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LONG-PERIOD LIMIT OF NONLINEAR DISPERSIVE WAVES: THE BBM-EQUATION

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Abstract. The focus of the present study is the standard BBM equation which models unidirectional propagation of small amplitude long waves in dispersive media. The equation is posed on the entire real line and the interest here is the relationship between two different types of solutions. The problem has been studied with initial data in various Sobolev spaces defined on $\mathbb{R}$ and for periodic initial data, say of period $2l$. The principal new result is an exact theory of convergence of the periodic solutions to the solutions in Sobolev spaces as $l \to \infty$.

1. Introduction

Considered here are small amplitude long waves on the surface of an ideal fluid of finite depth over a featureless, horizontal bottom under the force of gravity. When such wave motion is long crested, it may propagate essentially in, say, the x-direction and without significant variation in the y-direction of a standard xyz-Cartesian frame in which gravity acts in the negative z-direction. In this situation, the Korteweg de Vries equation

$$\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0 \tag{1.1}$$

was derived to approximate the full three-dimensional Euler equations. References can be found in Boussinesq (1877), Korteweg and de Vries (1895) and more modern works, eg. Bona, Chen and Saut (2002), Miura (1976), Whitham (1974).

In a little more detail, the variable x denotes the coordinate in the direction of propagation and h_0 the undisturbed depth, $h(x,t)$ is the depth of

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the liquid surface over the flat bottom at the spatial point x at time t. The crucial dependent variable $\eta(x,t) = h(x,t) - h_0$ represents the deviation of the water surface relative to the undisturbed surface. The KdV-equation was derived under the assumption that the waves propagate in the positive x-direction, that the amplitude a of the waves is small compared to the undisturbed depth h_0 and that typical wavelengths λ of the motion are long compared to h_0, so

$$\alpha = \frac{a}{h_0} << 1 \quad \text{and} \quad \beta = \frac{h_0}{\lambda} << 1.$$

Moreover, the Stokes number

$$S = \frac{\alpha}{\beta^2} = \frac{a\lambda^2}{h_0^3} \tag{1.2}$$

is assumed to be order one, which means nonlinear and dispersive effects are balanced. The regularized long wave equation, or the BBM-equation

$$\eta_t + \eta_x + \eta\eta_x - \eta_{xxt} = 0, \tag{1.3}$$

an alternative to the KdV-equation, was written by Peregrine (1967) in his study of bore propagation and first analyzed by Benjamin *et al.* (1972). Both these equations are written in nondimensional, laboratory coordinates. In (1.1) and (1.3), the Stokes number S is assumed to be exactly 1, the horizontal variable is scaled by λ, the vertical coordinate by h_0, the deviation η of the free surface by a and time by $(h_0/g)^{\frac{1}{2}}$.

It is standard to consider the pure initial-value problem for (1.1) and (1.3), that is, η is specified for all the relevant values of x at some moment t, normally taken to be $t = 0$, the so-called initial condition,

$$\eta(x,0) = f(x) \qquad \text{for} \qquad x \in \mathbb{R}. \tag{1.4}$$

Thus, values of $t > 0$ represent time elapsed since the inception of the motion as just described by (1.4). When the small amplitude and long wavelength presumptions need to be displayed explicitly, f can be taken in the form $f(x) = \alpha F(\beta x)$ where F is independent of the small parameters α and β introduced earlier. In this paper, we do not ask how the wave motion was truly initiated, but instead propose a scheme for approximating a localized disturbance by a spatially periodic evolution. More precisely, the goal in the present paper is to understand the BBM-equation (1.3) with two types of initial data, one vanishing at ∞ and the other periodic of large period.

The plan of the paper is as follows. In Section 2, we briefly review existing theory and then extend this theory in a way that is useful for the present purpose. The main comparison result is derived in Section 3.

To give the study focus, the main result is here stated informally. Detailed assumptions will be spelled out in Section 3.

Theorem 1.1. *Let u be the solution of the BBM-equation* (1.3) *corresponding to the initial condition $u(x,0) = \psi(x)$ which is sufficiently nice and decays to zero as $|x| \to \infty$. Let $\mathcal{P}_l(\psi)$ be an appropriately defined periodic version of ψ (see* (3.3)*) and let u_l be the solution of* (1.3) *with initial data $\mathcal{P}_l(\psi)$. Then, when both solutions are restricted on the spatial interval $(-l,l)$, their difference satisfies*

$$\lim_{l\to\infty} \|u_l(\cdot,t) - u(\cdot,t)\|_{W^{1,2}(-l,l)} = 0,$$

uniformly on compact time intervals.

2. The problem posed on the line

Throughout this paper, the symbol $\mathbb{C}$ represents the complex numbers. For a real number x, the notation $\lfloor x \rfloor$ represents the largest integer which is less than or equal to x. The real axis $(-\infty,\infty)$ is denoted by $\mathbb{R}$. $C(\mathbb{R})$ is the set of continuous functions defined on $\mathbb{R}$ and $C_b(\mathbb{R})$ is the subset of $C(\mathbb{R})$ of bounded functions. For $k \geq 1$, $C^k(\mathbb{R})$ consists of all continuous functions whose jth derivatives lie in $C(\mathbb{R})$ for $j = 1, \cdots, k$ and similarly for $C_b^k(\mathbb{R})$. For $p \geq 1$, $L_p(\mathbb{R})$ is the Lebesgue space with its usual norm denoted by $\|\cdot\|_{L_p}$. For any real number s, $H^s(\mathbb{R})$ is the usual L_2–based Sobolev space with its norm $\|f\|_{H^s(\mathbb{R})}$, abbreviated as $\|f\|_{H^s}$, defined as

$$\|f\|_{H^s} = \Big(\int_{-\infty}^{\infty} \big(1 + |2\pi\xi|^{2s}\big) |\widehat{f}(\xi)|^2 \, d\xi \Big)^{\frac{1}{2}}.$$

Similarly, if I is an interval of $\mathbb{R}$, $L_p(I)$ for $p \geq 1$ represents the usual Lebesgue space with the usual norm $\|\cdot\|_{L_p(I)}$. For any nonnegative integer m, the space

$$W^{m,2}(I) = \Big\{ f, f', \cdots, f^{(m)} \in L_2(I) : \int_I \big(|f(x)|^2 + f^{(m)}(x)|^2\big) \, dx < \infty \Big\}$$

is a Hilbert space with the norm defined as

$$\|f\|_{W^{m,2}(I)} = \Big(\int_I \Big(\big|f(x)\big|^2 + \big|f^{(m)}(x)\big|^2 \Big) \, dx \Big)^{\frac{1}{2}}.$$

We also introduce an analogous periodic function space

$$H_l^s = \Big\{ f(x) = \sum_{n=-\infty}^{\infty} f_n e^{i\frac{n\pi}{l}x} : \, f_n = \overline{f_{-n}} \in \mathbb{C}, \sum_{n=-\infty}^{\infty} (1+n^2)^s |f_n|^2 < \infty \Big\}$$

with the norm

$$\|f\|_{H_l^s} = \Big(2l \sum_{n=-\infty}^{\infty} \big(1 + |\frac{n\pi}{l}|^{2s}\big) |f_n|^2 \Big)^{\frac{1}{2}}.$$

In the situation where s is an integer, the norm $\|f\|_{H_l^s}$ has an alternative representation

$$\|f\|_{H_l^s} = \Big(\int_{-l}^{l} \Big(f^2(x) + \big(f^{(s)}(x)\big)^2 \Big) \, dx \Big)^{\frac{1}{2}}$$

which coincides with the norm $\|f\|_{W^{s,2}(-l,l)}$ when f is restricted to the interval $(-l, l)$.

Remark. Functions in H_l^s are real-valued functions.

Now consider the BBM-equation posed on the line

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x, 0) = \psi(x), & x \in \mathbb{R}, \end{cases} \tag{2.1}$$

repeated here for easy reference. Following Benjamin *el al.* (1972), write the last equation as

$$u_t - u_{xxt} = -u_x - uu_x,$$

and formally solve for u_t to obtain

$$u_t(x,t) = -\frac{1}{2} \int_{-\infty}^{\infty} e^{-|x-y|} \Big(u_y(y,t) + u(y,t) u_y(y,t) \Big) \, dy. \tag{2.2}$$

Integrating by parts on the right-hand side of (2.2) yields

$$u_t(x,t) = \frac{1}{2} \int_{-\infty}^{\infty} \operatorname{sgn}(x-y) e^{-|x-y|} \Big(u(y,t) + \frac{1}{2} u^2(y,t) \Big) \, dy. \tag{2.3}$$

Formally integrating with respect to the temporal variable over $[0, t]$, one obtains the integral equation

$$u(x,t) = \psi(x) + \int_0^t \int_{-\infty}^{\infty} K(x-y) \Big(u(y,s) + \frac{1}{2} u^2(y,s) \Big) \, dy \, ds \tag{2.4}$$

where

$$K(x) = \frac{1}{2} \operatorname{sgn}(x) \, e^{-|x|}. \tag{2.5}$$

Benjamin *et al.* (1972) established the following result.

Theorem 2.1. *The BBM-equation* (2.1) *is globally well posed in the space* $H^1(\mathbb{R})$ *in the sense that if the initial value* $\psi \in H^1(\mathbb{R})$*, then there is a unique distributional solution* $u \in C([0,\infty); H^1(\mathbb{R}))$ *which depends continuously in* $C([0,\infty); H^1(\mathbb{R}))$ *on* $\psi \in H^1(\mathbb{R})$*. Moreover,* u *is* C^∞ *in the temporal variable* t *and* $\partial_t^j u \in C([0,\infty); H^2(\mathbb{R}))$ *for* $j = 1, 2, \cdots$ *. If* $\psi \in C^k(\mathbb{R}) \cap H^1(\mathbb{R})$*, for some* $k \geq 2$*, then* $u \in C([0,\infty); H^1(\mathbb{R}))$ *is a classical solution of* (2.1) *on* $\mathbb{R} \times [0,\infty)$ *and* $\partial_t^j \partial_x^i u \in C([0,\infty); H^1(\mathbb{R}))$ *for* $j \geq 1$ *and* $0 \leq i \leq k$*.*

Their proof is made via the contraction mapping principle in the space $C([0,T]; H^1(\mathbb{R}))$, for suitably small values of T, applied to the operator which is defined by the right-hand side of (2.4). The conserved quantity

$$\|u(\cdot,t)\|_{H^1} = \|\psi\|_{H^1}$$

shows that the time interval can be extended to $[0,\infty)$.

The following theorem gives spatial decay estimates of solutions u of the Cauchy problem (2.1) corresponding to similar conditions on the initial condition ψ for the purpose of the present study.

Theorem 2.2. *In addition to the assumption that* $\psi \in H^1(\mathbb{R})$*, we further assume that* $r(x)\psi(x)$ *is uniformly bounded where* $r(x) = (1+x^2)^\sigma$ *with* $\sigma > 0$ *a constant, or* $r(x) = e^{\lambda|x|}$ *with* $\lambda \in (0,1)$*. Then, for any* $x \in \mathbb{R}$ *and* $t > 0$*, the solution* u *satisfies*

$$\big|r(x)u(x,t)\big| \leq \|r\psi\|_{L_\infty}\, e^{\kappa\left(1+\frac{\sqrt{2}}{2}\|\psi\|_{H^1}\right)t} \tag{2.6}$$

where

$$\kappa = \sup_{x\in\mathbb{R}} \int_{-\infty}^{\infty} \frac{r(x)}{r(y)} e^{-|x-y|}\, dy.$$

Proof. If $U(x,t) = r(x)u(x,t)$, then the integral equation (2.4) is equivalent to

$$U(x,t) = r(x)\psi(x) + r(x)\int_0^t \int_{-\infty}^{\infty} \frac{K(x-y)}{r(y)} \Big(U(y,s) + \tfrac{1}{2}u(y,s)U(y,s)\Big)\, dy\, ds, \tag{2.7}$$

where K is as in (2.5). Elementary considerations yield the inequality

$$|U(x,t)| \leq \|r\psi\|_{L_\infty} + \tfrac{1}{2}\int_0^t \int_{-\infty}^{\infty} \frac{r(x)}{r(y)} e^{-|x-y|} dy (1+\tfrac{1}{2}\|u(\cdot,\tau)\|_{L_\infty})\|U(\cdot,\tau)\|_{L_\infty} d\tau. \tag{2.8}$$

Let

$$\mu(x) = \frac{1}{2}\int_{-\infty}^{\infty} \frac{r(x)}{r(y)} e^{-|x-y|}\, dy.$$

In the case $r(x) = (1+x^2)^\sigma$ for some $\sigma > 0$,

$$\frac{r(x)}{r(y)} = \frac{(1+(x-y+y)^2)^\sigma}{(1+y^2)^\sigma} \le \frac{\big(1+2y^2+2(x-y)^2\big)^\sigma}{(1+y^2)^\sigma} < \big(2+2(x-y)^2\big)^\sigma,$$

so

$$\mu(x) \le \frac{1}{2}\int_{-\infty}^{\infty} 2\big(1+(x-y)^2\big)^\sigma e^{-|x-y|}\, dy = \frac{1}{2}\int_{-\infty}^{\infty} \big(2+2y^2\big)^\sigma e^{-|y|}\, dy < \infty.$$

In the case $r(x) = e^{\lambda|x|}$ for some $\lambda \in (0,1)$,

$$\frac{r(x)}{r(y)} = e^{\lambda(|x|-|y|)} = e^{\lambda(|x-y+y|-|y|)} \le e^{\lambda|x-y|},$$

and hence

$$\mu(x) \le \frac{1}{2}\int_{-\infty}^{\infty} e^{\lambda|x-y|-|x-y|}\, dy \le \frac{1}{1-\lambda}.$$

In either case, it is seen that $\mu(x)$ is uniformly bounded on $\mathbb{R}$. Let $\kappa = \sup_{x\in\mathbb{R}} \mu(x)$ and notice that since

$$\|u(\cdot,\tau)\|_{L^\infty} \le \frac{\sqrt{2}}{2}\|u(\cdot,\tau)\|_{H^1} = \frac{\sqrt{2}}{2}\|\psi\|_{H^1},$$

one obtains

$$\big\|U(\cdot,t)\big\|_{L_\infty} \le \|r\psi\|_{L^\infty} + \kappa\big(1+\frac{\sqrt{2}}{2}\|\psi\|_{H^1}\big)\int_0^t \|U(\cdot,s)\|_{L_\infty}\, ds. \tag{2.9}$$

Applying the Gronwall inequality gives

$$\big\|U(\cdot,t)\big\|_{L_\infty} \le \|r\psi\|_{L_\infty} e^{\kappa\left(1+\frac{\sqrt{2}}{2}\|\psi\|_{H^1}\right)t}$$

and the estimate (2.6) follows readily. □

We now review the problem (2.1) with periodic initial data. Write the periodic initial data ψ of period $2l$ in terms of its Fourier coefficients as

$$\psi(x) = \sum_{n=-\infty}^{\infty} \psi_n e^{i\frac{n\pi}{l}x},$$

where $\psi_n = \frac{1}{2l}\int_{-l}^{l} \psi(x) e^{-i\frac{n\pi x}{l}}\, dx$. Chen (2004) proved the following result.

Theorem 2.3. *If $\psi \in H_l^1$, or, equivalently, its Fourier coefficients ψ_n have the property*

$$\sum_{n=-\infty}^{\infty} \left(1 + \left|\frac{n\pi}{l}\right|^2\right) |\psi_n|^2 < \infty,$$

then there is a unique solution

$$u(x,t) = \sum_{n=-\infty}^{\infty} u_n(t) e^{i\frac{n\pi}{l}x}$$

of (2.1) *which lies in the space $C([0,\infty); H_l^1)$. It depends continuously on $\psi \in H_l^1$, is C^∞ in t and for every $j \geq 1$, $\partial_t^j u \in C([0,\infty); H_l^2)$. Moreover, if the Fourier coefficients ψ_n of ψ satisfy*

$$\sum_{n=-\infty}^{\infty} \left(1 + \left|\frac{n\pi}{l}\right|^k\right) |\psi_n| < \infty$$

for some $k \geq 2$, then the periodic solution u is a classical solution of the BBM-equation (2.1) *on $\mathbb{R} \times [0,\infty)$ and $\partial_t^j \partial_x^i u \in C([0,\infty); C_b(\mathbb{R}))$ for $0 \leq i \leq k$ and $j \geq 0$.*

The idea of the proof is to convert the differential equation (2.1) into a system of equations in terms of the Fourier coefficients u_n of u, *viz.*

$$u_n(t) = \psi_n + \int_0^t \frac{i\frac{n\pi}{l}}{1 + |\frac{n\pi}{l}|^2} \Big(u_n(\tau) + \frac{1}{2}\sum_{k=-\infty}^{\infty} u_{n-k}(\tau) u_k(\tau)\Big)\, d\tau \qquad (2.10)$$

for $n = 0, \pm 1, \cdots$. One applies the contraction mapping principle to the operator defined by the right-hand side of the system in the space $C([0,T]; H_l^1)$ for small $T > 0$ (see Chen 2004). Then, the conserved quantity

$$\int_{-l}^{l} \left(u^2(x,t) + \big(u'(x,t)\big)^2\right) dx \equiv \int_{-l}^{l} \left(\psi^2(x) + (\psi'(x))^2\right) dx,$$

or, in an alternative form,

$$2l \sum_{n=-\infty}^{\infty} \Big(1 + |\frac{n\pi}{l}|^2\Big) |u_n(t)|^2 = 2l \sum_{n=-\infty}^{\infty} \Big(1 + |\frac{n\pi}{l}|^2\Big) |\psi_n|^2,$$

shows that the time interval $[0,T]$ for the existence result can be extended to $[0,\infty)$. The condition $\sum_{n=-\infty}^{\infty}(1 + |\frac{n\pi}{l}|^k)\,|\psi_n| < \infty$ on the initial data is a sufficient condition for the initial data ψ to lie in $C_b^k(\mathbb{R})$. Hence, the regularity result follows from the argument of Benjamin *et al.* (1972).

The following theorem gives the estimates of the periodic solution u of (2.1) at $x = \pm\, l$.

Theorem 2.4. *In Cauchy problem* (2.1), *assume that the periodic initial data* $\psi \in H_l^1$ *and let*

$$D = \sum_{n=-\infty}^{\infty} \Big| \frac{\psi_n - \psi_{n-1}}{\frac{1}{l}} \Big| = l \sum_{n=-\infty}^{\infty} |\psi_n - \psi_{n-1}|,$$

where ψ_n *are Fourier coefficients of* ψ. *Denote*

$$A = \frac{1 + \beta(l)\|\psi\|_{H_l^1}}{2} \quad \textit{and} \quad B = \pi\beta(l)\|\psi\|_{H_l^1} + \frac{1}{2}\pi\beta(l)^2\|\psi\|_{H_l^1}^2, \tag{2.11}$$

where

$$\beta(l) = \Big(\frac{1}{2l} \sum_{n=-\infty}^{\infty} \big(1 + |\frac{n\pi}{l}|^2\big)^{-1} \Big)^{\frac{1}{2}}. \tag{2.12}$$

Then at $x = \pm\, l$, *the solution* $u(x,t) = \sum_{n=-\infty}^{\infty} u_n(t) e^{i\frac{n\pi}{l}x}$ *of* (2.1) *has the bound*

$$|l\, u(l,t)| = |l\, u(-l,t)| \le \frac{D}{2} e^{At} + \frac{B}{2A}\big(e^{At} - 1\big). \tag{2.13}$$

Proof. Let

$$q_n(t) = \frac{u_n(t) - u_{n-1}(t)}{\frac{1}{l}} = l(u_n(t) - u_{n-1}(t))$$

for $n = 0, \pm 1, \cdots$. Then $q_n(t)$ satisfies the following system of equations:

$$\begin{aligned} q_n(t) = {} & \frac{\frac{in\pi}{l}}{1 + \frac{n^2\pi^2}{l^2}} \int_0^t \Big(q_n(\tau) + \sum_{k=-\infty}^{\infty} u_{n-k}(\tau) q_k(\tau) \Big)\, d\tau + l(\psi_n - \psi_{n-1}) \\ & + \frac{i\pi\big(1 - \frac{n(n-1)\pi^2}{l^2}\big)}{\big(1 + \frac{(n-1)^2\pi^2}{l^2}\big)\big(1 + \frac{n^2\pi^2}{l^2}\big)} \int_0^t \big(u_{n-1}(\tau) + \tfrac{1}{2} \sum_{k=-\infty}^{\infty} u_{n-1-k}(\tau) u_k(\tau) \big)\, d\tau. \end{aligned} \tag{2.14}$$

Define

$$Q(t) = \sum_{n=-\infty}^{\infty} |q_n(t)|,$$

then the system (2.14) yields the following inequality

$$Q(t) \le \frac{1}{2}\int_0^t \big(1 + \sum_{n=-\infty}^{\infty} |u_n(\tau)|\big) Q(\tau)\, d\tau + D \\ + \pi \int_0^t \Big(\sum_{n=-\infty}^{\infty} |u_n(\tau)| + \frac{1}{2}\big(\sum_{n=-\infty}^{\infty} |u_n(\tau)|\big)^2 \Big)\, d\tau. \tag{2.15}$$

Note that

$$\sum_{n=-\infty}^{\infty} |u_n(\tau)| = \sum_{n=-\infty}^{\infty} \frac{1}{\big(1+|\frac{n\pi}{l}|^2\big)^{\frac{1}{2}}} \big(1+|\frac{n\pi}{l}|^2\big)^{\frac{1}{2}} |u_n(\tau)| \\ \le \Big(\frac{1}{2l} \sum_{n=-\infty}^{\infty} \frac{1}{\big(1+|\frac{n\pi}{l}|^2\big)}\Big)^{\frac{1}{2}} \Big(2l \sum_{n=-\infty}^{\infty} \big(1+|\frac{n\pi}{l}|^2\big)|u_n(t)|^2\Big)^{\frac{1}{2}} \\ = \beta(l)\|u(\cdot,t)\|_{H_l^1}, \tag{2.16}$$

where $\beta(l)$ is given in (2.12). Since $\|u(\cdot,t)\|_{H_l^1} = \|\psi\|_{H_l^1}$ for all $t > 0$, applying the Gronwall inequality to (2.15), it follows that

$$Q(t) \le De^{At} + \frac{B}{A}(e^{At} - 1)$$

where A and B are provided in (2.11).

Now consider values of the solution $u(x,t) = \sum_{n=-\infty}^{\infty} u_n(t) e^{i\frac{n\pi}{l}x}$ at $x = \pm\, l$; it is seen that

$$u(l,t) = u(-l,t) = \sum_{n=-\infty}^{\infty} (-1)^n u_n(t) = \sum_{n=-\infty}^{\infty} \big(u_{2n}(t) - u_{2n+1}\big) \\ = -\sum_{n=-\infty}^{\infty} \big(u_{2n+1}(t) - u_{2n}(t)\big), \tag{2.17}$$

and hence,

$$l|u(l,t)| = l|u(-l,t)| \le \frac{1}{2}Q(t) \le \frac{1}{2}De^{At} + \frac{B}{2A}\big(e^{At} - 1\big);$$

the estimate (2.13) is proved. □

Corollary 2.5. *In the last theorem, if* $D = D(l) = \sum_{n=-\infty}^{\infty} l|\psi_n - \psi_{n-1}|$ *and* $\|\psi\|_{H_l^1}$ *are uniformly bounded as* $l \to \infty$*, then on any compact interval* $[0,T]$*, there is a constant* C *independent of* l *such that*

$$|u(l,t)| = |u(-l,t)| \le \frac{C}{l} \quad \textit{for all } t \in [0,T].$$

3. Limiting Results

We repeat the BBM-equation (2.1) here for readers' convenience.

$$\begin{cases} u_t + u_x + uu_x - u_{xxt} = 0, & x \in \mathbb{R}, \quad t > 0, \\ u(x,0) = \psi(x), & x \in \mathbb{R}. \end{cases} \tag{3.1}$$

Suppose the initial data ψ is sufficiently nice (to be described in Theorem 3.2), so that its Fourier transform $\widehat{\psi}$ exists and is continuous. The functions ψ and $\widehat{\psi}$ are related by the usual formulas

$$\widehat{\psi}(\xi) = \int_{-\infty}^{\infty} \psi(x) e^{-2\pi i x\xi}\, dx \quad \text{and} \quad \psi(x) = \int_{-\infty}^{\infty} \widehat{\psi}(\xi) e^{2\pi i x\xi}\, d\xi.$$

Consider the periodic initial-value problem

$$\begin{cases} v_t + v_x + vv_x - v_{xxt} = 0, & x \in \mathbb{R}, \quad t > 0, \\ v(x,0) = \sum_{n=-\infty}^{\infty} \frac{1}{2l} \widehat{\psi}\big(\frac{n}{2l}\big) e^{i\frac{n\pi}{l}x}, & x \in \mathbb{R}. \end{cases} \tag{3.2}$$

The aim of this section is to develop estimates of the difference between the solution u of (3.1) and the solution v of (3.2) on the spatial interval $[-l, l]$ for sufficiently large values of l. Bearing this point in mind, we start with some preliminary comments.

Write ϕ in terms of its Fourier Transform, $\widehat{\phi}$ say; that is,

$$\phi(x) = \int_{-\infty}^{\infty} \widehat{\phi}(\xi) e^{2\pi i x\xi}\, d\xi.$$

Naturally it is assumed that $\widehat{\phi}(-\xi) = \overline{\widehat{\phi}(\xi)}$ because ϕ is a real-valued function. Supposing $\widehat{\phi}$ to be continuous on $\mathbb{R}$, for $l > 0$ we introduce a transform $\mathcal{P}_l$ as follows;

$$\mathcal{P}_l(\phi)(x) = \sum_{n=-\infty}^{\infty} \frac{1}{2l} \widehat{\phi}\Big(\frac{n}{2l}\Big) e^{i\frac{n\pi}{l}x}. \tag{3.3}$$

The function $\mathcal{P}_l(\phi)$ is formally a real periodic function of period $2l$ because $\widehat{\phi}\big(\frac{n}{2l}\big)$ and $\widehat{\phi}\big(\frac{-n}{2l}\big)$ are complex conjugates for every n.

Proposition 3.1. *Let m be a positive integer. If $\phi \in H^m = H^m(\mathbb{R})$ and $\phi, \phi^{(m)}, \widehat{\phi^{(m)}} \in L_1 = L_1(\mathbb{R})$, then for any $l > 0$, the periodic function $\mathcal{P}_l(\phi) \in H_l^m$, and for any given $\epsilon > 0$, there is an $l_\epsilon > 0$ sufficiently large such that when both $\mathcal{P}_l(\phi)$ and ϕ are restricted to the interval $(-l, l)$,*

$$\|\mathcal{P}_l(\phi) - \phi\|_{W^{m,2}(-l,l)} < \epsilon \quad \textit{for all } l \geq l_\epsilon. \tag{3.4}$$

As a reminder, the space H_l^m consists of periodic functions which are of period $2l$ and for every $f \in H_l^m$, f and its derivatives $f', \cdots, f^{(m)}$ are square integrable on each period interval $(-l, l)$.

Before we prove the proposition, the following small lemma is worth singling out.

Lemma 3.2. *If $f \in L_1(\mathbb{R}) \cap C_b(\mathbb{R})$ and $f(x) \geq 0$ for $x \in \mathbb{R}$, then*

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{l\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{2l} f\Big(\frac{n}{2l}\Big). \tag{3.5}$$

Proof. Since $f \in L_1(\mathbb{R})$ and $f \geq 0$, for any $\epsilon > 0$, there is $R_\epsilon > 0$ such that

$$\int_{-R_\epsilon}^{R_\epsilon} f(x)\,dx - \epsilon \leq \int_{-\infty}^{\infty} f(x)\,dx \leq \int_{-R_\epsilon}^{R_\epsilon} f(x)\,dx + \epsilon. \tag{3.6}$$

Since f is continuous, the Lebesgue integral $\int_{-R_\epsilon}^{R_\epsilon} f(x)\,dx$ is equal to the Riemann integral, *viz.* $(R)\int_{-R_\epsilon}^{R_\epsilon} f(x)\,dx$. Temporarily, we turn our attention to this Riemann integral. For $l > 0$, decompose the interval $[-R_\epsilon, R_\epsilon]$ into a union of disjoint intervals as follows $[-R_\epsilon, R_\epsilon] = \cup_{n=-N-1}^{N+1} I_n$, where $N = N_\epsilon = \lfloor 2lR_\epsilon - \frac{1}{2} \rfloor$, the largest integer which is less than or equal to the number enclosed, and $I_0 = [-\frac{1}{2l}, \frac{1}{2l}]$,

$$I_n = \Big(\frac{n-\frac{1}{2}}{2l}, \frac{n+\frac{1}{2}}{2l}\Big] \text{ for } 1 \leq n \leq N,$$

$$I_n = \Big[\frac{n-\frac{1}{2}}{2l}, \frac{n+\frac{1}{2}}{2l}\Big) \text{ for } -N \leq n \leq -1,$$

and

$$I_{N+1} = \Big(\frac{N+\frac{1}{2}}{2l}, R_\epsilon\Big], \quad I_{-N-1} = \Big[-R_\epsilon, \frac{-N-\frac{1}{2}}{2l}\Big).$$

By definition, the Riemann sum

$$\sum_{n=-N}^{N} f\Big(\frac{n}{2l}\Big)\frac{1}{2l} + f(R_\epsilon)\Big(R_\epsilon - \frac{N+\frac{1}{2}}{2l}\Big) + f(-R_\epsilon)\Big(R_\epsilon - \frac{N+\frac{1}{2}}{2l}\Big)$$

has limit

$$(R)\int_{-R_\epsilon}^{R_\epsilon} f(x)\,dx$$

as $l \to \infty$. Note that the last two terms in the Riemann sum have limit zero because f is bounded. Therefore,

$$\int_{-R_\epsilon}^{R_\epsilon} f(x)\,dx = (R)\int_{-R_\epsilon}^{R_\epsilon} f(x)\,dx = \lim_{l\to\infty}\sum_{n=-N}^{N} f\Big(\frac{n}{2l}\Big)\frac{1}{2l}.$$

Substituting this result into (3.6) and noticing how $N = N_\epsilon$ depends on ϵ, we have the equality (3.5) immediately by letting $\epsilon \to 0$. $\square$

We now prove Proposition 3.1.

Proof. The condition $\phi \in L_1(\mathbb{R})$ implies that $\widehat{\phi}$ is a bounded continuous function on $\mathbb{R}$. Hence $(1+|2\pi\xi|^{2m})|\widehat{\phi}(\xi)|^2$ is non-negative and lies in $L_1(\mathbb{R})\cap C_b(\mathbb{R})$. From Lemma 3.2, one obtains

$$\int_{-\infty}^{\infty}(1+|2\pi\xi|^{2m})|\widehat{\phi}(\xi)|^2\,d\xi = \lim_{l\to\infty}\sum_{n=-\infty}^{\infty}\frac{1}{2l}\Big(1+\Big|\frac{n\pi}{l}\Big|^{2m}\Big)\Big|\widehat{\phi}\Big(\frac{n}{2l}\Big)\Big|^2. \quad (3.7)$$

Notice that the left-hand side of (3.7) is equal to $\|\phi\|^2_{H^m(\mathbb{R})}$. It is thus deduced that for any $l > 0$, $\mathcal{P}_l(\phi) \in H^m_l$. Now, we demonstrate the last part of the proposition.

$$\begin{aligned}
&\|\mathcal{P}_l(\phi)-\phi\|^2_{W^{m,2}(-l,l)} = \|\mathcal{P}_l(\phi)\|^2_{W^{m,2}(-l,l)} + \|\phi\|^2_{W^{m,2}(-l,l)}\\
&\quad - 2\int_{-l}^{l}\Big(\mathcal{P}_l(\phi)(x)\phi(x) + \mathcal{P}_l^{(m)}(\phi)(x)\phi^{(m)}(x)\Big)\,dx\\
&= 2l\sum_{n=-\infty}^{\infty}\Big(1+\Big|\frac{n\pi}{l}\Big|^{2m}\Big)\Big|\frac{1}{2l}\widehat{\phi}\Big(\frac{n}{2l}\Big)\Big|^2 + \int_{-l}^{l}\Big(\phi^2(x)+\big(\phi^{(m)}(x)\big)^2\Big)\,dx\\
&\quad - 2\int_{-l}^{l}\phi(x)\sum_{n=-\infty}^{\infty}\frac{1}{2l}\widehat{\phi}\Big(\frac{n}{2l}\Big)e^{i\frac{n\pi}{l}x}\,dx\\
&\quad - 2\int_{-l}^{l}\phi^{(m)}(x)\sum_{n=-\infty}^{\infty}\Big(i\frac{n\pi}{l}\Big)^m\frac{1}{2l}\widehat{\phi}\Big(\frac{n}{2l}\Big)e^{i\frac{n\pi}{l}x}\,dx.
\end{aligned}$$

Treat each integral $\int_{-l}^{l}[\cdot]\,dx$ as $\int_{-\infty}^{\infty}[\cdot]\,dx - \int_{|x|>l}[\cdot]\,dx$, and notice the relations

$$\int_{-\infty}^{\infty}\Big(\phi^2(x)+\big(\phi^{(m)}(x)\big)^2\Big)\,dx = \int_{-\infty}^{\infty}\big(1+(2\pi\xi)^{2m}\big)|\widehat{\phi}(\xi)|^2\,d\xi,$$

$$\int_{-\infty}^{\infty}\phi(x)e^{i\frac{n\pi}{l}}\,dx = \widehat{\phi}\Big(\frac{-n\pi}{2l}\Big) = \overline{\widehat{\phi}\Big(\frac{n\pi}{2l}\Big)}$$

and

$$\int_{-\infty}^{\infty} \phi^{(m)}(x) e^{i\frac{n\pi}{l}}\, dx = (-1)^m \Big(i\frac{n\pi}{l}\Big)^m \overline{\widehat{\phi}\Big(\frac{n\pi}{2l}\Big)}.$$

It follows that

$$\begin{aligned}
&\|\mathcal{P}_l(\phi) - \phi\|^2_{W^{m,2}(-l,l)} \\
&= \frac{1}{2l} \sum_{n=-\infty}^{\infty} \Big(1 + \Big|\frac{n\pi}{l}\Big|^{2m}\Big) \Big|\widehat{\phi}\Big(\frac{n}{2l}\Big)\Big|^2 + \int_{-\infty}^{\infty} \big(1 + (2\pi\xi)^{2m}\big) |\widehat{\phi}(\xi)|^2\, d\xi \\
&\quad - \frac{1}{l} \sum_{n=-\infty}^{\infty} \Big(1 + \Big|\frac{n\pi}{l}\Big|^{2m}\Big) \Big|\widehat{\phi}\Big(\frac{n}{2l}\Big)\Big|^2 + \Delta_l \qquad (3.8) \\
&= -\frac{1}{2l} \sum_{n=-\infty}^{\infty} \Big(1 + \Big|\frac{n\pi}{l}\Big|^{2m}\Big) \Big|\widehat{\phi}\Big(\frac{n}{2l}\Big)\Big|^2 + \int_{-\infty}^{\infty} \big(1 + (2\pi\xi)^{2m}\big) |\widehat{\phi}(\xi)|^2\, d\xi + \Delta_l,
\end{aligned}$$

where

$$\begin{aligned}
\Delta_l = &\int_{|x|>l} \Big(\phi^2(x) + \big(\phi^{(m)}(x)\big)^2\Big)\, dx - 2\int_{|x|>l} \phi(x) \sum_{n=-\infty}^{\infty} \frac{1}{2l}\widehat{\phi}\Big(\frac{n}{2l}\Big) e^{i\frac{n\pi}{l}x}\, dx \\
&- 2\int_{|x|>l} \phi^{(m)}(x) \sum_{n=-\infty}^{\infty} \Big(i\frac{n\pi}{l}\Big)^m \frac{1}{2l}\widehat{\phi}\Big(\frac{n}{2l}\Big) e^{i\frac{n\pi}{l}x}\, dx.
\end{aligned}$$

Applying Lemma 3.2 again, it follows that

$$\int_{-\infty}^{\infty} \big(1 + (2\pi\xi)^{2m}\big) |\widehat{\phi}(\xi)|^2\, d\xi = \lim_{l\to\infty} \frac{1}{2l} \sum_{n=-\infty}^{\infty} \Big(1 + \Big|\frac{n\pi}{l}\Big|^{2m}\Big) \Big|\widehat{\phi}\Big(\frac{n}{2l}\Big)\Big|^2.$$

This is to say, the sum of the first two terms on the right-hand side of (3.8) tends to zero as l approaches infinity. Thus, it is sufficient to show $\lim_{l\to\infty} \Delta_l = 0$ to prove (3.4).

Since $\phi \in H^m(\mathbb{R})$,

$$\lim_{l\to\infty} \int_{|x|>l} \Big(\phi^2(x) + \big(\phi^{(m)}(x)\big)^2\Big)\, dx = 0.$$

The assumption $\phi, \phi^{(m)} \in L_1(\mathbb{R})$ implies

$$\lim_{l\to\infty} \int_{|x|>l} \big(|\phi(x)| + |\phi^{(m)}(x)|\big)\, dx = 0.$$

Because $\phi \in H^m(\mathbb{R})$, it is automatically true that $\widehat{\phi^{(j)}} \in L_1(\mathbb{R})$ for $0 \le j < m$. Together with $\widehat{\phi^{(m)}} \in L_1(\mathbb{R}) \cap C_b(\mathbb{R})$, this gives

$$\lim_{l\to\infty} \sum_{n=-\infty}^{\infty} \frac{1}{2l}\Big(1+\Big|\frac{n\pi}{l}\Big|^m\Big)\Big|\widehat{\phi}\Big(\frac{n}{2l}\Big)\Big| = \int_{-\infty}^{\infty}\big(1+(2\pi\xi)^m\big)|\widehat{\phi}(\xi)|\,d\xi = |\widehat{\phi}|_1 + |\widehat{\phi^{(m)}}|_1.$$

In consequence, it is deduced that

$$\lim_{l\to\infty} \Delta_l = 0,$$

from which it follows that

$$\lim_{l\to\infty} \|\mathcal{P}_l(\phi) - \phi\|_{W^{m,2}(-l,l)} = 0.$$

The proof is complete. □

Here we come to our main result on comparisons between the solutions u and v of (3.1) and (3.2), respectively.

Theorem 3.3. *Consider the Cauchy problem* (3.1)*, assume that the initial data* ψ *satisfies:* $\psi \in H^1(\mathbb{R})$, $|\widehat{x\psi(x)}| \in L_1(\mathbb{R})$, *and there is an* $s > 0$ *such that* $(1+x^2)^{\frac{s}{2}}\psi(x) \in L_\infty(\mathbb{R})$; *moreover, its derivative* $\psi' \in L_1(\mathbb{R})$ *and its Fourier transform* $\widehat{\psi}$ *has the property that* $(1+\xi^2)^{\frac{1}{2}}\widehat{\psi} \in L_1(\mathbb{R})$. *Then, the solutions* u_l *of* (3.2) *and the solution* u *of* (3.1)*, when restricted on the spatial interval* $(-l,l)$*, have the relation*

$$\lim_{l\to\infty} \|u_l(\cdot,t) - u(\cdot,t)\|_{W^{1,2}(-l,l)} = 0.$$

This convergence is uniform on any compact set $[0,T]$.

Proof. The assumptions on the initial data ψ guarantee that

$$\mathcal{P}_l(\psi) = \sum_{n=-\infty}^{\infty} \frac{1}{2l}\widehat{\psi}\Big(\frac{n}{2l}\Big)e^{i\frac{n\pi}{l}x}$$

is well defined and belongs to the periodic Sobolev space H_l^1 and both solutions u of (3.1) and u_l of (3.2) exist and lie in $C([0,\infty); H^1(\mathbb{R}))$ and $C([0,\infty); H_l^1)$, respectively. Since $H^1(\mathbb{R}) \subset C_b(\mathbb{R})$, u is well defined pointwise. Introduce new dependent variables w and z as

$$z(x,t) = u_l(x,t) - u(x,t), \quad w(x,t) = z(x,t) - z(l,t)\varphi_+(x) - z(-l,t)\varphi_-(x), \tag{3.9}$$

where

$$\varphi_-(x) = \frac{e^{l-x} - e^{-l+x}}{e^{2l} - e^{-2l}}, \quad \varphi_+(x) = \varphi_-(-x). \tag{3.10}$$

Then, w is C^∞ in the temporal variable $t \in [0, \infty)$, both w and w_x are square integrable on the interval $(-l, l)$ in the spatial variable x, that is $w(\cdot, t) \in W^{1,2}(-l, l)$ when the spatial variable x is restricted to $(-l, l)$, and w satisfies the initial-boundary-value problem

$$\begin{cases} w_t + w_x + ww_x - w_{xxt} = \big(\varphi + \varphi u - \frac{1}{2}\varphi^2 - (u-\varphi)w\big)_x, \\ \qquad\qquad\qquad\qquad\qquad\qquad -l \le x \le l, \quad t > 0, \\ w(-l,t) = w(l,t) = 0, \qquad t \ge 0, \\ w(x,0) = w_0(x) = \mathcal{P}_l(\psi)(x) - \psi(x) - \big(\mathcal{P}_l(\psi)(l) - \psi(l)\big)\varphi_+(x) \\ \qquad\qquad -\big(\mathcal{P}_l(\psi)(x) - \psi(x)\big)\varphi_-(x), \quad -l \le x \le l, \quad t \ge 0, \end{cases} \tag{3.11}$$

where $\varphi(x,t) = -z(l,t)\varphi_+(x) - z(-l,t)\varphi_-(x)$. Multiply (3.11) by $2w$ and integrate over $[-l, l]$ with respect to x; after integrations by parts there appears

$$\begin{aligned} &\frac{d}{dt}\|w(\cdot,t)\|^2_{W^{1,2}(-l,l)} = \frac{d}{dt}\int_{-l}^{l}\Big(w^2(x,t) + w_x^2(x,t)\Big)\,dx \\ &= 2\int_{-l}^{l}(u-\varphi)ww_x\,dx + \int_{-l}^{l} 2w\Big(\varphi + \varphi u - \frac{1}{2}\varphi^2)\Big)_x\,dx \\ &\le \|u(\cdot,t) - \varphi(\cdot,t)\|_{L_\infty(-l,l)}\|w(\cdot,t)\|^2_{W^{1,2}(-l,l)} \\ &\quad + 2\|\varphi(\cdot,t) - \frac{1}{2}\varphi^2(\cdot,t) + \varphi u(\cdot,t)\|_{L_2(-l.l)}\,\|w_x(\cdot,t)\|_{L_2(-l,l)} \\ &\le \|u(\cdot,t) - \varphi(\cdot,t)\|_{L_\infty(-l,l)}\|w(\cdot,t)\|^2_{W^{1,2}(-l,l)} \\ &\quad + 2\|\varphi(\cdot,t)\|_{L_2(-l,l)}\,\|1 - \frac{1}{2}\varphi(\cdot,t) + u(\cdot,t)\|_{L_\infty(-l,l)}\|w(\cdot,t)\|_{W^{1,2}(-l,l)}. \end{aligned} \tag{3.12}$$

Since $u_l(l,t) = u_l(-l,t)$ and $0 \le \varphi_+(x) + \varphi_-(x) < 1$,

$$\begin{aligned} &u(x,t) - \varphi(x,t) \\ &= u(x,t) - u(l,t)\varphi_+(x) - u(-l,t)\varphi_-(x) + u_l(l,t)\big(\varphi_+(x) + \varphi_-(x)\big), \end{aligned}$$

and we have

$$\|u(\cdot,t) - \varphi(\cdot,t)\|_{L_\infty(-l,l)} \le 2\|u(\cdot,t)\|_{L_\infty(-l,l)} + |u_l(l,t)|.$$

Note that

$$\|u(\cdot,t)\|_{L_\infty(-l,l)} \le \|u(\cdot,t)\|_{L_\infty} \le \frac{1}{\sqrt{2}}\|u(\cdot,t)\|_{H^1} = \frac{1}{\sqrt{2}}\|\psi\|_{H^1}$$

and for any x,

$$
\begin{aligned}
|u_l(x,t)| &\le \sum_{n=-\infty}^{\infty} |u_n(t)| \\
&\le \Big(\frac{1}{2l}\sum_{n=-\infty}^{\infty}\big(1+(\frac{n\pi}{l})^2\big)^{-1}\Big)^{\frac12}\Big(2l\sum_{n=-\infty}^{\infty}(1+(\frac{n\pi}{l})^2|u_n(t)|^2\Big)^{\frac12} \\
&= \beta(l)\|u_l(\cdot,t)\|_{H_l^1} = \beta(l)\|\mathcal{P}_l(\psi)\|_{H_l^1},
\end{aligned}
$$

where

$$
\beta(l) = \Big(\frac{1}{2l}\sum_{n=-\infty}^{\infty}\big(1+(\frac{n\pi}{l})^2\big)^{-1}\Big)^{\frac12}.
$$

Whence,

$$
\|u(\cdot,t)-\varphi(\cdot,t)\|_{L_\infty(-l,l)} \le \sqrt{2}\|\psi\|_{H^1} + \beta(l)\|\mathcal{P}_l(\psi)\|_{H_l^1}. \tag{3.13}
$$

By straightforward calculations, it follows that

$$
\begin{aligned}
\|\varphi(\cdot,t)\|_{L_2(-l,l)} &\le \big(\max\{|u(l,t)|,\,|u(-l,t)|\} + |u_l(l,t)|\big)\,\|\varphi_+ + \varphi_-\|_{L_2(-l,l)} \\
&\le |u(l,t)| + |u(-l,t)| + |u_l(l,t)| \quad \text{for } l>1
\end{aligned} \tag{3.14}
$$

and

$$
\begin{aligned}
\|1-\frac12\varphi(\cdot,t)+u(\cdot,t)\|_{L_\infty(-l,l)} &\le 1+|u_l(l,t)| + 2\|u(\cdot,t)\|_{L_\infty} \\
&\le 1+\beta(l)\|\mathcal{P}_l(\psi)\|_{H_l^1} + \sqrt{2}\|\psi\|_{H^1}.
\end{aligned} \tag{3.15}
$$

Denote

$$
\gamma = \frac{\sqrt{2}\|\psi\|_{H^1} + \beta(l)\|\mathcal{P}_l(\psi)\|_{H_l^1}}{2}
$$

and $U(l,\tau) = |u(l,\tau)| + |u(-l,\tau)| + u_l(l,\tau)|$. Then (3.12) can be rewritten as

$$
\frac{d}{dt}\|w(\cdot,t)\|^2_{W^{1,2}(-l,l)} \le 2\gamma\|w(\cdot,t)\|^2_{W^{1,2}(-l,l)} + 2(1+2\gamma)U(l,t)\|w(\cdot,t)\|_{W^{1,2}(-l,l)}.
$$

Hence, the Gronwall inequality provides the following estimate,

$$
\|w(\cdot,t)\|_{W^{1,2}(-l,l)} \le \|w_0\|_{W^{1,2}(-l,l)}e^{\gamma t} + \big(1+2\gamma\big)\int_0^t U(l,\tau)e^{\gamma(t-\tau)}\,d\tau. \tag{3.16}
$$

By definition of w, we have

$$
\begin{aligned}
\|u_l(\cdot,t)-u(\cdot,t)\|_{W^{1,2}(-l,l)} &= \|z(\cdot,t)\|_{W^{1,2}(-l,l)} \\
&\le \|\varphi\|_{W^{1,2}(-l,l)} + \|w(\cdot,t)\|_{W^{1,2}(-l,l)},
\end{aligned}
$$

where

$$\|\varphi(\cdot,t)\|_{W^{1,2}(-l,l)} \le |z(l,t)|\,\|\varphi_+\|_{W^{1,2}(-l,l)} + |z(-l,t)|\,\|\varphi_-\|_{W^{1,2}(-l,l)}$$
$$\le \frac{(e^{4l}-e^{-4l})^{\frac{1}{2}}}{e^{2l}-e^{-2l}}(|z(l,t)|+|z(-l,t)|) \le 2(|z(l,t)|+|z(-l,t)|)$$

for $l > 1$. Therefore,

$$\begin{aligned}\|u_l(\cdot,t)-u(\cdot,t)\|_{W^{1,2}(-l,l)} &\le 2(|z(l,t)|+z(-l,t)|)\\ &+\Big(\|\mathcal{P}_l(\psi)-\psi\|_{W^{2,1}(-l,l)} + 2\big(|\psi(l)|+|\psi(-l)|\big)\Big)e^{\gamma t}\\ &+\big(1+2\gamma\big)\int_0^t U(l,\tau)e^{\gamma(t-\tau)}\,d\tau.\end{aligned} \tag{3.17}$$

By definition, $\beta(l)$ tends to the integral $\left(\int_{-\infty}^{\infty}(1+(2\pi\xi)^2)^{-1}\,d\xi\right)^{\frac{1}{2}}$ as $l\to\infty$. Proposition 3.1 shows that $\|\mathcal{P}_l(\psi)-\psi\|_{W^{1,2}(-l,l)}$ tends to zero as $l\to\infty$. $(1+l^2)^{\frac{s}{2}}u(l,t)$ is bounded on any compact temporal sets as $l\to\infty$ by Theorem 2.2. The condition $\widehat{x\phi}\in L_\infty(\mathbb{R})$ guarantees that $D = l\sum_{n=-\infty}^{\infty}|\widehat{\psi}(\frac{n\pi}{l} - \widehat{\psi}(\frac{(n-1)\pi}{l}|$ is bounded as $l\to\infty$; this implies that $u_l(l,t) = u_l(-l,t)$ is bounded by $\frac{C}{l}$ uniformly on compact sets $[0,T]$ by Corollary 2.5. Therefore, for any given $t>0$, the last inequality implies

$$\lim_{l\to\infty}\|u_l(\cdot,t)-u(\cdot,t)\|_{W^{1,2}(-l,l)} = 0.$$

The convergence is uniform on any compact set $t\in[0,T]$. The theorem is proved. □

Remark. This convergence theory provides a discrete method to approximate the solutions of the BBM-equation (2.1) when the initial data lies in some Sobolev space on the line.

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References

[1] T.B. Benjamin, *Lectures on Nonlinear Wave Motion*, In Lect. Appl. Math. Vol. **15**, (ed. A. Newell) American. Math. Soc.: Providence, RI (1974), 3–47.

[2] T.B. Benjamin, J.L. Bona, and J.J. Mahony, *Model equations for long waves in non-linear dispersive systems*, Philos. Trans. Royal Soc. London Series A, **272** (1972), 47–78.

[3] J.L. Bona, *Convergence of periodic wave trains in the limit of large wavelength,* Appl. Sci. Res., **37** (1981), 21–30.

[4] J.L. Bona and H. Chen, *Comparison of model equations for small-amplitude long waves.* Nonlinear Analysis TMA Series B, **38** (1999), 625–647.

[5] J.L. Bona, H. Chen, S. Sun, and B.-Y. Zhang, *Comparison of quarter-plane and two-point boundary value problems: the BBM-equation,* Discrete and Continuous Dynamical Systems, **13** (2005), 921–940.

[6] J.L. Bona, M. Chen, and J.-C. Saut, *Boussinesq equations and other systems for small amplitude long waves in nonlinear dispersive media. I: derivation and linear theory,* J. Nonlinear Sci., **12** (2002), 283–318.

[7] J.L. Bona and V. Dougalis, *An initial and boundary value problem for a model equation for propagation of long waves*, J. Math. Anal. Appl., **75** (1980), 501–522.

[8] J.L. Bona and L. Luo, *Initial-boundary value problems for model equations for the propagation of long waves*. In Evolution Equations (ed. G. Gerreyra, G. Goldstein and F. Neubrander), Lecture Notes in Pure and Appl. Math., Marcel Dekker: New York, **168** (1995),. 65–94.

[9] J.L. Bona, S. Sun, and B.-Y. Zhang, *A nonhomogeneous boundary-value problem for the Korteweg-de Vries equation posed on a finite domain*, Comm. Partial Differential Equations, **28** (2003), 1391–1436.

[10] J. Boussinesq, *Essai sur la Théorie des Eaux Courantes*, Mémoires de l'Académie des Sciences, T. 23 et 24 (1877).

[11] H. Chen, Periodic initial-value problem for the BBM-equation, *Computers and Mathematics with Applications, Special Issue on "Computational Methods in Analysis"* **48** (2004), 1305–1318.

[12] J. Hammack and H. Segur, *The Korteweg-de Vries equation and water waves,* part 2. comparison with experiments, J. Fluid Mech., **65** (1974), 237–246.

[13] D.J. Korteweg and G. de Vries, *On the change of form of long waves advancing in a rectangular canal and on a new type of long stationary waves*, Philosophical Magazine, 5th series, **36** (1895), 422–443.

[14] E.H. Lieb and M. Loss, "Analysis" (Second Edition), Amer. Math. Soc.: Providence, RI (2001).

[15] D.H. Peregrine, *Long waves on a beach*, J. Fluid Mech., **27** (1967), 815–827.

[16] R. Miura, *The Korteweg-de Vries equation: a survey of results,* SIAM Review, **18** (1976), 412–459.

[17] G. Whitham, "Linear and Nonlinear Waves," John Wiley, New York (1974).