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ON THE EXISTENCE AND REGULARITY OF SOLUTIONS FOR DEGENERATE POWER-LAW FLUIDS

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Abstract. We study time-dependent flows of incompressible *degenerate* power-law fluids characterized by the power-law index $p-2$ with $p>2$. In this case, the generalized viscosity vanishes as (the modulus of) the shear rate tends to zero. We prove global-in-time existence of a weak solution if $p > \max\{\frac{3d-4}{d}, 2\}$. This improves the range $p > \frac{3d+2}{d+2}$ for which the existence result was obtained by O.A. Ladyzhenskaya and J.L. Lions, via standard monotone operator theory. Since we apply higher differentiability techniques, certain regularity results are also established. The key step of the proof is an estimate of the velocity gradient in a suitable Nikol′skiĭ space. To make the presentation of the method transparent, we restrict ourselves to the spatially periodic problem. A possible extension of the approach to no-slip boundary conditions is however discussed as well.

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1. Introduction

In this paper, we are concerned with the question of global existence of a weak solution to the system

$$\begin{aligned} \partial_t \boldsymbol{v} + \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v}) - \operatorname{div} \boldsymbol{T}(D\boldsymbol{v}) + \nabla \pi = \boldsymbol{f}, \\ \operatorname{div} \boldsymbol{v} = 0, \\ \boldsymbol{v}|_{t=0} = \boldsymbol{v}_0, \end{aligned} \tag{1.1}$$

where

$$\boldsymbol{T}(D\boldsymbol{v}) = \nu |D\boldsymbol{v}|^{p-2} D\boldsymbol{v}, \qquad \nu > 0. \tag{1.2}$$

The system comes from non-Newtonian fluid mechanics: $\boldsymbol{v} : \Omega \times (0,T) \to \mathbb{R}^d$ and $\pi : \Omega \times (0,T) \to \mathbb{R}$ are the unknown velocity and pressure, while the external forces $\boldsymbol{f} : \Omega \times (0,T) \to \mathbb{R}^d$ and the initial velocity $\boldsymbol{v}_0 : \Omega \to \mathbb{R}^d$ are given.

In the present paper we only consider the spatially periodic setting, i.e., we eliminate the presence of the boundary by setting $\Omega = (0,L)^d$ and assuming that all the functions are L-periodic in every direction and have zero mean value. Possible extensions to other boundary conditions are discussed in Section 4. In (1.2), $D\boldsymbol{v} = \frac{1}{2}(\nabla \boldsymbol{v} + \nabla \boldsymbol{v}^T)$ denotes the symmetric part of the velocity gradient.

The system (1.1) describes unsteady flow of an incompressible fluid whose rheological properties are encoded in (1.2). This constitutive equation has the structure

$$\boldsymbol{T}(D\boldsymbol{v}) = a(|D\boldsymbol{v}|) D\boldsymbol{v} \qquad \text{with} \qquad a(s) = \nu s^{p-2}, \tag{1.3}$$

and thus, for $p \neq 2$, the model (1.2) falls into the class of fluids with shear dependent viscosity. Note that for $p = 2$, we obtain the Navier-Stokes equations. Fluids (1.2) are called power-law fluids, the exponent $p - 2$ being then the power-law index. If $p > 2$ the fluid has the ability to shear thicken (such fluids are then called shear thickening or dilatant fluids). If $p < 2$, the fluid has the ability to shear thin (these fluids are called shear thinning or pseudoplastic fluids). Note that for $p > 2$, the generalized viscosity $\nu |D\boldsymbol{v}|^{p-2}$ vanishes (degenerates) as $|D\boldsymbol{v}| \to 0$, while for $p < 2$ the viscosity tends to $+\infty$ (i.e., becomes singular). A more detailed exposition of incompressible fluids with shear dependent viscosity can be found in [12], for example.

In this article, we deal with power-law fluids characterized by (1.2) with positive power-law index, i.e., $p > 2$. We clarify reasons for this choice in what follows.

There are quite a number of papers devoted to questions of existence, uniqueness, regularity and further properties (as long time behaviour) of solutions to the system of equations (1.1) with (1.3), where the constitutive equation (1.2) is a special case. We refer to the survey article [13] where an overview and plenty of references are provided. Yet, even the question of (global) existence of solutions is not completely solved (meaning for all $p \neq 2$), though a variety of techniques have been developed, depending on whether $p < 2$ or $p > 2$; see for example [4] for an overview of available techniques which we also briefly sketch below.

The usual strategy of the existence theory is to find $\boldsymbol{v}^n$, solutions of a suitably chosen approximating problem (a Galerkin approximation, for example). With the help of uniform estimates in proper spaces one then obtains $\boldsymbol{v}$, a solution to (1.1), as a "weak" limit of a suitable chosen subsequence to $\boldsymbol{v}^n$.

The central problem is the passage to the limit in the nonlinear terms. From standard energy estimates one concludes (regardless of the dimension d) that

$$\begin{aligned} \boldsymbol{v}^n \text{ is bounded in } \quad & L^p(0,T;W^{1,p}_{div}) \cap L^\infty(0,T;L^2_{div}), \\ \partial_t \boldsymbol{v}^n \text{ is bounded in } \quad & L^2(0,T;[W^{3,2}_{div}]'). \end{aligned} \tag{1.4}$$

This implies weak convergence of $\nabla \boldsymbol{v}^n$ and strong convergence of $\boldsymbol{v}^n$, which is enough to handle the quadratic convective term $\operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})$ provided that $p > \frac{2d}{d+2}$, but would not suffice to deal with the stress tensor $\boldsymbol{T}(D\boldsymbol{v})$.

The fact that $\boldsymbol{T}$ forms a monotone operator can help. The first method (Approach 1) considered in Ladyzhenskaya [6, 7] and Lions [8], where all the details can be found, is based on Minty's trick implying that $\boldsymbol{T}(D\boldsymbol{v}^n) \to \boldsymbol{T}(D\boldsymbol{v})$. This, however, requires that one can take as a test function in (1.1) the difference $\boldsymbol{v}^n - \boldsymbol{v}$, having only those regularity properties stated in (1.4). Here, the convective term gives a lower bound $p \geq \frac{3d+2}{d+2}$ which, unfortunately, is strictly greater than 2 if $d \geq 3$.

One can overcome this difficulty by testing with a suitable truncation of $\boldsymbol{v}^n - \boldsymbol{v}$. In [5] this method (Approach 2) is presented and the existence theory is extended up to $p > \frac{2(d+1)}{d+2}$. The disadvantage of this result is its unclear extension to homogeneous Dirichlet (no-slip) boundary conditions. The method also provides no regularity statements.

In this paper we follow another method (Approach 3) based on higher differentiability techniques, introduced to non-degenerate systems with spatially period setting in [10] and [1], and presented with all the details in [9]. Roughly speaking, the aim is to obtain estimates for $\boldsymbol{v}^n$ in $L^{\tilde{p}}(0,T;W^{1+\sigma,p})$ with suitable $\sigma > 0$ and $\tilde{p} \in (1,p)$. By standard embedding one obtains then strong convergence of $\nabla \boldsymbol{v}^n$, which enables us to identify the limit of $\boldsymbol{T}(D\boldsymbol{v}^n)$.

In order to obtain this higher-order derivative estimate, we simply test the equation by a second difference of $\boldsymbol{v}^n$, which thanks to the absence of the boundary simplifies to testing by $-\Delta \boldsymbol{v}^n$.

This approach, however, has to overcome two difficulties.

(1) For power-law fluids, taking the scalar product of $\boldsymbol{T}(D\boldsymbol{v})$ with $-\Delta\boldsymbol{v}$ leads to the term

$$I_p(\boldsymbol{v}) := \int_\Omega |D\boldsymbol{v}|^{p-2}|\nabla D\boldsymbol{v}|^2. \tag{1.5}$$

It is not completely obvious how this quantity ensures the compactness of gradients if $p > 2$. On the contrary, the "singular" case $p < 2$ can be treated easily since $I_p(\boldsymbol{v}) \geq c\|D(\nabla\boldsymbol{v})\|_p^2$; see [9] for details. This is the reason why we concentrate on the "degenerate" case $p > 2$ here. Note also that this difficulty would not occur in the case of non-degenerate stresses of the type $\boldsymbol{T}(D\boldsymbol{v}) = \nu(1 + |D\boldsymbol{v}|^{p-2})D\boldsymbol{v}$. In such a case one directly comes to $I_p(\boldsymbol{v}) \geq c\|\nabla D\boldsymbol{v}\|_2^2$. See again [9] for details. It is worth mentioning that this procedure can be, at least for $p > 2$, extended to the homogeneous Dirichlet boundary value problem, see [11].

(2) The second problem comes from the convective term. After taking the scalar product of the convective term with $-\Delta\boldsymbol{v}$ one obtains $\|\nabla\boldsymbol{v}\|_3^3$ which is not integrable if $p < 3$. The standard tricks here give the same lower bound $p \geq \frac{3d+2}{d+2}$ that follow from the monotone operator theory.

To overcome the first difficulty one observes that for $p > 2$ the integral $I_p(\boldsymbol{v})$ defined in (1.5) estimates the norm of $D\boldsymbol{v}$ in a certain Nikol′skiĭ space. This gives the desired compact embedding. For a more general development of this idea treating the p-Laplacian using the Nikol′skiĭ spaces, see [3].

The second difficulty is fixed using the technique from [9]. Roughly speaking, the equation can be integrated after dividing by $(1 + \|\nabla\boldsymbol{v}\|_2^2)^\lambda$ with a suitable $\lambda > 1$. This, of course, weakens our estimate coming from (1.5). Still, this weakened estimate keeps enough information to conclude the compactness of gradients.

To clearly formulate the novelty of this article we summarize the above discussion for the most interesting case $d = 3$.

The theory developed by Ladyzhenskaya gives the existence to (1.1)-(1.3) for $p \geq \frac{11}{5}$ both for the spatially periodic and the homogeneous Dirichlet problem. The long-time and large-data existence for $p \in [2, \frac{11}{5})$ is treated in [5] in the case of the spatially periodic problem (in fact the case $p \in (\frac{8}{5}, \frac{11}{5})$ is treated therein). The extension of the result from [5] to the homogeneous Dirichlet problem is open.

In [9], the spatially periodic problem for $p \in [2, \frac{11}{5})$ is analyzed via a higher-differentiability approach, however, for non-degenerate operators only. It is worth mentioning that this approach has been already extended, again only for non-degenerate operators, to the case of the homogeneous Dirichlet problem; see [11] for details.

In the here presented article, dealing with a degenerate elliptic operator for the spatially periodic problem, we establish an existence theory for $p \in (2, \frac{11}{5})$ using Approach 3 and obtain new fractional estimates; if $p \geq \frac{11}{5}$, we even strengthen the regularity results proved in [9] and show that for all $\epsilon > 0$ $\boldsymbol{v} \in L^p(0,T; W^{1+\frac{2}{p}-\epsilon,p}(\Omega))$. Because of the result stated in [11] we think Approach 3 (higher differentiability) is more suitable for possible extension to Dirichelt boundary conditions than Approach 2 (strict monotonicity combined with truncation operators).

The paper is organized as follows: in Section 2, we introduce some function spaces and establish several auxiliary lemmas. In Section 3 we state and prove our main result, Theorem 3.1. Concluding remarks, in particular those related to other boundary conditions, are presented in the last Section 4.

2. Preliminaries

Set

$$\mathcal{V} = \big\{\phi \in C^\infty(\mathbb{R}^d) : \phi(x + Le_j) = \phi(x)\ \forall j,\ \int_\Omega \phi = 0\big\}.$$

Then L^p, $W^{s,p}$ are the closures of $\mathcal{V}$ with respect to the corresponding norms $\|\cdot\|_p$, $\|\cdot\|_{s,p}$, restricted (thanks to periodicity) to $\Omega = (0, L)^d$. We allow also for spaces $W^{s,p}$ with a noninteger s and the functions can be scalar or vector valued. In the latter case the subscript div as in L^p_{div} indicates that the functions are free of divergence.

It is worth recalling that for any $\epsilon > 0$, $W^{s+\epsilon,p} \hookrightarrow\hookrightarrow W^{s,p} \hookrightarrow L^q$, provided that $\frac{1}{q} = \frac{1}{p} - \frac{s}{d}$ and $sp < d$.

Further, for $p \in [1, \infty)$ and $s = m + \sigma$, where $m \geq 0$ is an integer and $\sigma \in (0, 1)$ we introduce the Nikol′skiĭ space $\mathcal{N}^{s,p}$ as the subspace of L^p-functions for which the norm

$$\begin{aligned}\|u\|^p_{\mathcal{N}^{s,p}} &= \|u\|^p_p + |u|^p_{\mathcal{N}^{s,p}} \\ &= \|u\|^p_p + \sum_{|\alpha|=m} \sup_{0<|h|<\delta} \int \frac{|\partial^\alpha u(x+h) - \partial^\alpha u(x)|^p}{|h|^{\sigma p}}\, dx\end{aligned}$$

is finite. Here $\delta > 0$ is fixed. We have for any $\epsilon > 0$ the embeddings (see [15])

$$\mathcal{N}^{s,p} \hookrightarrow W^{s-\epsilon,p} \hookrightarrow \mathcal{N}^{s-\epsilon,p}. \tag{2.1}$$

Note that thanks to the zero mean condition, one can take the highest order derivative seminorm as an equivalent norm in each of the above spaces.

We complete this section with several auxiliary lemmas. The first lemma is a key step in exploiting the estimates of $I_p(\boldsymbol{v})$ defined in (1.5) in terms of the norm in a Nikol′skiĭ space; see [3, Eq. (3.7)] for its discrete analogue.

Lemma 2.1. *Let $u \in W^{1,1}$ be a scalar or vector-valued function and $p > 2$. Let*

$$I_p(u) = \int_\Omega |u|^{p-2} |\nabla u|^2 < \infty\,.$$

Then $u \in \mathcal{N}^{\frac{2}{p},p}$ and

$$\|u\|^p_{\mathcal{N}^{\frac{2}{p},p}} \le c\, I_p(u)$$

with c depending only on p and Ω.

Proof. For $a \ge 1$, we start with the inequality

$$|u - v|^a \le c_1\, \big||u|^{a-1}u - |v|^{a-1}v\big| \qquad \text{for all } u,\, v \in \mathbb{R}^d \tag{2.2}$$

which holds with a suitable $c_1 = c_1(a)$. Inequality (2.2) follows from

$$\begin{aligned} |u - v|^{a+1} &\le c_1 \left\langle |u|^{a-1}u - |v|^{a-1}v, u - v \right\rangle \\ &\le c_1\, \big||u|^{a-1}u - |v|^{a-1}v\big|\; |u - v|\,, \end{aligned}$$

whereas the first inequality is proven in [2, chapter I, Lemma 4.4, page 13].

Taking $\delta > 0$ fixed and $x \in \Omega$, and considering $h \in \mathbb{R}^d$ such that $0 < |h| < \delta$ we obtain using inequality (2.2)

$$\begin{aligned} |u(x+h) - u(x)|^{\frac{p}{2}} &\le c_1 \big||u(x+h)|^{\frac{p}{2}-1}u(x+h) - |u(x)|^{\frac{p}{2}-1}u(x)\big| \\ &= c_1 \Big| \int_0^1 \frac{\partial}{\partial s} \Big\{ |u(x+sh)|^{\frac{p}{2}-1} u(x+sh) \Big\}\, dt \Big| \\ &\le c_2 |h| \int_0^1 |u(x+sh)|^{\frac{p}{2}-1} |\nabla u(x+sh)|\, ds\,. \end{aligned}$$

By Hölder's inequality we conclude that

$$|u(x+h) - u(x)|^p \le c_2^2 |h|^2 \int_0^1 |u(x+sh)|^{p-2} |\nabla u(x+sh)|^2\, ds\,.$$

Integrating the result over $x \in \Omega$, and applying then Fubini's theorem we come to the inequality

$$\int_\Omega \frac{|u(x+h)-u(x)|^p}{|h|^{\frac{2}{p}\cdot p}} \le c_3\, I_p(u)\,.$$

Now the left-hand side is the seminorm of $\mathcal{N}^{\frac{2}{p},p}$, which is enough to finish the proof thanks to the spatially periodic setting. □

The well-known Aubin-Lions lemma about the compact embedding of Bochner spaces will be also needed.

Lemma 2.2. *Let $Y \hookrightarrow\hookrightarrow X \hookrightarrow Z$ be Banach spaces and let X be reflexive. Let $p > 1$, $q \in [1, \infty]$. Then*

$$\left\{u \in L^p(0,T;Y) : \partial_t u \in L^q(0,T;Z)\right\} \hookrightarrow\hookrightarrow L^p(0,T;X)$$

Proof. See [16], for example. □

We also need a generalized version of the well-known Korn-inequality.

Lemma 2.3. *Let (vector-valued) $\boldsymbol{v} \in W^{s,p}$, $p \in (1,\infty)$. Then for any $s \in [0,1]$ one has*

$$\|\nabla \boldsymbol{v}\|_{s,p} \le c\,\|D\boldsymbol{v}\|_{s,p},$$

where $D\boldsymbol{v} = \frac{1}{2}(\nabla\boldsymbol{v} + \nabla\boldsymbol{v}^T)$ and c depends only on s, p and Ω.

Proof. For $s = 0$ (the standard version) see Nečas [14] or for example [9] and the references therein. The case $s = 1$ is in fact elementary as every second derivative of $\boldsymbol{v}$ can be expressed in terms of first derivatives of $D\boldsymbol{v}$:

$$\frac{\partial^2 v_j}{\partial x_i \partial x_k} = \frac{\partial}{\partial x_k} D_{ij}(\boldsymbol{v}) + \frac{\partial}{\partial x_i} D_{kj}(\boldsymbol{v}) - \frac{\partial}{\partial x_j} D_{ik}(\boldsymbol{v}) \qquad i,j,k = 1,\dots,d.$$

The general case is then obtained by interpolation. □

Finally, the following lemma concerning the passage to the limit under the integral sign will be needed.

Lemma 2.4. *Let $M \subset \mathbb{R}^m$ be measurable and bounded. Let the sequence $\{f^n\}_{n\in\mathbb{N}}$ be uniformly bounded in $L^q(M)$ for some $q > 1$. Finally, let $f^n \to f$ almost everywhere in M for some $f \in L^q(M)$. Then*

$$\int_M f^n \to \int_M f\,.$$

Proof. It is a straightforward consequence of Vitali's theorem. □

3. Main theorem

In this section we formulate and prove our main result.

Theorem 3.1. *Let $p \geq 2$ and $d \geq 2$. Assume that $\boldsymbol{v}_0 \in L^2_{div}$ and $\boldsymbol{f} \in L^{p'}(0,T;W^{1,\frac{dp}{dp-d+2}})$. Then the following hold:*

(i) If

$$p > \frac{3d-4}{d} = 3 - \frac{4}{d}\,,$$

then there exist

$$\boldsymbol{v} \in L^p(0,T;W^{1,p}_{div}) \cap L^\infty(0,T;L^2_{div}) \quad \text{and} \quad \pi \in L^{\frac{(d+2)p}{2d}}(0,T;L^{\frac{(d+2)p}{2d}})$$

being together a weak solution to (1.1) with $\boldsymbol{v}(0) = \boldsymbol{v}_0$.

(ii) If moreover p is such that

$$\varrho := \frac{p^2(dp-3d+4)}{p^2d-3dp+12} \geq 1,$$

then $\boldsymbol{v} \in L^\varrho(0,T;W^{1+\sigma,p})$ with $\sigma > 0$ fulfilling the relation (3.9) below.

(iii) Finally, if

$$p \geq \frac{3d+2}{d+2}\,,$$

then $\boldsymbol{v} \in L^\infty(\eta,T;W^{1,2}) \cap L^p(\eta,T;\mathcal{N}^{1+s,p})$ for any $s \in (0,\frac{2}{p})$ and any $\eta \in (0,T)$. One can take $\eta = 0$ if $\boldsymbol{v}_0 \in W^{1,2}_{div}$.

Proof. Assume $\boldsymbol{v}^n$ solve the Galerkin system related to (1.1) as described in [9, chapter 5, page 207]. Note that the functions $\boldsymbol{v}^n$ are based on the eigenfunctions of the Stokes operator, which justifies the use of $-\Delta\boldsymbol{v}^n$ as a test function.

Let us first test by $\boldsymbol{v}^n$. Since the pressure and the convective term cancel out, one obtains

$$\frac{1}{2}\frac{d}{dt}\|\boldsymbol{v}^n\|_2^2 + \nu\|D\boldsymbol{v}^n\|_p^p \leq \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}^n\,,$$

which by Lemma 2.3 and some standard estimates gives

$$\sup_{t\in(0,T)} \|\boldsymbol{v}^n(t)\|_2^2 + \int_0^T \|D\boldsymbol{v}^n\|_p^p + \int_0^T \|\nabla\boldsymbol{v}^n\|_p^p \leq K\,. \tag{3.1}$$

This together with the equation implies

$$\int_0^T \|\partial_t\boldsymbol{v}^n\|^2_{(W^{3,2}_{div})'} \leq K\,. \tag{3.2}$$

See [9, page 207 ff.] for details. Here and in what follows K stands for a generic constant that can depend on $\|\boldsymbol{v}^n(0)\|_2^2$ and a suitable norm of $\boldsymbol{f}$, but which is independent of n.

As a consequence of the estimates (3.1) and (3.2), there exists $\boldsymbol{v}$, belonging to the corresponding spaces, such that $\boldsymbol{v}^n \to \boldsymbol{v}$ weakly in $L^p(0,T;W^{1,p}_{div})$ and by Lemma 2.2 also strongly in $L^p(0,T;L^p_{div})$, for example. This is enough, as $p > 2$, to conclude that

$$\operatorname{div}(\boldsymbol{v}^n \otimes \boldsymbol{v}^n) \to \operatorname{div}(\boldsymbol{v} \otimes \boldsymbol{v})$$

at least in $\mathcal{D}'(\Omega \times (0,T))$. A more difficult problem, however, is whether also

$$\boldsymbol{T}(D\boldsymbol{v}^n) \to \boldsymbol{T}(D\boldsymbol{v}) \quad \text{in} \quad \mathcal{D}'(\Omega \times (0,T)) \,. \tag{3.3}$$

As already mentioned, this can proved using monotone operator theory (Minty's trick) provided that $p \geq \frac{3d+2}{d+2}$, see e.g. [7, 8] for details.

Our first aim is to improve this lower bound. Consequently, we can restrict ourselves to the case $p < \frac{3d+2}{d+2} = 2 + \frac{d-2}{d+2}$. Since $\boldsymbol{T}$ is of the form (1.2) we only consider $p > 2$ and thus we take $d \geq 3$.

Our intermediate goal is to obtain the estimate

$$\int_0^T \|\nabla \boldsymbol{v}^n\|^r_{\sigma,p} \leq K \,, \qquad \text{with some} \qquad r > 1, \sigma > 0. \tag{3.4}$$

Let us first observe that (3.4) together with (3.2) implies (3.3). Indeed, by Lemma 2.2 with $X = W^{1,p}$, $Y = W^{1+\sigma,p}$, $Z = (W^{3,2}_{div})'$, $p = r$ and $q = 2$ one obtains that $\nabla \boldsymbol{v}^n \to \nabla \boldsymbol{v}$ (strongly) in $L^r(0,T;L^p)$. In particular, one can assume $D\boldsymbol{v}^n \to D\boldsymbol{v}$ a.e. in $\Omega \times (0,T)$.

Let now $\phi \in \mathcal{D}(\Omega \times (0,T))$ be arbitrary. One has

$$\int_{\Omega \times (0,T)} |\boldsymbol{T}(D\boldsymbol{v}^n) \cdot \nabla \phi|^{p'} \leq c \int_{\Omega \times (0,T)} |D\boldsymbol{v}^n|^p |\nabla \phi|^{p'} \leq K$$

independently of n. Hence by Lemma 2.4 with $M = \Omega \times (0,T)$, $f^n = \boldsymbol{T}(D\boldsymbol{v}^n) \cdot \nabla \phi$ and $q = p'$ one sees that

$$\int_{\Omega \times (0,T)} \boldsymbol{T}(D\boldsymbol{v}^n) \cdot \nabla \phi \to \int_{\Omega \times (0,T)} \boldsymbol{T}(D\boldsymbol{v}) \cdot \nabla \phi$$

and (3.3) holds.

To obtain (3.4) we incorporate $-\Delta \boldsymbol{v}^n$ as a test function in the relevant Galerkin system and perform the following operation with the particular terms separately. Note that we strongly rely on the fact that all functions are spatially periodic. (From now on, we drop the index n for simplicity.) We have

$$\begin{aligned}\int_\Omega \{\boldsymbol{T}(D\boldsymbol{v})\} : D(-\Delta \boldsymbol{v}) &= \int_\Omega |D\boldsymbol{v}|^{p-2} D\boldsymbol{v} : D(-\Delta\boldsymbol{v}) \\ &= \int_\Omega \nabla\big(|D\boldsymbol{v}|^{p-2}D\boldsymbol{v}\big) : D(\nabla\boldsymbol{v}) \\ &= \int_\Omega |D\boldsymbol{v}|^{p-2}|D(\nabla\boldsymbol{v})|^2 + (p-2)\int_\Omega |D\boldsymbol{v}|^{p-4}(D\boldsymbol{v} : D(\nabla\boldsymbol{v}))^2 \\ &\geq I_p(D\boldsymbol{v}), \quad (p>2),\end{aligned}$$

where

$$I_p(D\boldsymbol{v}) = \int_\Omega |D\boldsymbol{v}|^{p-2}|\nabla(D\boldsymbol{v})|^2 .$$

This term gives us two important estimates. On one hand, by Lemmas 2.1 and 2.3 and the embedding (2.1) one has

$$I_p(D\boldsymbol{v}) \geq c\|D\boldsymbol{v}\|^p_{\mathcal{N}^{\frac{2}{p},p}} \geq c\|D\boldsymbol{v}\|^p_{s,p} \geq c\|\nabla\boldsymbol{v}\|^p_{s,p}$$

where $s \in (0, \frac{2}{p})$ is an arbitrary (but from now on a fixed) number.

On the other hand, as

$$\frac{\partial}{\partial x_k}|D\boldsymbol{v}|^{p/2} = \frac{p}{2}|D\boldsymbol{v}|^{\frac{p}{2}-2} D_{ij}(\frac{\partial \boldsymbol{v}}{\partial x_k})D_{ij}(\boldsymbol{v})\,,$$

we have

$$I_p(D\boldsymbol{v}) \geq c\int_\Omega \big|\nabla|D\boldsymbol{v}|^{\frac{p}{2}}\big|^2 = c\|\nabla|D\boldsymbol{v}|^{\frac{p}{2}}\|_2^2\,.$$

Adding to both sides the term $c\|D\boldsymbol{v}\|_p^p = c\||D\boldsymbol{v}|^{\frac{p}{2}}\|_2^2$ and using then the embedding $W^{1,2} \hookrightarrow L^{\frac{2d}{d-2}}$ (recall that $d \geq 3$) and Lemma 2.3, we obtain

$$c\|D\boldsymbol{v}\|_p^p + I_p(D\boldsymbol{v}) \geq c\,\|\nabla\boldsymbol{v}\|^p_{\frac{dp}{d-2}}\,,$$

which together with (1.5) implies

$$I_p(D\boldsymbol{v}) \geq c\,\|\nabla\boldsymbol{v}\|^p_{\frac{dp}{d-2}} - K\,.$$

From the convective term one has

$$\begin{aligned}-\int_\Omega v_j\frac{\partial v_i}{\partial x_j}\Delta v_i &= \int_\Omega v_j\frac{\partial}{\partial x_j}\big(\nabla\boldsymbol{v}\big)\cdot\nabla\boldsymbol{v} + \int_\Omega \frac{\partial v_j}{\partial x_k}\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_k} \\ &= \frac{1}{2}\int_\Omega v_j\frac{\partial}{\partial x_j}|\nabla\boldsymbol{v}|^2 + \int_\Omega \frac{\partial v_j}{\partial x_k}\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_k}\end{aligned}$$

$$= -\frac{1}{2}\int_\Omega \operatorname{div} \boldsymbol{v}|\nabla \boldsymbol{v}|^2 + \int_\Omega \frac{\partial v_j}{\partial x_k}\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_k} = \int_\Omega \frac{\partial v_j}{\partial x_k}\frac{\partial v_i}{\partial x_j}\frac{\partial v_i}{\partial x_k} \le \|\nabla \boldsymbol{v}\|_3^3 \,.$$

Since $\Delta\boldsymbol{v}$ is divergence-free, the term involving the pressure vanishes. Finally, the right-hand side gives

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot(-\Delta\boldsymbol{v}) = \int_\Omega \nabla\boldsymbol{f}\cdot\nabla\boldsymbol{v} &\le \|\nabla\boldsymbol{f}\|_{\frac{dp}{dp-d+2}}\|\nabla\boldsymbol{v}\|_{\frac{dp}{d-2}}\\ &\le C(\varepsilon)\|\nabla\boldsymbol{f}\|^{p'}_{\frac{dp}{dp-d+2}} + \varepsilon\|\nabla\boldsymbol{v}\|^p_{\frac{dp}{d-2}}\,.\end{aligned}$$

Altogether we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{v}\|_2^2 + c_1\|\nabla\boldsymbol{v}\|^p_{s,p} + (c_1-\varepsilon)\|\nabla\boldsymbol{v}\|^p_{\frac{dp}{d-2}} \le \|\nabla\boldsymbol{v}\|_3^3 + C(\varepsilon)\|\nabla\boldsymbol{f}\|^{p'}_{\frac{dp}{dp-d+2}}\,.$$

The last term is integrable over $(0,T)$ by our assumptions and to simplify the subsequent formulas we put $\boldsymbol{f}\equiv\boldsymbol{0}$. Hence we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{v}\|_2^2 + c_1\|\nabla\boldsymbol{v}\|^p_{s,p} + c_2\|\nabla\boldsymbol{v}\|^p_{\frac{dp}{d-2}} \le \|\nabla\boldsymbol{v}\|_3^3\,. \tag{3.5}$$

Considering $\|\nabla\boldsymbol{v}\|_3^3 = \|\nabla\boldsymbol{v}\|_3^{3\alpha}\|\nabla\boldsymbol{v}\|_3^{3(1-\alpha)}$ and using the interpolation inequalities[2]

$$\|z\|_3 \le \|z\|_2^\beta \;\; \|z\|^{1-\beta}_{\frac{dp}{d-2}} \qquad\text{and}\qquad \|z\|_3 \le \|z\|_p^\gamma \;\; \|z\|^{1-\gamma}_{\frac{dp}{d-2}}\,,$$

and Young's inequality, we arrive at (see [9, pages 234-235])

$$\frac{1}{2}\frac{d}{dt}\|\nabla\boldsymbol{v}\|_2^2 + c_1\|\nabla\boldsymbol{v}\|^p_{s,p} + \frac{c_2}{2}\|\nabla\boldsymbol{v}\|^p_{\frac{dp}{d-2}} \le c_3\|\nabla\boldsymbol{v}\|_2^{2\lambda}\|\nabla\boldsymbol{v}\|_p^p\,, \tag{3.6}$$

where

$$\lambda = \frac{2(3-p)}{dp-3d+4}\,.$$

Note that here the condition $p > \frac{3d-4}{d}$ comes out. Note also that $\lambda \le 1 \iff p \ge \frac{3d+2}{d+2}$. Thus part (iii) of the theorem follows by Gronwall's lemma.

As $p < \frac{3d+2}{d+2}$ implies $\lambda > 1$, a trick to obtain uniform estimates from (3.6) (described again in [9]) is to divide (3.6) by $\|\nabla\boldsymbol{v}\|_2^{2\lambda}$. Rewriting the last inequality as

$$\frac{d}{dt}\big(1+\|\nabla\boldsymbol{v}\|_2^2\big) + c_1\|\nabla\boldsymbol{v}\|^p_{s,p} \le c_3\big(1+\|\nabla\boldsymbol{v}\|_2^2\big)^\lambda\|\nabla\boldsymbol{v}\|_p^p\,,$$

[2]Here, $\beta = \frac{2((d+3)p-2d)}{3((d+2)p-2d)}$ and $\gamma = \frac{(d+3)p-3d}{3p}$.

and dividing the result by $(1+\|\nabla \boldsymbol{v}\|_2^2)^\lambda$ leads to

$$\frac{d}{dt}A(t) + c_1\|\nabla \boldsymbol{v}\|_{s,p}^p (1+\|\nabla \boldsymbol{v}\|_2^2)^{-\lambda} \le c_3 \|\nabla \boldsymbol{v}\|_p^p,$$

where $A(t) = (1-\lambda)^{-1}(1+\|\nabla \boldsymbol{v}(t)\|_2^2)^{1-\lambda}$. Note that $A(t)$ is bounded; in particular, one does not need uniform bounds for $\|\nabla \boldsymbol{v}^n(0)\|_2$.

Integrating the last inequality over $(0,T)$ and using (3.1) gives

$$\int_0^T \|\nabla \boldsymbol{v}\|_{s,p}^p \big(1+\|\nabla \boldsymbol{v}\|_2^2\big)^{-\lambda} < K\,. \tag{3.7}$$

Similarly, as in [9] we derive (3.4) from (3.7). Recall that $\lambda = \frac{2(3-p)}{dp-3d+4}$, $\lambda > 1$ while $s \in (0, \frac{2}{p})$ is an arbitrary, fixed number.

By Hölder's inequality one has for $\beta \in (0,1)$

$$\begin{aligned}\int_0^T \|\nabla \boldsymbol{v}\|_{s,p}^{\beta p} &= \int_0^T \|\nabla \boldsymbol{v}\|_{s,p}^{\beta p}\big(1+\|\nabla \boldsymbol{v}\|_2^2\big)^{-\beta\lambda}\big(1+\|\nabla \boldsymbol{v}\|_2^2\big)^{\beta\lambda} \\ &\le \Big[\int_0^T \|\nabla \boldsymbol{v}\|_{s,p}^{p}\big(1+\|\nabla \boldsymbol{v}\|_2^2\big)^{-\lambda}\Big]^{\beta}\Big[\int_0^T \big(1+\|\nabla \boldsymbol{v}\|_2^2\big)^{\frac{\beta\lambda}{1-\beta}}\Big]^{1-\beta}.\end{aligned}$$

The first integral is bounded by (3.7). The second is bounded by (3.1) for the largest possible value β given through the relation

$$\frac{2\beta\lambda}{1-\beta} = p \qquad \Longleftrightarrow \qquad \beta = \frac{p(dp-3d+4)}{4(3-p)+p(dp-3d+4)}\,.$$

Hence

$$\int_0^T \|\nabla \boldsymbol{v}\|_{s,p}^{\beta p} < K\,. \tag{3.8}$$

Note that $p\beta = \varrho$ as stated in part (ii) of the theorem, which thus follows. Also, for $p\beta > 1$ (3.8) is just (3.4), and the proof is completed.

It remains to treat the case $p\beta \le 1$. Fix $r \in (1,p)$ arbitrary. Due to the interpolation

$$\|u\|_{\sigma,p} \le c\|u\|_p^{1-\frac{\sigma}{s}}\|u\|_{s,p}^{\frac{\sigma}{s}}$$

and Hölder's inequality we have

$$\begin{aligned}\int_0^T \|\nabla \boldsymbol{v}\|_{\sigma,p}^r &\le c\int_0^T \|\nabla \boldsymbol{v}\|_p^{r(1-\frac{\sigma}{s})}\|\nabla \boldsymbol{v}\|_{s,p}^{r\frac{\sigma}{s}} \\ &\le \Big[\int_0^T \|\nabla \boldsymbol{v}\|_p^{\delta r(1-\frac{\sigma}{s})}\Big]^{\frac{1}{\delta}}\Big[\int_0^T \|\nabla \boldsymbol{v}\|_p^{\delta' r\frac{\sigma}{s}}\Big]^{\frac{1}{\delta'}}\,.\end{aligned}$$

Thus, by virtue of (3.1) and (3.8) we obtain (3.4) provided that δ, $\delta' > 1$ are such that

$$\delta r\big(1 - \frac{\sigma}{s}\big) = p \qquad \delta' r\frac{\sigma}{s} = \beta p\,.$$

This means

$$1 = \frac{1}{\delta} + \frac{1}{\delta'} = \frac{\sigma}{s}\cdot\frac{r}{\beta p} + \big(1 - \frac{\sigma}{s}\big)\frac{r}{p}\,. \tag{3.9}$$

Since $p > r > 1 \geq \beta p$, there exists a uniquely determined $\sigma \in (0, s)$ such that this equality holds.

The proof of Theorem 3.1 is finished. □

4. Concluding remarks

Let us first summarize what is the novelty of the present paper concerning the spatially-periodic problem.

- The existence theory covers degenerate viscosities for a significantly larger range of the parameter p. In particular, in three spatial dimensions, all $p > 2$ are included. Earlier results based on monotone operator theory were established only for $p > \frac{11}{5}$.
- Part (ii) of Theorem 3.1 is new both for the degenerate and the non-degenerate case. The use of Nikol′skiĭ spaces is essential in both of these improvements.

Next, we comment on possible extensions of our result to the homogeneous Dirichlet problem. In [11], the authors treat the non-degenerate case and prove global-in-time existence of weak solutions for $p \geq 2$, and regularity results similar to part (iii) of Theorem 3.1 for $p \geq \frac{9}{4}$. In particular, for $\boldsymbol{T}(D\boldsymbol{v}) := (1 + |D\boldsymbol{v}|^2)^{\frac{p-2}{2}} D\boldsymbol{v}$ the interior estimate

$$\int_0^T (1 + \|\nabla \boldsymbol{v}\|_2^2)^{-\lambda} \int_{\Omega_0} (1 + |D\boldsymbol{v}|^2)^{\frac{p-2}{2}} |D(\nabla \boldsymbol{v})|^2 \, dx\, dt < \infty \tag{4.1}$$

for any $\Omega_0 \subset\subset \Omega$ is established.

Assume that, using similar investigations as in [11], it is possible to show that for every $\epsilon > 0$ and for $\boldsymbol{T}_\epsilon(D\boldsymbol{v}) := (\epsilon + |D\boldsymbol{v}|^2)^{\frac{p-2}{2}} D\boldsymbol{v}$, the corresponding weak solutions $\boldsymbol{v}^\epsilon$ fulfill

$$\int_0^T (1 + \|\nabla \boldsymbol{v}^\epsilon\|_2^2)^{-\lambda} \int_{\Omega_0} (\epsilon + |D\boldsymbol{v}^\epsilon|^2)^{\frac{p-2}{2}} |D(\nabla \boldsymbol{v}^\epsilon)|^2 \, dx\, dt \leq C < \infty\,, \tag{4.2}$$

with C independent of $\epsilon > 0$. Then, neglecting ϵ in (4.2), we can proceed as in the spatially periodic case, this time, however, only locally for any $\Omega_0 \subset\subset \Omega$.

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