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RENORMALIZED VARIATIONAL PRINCIPLES AND HARDY-TYPE INEQUALITIES

Satyanad Kichenassamy
Laboratoire de Mathématiques (UMR 6056)
CNRS & Université de Reims Champagne-Ardenne
Moulin de la Housse, B.P. 1039, F-51687 Reims Cedex 2, France

(Submitted by: Haim Brezis)

Abstract. Let $\Omega \subset \mathbb{R}^2$ be a bounded domain on which Hardy's inequality holds. We prove that $[\exp(u^2)-1]/\delta^2 \in L^1(\Omega)$ if $u \in H_0^1(\Omega)$, where δ denotes the distance to $\partial\Omega$. The corresponding higher-dimensional result is also given. These results contain both Hardy's and Trudinger's inequalities, and yield a new variational characterization of the maximal solution of the Liouville equation on smooth domains, in terms of a renormalized functional. A global H^1 bound on the difference between the maximal solution and the first term of its asymptotic expansion follows.

1. Introduction

1.1. **Hardy's and Trudinger's inequalities.** Let Ω be an arbitrary domain in $\mathbb{R}^N$, $N \geq 2$; let $\delta(x)$ denote the distance of x from $\partial\Omega$.

If Ω is bounded with Lipschitz boundary, and $u \in H_0^1(\Omega)$, the generalized Hardy's inequality states that

$$\|\frac{u}{\delta}\|_{L^2(\Omega)} \leq H\|\nabla u\|_{L^2(\Omega)}.$$

The optimal value of the "Hardy constant" H, as well as possible generalizations and improvements of this inequality, have been the object of much attention, see [13, 15, 17]. Hardy's inequality arises naturally in several variational problems of recent interest, as well as in the proof of decay estimates [3, 7].

On the other hand, if $N = 2$, Trudinger's inequality implies that $\exp(u^2)-1$ is integrable. This suggests our first result:

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Theorem 1. *Let $\Omega \subset \mathbb{R}^2$ be such that Hardy's inequality holds. Then, for any $u \in H_0^1(\Omega)$,*

$$[\exp(u^2) - 1]/\delta^2 \in L^1(\Omega).$$

This theorem will be derived from a result of independent interest:[1]

Theorem 2. *Let $\Omega \subset \mathbb{R}^2$ be an arbitrary domain with $\partial\Omega \neq \emptyset$. Then, if $u \in H_0^1(\Omega)$, and $q > 2$,*

$$\Big(\int_\Omega \frac{u^q}{\delta^2} \Big)^{1/q} \leq \Sigma_q \Big(\int_\Omega |\nabla u|^2 + \frac{u^2}{\delta^2} \Big)^{1/2},$$

where $\Sigma_q = O(q^{1/2+1/q})$ as $q \to \infty$.

Remark 1. These results admit natural generalizations to higher dimensions, which are stated and proved in Section 2. Theorem 2 may be considered as trivially true if $\partial\Omega = \emptyset$ with the convention that $\delta \equiv +\infty$ in this case. For background results on Trudinger's inequality, see [20, 16, 1, 4]; we merely note that our argument is closer to Trudinger's than to Moser's, because the distance function does not transform in a convenient manner under symmetrization.

Remark 2. No regularity or boundedness assumptions on Ω are required. This is somewhat surprising in view of the fact [1, page 120] that elements of $H^1(\Omega)$ are not necessarily L^q for $q > 2$, if Ω is unbounded and with finite volume. Note that for domains with thin "ends" at infinity, δ is very small, and the right-hand side of our inequality is not equivalent to the H^1 norm.

In the situation of Theorem 1, we find in particular that

$$\frac{e^{2u} - 1 - 2u}{\delta^2} \in L^1(\Omega). \tag{1.1}$$

As an application of this result, we present next a variational characterization of the maximal solution of an elliptic problem with monotone nonlinearity. The question of constructing a renormalized energy in this context was raised by H. Brezis last year in the Nonlinear Analysis Seminar of Paris VI.

1.2. **A renormalized energy for boundary blow-up.** Let $\Omega \subset \mathbb{R}^2$ be a bounded domain of class $C^{2+\alpha}$, with $0 < \alpha < 1$. The distance function $\delta(x)$ is of class $C^{2+\alpha}$ near and up to the boundary, but is only Lipschitz over Ω in

[1] The Lebesgue measure dx is understood in all integrals in this paper.

general. It is therefore convenient to introduce a function $d(x) \in C^{2+\alpha}(\overline{\Omega})$, which coincides with $\delta(x)$ near $\partial\Omega$, and is positive inside Ω.[2]

Consider the maximal solution u_Ω of the Liouville equation

$$-\Delta u + 4e^{2u} = 0 \tag{1.2}$$

in Ω. It is known that u_Ω is the supremum of all solutions of the Dirichlet problem with smooth boundary data, and that it is equivalent to $-\ln(2d)$ near the boundary.[3]

Even though the equation is formally the Euler-Lagrange equation derived from the Lagrangian $L[u] := |\nabla u|^2 + 4e^{2u}$, a direct variational approach is impossible, because $L[u_\Omega] \notin L^1(\Omega)$. Nevertheless, Fuchsian Reduction [8, 9, 11] enables one to decompose u_Ω into an explicit singular part and a C^1 function:

$$u_\Omega = v + w,$$

where the following properties hold

(P1) $w \in C^{1+\alpha}(\overline{\Omega}) \cap C^2(\Omega)$;
(P2) $w = O(d)$ as $d \to 0$;
(P3) $e^v = O(1/d)$ as $d \to 0$;
(P4) $r[v] := -\Delta v + 4e^{2v} = O(1/d)$ as $d \to 0$.

One may, for instance, take $v = -\ln(2d)$ [9].

For $\phi \in H_0^1$, let us define

$$R[\phi, v] := \int_\Omega |\nabla\phi|^2 + 4e^{2v}[e^{2\phi} - 1 - 2\phi] + 2r[v]\phi, \tag{1.3}$$

which is well-defined thanks to equation (1.1), properties (P3-P4), and Hardy's inequality. We then have a variational characterization of u_Ω:

Theorem 3. *The infimum*

$$\text{Inf}\{R[\psi - v, v] : \psi \in v + H_0^1(\Omega)\}$$

is attained precisely for $\psi = u_\Omega$.

Since v is given, this provides a characterization of u_Ω.

Remark 3. The result may be stated equivalently as follows: if $\phi \in H_0^1$,

$$R[\phi, v] \geq R[w, v], \tag{1.4}$$

[2] One may simply take $d = F(\delta)$ for an appropriate F.

[3] For background results and applications, see *e.g.* [2, 6, 12, 9, 14, 18]. We merely recall that if Ω is simply connected, e^{-u} is the "mapping radius," or "conformal radius" function of Ω.

with equality if and only if $\phi = w$.

As a consequence of the variational characterization, we derive a new global *a priori* bound on u_Ω:

Corollary 4. *The maximal solution* u_Ω *of Liouville's equation satisfies*

$$\|u_\Omega + \ln(2d)\|_{H^1_0} \le 2H\|\Delta d\|_{L^2(\Omega)}.$$

1.3. **Organization of the paper.** Section 2 states and proves two results (Theorems 5 and 6), of which Theorems 1 and 2 are special cases. Theorem 3 and Corollary 4 are both proved in Section 3. The proofs of Section 2 require the construction of a partition of unity with special properties, carried out in Section 4. Comments on the rationale leading to the renormalized functional are given as concluding remarks.

2. Hardy-Trudinger inequalities

In this section, $\Omega \subset \mathbb{R}^N$ is an arbitrary domain with $\partial\Omega \neq \emptyset$. The N-dimensional analogues of Theorems 1 and 2 are stated in Section 2.1, and proved in Sections 2.2 and 2.3 respectively.

2.1. **A synthesis of Hardy's and Trudinger's inequalities.** Let $N' = N/(N-1)$ and define

$$\Phi_N(u) := \sum_{k \ge N-1} \frac{|u|^{kN'}}{k!},$$

and, for $1 \le p < \infty$,

$$M_p(u) := \Big(\int_\Omega |\nabla u|^p + \frac{|u|^p}{\delta^p} \Big)^{1/p}.$$

We prove:

Theorem 5. *If* $\partial\Omega \neq \emptyset$, *there are constants* c_1 *and* c_2, *which only depend on the dimension* N, *such that, for any* $u \in W^{1,N}_0(\Omega)$ *with* $M_N(u) = 1$,

$$\int_\Omega \frac{\Phi_N(u/c_1)}{\delta^N} \le c_2.$$

Remark 4. Note that no smoothness or boundedness assumptions on Ω are required, and that $M_N(u)$ is not necessarily equivalent to the $W^{1,N}_0$ norm. This result implies Theorem 2.

Remark 5. If Ω is bounded and Lipschitz, Hardy's inequality holds, and we claim that $\Phi_N(u)$ is integrable for any $u \in W_0^{1,N}(\Omega)$: write $u = f + g$ where f is smooth with compact support; since $(|f| + |g|)^{kN'} \le 2^{kN'}(|f|^{kN'} + |g|^{kN'})$, we have $\Phi_N(f + g) \le \Phi_N(2f) + \Phi_N(2g)$. The result follows if g is small in $W_0^{1,N}$. For $N = 2$ and $u \in H_0^1(\Omega)$, we recover Theorem 1.

2.2. **An auxiliary result.** Let $1 \le p \le N$, $p^* = Np/(N - p)$ if $p < N$ (resp. $p^* = +\infty$ if $p = N$). We prove:

Theorem 6. *If $N \ge 2$, Ω is a domain in $\mathbb{R}^N$, $1 \le q < \infty$ and $1 \le p < q \le p^*$, there is a constant $\Sigma_q(N, p)$ such that*

$$\Big(\int_\Omega \frac{|u|^q}{\delta^N} \Big)^{1/q} \le \Sigma_q \Big(\int_\Omega \frac{|\nabla u|^p}{\delta^{N-p}} + \frac{|u|^p}{\delta^N} \Big)^{1/p}$$

for any $u \in W^{1,p}(\Omega)$.

Remark 6. *In general, the right-hand side may be infinite. If $p = N$ and Hardy's inequality holds in Ω, the right-hand side is finite for $u \in W_0^{1,p}(\Omega)$.*

2.3. **Proof of Theorem 6. Step 1: Partition of unity.** Denote by $Q(x, s)$ the cube of center x and side s. We prove in Section 4 that there is a smooth partition of unity $(\phi_k)_{k \ge 0}$ in Ω with the following properties:

- (PU1) For every k, ϕ_k is supported in a cube $Q_k = Q(x_k, s_k) \subset \Omega$.
- (PU2) For every k, $0 \le \phi_k \le 1$ and $|\nabla \phi_k| \le c_3/s_k$, where c_3 only depends on N.
- (PU3) There are two positive constants λ and μ such that, on $\operatorname{supp} \phi_k$, $\lambda \le \delta/s_k \le \mu$.
- (PU4) There is a number P which only depends on the dimension N, such that, for every $x \in \Omega$, at most P among the numbers $\phi_k(x)$ are non-zero.

Simple consequences of these properties are:

(1) For every $q \ge 1$, $\sum_k \phi_k^q \le 1$.
(2) $\sum_k \chi_{Q_k} \le P(N)$, where χ_{Q_k} denotes the characteristic function of Q_k.
(3) $(\sum_k \phi_k)^q \le P^q \sum_k \phi_k^q$. Indeed, for any x, $\phi_k(x) \ne 0$ for at most P values of k, so that $(\sum_k \phi_k(x))^q \le P^q \max_k \phi_k(x)^q$.

We also need an elementary observation: for any collection of nonnegative numbers b_k, and any $r \ge 1$,

$$\sum_k b_k^r \le \Big(\sum_k b_k \Big)^r. \tag{2.1}$$

This may be seen for finite sequences by induction, starting from the inequality: $x^r + y^r \le (x+y)^r$. Recall also that $(x+y)^r \le 2^r(x^r+y^r)$.

Step 2: Decomposition of u. For any $u \in W^{1,p}(\Omega)$, we have

$$\begin{aligned}\int_\Omega \frac{|u|^q}{\delta^N} &= \int_\Omega |\sum_k u\phi_k|^q \delta^{-N} \\ &\le P^q \sum_k \int_{Q_k} |u\phi_k|^q \delta^{-N} \le P^q \sum_k (\lambda s_k)^{-N} \|u\phi_k\|^q_{L^q(Q_k)}.\end{aligned}$$

Write $Q(s)$ for $Q(0,s)$, and let $S_q = S_q(N,p)$ denote the norm of the embedding of $W_0^{1,p}(Q(1))$ into $L^q(Q(1))$. If $v \in W_0^{1,p}(Q(s))$, one finds, applying the Sobolev inequality to $v(sx) \in W_0^{1,p}(Q(1))$,

$$\|v\|_{L^q(Q(s))} \le S_q s^{\frac{N}{q}+1-\frac{N}{p}} \|\nabla v\|_{L^p(Q(s))}. \tag{2.2}$$

It follows that, for every k,

$$(\lambda s_k)^{-N} \|u\phi_k\|^q_{L^q(Q_k)} \le S_q^q \lambda^{-N} s_k^{q(p-N)/p} \|\nabla(u\phi_k)\|^q_{L^p(Q_k)}.$$

It follows that

$$\begin{aligned}\Big(\int_\Omega \frac{|u|^q}{\delta^N}\Big)^{p/q} &\le (PS_q)^p \Big(\sum_k \lambda^{-N} s_k^{q(p-N)/p} \|\nabla(u\phi_k)\|^q_{L^p(Q_k)}\Big)^{p/q} \\ &\le (PS_q\lambda^{-N/q})^p \sum_k s_k^{p-N} \|\nabla(u\phi_k)\|^p_{L^p(Q_k)} \\ &\le (2PS_q\lambda^{-N/q})^p \sum_k s_k^{p-N} \left[\|\phi_k \nabla u\|^p_{L^p(Q_k)} + \|u\nabla\phi_k\|^p_{L^p(Q_k)}\right],\end{aligned}$$

where we used equation (2.1) to obtain the second inequality. Now,

$$\int_\Omega \sum_k s_k^{p-N} \phi_k^p |\nabla u|^p \le \mu^{N-p} \int_\Omega \frac{|\nabla u|^p}{\delta^{N-p}},$$

and

$$\sum_k s_k^{p-N} \|u\nabla\phi_k\|^p_{L^p(Q_k)} \le c_3^p \int_\Omega (\sum_k \chi_{Q_k}(x)) \frac{|u|^p}{s_k^N} \le P c_3^p \mu^N \int_\Omega \frac{|u|^p}{\delta^N}.$$

We have therefore the desired inequality, with

$$\Sigma_q = 2PS_q(N,p)\lambda^{-N/q}[\mu^{N-p} + Pc_3^p\mu^N]^{1/p}. \tag{2.3}$$

This completes the proof.

2.4. **Proof of Theorem 5.** We now consider the case $p = N$, so that $p^* = +\infty$, and q can take arbitrarily large values. From [5, Lemma 7.12 and equation (7.37)], it follows that

$$S_q(N,N) \le (\omega_N q)^{1-1/N+1/q} \text{ if } q \ge N.$$

If $q \ge N-1$, we have $N'q \ge N$, and therefore,

$$S_{N'q}(N,N)^{N'q} \le (N'q\omega_N)^{q+1} \text{ if } q \ge N-1.$$

For any $c_1 > 0$, we therefore find

$$\int_\Omega \sum_{q\ge N-1} \frac{|u|^{N'q}}{q! c_1^{N'q}\delta^N} \le c_2 := \sum_{q\ge N-1} \lambda^{-N} N'\omega_N \Big(\frac{N'\omega_N A}{c_1^{N'}}\Big)^q \frac{q^q}{(q-1)!},$$

where $A = \{2P[1 + P(c_3\mu)^N]^{1/N}\}^{N'}$. The series defining c_2 converges if $c_1^{N'} > e\omega_N N'A$.

This completes the proof.

3. VARIATIONAL CHARACTERIZATION OF SOLUTIONS WITH BOUNDARY BLOW-UP

3.1. **Proof of Theorem 3.** Let $\phi \in H_0^1(\Omega)$. We wish to prove inequality (1.4). First observe that since $u_\Omega = v + w$ solves Liouville's equation,

$$\Delta w = -\Delta v + 4e^{2v} + 4e^{2v}(e^{2w} - 1) = r[v] + 4e^{2v}(e^{2w} - 1). \tag{3.1}$$

It follows from (P2-P4) that

$$\Delta w = O(1/d).$$

Since $w \in C_0^1(\overline{\Omega}) \cap C^2(\Omega)$, we have, for any $\psi \in C_0^1(\overline{\Omega})$ and any $\varepsilon > 0$ small enough,

$$\int_{d>\varepsilon} \nabla\psi \cdot \nabla w = -\int_{d=\varepsilon} \psi(\nabla w \cdot \nabla d)\, ds - \int_{d>\varepsilon} \psi\Delta w.$$

Letting first $\varepsilon \to 0$, and then approximating ϕ by ψ in the H_0^1 norm, we find

$$\int_\Omega \nabla\phi \cdot \nabla w + \int_\Omega \phi\Delta w = 0. \tag{3.2}$$

Using equation (3.1), we find

$$\int_\Omega \nabla\phi \cdot \nabla w + 4e^{2v}(e^{2w} - 1)\phi + r[v]\phi = 0. \tag{3.3}$$

Since

$$R[\phi+w,v]-R[w,v]=\int_\Omega |\nabla\phi|^2+2\nabla\phi\cdot\nabla w+4e^{2v}[e^{2w+2\phi}-e^{2w}-2\phi]+2r[v]\phi,$$

we find

$$R[\phi+w,v]-R[w,v]=\int_\Omega |\nabla\phi|^2+4e^{2(v+w)}(e^{2\phi}-1-2\phi), \tag{3.4}$$

which is manifestly nonnegative, and vanishes precisely if $\phi=0$. Q.E.D.

Remark 7. Since $v+w=u_\Omega$ and $r[u_\Omega]=0$, the right-hand side of equation (3.4) is equal to $R[\phi,u_\Omega]$.

3.2. **Proof of Corollary 4.** Property (P4) and Hardy's inequality ensure that there is a constant $K[v]$ such that

$$-\int_\Omega 2r[v]\phi \le K[v]\|\phi\|_{H_0^1}.$$

Expressing that $R[w,v]\le R[0,v]=0$, we find

$$\|w\|_{H_0^1}\le K[v].$$

If $v=-\ln(2d)$, one finds $r[-\ln(2d)]=(\Delta d)/d$. Hölder's and Hardy's inequalities yield $K[v]\le 2H\|\Delta d\|_{L^2}$. Since $w=u_\Omega-v$, the announced *a priori* H^1 bound on u_Ω follows.

4. Construction of the partition of unity

We construct the partition of unity used in the proof of Theorem 6; the basic ideas go back to Whitney [21], and many variants may be found in the literature, see e.g. [19, 15].

First of all, choose two constants η and η' such that

$$\eta/\sqrt{N}>\eta'>1.$$

Recall that $Q(x,s)$ is the closed cube of center x and side s. For any $\sigma>0$, we write $Q_\sigma(x,s)$ for $Q(x,\sigma s)$. The following observation will be useful: for any $x\in\Omega$,

$$\delta(x)>\frac{s}{2}\sqrt{N}\Rightarrow Q(x,s)\subset\Omega\Rightarrow\delta(x)>\frac{s}{2}.$$

Conversely,

$$\delta(x)\le\frac{s}{2}\Rightarrow Q(x,s)\not\subset\Omega\Rightarrow\delta(x)\le\frac{s}{2}\sqrt{N}.$$

4.1. **Covering by dyadic cubes.** For $k \in \mathbb{Z}$, let $\mathcal{F}_k$ denote the family of closed cubes of the form

$$[0, 2^{-k}]^N + (m_1, \dots, m_N)2^{-k},$$

where the m_j are signed integers. The cubes in $\mathcal{F}_{k+1}$ are obtained by dyadic subdivision of the cubes in $\mathcal{F}_k$; in particular, for any k, every $Q(x,s) \in \mathcal{F}_{k+1}$ is included in a unique cube $\tilde{Q}(\tilde{x}, \tilde{s}) \in \mathcal{F}_k$, with $\tilde{s} = 2s$. Since Q is obtained by dyadic division of $\tilde{Q}$, $\tilde{x}$ must be a vertex of Q; it follows that $|x - \tilde{x}| = \frac{1}{2}s\sqrt{N}$.

Let $\mathcal{F} = \bigcup_{k=-\infty}^{+\infty} \mathcal{F}_k$. Define a set $\mathcal{Q} \subset \mathcal{F}$ as follows: $Q \in \mathcal{Q}$ if and only if

$$Q_\eta \subset \Omega \text{ and } \tilde{Q}_\eta \not\subset \Omega. \tag{4.1}$$

$\mathcal{F}$ is not empty since $\partial\Omega \neq \emptyset$ by assumption.

Lemma 7. *Ω is the union of the cubes $Q \in \mathcal{Q}$.*

Proof. Let $y \in \Omega$. Consider the set of numbers k for which there is a cube $Q(x, 2^{-k}) \in \mathcal{F}_k$ which contains y and which satisfies $Q_\eta \subset \Omega$. This set is not empty: if k is large enough, we have $\delta(x) \geq \delta(y) - \frac{1}{2}2^{-k}\sqrt{N} > \frac{1}{2}2^{-k}\eta\sqrt{N}$, and $Q_\eta \subset \Omega$. It is bounded below because $\partial\Omega \neq \emptyset$. Let k_0 be the smallest integer in that set, and consider a cube Q with the above property with $k = k_0$. Since k_0 is minimal, $\tilde{Q}_\eta \not\subset \Omega$. Therefore, $y \in Q \in \mathcal{Q}$, as desired. □

Since, for each $Q \in \mathcal{Q}$, $Q \subset Q_{\eta'} \subset \Omega$, we have *a fortiori*

$$\Omega = \bigcup_{Q \in \mathcal{Q}} Q_{\eta'}.$$

4.2. **Properties of the cube decomposition.** We now prove that the covering of Ω by the cubes $Q_{\eta'}$ has the additional property that, on each of them, the function δ is comparable to the side of Q:

Lemma 8. *There are positive constants c_4 and c_5, independent of Ω, such that, if $Q(x,s) \in \mathcal{Q}$ and $y \in Q_{\eta'}$, then*

$$c_4 \leq \frac{\delta(y)}{s} \leq c_5.$$

Proof. Since $Q_\eta \subset \Omega$, $\delta(x) > \eta s/2$. If $y \in Q_{\eta'}$, $\delta(y) \geq \delta(x) - \frac{1}{2}\eta' s\sqrt{N} > \frac{1}{2}(\eta - \eta'\sqrt{N})s$. Therefore,

$$\frac{1}{2}(\eta - \sqrt{N}\eta') < \frac{\delta(y)}{s}.$$

To establish an upper bound, we first estimate $\delta(x)/s$. Since $Q(x,\eta s)\subset\Omega$ and $\tilde{Q}(\tilde{x},2\eta s)\not\subset\Omega$, $\delta(x)>\frac{1}{2}\eta s$ and $\delta(\tilde{x})\le\frac{1}{2}(2\eta s)\sqrt{N}$. Therefore,

$$\delta(x)\le\delta(\tilde{x})+|x-\tilde{x}|\le(\eta+\frac{1}{2})s\sqrt{N}.$$

Therefore, if $Q(x,s)\in\mathcal{Q}$,

$$\frac{1}{2}\eta<\frac{\delta(x)}{s}\le(\eta+\frac{1}{2})\sqrt{N}. \tag{4.2}$$

We now estimate $\delta(y)$:

$$\delta(y)\le\delta(x)+\frac{1}{2}\eta' s\sqrt{N}\le(\eta+\frac{1}{2}+\frac{1}{2}\eta')s\sqrt{N}.$$

It follows that $\delta(y)/s$ lies between positive bounds which do not depend on Ω, as desired. □

4.3. **Finite intersection property.** The cubes $Q_{\eta'}$ are not disjoint; nevertheless, two such cubes may intersect only if the ratio of their sides lies between fixed bounds; this implies that a given cube may only intersect finitely many others. More precisely,

Lemma 9. *There is a constant c_6, independent of Ω, such that, if $Q(x,s)$ and $Q'(x',s')$ belong to $\mathcal{Q}$, and if $Q(x,\eta' s)\cap Q'(x',\eta' s')\neq\emptyset$, with $s<s'$, necessarily*

$$s<s'<c_6 s.$$

Furthermore, there is a number P, independent of Ω, such that given $Q\in\mathcal{Q}$, there are at most P cubes $Q'\in\mathcal{Q}$ such that $Q_{\eta'}\cap Q'_{\eta'}\neq\emptyset$.

Proof. Let $y\in Q_{\eta'}\cap Q'_{\eta'}$. We have $|x-x'|\le|x-y|+|y-x'|\le\frac{1}{2}(s+s')\eta'\sqrt{N}$. Since Q and Q' both belong to $\mathcal{Q}$, inequality (4.2) yields $\delta(x')>\frac{1}{2}\eta s'$ and $\delta(x)\le(\frac{1}{2}+\eta)s\sqrt{N}$. Since δ is 1-Lipschitz,

$$\frac{1}{2}\eta s'<\delta(x')\le\delta(x)+|x-x'|\le[(\frac{1}{2}+\eta+\frac{1}{2}\eta')s+\frac{1}{2}\eta' s']\sqrt{N}.$$

It follows that $s'<c_6 s$ with

$$c_6=\frac{2\eta+1+\eta'}{\eta-\eta'\sqrt{N}}\sqrt{N}.$$

Therefore, if $s=2^{-k}$ and $s'=2^{-k'}$, we have $|k-k'|\le J_1:=\ln c_6/\ln 2$. On the other hand, $|x-x'|\le\frac{1}{2}(s+s')\eta'\sqrt{N}\le J_2 s$, with $J_2=\frac{1}{2}(1+c_6)\eta'\sqrt{N}$. Scaling the cubes by the factor $1/s$, and taking x as the origin of coordinates, we are led to the question: "how many cubes $Q(x',2^j)$ can one find subject to the restrictions: $|j| \leq J_1$ and $|x'| \leq J_2$?" The answer is a finite number P, which depends only on N, η and η'. □

4.4. **Constructing the partition of unity.** Take a smooth function $\varphi(x)$ with support in $Q(0, \eta')$, equal to 1 on $Q(0, 1)$, and such that $0 \leq \varphi(x) \leq 1$ for all x. Consider

$$\psi(x) = \sum_{Q=Q(x_Q, s_Q) \in \mathcal{Q}} \varphi(\frac{x - x_Q}{s_Q}).$$

Since the cubes $Q \in \mathcal{Q}$ cover Ω, $\psi(x) \geq 1$ for all $x \in \Omega$. Since any point belongs to at most P of the cubes $Q_{\eta'}$, we have $\psi(x) \leq P$ for all $x \in \Omega$. It follows that the functions $\phi_Q(x) := \varphi((x - x_Q)/s_Q)/\psi(x)$ form a partition of unity, and are supported in the cubes $Q_{\eta'}$.

We now prove that properties (PU1–4) of Section 2.3 hold. (PU1) and (PU2) are immediate. If $x \in Q_{\eta'}$, Lemma 8 ensures that $\delta(x)/s_Q$ is bounded above and below by positive bounds which only depend on N. This proves property (PU3). Finally, since $\varphi((x - x_Q)/s_Q)$ is supported in Q'_η, we find, thanks to Lemma 9, that (PU4) holds.

This completes the construction of the partition of unity.

5. Concluding remarks

The boundary blow-up problem appears at first sight to be beyond the reach of critical-point theory. We have shown that it is in fact equivalent to the minimization of a convex functional in H_0^1.

How is this functional related to the usual functional $E[u] := \int_\Omega L[u]$? Take $v = -\ln(2d)$ to fix ideas. Even though $E[u]$ is infinite for $u = u_\Omega$, it is easy to see that, if ϕ is smooth and sufficiently flat near the boundary, $\tilde{E}[\phi] := \int_\Omega L[\phi + v] - L[v]$ is well-defined. But this does not provide a satisfactory variational principle for two reasons: (i) $u_\Omega - v$ is not very flat at the boundary: it is only $O(d)$; (ii) $\tilde{E}[\phi]$ is not well-defined if ϕ is merely $O(d)$ (indeed, the term $2\nabla\phi \cdot \nabla v$ is not necessarily integrable). However, subtracting $\int_\Omega 2 \operatorname{div}(\phi \nabla v)$ from $\tilde{E}[\phi]$, and using the equation satisfied by v, one recovers an expression equivalent to our functional R. Note that the integral of this divergence term is not zero, even if ϕ is smooth, because ∇v blows up at the boundary.

Similar ideas are applicable in other situations, as soon as one can decompose the singular solution into the sum $v + w$ of an explicit singular function, and a remainder of controlled regularity. Fuchsian reduction provides a systematic procedure for achieving this decomposition.

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