is bounded in $g\in H(f)$.

2) *Let* $u^1_\delta(t,x;g)$, $u^2_\delta(t,x;g)$ *be solutions of* $(1.4)_g$ *with*

$$u^1_\delta(0,x;g) = u^0(g) + \delta H(x)$$

and

$$u^2(0,x;g) = u^0(g) + \delta(-1+2H(x)).$$

There is $\delta_1 > 0$ *such that for* $0 < \delta \le \delta_1$,

$$u^1_\delta(\tau,\chi(g),g) \ge u^0(\sigma_\tau g) + 2\delta, \qquad u^2_\delta(\tau,\chi(g);g) \le u^0(\sigma_\tau g) - 2\delta.$$

3) *Let* $u^3_\delta(t,x;g)$, $u^4_\delta(t,x;g)$ *be solutions of* $(1.4)_g$ *with*

$$u^3_\delta(0,x;g) = u^0(g) + \delta H(x) - (u^0(g) - u^-(g))H(-h-x),$$

$$u^4_\delta(0,x;g) = u^0(g) + \delta(-1+2H(x)) + (u^+(g) - u^0(g) - \delta)H(x-h).$$

Let δ_1 *be as in 2). Then for any* $0 < \delta \le \delta_1$, *there is* $h_1(\delta)$ *such that for any* $h \ge h_1(\delta)$,

$$u^3_\delta(\tau,\chi(g);g) \ge u^0(\sigma_\tau g) + \delta, \qquad u^4_\delta(\tau,\chi(g),g) \le u^0(\sigma_\tau g) - \delta.$$

Proof. 1) Fix $g \in H(f)$. Denote $v(t,\cdot;v_0,g)$ as the solution of (4.1) with $v(0,\cdot;v_0,g) = v_0(\cdot)$. Then $v(t,x) = v(t,x;0,g) \equiv 0$ and

$$0 \le v^1(t,x;g) \le v(t,x;1,g),$$

$$v^2(t,x;g) = -v(t,x;1,g) + 2v^1(t,x;g).$$

Take a $\delta_0^* > 0$. Let $g(t) = g_u(t,u^0(\sigma_t g))$ and $g_+(t)$ be a bounded smooth function satisfying that $g_+(t) \ge \max\{g(t),\delta_0^*\}$ for any $t \in \mathbb{R}$. Let

$$\gamma_g(t) = \int_0^t g(s)ds$$

and

$$\gamma_g^+(t) = \int_0^t g_+(s)ds.$$

We claim that

$$v^1(t,-\infty;g) = 0, \quad v^1(t,\infty;g) \ge e^{\gamma_g(t)}.$$

In fact, let $w^+(t,x) = \rho(\epsilon)e^{2\gamma_g^+(t)} + \eta(\epsilon x)e^{\gamma_g^+(t)}$. Then

$$\begin{aligned} w_t^+ - w_{xx}^+ - g_u(t,u^0(\sigma_t g))w^+ &\ge w_t^+ - w_{xx}^+ - g_+(t)w^+ \\ &= (2\rho(\epsilon)e^{2\gamma_g^+(t)} + \eta(\epsilon x)e^{\gamma_g^+(t)})g_+(t) - \epsilon^2\eta''(\epsilon x)e^{\gamma_g^+(t)} \\ &\quad - g_+(t)\rho(\epsilon)e^{2\gamma_g^+(t)} - \eta(\epsilon x)g_+(t)e^{\gamma_g^+(t)} \\ &= \rho(\epsilon)e^{2\gamma_g^+(t)}g_+(t) - \epsilon^2\eta''(\epsilon x)e^{\gamma_g^+(t)}. \end{aligned}$$

Let

$$\rho(\epsilon) = \sup_{t,x\in\mathbb{R}} \frac{|\epsilon^2\eta''(\epsilon x)|}{g_+(t)}.$$

Then $\rho(\epsilon) > 0$ and $\rho(\epsilon) \to 0$ as $\epsilon \to 0$. It then follows that $w^+(t, x + \frac{4}{\epsilon})$ is a supersolution of (4.1). Note that $v^1(0, x; g) \le w^+(0, x + \frac{4}{\epsilon})$. By the comparison principle for parabolic equations, we have

$$v^1(t, x; g) \le w^+(t, x + \frac{4}{\epsilon}) = \rho(\epsilon)e^{2\gamma_g^+(t)} + \eta(\epsilon x + 4)e^{\gamma_g^+(t)}. \tag{4.2}$$

This implies that

$$\lim\sup_{x\to-\infty} v^1(t, x; g) \le \rho(\epsilon)e^{2\gamma_g^+(t)}$$

for any $\epsilon > 0$. Letting $\epsilon \to 0$, we have $v^1(t, -\infty; g) = 0$.

Next, note that $-w^+(t, -x)$ is a subsolution of (4.1) and $e^{\gamma_g(t)}$ is a solution of (4.1). Hence, $w^-(t, x) = e^{\gamma_g(t)} - w^+(t, -x)$ is a subsolution of (4.1). Clearly $v^1(0, x; g) \ge w^-(0, x - \frac{4}{\epsilon})$. Hence by the comparison principle for parabolic equations again,

$$v^1(t, x; g) \ge w^-(t, x - \frac{4}{\epsilon}) = e^{\gamma_g(t)} - \rho(\epsilon)e^{2\gamma_g^+(t)} - \eta(-\epsilon x + 4)e^{\gamma_g^+(t)}. \tag{4.3}$$

This implies that

$$\lim\inf_{x\to\infty} v^1(t, x; g) \ge e^{\gamma_g(t)} - \rho(\epsilon)e^{2\gamma_g^+(t)}$$

for any ϵ. Letting $\epsilon \to 0$, we have

$$\lim\inf_{x\to\infty} v^1(t, x; g) \ge e^{\gamma_g(t)} \quad \text{for } t > 0.$$

Now by **(H2)**, $\gamma_g(t) \ge \delta^0 t$ for $t \ge T^0$ and $g \in H(f)$. Clearly there is τ such that $e^{\gamma_g(\tau)} \ge 9$ for any $g \in H(f)$. Then $v(\tau, x; 1, g) \ge 9$. Since for $x \ll -1$, $v^1(\tau_0, x; g) < 3$ and for $x \gg 1$, $v^1(\tau, x, g) > 3$. Hence there is $\chi(g)$ such that $v^1(\tau, \chi(g); g) = 3$ and then

$$v^2(\tau, \chi(g); g) = -v(\tau, \chi(g); 1, g) + 2v^1(\tau, \chi(g), g) \le -9 + 6 = -3.$$

Moreover, by (4.2) and (4.3), $\chi(g)$ in bounded in $g \in H(f)$.

2) Let $w^1(t, x) = u^0(\sigma_t g) + \delta v^1(t, x; g) - \delta^2 e^{K_1 t}$, where v^1 is as in 1), K_1 and δ will be determined later. Then

$$\begin{aligned} & w_t^1 - w_{xx}^1 - g(t, w^1(t, x)) \\ & = g(t, u_0(\sigma_t g)) - g(t, u^0(\sigma_t g) + \delta v^1(t, x; g) - \delta^2 e^{K_1 t}) \\ & \quad + \delta g_u(t, u^0(\sigma_t g))v^1(t, x; g) - \delta^2 K_1 e^{K_1 t} \end{aligned}$$

$$= \delta^2\Big(g_u(t,u^0(\sigma_t g))e^{K_1 t} - K_1 e^{K_1 t} - \frac{1}{2} g_{uu}(t,u^*(t,x))(v^1(t,x;g) - \delta e^{K_1 t})^2\Big),$$

where $u^*(t,x;g)$ lies between $u^0(\sigma_t g)$ and $w^1(t,x)$. Then for K_1 large enough, δ_1 small enough such that $\delta_1 e^{K_1\tau} \le 1$,

$$w_t^1 - w_{xx}^1 - g(t,w^1) \le 0$$

for $0 \le t \le \tau$ and $0 < \delta \le \delta_1$. Note that $u_\delta^1(0,x;g) \ge w^1(0,x)$. Hence, $u_\delta^1(\tau,x;g) \ge w^1(\tau,x)$. For $0 < \delta \le \delta_1$, we have

$$u_\delta^1(\tau,\chi(g);g) \ge u^0(\sigma_\tau g) + 2\delta.$$

Similarly, we can prove that

$$u_\delta^2(\tau,\chi(g);g) \le u^0(\sigma_\tau g) - 2\delta.$$

3) Let $0 < \delta \le \delta_1$ and $u_\delta^1(t,x;g)$ be as in 2). For given positive constants ϵ, K_2, C, and $M \ge \sup_{g\in H(f)} u^+(g) - u^-(g)$, define

$$\begin{aligned} w^2(t,x;g) &= u_\delta^1(t,x;g) - \rho(\epsilon)e^{2K_2 t} \\ &\quad - M\eta\big(-\epsilon(x-\chi(g)-C(t-\tau))\big)\big(1-\eta(\epsilon(x-\chi(g)-C(t-\tau)))\big). \end{aligned}$$

Then

$$\begin{aligned} & w_t^2 - w_{xx}^2 - g(t,w^2(t,x;g)) \\ &= g(t,u_\delta^1(t,x;g)) - g(t,w^2(t,x;g)) - 2K_2\rho(\epsilon)e^{2K_2 t} \\ &\quad - \epsilon CM\Big[\eta'(-y)(1-\eta(y)) + \eta(-y)\eta'(y)\Big]_{y=\epsilon(x-\chi(g)-C(t-\tau))} \\ &\quad - \epsilon^2 M\Big[\eta''(-y)(1-\eta(y)) + 2\eta'(-y)\eta'(y) - \eta(-y)\eta''(y)\Big]_{y=\epsilon(x-\chi(g)-C(t-\tau))}. \end{aligned}$$

Let

$$\rho(\epsilon) = \epsilon^2 \sup_{y\in\mathbb{R}}\Big[\eta''(-y)(1-\eta(y)) + 2\eta'(-y)\eta'(y) - \eta(-y)\eta''(y)\Big],$$

$$K_2 = (1+M)\sup\{1 + |g_u(t,u)| : \hat{u}_\delta^1(t,x;g) \le u \le u_\delta^1(t,x;g), 0 \le t \le \tau\},$$

where $\hat{u}_\delta^1(t,x;g) = u_\delta^1(t,x;g) - 1 - M$. Then

$$-K_2\rho(\epsilon)e^{2K_2 t} - \epsilon^2 M\Big[\eta''(-y)(1-\eta(y)) + 2\eta'(-y)\eta'(y) - \eta(-y)\eta''(y)\Big] \le 0 \tag{4.4}$$

for any $0 \le t \le \tau$ and $x \in \mathbb{R}$, where $y = \epsilon(x-\chi(g)-C(t-\tau))$, and

$$g(t,u_\delta^1(t,xg)) - g(t,w^2(t,x;g)) - K_2\rho(\epsilon)e^{2K_2 t} \le 0 \tag{4.5}$$

for (t,x) with $0 \le t \le \tau$ and $\eta(-\epsilon(x-\chi(g)-C(t-\tau))$ or $1-\eta(-\epsilon(x-\chi(g)-C(t-\tau)) \le \rho(\epsilon)$. Now let

$$\gamma = \min\{\eta^{'}(-y)(1-\eta(y)) + \eta(-y)\eta^{'}(y) : \rho(\epsilon) \le \eta(-y) \le 1-\rho(\epsilon)\}$$

and $C = \frac{K_2}{\gamma\epsilon(1+M)}$. Let $\epsilon = \epsilon(\delta)$ be such that $\rho(\epsilon(\delta))e^{2K_2\tau} \le \delta$. Then

$$\begin{aligned} &g(t, u_\delta^1(t,x;g)) - g(t, w^2(t,x,;g)) \\ &\quad -\rho(\epsilon)K_2e^{2K_2t} - \epsilon CM[\eta^{'}(-y)(1-\eta(y)) + \eta(-y)\eta^{'}(y)] \le 0, \end{aligned} \tag{4.6}$$

where $y = \epsilon(x-\chi(g)-C(t-\tau))$ and (t,x) is such that $0 \le t \le \tau$, $\rho(\epsilon) \le \eta(-y) \le 1-\rho(\epsilon)$. By (4.4) - (4.6),

$$w_t^2 - w_{xx}^2 - g(\sigma_t g, w^2(t,x;g)) \le 0$$

for $0 \le t \le \tau$ and $x \in \mathbb{R}$. Let $h_1(\delta) = \frac{4}{\epsilon(\delta)} + \sup_{g\in H(f)} |\chi(g)| + C\tau$. Then $u_\delta^3(0,x;g) \ge w^2(0,x;g)$. By the comparison principle for parabolic equations, we have $u_\delta^3(t,x;g) \ge w^2(t,x;g)$ for $0 \le t \le \tau$ and $x \in \mathbb{R}$. In particular,

$$\begin{aligned} u_\delta^3(\tau,\chi(g);g) &\ge w^2(\tau,\chi(g);g) = u_\delta^1(\tau,\chi(g);g) - \rho(\epsilon)e^{2K_2\tau} \\ &\ge u^0(\sigma_\tau g) + 2\delta - \delta = u^0(\sigma_\tau g) + \delta. \end{aligned}$$

Similarly, we can prove that

$$u_\delta^4(\tau,\chi(g);g) \le u^0(\sigma_\tau) - \delta.$$

□

Lemma 4.4. *Let τ be as in Lemma 4.3.*

1) *Let $\tilde v^1(t,x;g)$, $\tilde v^2(t,x;g)$ be the solutions of (4.1) with $\tilde v^1(t,0;g) = -1 + H(x)$, $\tilde v^2(0,x;g) = -1 + 2H(x)$. Then there is $\tilde\chi(g) \in \mathbb{R}$ such that*

$$\tilde v^1(\tau,\tilde\chi(g);g) \le -3, \quad \tilde v^2(\tau,\tilde\chi(g);g) \ge 3.$$

Moreover, $\tilde\chi(g)$ is bounded in $g \in H(f)$.

2) *Let $\tilde u_\delta^1(t,x;g)$, $\tilde u_\delta^2(t,x;g)$ be solutions of $(1.4)_g$ with*

$$\tilde u_\delta^1(0,x;g) = u^0(g) + \delta(-1 + H(x))$$

and

$$\tilde u^2(0,x;g) = u^0(g) + \delta(-1 + 2H(x)).$$

There is $\delta_2 > 0$ such that for $0 < \delta \le \delta_2$,

$$\tilde u_\delta^1(\tau,\tilde\chi(g),g) \le u^0(\sigma_\tau g) - 2\delta, \qquad \tilde u_\delta^2(\tau,\tilde\chi(g);g) \ge u^0(\sigma_\tau g) + 2\delta.$$

3) *Let $\tilde u_\delta^3(t,x;g)$, $\tilde u_\delta^4(t,x;g)$ be solutions of $(1.4)_g$ with*

$$\tilde u_\delta^3(0,x;g) = u^0(g) + \delta(-1 + H(x)) + (u^+(g) - u^0(g))H(x-h),$$

$$\tilde{u}_\delta^4(0,x;g) = u^0(g) + \delta(-1+2H(x)) - (u^0(g) - u^-(g) - \delta)H(-x-h).$$

Let δ_2 be as in 2). Then for any $0<\delta\le\delta_2$, there is $h_2(\delta)$ such that for any $h\ge h_2(\delta)$,

$$\tilde{u}_\delta^3(\tau,\tilde{\chi}(g);g) \le u^0(\sigma_\tau g) - \delta, \qquad \tilde{u}_\delta^4(\tau,\tilde{\chi}(g),g) \ge u^0(\sigma_\tau g) + \delta.$$

Proof. It can be proved by similar arguments as in Lemma 4.3. □

Lemma 4.5. *For any $0<\delta\le\delta^*$ ($\delta^*=\min(\delta_1,\delta_2)$), there is $\epsilon^*(\delta)$ such that*

$$0 < \xi_+(t,g,\delta) - \xi_-(t,g,\delta) \le \xi_+(0,g,\delta) - \xi_-(0,g,\delta) + \epsilon^*(\delta),$$

where $t\ge 0$ and $\xi_\pm(t,g,\delta)$ satisfies

$$u(t,\xi_\pm(t,g,\delta);u_0^g,g) = u^0(\sigma_t g) \pm \delta.$$

Proof. First, by the monotonicity of u_0^g and Lemma 2.1, there are unique $\xi_\pm(t,g,\delta)$ such that

$$u(t,\xi_\pm(t,g,\delta);u_0^g,g) = u^0(\sigma_t g) \pm \delta$$

for $t\ge 0$. Without loss of generality, suppose that

$$\xi_-(0,g,\delta) < 0 < \xi_+(0,g,\delta), \quad u_0^g(0) = u^0(g).$$

Then one of the following must hold:

$$\xi_+(0,g,\delta) \ge h^*(\delta), \tag{4.7}$$

$$-\xi_-(0,g,\delta) \ge h^*(\delta), \tag{4.8}$$

or

$$h^*(\delta) \ge \max\{\xi_+(0,g,\delta), -\xi_-(0,g,\delta)\}, \tag{4.9}$$

where $h^*(\delta)=\max\{h_1(\delta),h_2(\delta)\}$. Suppose that (4.7) holds. Then

$$u(0,x+\xi_+(0,g,\delta);u_0^g,g) \ge u_\delta^3(0,x;g),$$
$$u(0,x+\xi_-(0,g,\delta),u_0^g,g) \le u_\delta^4(0,x;g).$$

By the comparison principle for parabolic equations,

$$u(\tau,x+\xi_+(0,g,\delta);u_0^g,g) \ge u_\delta^3(\tau,x;g),$$
$$u(\tau,x+\xi_-(0,g,\delta);u_0^g,g) \le u_\delta^4(\tau,x;g).$$

This implies that

$$\xi_+(\tau,g,\delta) \le \chi(g) + \xi_+(0,g,\delta)$$

and

$$\xi_-(\tau,g,\delta) \ge \chi(g) + \xi_-(0,g,\delta).$$

Hence,

$$\xi_+(\tau,g,\delta) - \xi_-(\tau,g,\delta) \le \xi_+(0,g,\delta) - \xi_-(0,g,\delta).$$

Similarly, if (4.8) holds, then the above inequality holds.

Now if (4.9) holds, then

$$u(0, x + h^*(\delta); u_0^g, g) \geq u_\delta^3(0, x; g),$$

$$u(0, x - h^*(\delta); u_0^g, 0) \leq u_\delta^4(0, x; g).$$

It then follows from Lemma 4.3 that

$$\xi_+(\tau, g, \delta) \leq \chi(g) + h^*(\delta)$$

and

$$\xi_-(\tau, g, \delta) \geq \chi(g) - h^*(\delta).$$

Hence

$$\xi_+(\tau, g, \delta) - \xi_-(\tau, g, \delta) \leq 2h^*(\delta).$$

Next, it is not difficult to see that for any $0 < \delta \leq \delta^*$, there is $\epsilon_1(\delta)$ such that

$$\xi_+(t, g, \delta) - \xi_-(t, g, \delta) \leq \xi_+(0, g, \delta) - \xi_-(0, g, \delta) + \epsilon_1(\delta)$$

for $t \in [0, \tau]$ and $g \in H(f)$. Let $\epsilon^*(\delta) = \max\{\epsilon_1(\delta), 2h^*(\delta)\}$. We have that for any $t \geq 0$,

$$\xi_+(t, g, \delta) - \xi_-(t, g, \delta) \leq \xi_+(0, g, \delta) - \xi_-(0, g, \delta) + \epsilon^*(\delta).$$

□

The next lemma can be proved by direct computation.

Lemma 4.6. *For any given $M > 0$, there is $C \geq 0$ such that for any α_-, α_+ with $-M \leq \alpha_- < \alpha_+ \leq M$, any $c \geq C$ and $g \in H(f)$,*

1) v^+ and v^- are super- and sub-solutions of $(1.4)_g$, respectively, where

$$v^+(t, x) = u(t, x; \alpha_+, g)\zeta(x + ct) + u(t, x; \alpha_-, g)(1 - \zeta(x + ct)),$$

$$v^-(t, x) = u(t, x; \alpha_-, g)\zeta(x + ct) + u(t, x; \alpha_+, g)(1 - \zeta(x + ct)).$$

2) w^+ and w^- are also super- and sub-solutions of $(1.4)_g$, respectively, where

$$w^+(t, x) = u(t, x; \alpha_-, g)\zeta(x - ct) + u(t, x; \alpha_+, g)(1 - \zeta(x - ct)),$$

$$w^-(t, x) = u(t, x; \alpha_+, g)\zeta(x - ct) + u(t, x; \alpha_-, g)(1 - \zeta(x - ct)).$$

Proof of Theorem 3.1. First, let δ^*, $\epsilon^*(\delta)$, and $\xi_\pm(t, \delta, g)$ be as in Lemma 4.5. Then we have for any $0 < \delta \leq \delta^*$,

$$\xi_+(t, \delta, g) - \xi_-(t, \delta, g) \leq \xi_+(0, \delta, g) - \xi_-(0, \delta, g) + \epsilon^*(\delta)$$

for all $t \geq 0$.

Next, by Lemma 2.1, there are unique $\tilde{\xi}_\pm(t,\delta,g)$ $(t \ge 0)$ such that

$$u(t,\tilde{\xi}_\pm(t,\delta,g);u_0^g,g) = u^\pm(\sigma_t g) \mp \delta.$$

We claim that there is $c > 0$, $\tilde{\epsilon}_1(\delta) > 0$, $\tilde{T}(\delta) > 0$ $(0 < \delta \le \delta^*)$ such that for any $g \in H(f)$,

$$\tilde{\xi}_+(t,g,\delta) - \tilde{\xi}_-(t,\delta,g) \le \xi_+(0,\delta,g) - \xi_-(0,\delta,g) + \tilde{\epsilon}_1(\delta) + 2ct$$

for all $t \ge \tilde{T}(\delta)$. To this end, for given $g \in H(f)$, $0 < \delta \le \delta^*$, define

$$w^+(t,x;g) = u(t,x;u^0(g)-\delta,g)(1-\zeta(x+ct)) + u(t,x;u^+(g)+\delta,g)\zeta(x+ct)$$

and

$$w^-(t,x;g) = u(t,x;u^-(g)-\delta,g)(1-\zeta(x-ct)) + u(t,x;u^0(g)+\delta,g)\zeta(x-ct).$$

Then by Lemma 4.6, when $c \gg 1$, w^+ and w^- are super- and sub-solutions, respectively. Clearly, there is $\chi^*(\delta)$ such that

$$w^+(0,\chi^*(\delta);g) \ge u^+(g), \quad w^-(0,-\chi^*(\delta),g) \le u^-(g).$$

Hence,

$$w^-(0,x-\xi_+(0,\delta,g)-\chi^*(\delta);g) \le u_0^g(x) \le w^+(x-\xi_-(0,\delta,g)+\chi^*(\delta);g).$$

This implies that

$$\begin{aligned} w^-(t,x-\xi_+(0,\delta,g)-\chi^*(\delta);g) &\le u(t,x;u_0^g,g) \\ &\le w^+(t,x-\xi_-(0,\delta,g)+\chi^*(\delta),;g). \end{aligned}$$

By the stability of $u^\pm(\sigma_t g)$ (see **(H2)** ii)), there is $\tilde{T}(\delta) > 0$, $\tilde{\chi}^*(\delta) > 0$ such that

$$w^+(t,-\tilde{\chi}^*(\delta)-ct;g) \le u^-(\sigma_t g) + \delta$$

and

$$w^-(t,\tilde{\chi}^*(\delta)+ct;g) \ge u^+(\sigma_t g) - \delta$$

for $t \ge \tilde{T}(\delta)$. Hence,

$$\tilde{\xi}_+(t,\delta,g) \le \xi_+(0,\delta,g) + \chi^*(\delta) + \tilde{\chi}^*(\delta) + ct$$

and

$$\tilde{\xi}_-(t,\delta,g) \ge \xi_-(0,\delta,g) - \chi^*(\delta) - \tilde{\chi}^*(\delta) - ct$$

for $t \ge \tilde{T}(\delta)$. This implies the claim holds with $\tilde{\epsilon}_1(\delta) = 2\chi^*(\delta) + 2\tilde{\chi}^*(\delta)$. It then follows from Lemma 3.5 that

$$\tilde{\xi}_+(t,\delta,g) - \tilde{\xi}_-(t,\delta,g) \le \xi_+(0,\delta,g) - \xi_-(0,\delta,g) + \tilde{\epsilon}_1(\delta) + \epsilon^*(\delta) + 2c\tilde{T}(\delta)$$

for all $t \geq \tilde{T}(\delta)$. Clearly, there is $\tilde{\epsilon}_2(\delta) > 0$ such that

$$\tilde{\xi}_+(t,\delta,g) - \tilde{\xi}_-(t,\delta,g) \leq \tilde{\epsilon}_2(\delta)$$

for $0 \leq t \leq \tilde{T}(\delta)$. This implies that

$$\tilde{\xi}_+(t,\delta,g) - \tilde{\xi}_-(t,\delta,g) \leq \xi_+(0,\delta,g) - \xi_-(0,\delta,g) + \tilde{\epsilon}^*(\delta)$$

for all $t \geq 0$, $0 < \delta \leq \delta^*$, where

$$\tilde{\epsilon}^*(\delta) = \max\{\tilde{\epsilon}_2(\delta), \tilde{\epsilon}_1(\delta) + \epsilon^*(\delta) + 2c\tilde{T}(\delta)\}.$$

The theorem then follows. □

5. Uniqueness and Stability of Traveling Wave Solutions in Bistable equations

In this section, we consider the uniqueness and stability of traveling waves of (1.4) when f is bistable and recurrent and prove Theorem B 2)-4). By Corollary 4.2, if f is bistable, then (1.4) has a critical monotone traveling wave solution.

Theorem 5.1. *Suppose that f is bistable and recurrent, and $U : H(f) \to X$ generates a critical monotone traveling wave solution of* (1.4)*. Then*

1) (*Stability*) *For any $g \in H(f)$ and $u_0 \in X$ with*

$$u^-(g) \leq u_0(x) \leq u^+(g) \quad \text{for} \quad x \in \mathbb{R}$$

and $\lim_{x\to\pm\infty} u_0(x) = u^\pm(g)$, there is $z^*(g,u_0)$ such that

$$u(t,x;u_0,g) - u(t,x+z^*(g,u_0);U(\cdot,g),g) \to 0 \quad \textit{as } t \to \infty.$$

2) (*Continuity*) *$U : H(f) \to X$ is continuous at any $g \in H(f)$.*

3) (*Uniqueness*) *If $V : H(f) \to X$ is continuous at any $g \in H(f)$ and generates a traveling wave solution, then there is $\xi : H(f) \to \mathbb{R}$ such that*

$$V(x,g) = U(x+\xi(g),g)$$

and $\xi(g)$ is continuous at any $g \in H(f)$.

Proof. Without loss of generality, we assume that $U(0,g) = u^0(g)$ (i.e. $\tilde{u}^0(g) = u^0(g)$). Then $u(t,c(t,g);U(\cdot,g),g) = u^0(\sigma_t g)$.

1) We divide the proof into three steps.

Step 1. In this step, we shall show that there are δ_0, K_0 and $\rho_0 \in \mathbb{R}$ such that for any $g \in H(f)$ and $u_0 \in X$, if for some $\delta \in (0,\delta_0]$ and $\hat{z} \in \mathbb{R}$,

$$u_0(\cdot) \leq U(\cdot + \hat{z}, g) + \delta \tag{5.1}$$

or

$$u_0(\cdot) \geq U(\cdot + \hat{z}, g) - \delta, \tag{5.2}$$

then for all $t \geq 0$,

$$u(t, x; u_0, g) \leq u(t, x + \hat{z} + K_0\delta; U(\cdot, g), g) + K_0\delta e^{-\rho_0 t} \tag{5.3}$$

or

$$u(t, x; u_0, g) \geq u(t, x + \hat{z} - K_0\delta; U(\cdot, g), g) - K_0\delta e^{-\rho_0 t}. \tag{5.4}$$

We prove that (5.1) implies (5.3). The other part can be proved by similar arguments.

Let

$$\nu^{\pm} = -\inf_{g \in H(f)} \limsup_{t \to \infty} \frac{1}{t} \int_0^t g_u(s, u^{\pm}(\sigma_s g))ds$$

and

$$a^{\pm}(t) = \exp\Big(\frac{\nu^{\pm}}{2} t + \int_0^t g_u(s, u^{\pm}(\sigma_s g))ds\Big).$$

Then $\nu^{\pm} > 0$ and $a^{\pm}(t) \to 0$ as $t \to \infty$ uniformly in $g \in H(f)$.

Let $\zeta(\cdot) \in C^2(\mathbb{R})$ be a function satisfying that $\zeta(x) = 1$ for $x \in [3, \infty)$, $\zeta(x) = 0$ for $x \in (-\infty, 0]$, and $0 \leq \zeta^{'}(x) \leq 1$, $|\zeta^{''}(x)| \leq 1$ for $x \in \mathbb{R}$.

Define

$$A(t, x) = \zeta(x - c(t, g))a^{+}(t) + (1 - \zeta(x - c(t, g)))a^{-}(t),$$

$$B(t) = \int_0^t \max(a^{+}(s), a^{-}(s))ds,$$

and

$$v(t, x) = u(t, x + \hat{z} + K\delta B(t); U(\cdot, g), g) + \delta A(t, x).$$

Denote $u(t, x+\hat{z}+K\delta B(t))$ as $u(t, x+\hat{z}+K\delta B(t); U(\cdot, g), g)$. Then $v(0, x) \geq u_0(x)$ and

$$\begin{aligned}
&v_t - v_{xx} - g(t, v)\\
&= u_t(t, x + \hat{z} + K\delta B(t)) + K\delta u_x(t, x + \hat{z} + K\delta B(t))B^{'}(t) + \delta A_t(t, x)\\
&\quad - u_{xx}(t, x + \hat{z} + K\delta B(t)) - \delta A_{xx}(t, x) - g(t, v(t, x))\\
&= K\delta u_x(t, x + \hat{z} + K\delta B(t))B^{'} + \delta A_t(t, x) - \delta A_{xx}(t, x)\\
&\quad + g(t, u(t, x + \hat{z} + K\delta B(t))) - g(t, u(t, x + \hat{z} + K\delta B(t)) + \delta A(t, x))\\
&= K\delta u_x(t, x + \hat{z} + K\delta B(t)) \max(a^{+}(t), a^{-}(t))\\
&\quad + \delta\zeta(x - c(t, g))\Big(\frac{\nu^{+}}{2} + g_u(t, u^{+}(\sigma_t g))\Big)a^{+}(t)
\end{aligned}$$

$$+ \delta(1-\zeta(x-c(t,g))\Big(\frac{\nu^-}{2}+g_u(t,u^-(\sigma_t g))\Big)a^-(t)$$
$$- \delta\zeta^{'}(x-c(t,g))c^{'}(t,g)(a^+(t)-a^-(t))$$
$$- \delta\zeta^{''}(x-c(t,g))(a^+(t)-a^-(t))$$
$$+ g(t,u(t,x+\hat{z}+K\delta B(t))) - g(t,u(t,x+\hat{z}+K\delta B(t))$$
$$+ \delta\zeta(x-c(t,g))a^+(t) + \delta(1-\zeta(x-c(t,g))a^-(t)).$$

Let $\xi_0 > 0$ be large enough and $\tilde{\delta}_0 > 0$ be small enough that

$$|g_u(t,u^+(\sigma_t g)) - g_u(t,u)| < \frac{\nu^+}{2} \quad \text{if} \quad |u-u^+(\sigma_t g)| < \tilde{\delta}_0,$$
$$|g_u(t,u^-(\sigma_t g)) - g_u(t,u)| < \frac{\nu^-}{2} \quad \text{if} \quad |u-u^-(\sigma_t g)| < \tilde{\delta}_0,$$

and

$$|u(t,x;U(\cdot,g),g) - u^-(\sigma_t g)| < \tilde{\delta}_0 \quad \text{if} \quad x-c(t,g) < -\xi_0,$$
$$|u(t,x;U(\cdot,g),g) - u^+(\sigma_t g)| < \tilde{\delta}_0 \quad \text{if} \quad x-c(t,g) > \xi_0.$$

Note that there is $\alpha_0 > 0$ such that $u_x(t,x;U(\cdot,g),g) \geq \alpha_0$ for $|x-c(t,g)| \leq \xi_0$ and

$$\zeta^{'}(x-c(t,g)) = 0, \quad \zeta^{''}(x-c(t,g)) = 0$$

for $|x-c(t,g)| > \xi_0$. Hence by letting $K > 0$ be large enough and $\delta > 0$ be small enough, we have

$$v_t - v_{xx} - g(t,v) \geq 0.$$

Then by the comparison principle for parabolic equations,

$$u(t,x;u_0,g) \leq u(t,x+\hat{z}+K\delta B(t);U(\cdot,g),g) + \delta A(t,x).$$

(5.3) then follows.

Step 2. In this step, we shall show that for any $g \in H(f)$, $u_0 \in X$ with

$$U(x-\hat{z},g) - \delta \leq u_0(x) \leq U(x+\hat{z},g) + \delta \tag{5.5}$$

for some $0 < \delta \leq \delta_0$, there is $z^*(g,u_0)$ such that

$$u(t,x;u_0,g) - u(t,x+z^*(g,u_0);U(\cdot,g),g) \to 0 \ \text{ as } t \to \infty.$$

First of all, by (5.5) and the arguments in the first step,

$$\begin{aligned} &u(t,x-\hat{z}-K_0\delta;U(\cdot,g),g) - K_0\delta e^{-\rho_0 t} \leq u(t,x;u_0,g) \\ &\quad \leq u(t,x+\hat{z}+K_0\delta;U(\cdot,g),g) + K_0\delta e^{-\rho_0 t}. \end{aligned} \tag{5.6}$$

Let $\xi_0(t)$ be such that

$$u(t,\xi_0(t);u_0,g) = u^0(\sigma_t g) \tag{5.7}$$

and

$$u(t, x; u_0, g) > u^0(\sigma_t g) \tag{5.8}$$

for $x > \xi_0(t)$. Then

$$u^0(\sigma_t g) - K_0\delta e^{-\rho_0 t} \leq u(t, \xi_0(t) + \hat{z} + K_0\delta; U(\cdot, g), g)$$

and

$$u(t, x + \xi_0(t) - \hat{z} - K_0\delta; U(\cdot, g), g) \leq u^0(\sigma_t g) + K_0\delta e^{-\rho_0 t}.$$

Hence there is $M_0 > 0$ such that $|\xi_0(t) - c(t, g)| \leq M_0$ for $t \gg 1$. Denote $\tilde{U}(t, x) = u(t, x + \xi_0(t); u_0, g)$. Let $\{t_n\}$ be such that $t_n \to \infty$, $t_{n+1} - t_n \to \infty$, and $\sigma_{t_n} g \to g^* \in H_0(f)$ (such $\{t_n\}$ exists due to the recurrence of f). Without loss of generality, by (5.6), we may assume that $\tilde{U}(t_n, x) \to U^*(x)$ uniformly for $x \in \mathbb{R}$ and

$$\xi_0(t_n) - c(t_n, g) \to \xi^*. \tag{5.9}$$

Note that

$$u(t_n, x + c(t_n, g); U(\cdot, g), g) \to U(x, g^*).$$

Hence,

$$U(x + \xi^* - \hat{z} - K_0\delta, g^*) \leq U^*(x) \leq U(x + \xi^* + \hat{z} + K_0\delta, g^*).$$

We show now that there is $z^*(g, u_0)$ such that

$$U^*(x) = U(x + \xi^* + z^*(g, u_0), g^*).$$

In order to do so, let

$$z^*(g, u_0) = \inf\{z : U^*(x) \leq U(x + \xi^* + z, g^*)\}.$$

Then

$$U^*(x) \leq U(x + \xi^* + z^*(g, u_0), g^*). \tag{5.10}$$

We claim that $U^*(x) = U(x + \xi^* + z^*(g, u_0), g^*)$. For simplicity, denote z^* as $z^*(g, u_0)$. Note that

$$u(-s, \cdot; U^*(\cdot), g^*) = \lim_{n\to\infty} u(t_n - s, x + \xi_0(t_n); u_0, g)$$

exists for any $s \in \mathbb{R}$, where the limit is taken in the uniform convergence topology. Hence

$$U^*(x) = u(s, x; u(-s, \cdot; U^*(\cdot), g), \sigma_{-s} g^*) \tag{5.11}$$

for any $s \geq 0$. Similarly,

$$U(x + \xi^* + z^*, g^*) = u(s, x + \xi^* + z^*; u(-s, \cdot; U(\cdot, g^*), g^*), \sigma_{-s} g^*) \tag{5.12}$$

for any $s \geq 0$. Note also that

$$u(-s, x + \xi^* + z^*; U(\cdot, g^*), g^*) = U(x + \xi^* + z^* - c(-s, g^*), \sigma_{-s} g^*)$$

$$= \lim_{t\to\infty} u(t,x;u_0^*(\sigma_{-(t+s)}g^*)(\cdot + y(t,\sigma_{-s}g^*) + c(-s,g^*) + \xi^* + z^*), \sigma_{-(t+s)}g^*)$$

and

$$\begin{aligned} u(-s,x;U^*(\cdot),g^*) &= u(t,x;u(-t,\cdot;u(-s,\cdot;U^*(\cdot),g^*),\sigma_{-s}g^*),\sigma_{-(t+s)}g^*) \\ &= u(t,x;u(-t-s,\cdot;U^*(\cdot),g^*),\sigma_{-(t+s)}g^*). \end{aligned}$$

Then by Lemma 2.6, there is ξ_1 such that

$$u(-s,x+\xi^*+z^*;U(\cdot,g^*),g^*) \begin{cases} \geq u(-s,x;U^*(\cdot),g^*) & \text{for} \quad x \geq \xi_1 \\ \leq U(-s,x;U^*(\cdot),g^*) & \text{for} \quad x \leq \xi_1. \end{cases}$$

Suppose that $U^*(\cdot) \not\equiv U(\cdot + \xi^* + z^*, g^*)$. It then follows from Lemma 2.5 and the comparison principle for parabolic equations that there is ξ_2 such that

$$U(x+\xi^*+z^*,g^*) \begin{cases} > U^*(x) & \text{for} \quad x > \xi_2 \\ < U^*(x) & \text{for} \quad x < \xi_2. \end{cases}$$

By (5.10), we must have $\xi_2 = -\infty$ and then

$$U^*(x) < U(x+\xi^*+z^*,g^*) \quad \text{for} \quad x \in \mathbb{R}.$$

Now let $M > 0$ be such that

$$U_x(x+\xi^*+z^*,g^*) < \frac{1}{2K_0}$$

for $|x| \geq M$. Note that there is $\epsilon > 0$ such that

$$U^*(x) \leq U(x+\xi^*+z^*-\epsilon,g^*) \quad \text{for} \quad |x| \leq M+1$$

and

$$U(x+\xi^*+z^*,g^*) - U(x+\xi^*+z^*-\epsilon,g^*) < \frac{\epsilon}{2K_0}$$

for $|x| \geq M+1$. Hence

$$U^*(x) \leq U(x+\xi^*+z^*-\epsilon,g^*) + \frac{\epsilon}{2K_0}.$$

Therefore, for $n \gg 1$,

$$u(t_n,x+\xi_0(t_n);u_0,g) \leq U(x+\xi^*+z^*-\epsilon,\sigma_{t_n}g) + \frac{5\epsilon}{6K_0}.$$

By the arguments in step 1,

$$\begin{aligned} &u(t+t_n,x+\xi(t_n);u_0,g) = u(t,x;u(t_n,\cdot+\xi_0(t_n);u_0,g),\sigma_{t_n}g) \\ &\leq u(t,x+\xi^*+z^*-\frac{\epsilon}{6};U(\cdot,\sigma_{t_n}g),\sigma_{t_n}g) + \frac{5\epsilon}{6}e^{-\rho_0 t} \end{aligned}$$

$$= u(t+t_n, x+\xi^*+z^* - \frac{\epsilon}{6} + c(t_n,g); U(\cdot,g),g) + \frac{5\epsilon}{6}e^{-\rho_0 t}$$
$$= U(x+\xi^*+z^* - \frac{\epsilon}{6} + c(t_n,g) - c(t+t_n,g)), \sigma_{t+t_n}g) + \frac{5\epsilon}{6}e^{-\rho_0 t}.$$

Therefore,

$$u(t_{n+1}, x+\xi_0(t_{n+1}); u_0, g) \le U(x+\xi^*+z^* - \frac{\epsilon}{6} + c(t_n,g) - c(t_{n+1},g)$$
$$+ \xi_0(t_{n+1}) - \xi_0(t_n), \sigma_{t_{n+1}}g) + \frac{5\epsilon}{6}e^{-\rho_0(t_{n+1}-t_n)}.$$

By (5.9), we have

$$\lim_{n\to\infty}\big(c(t_n,g) - c(t_{n+1},g) + \xi_0(t_{n+1}) - \xi_0(t_n)\big) = 0.$$

Hence $U^*(x) \le U(x+\xi^*+z^*-\frac{\epsilon}{6}, g^*)$, which contradicts the definition of $z^* = z^*(g,u_0)$.

Finally, we prove that

$$u(t,x;u_0,g) - u(t,x+\xi^*+z^*;U(\cdot,g),g) \to 0$$

as $t\to\infty$. To do so, we note that for any $\epsilon>0$, there is N such that for $n \ge N$,

$$U(x+\xi^*+z^*,\sigma_{t_n}g) - \epsilon \le u(t_n, x+\xi_0(t_n);u_0,g) \le U(x+\xi^*+z^*,\sigma_{t_n}g)+\epsilon.$$

This together with step 1 implies that

$$u(t,x+\xi^*+z^*-K_0\epsilon; U(\cdot,\sigma_{t_n}g),\sigma_{t_n}g) - \epsilon k_0 e^{-\rho_0 t}$$
$$\le u(t,x;u(t,\cdot;u(t_n,\cdot+\xi_0(t_n);u_0,g),\sigma_{t_n}g))$$
$$\le u(t,x+\xi^*+z^*+K_0\epsilon; U(\cdot,\sigma_{t_n}g),\sigma_{t_n}g) + \epsilon K_0 e^{-\rho_0 t}$$

for $t\ge 0$. Therefore,

$$u(t+t_n, x+\xi^*+z^*-K_0\epsilon + c(t_n,g_0); U(\cdot,g),g) - \epsilon K_0 e^{-\rho_0 t}$$
$$\le u(t+t_n, x+\xi_0(t_n); u_0, g)$$
$$\le u(t+t_n, x+\xi^*+z^*+K_0\epsilon + c(t_n,g); U(\cdot,g),g) + \epsilon K_0 e^{-\rho_0 t}.$$

Hence,

$$u(t+t_n, x+\xi^*+z^*-K_0\epsilon + c(t_n,g) - \xi_0(t_n); U(\cdot,g),g) - \epsilon K_0 e^{-\rho_0 t}$$
$$\le u(t+t_n, x; u_0, g)$$
$$\le u(t+t_n, x+\xi^*+z^*+K_0\epsilon + c(t_n,g) - \xi_0(t_n); U(\cdot,g),g) + \epsilon K_0 e^{-\rho_0 t}.$$

For $n \gg 1$, $|\xi^* + c(t_n,g) - \xi_0(t_n)| \ll 1$. Hence there is $M_0>0$ such that

$$U(t+t_n, x+z^*; U(\cdot,g),g) - M_0\epsilon - \epsilon K_0 e^{-\rho_0 t} \le u(t+t_n,x;u_0,g)$$

$$\leq u(t+t_n, x+z^*; U(\cdot, g), g) + M_0\epsilon + \epsilon K_0 e^{-\rho_0 t}.$$

It then follows that

$$\lim_{t\to\infty}\big(u(t,x;u_0,g) - u(t,x+z^*;U(\cdot,g),g)\big) = 0.$$

Step 3. Note that for any $g \in H(f)$ and $u_0 \in X$ with $u^-(g) \leq u_0(x) \leq u^+(g)$ and $u_0(\pm\infty) = u^{\pm}(g)$, for any $\delta > 0$, there is $\hat{z}$ such that

$$U(x-\hat{z},g) - \delta \leq u_0(x) \leq U(x+\hat{z},g) + \delta.$$

The statement 1) then follows from steps 1 and 2.

2) First we show that $\bar{\Pi}_t$,

$$\bar{\Pi}_t(V,g) = \big(u(t,x+\xi(t,V,g);V(\cdot),g\big),\sigma_t g),$$

defines a flow on Y, where $Y = cl\{U(\cdot,g),g) : g \in H(f)\}$, and $\xi(t,V,g)$ is such that

$$u(t,\xi(t,V,g);V(\cdot),g) = u^0(\sigma_t g).$$

First of all, suppose that $(V,g) \in Y$ and

$$(V(x),g) = \lim_{n\to\infty}(U(x,g_n),g_n).$$

Then

$$u(t,x+\xi(t,V,g);V(\cdot),g) = \lim_{n\to\infty} u(t,x+\xi(t,V,g);U(\cdot,g_n),g_n).$$

Note that

$$u(t,x+c(t,g_n);U(\cdot,g_n),g_n) = U(x,\sigma_t g_n).$$

Hence

$$\lim_{n\to\infty} u(t,\xi(t,V,g);U(\cdot,g_n),g_n) = u^0(\sigma_t g)$$

and

$$\lim_{n\to\infty} u(t,c(t,g_n);U(\cdot,g_n),g_n) = u^0(\sigma_t g).$$

Therefore, we must have

$$\lim_{n\to\infty} \xi(t,V,g) - c(t,g_n) = 0$$

and then

$$\begin{aligned} u(t,x+\xi(t,V,g);V(\cdot),g) &= \lim_{n\to\infty} u(t,x+\xi(t,V,g);U(\cdot,g_n),g_n) \\ &= \lim_{n\to\infty} u(t,x+c(t,g_n)+\xi(t,V,g)-c(t,g_n);U(\cdot,g_n),g_n) \\ &= \lim_{n\to\infty} u(t,x+c(t,g_n);U(\cdot,g_n),g_n) \ = \lim_{n\to\infty} U(x,\sigma_t g_n). \end{aligned}$$

Hence, $\bar{\Pi}_t(V,g) \in Y$.

Next, it is clear that $\bar{\Pi}_t(V,g)$ is continuous in t, V, and g.

We show now that

$$\bar{\Pi}_{t+s}(V,g)=\bar{\Pi}_t(\bar{\Pi}_s(V,g)).$$

Observe that

$$\bar{\Pi}_{t+s}(V,g)=\big(u(t,x+\xi(t+s,V,g);V(\cdot),g),\sigma_{t+s}g\big)$$

and

$$\begin{aligned}&\bar{\Pi}_t(\bar{\Pi}_s(V,g))\\&=u(t,x+\xi(t,u(s,\cdot+\xi(s,V,g);V,g),\sigma_s g);u(s,\cdot+\xi(s,V,g);V,g),\sigma_s g),\sigma_{t+s}g)\\&=(u(t+s,x+\xi(t,u(s,\cdot+\xi(s,V,g);V,g),\sigma_s g)+\xi(s,V,g);V,g).\end{aligned}$$

Note that

$$\begin{aligned}u^0(\sigma_{t+s}g)&=u(t+s,\xi(t,u(s,\cdot+\xi(s,V,g);V,g),\sigma_s g)+\xi(s,V,g);V,g)\\&=u(t+s,\xi(t+s,V,g);V,g).\end{aligned}$$

We then must have

$$\xi(t,u(s,\cdot+\xi(s,V,g);V,g),\sigma_s g)+\xi(s,V,g)=\xi(t+s,V,g)$$

and hence $\bar{\Pi}_{t+s}(V,g)=\bar{\Pi}_t(\bar{\Pi}_s(V,g))$. This proves that $\bar{\Pi}_t:Y\to Y$ is a flow.

We show then that for any $g\in H(f)$, $P^{-1}(g)=\{U(\cdot,g),g)\}$, where $P:Y_xH(f)\to H(f)$, $P(V,g)=g$. First, by Theorem 3.1, there is $g^*\in H(f)$ such that $P^{-1}(g^*)=\{U(\cdot,g^*),g^*)\}$.

Next, for any $g\in H(f)$ and $(V,g)\in Y$, by the recurrence of f, there is $t_n\to\infty$ such that

$$\lim_{n\to\infty}\sigma_{-t_n}g=g^*.$$

Then

$$\lim_{n\to\infty}\bar{\Pi}_{-t_n}(V,g)=(U(\cdot,g^*),g^*)$$

and

$$\lim_{n\to\infty}(U(\cdot,\sigma_{-t_n}g),\sigma_{-t_n}g)=(U(\cdot,g^*),g^*).$$

It then follows that for any $\epsilon>0$,

$$|u(-t_n,x+\xi(-t_n,V,g);V,g)-U(x,\sigma_{-t_n}g)|<\epsilon$$

for $n\gg 1$. This together with the arguments in 1) implies that

$$\begin{aligned}&u(t,x-\epsilon K_0;U(\cdot,\sigma_{-t_n}g),\sigma_{-t_n}g)-\epsilon K_0e^{-\rho_0 t}\\&\le u(t,x;u(-t_n,\cdot+\xi(-t_n,V,g);V,g),\sigma_{-t_n}g)\\&\le u(t,x+\epsilon K_0;U(\cdot,\sigma_{-t_n},\sigma_{-t_n}g)+\epsilon K_0e^{-\rho_0 t}.\end{aligned}$$

Hence,

$$\begin{aligned}&U(x-\epsilon K_0-c(t_n,\sigma_{-t_n}g),g)-\epsilon K_0e^{-\rho_0 t}\\&\le V(x-\xi(t_n,u(-t_n,\cdot+\xi(-t_n,V,g);V,g))\\&\le U(x+\epsilon K_0-c(t_n,\sigma_{-t_n}g),g)+\epsilon K_0e^{-\rho_0 t}.\end{aligned}$$

This implies that

$$|\xi(t_n,u(-t_n,\cdot+\xi(-t_n,V,g);V,g)-c(t_n,\sigma_{-t_n}g)|\ll 1$$

for $n\gg 1$. Hence, we must have $V(\cdot)=U(\cdot,g)$ and then $P^{-1}(g)=\{(U(\cdot,g),g)\}$ is a singleton. Therefore, $U(\cdot,g)\in X$ is continuous at any $g\in H(f)$.

3) Suppose that $V(\cdot,g)$ also generates a continuous traveling wave solution: $V(\cdot,g)$ is continuous at any $g\in H(f)$ and

$$u(t,x;V,g)=V(x-\tilde c(t,g),\sigma_t g).$$

Let $\tilde\xi^{\pm}(g)$ be such that $V(\tilde\xi^{\pm}(g),g)=u^0(g)$ and

$$V(x,g)>u^0(g)\quad\text{for}\quad x>\tilde\xi^{+}(g),\quad V(x,g)<u^0(g)\quad\text{for}\quad x<\tilde\xi^{-}(g).$$

By the continuity of $V(\cdot,g)$ in g, $\tilde\xi^{\pm}(g)$ are bounded for $g\in H(f)$.

Now for any $g\in H(f)$ and any $\delta\in(0,\delta_0]$, there is $\hat z$ such that

$$U(x-\hat z,g)-\delta\le V(x,g)\le U(x+\hat z,g)+\delta.$$

By 1), there is z^* such that

$$\lim_{t\to\infty}u(t,x;V(\cdot,g),g)-u(t,x+z^*;U(\cdot,g),g)=0.$$

Let $t_n\to\infty$ be such that $\sigma_{t_n}g\to g$. Then

$$V(x-\tilde c(t_n,g),\sigma_{t_n}g)-U(x+z^*-c(t_n,g),\sigma_{t_n}g)\to 0$$

or

$$V(x+c(t_n,g)-z^*-c(t_n,g),\sigma_{t_n}g)-U(x,\sigma_{t_n}g)\to 0$$

as $n\to\infty$. By $U(0,\sigma_{t_n}g)=u^0(\sigma_{t_n}g)$, we have

$$\tilde\xi^{-}(\sigma_{t_n}g)\le c(t_n,g)-z^*-\tilde c t_n,g)\le\tilde\xi^{+}(\sigma_{t_n}g).$$

Therefore we must have $c(t_n,g)-z^*-\tilde c(t_n,g)$ is bounded and converges as $n\to\infty$. Suppose that

$$c(t_n,g)-z^*-\tilde c(t_n,g)\to\xi(g).$$

Then we have $V(x+\xi(g),g)=U(x,g)$. Note that $V(\xi(g),g)=u^0(g)$. We must also have that $\xi(g)$ is continuous in g. □

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