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PERTURBING EVOLUTIONARY SYSTEMS ON DUAL SPACES BY CUMULATIVE OUTPUTS

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Abstract. Evolutionary systems on dual Banach spaces $X = Z^*$ are perturbed by cumulative outputs. The perturbation procedure involves solving Stieltjes operator integral equations. While either of the two operator families may or may not be formed by dual operators, the case that the output family does not consist of dual operators receives particular attention. Topologies are identified in which both the unperturbed and the perturbed evolutionary system are continuous functions of the time variables. The perturbation of the associated evolution semigroups and integrated semigroups is an important intermediate step. The perturbation results are applied to evolutionary systems acting on spaces of Borel measures on certain locally compact Hausdorff spaces and to the associated transition functions.

1. Introduction

Evolutionary systems are a mathematical way of describing time-heterogeneous horizontal processes in structured space, whether they are due to physical movement or growth in age, size or another characteristic. Vertical processes like deaths or births in a population can be formulated as cumulative outputs. Horizontal and vertical processes can be combined through a perturbation procedure involving Stieltjes operator integral equations which leads to new evolutionary systems.

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1.1. **Outline of main results.** A dual (forward) evolutionary system is a two-parameter family $U_0(t,r)$, $0 \le r \le t < \tau < \infty$, defined on the dual $X = Z^*$ of a Banach space Z that is associated with a strongly continuous backward evolutionary system $U_\diamond(r,t)$ on Z. More precisely, each operator $U_0(t,r)$ is the dual of the bounded linear operator $U_\diamond(r,t)$,

$$U_0(t,r) = (U_\diamond(r,t))^* =: U_\diamond^*(t,r), \qquad 0 \le r \le t < \tau < \infty.$$

We assume that

$$\sup_{0 \le s \le t < \tau} \|U_0(t,s)\| < \infty. \tag{1.1}$$

A two-parameter family $V_0(t,r) : X \to X$, $0 \le r \le t \le \tau$, is called a *cumulative output* for U_0, if

$$V_0(t,s)U_0(s,r) = V_0(t,r) - V_0(s,r), \qquad 0 \le r \le s \le t < \tau. \tag{1.2}$$

A two-parameter family V_0 is called a (forward) *Volterra-Stieltjes kernel*, if the following properties hold:

(H.1.1) (a) $V_0(s,s) = 0$ for all $s \in [0,\tau)$.

(b) Whenever $0 \le r \le t < \tau$, then $V_0(t+s, r+s)$ is operator norm continuous in $s \in [0, \tau - t)$.

(c) For every $z \in Z$, $V_0^*(s, s+h)z \to 0$ as $h \to 0$, locally uniformly in $s \in [0,\tau)$,
i.e., for every $z \in Z$, $\epsilon > 0$ and $\tau' \in (0,\tau)$, there exists some $\delta > 0$ such that $\tau' + \delta < \tau$ and

$$\|V_0^*(s, s+h)z\| \le \epsilon \text{ whenever } 0 \le s \le \tau', 0 \le h \le \delta.$$

(d) $V_0(\cdot, s)x$ is of bounded variation on $[s,\tau)$ for all $x \in X$, $s \in [0,\tau)$ and the variations satisfy

$$\sup_{0 \le s < \tau} \mathbf{v}^\bullet\big(V_0(\cdot,s)x; [s,\tau)\big) < \infty.$$

Recall that a function $f : [a,b) \to X$ is of bounded variation if

$$\mathbf{v}^\bullet(f; [a,b)) := \sup\left\{\sum_{j=1}^n \|f(t_j) - f(t_{j-1})\|\right\} < \infty, \tag{1.3}$$

where the supremum is taken over all partitions $a = t_0 < \ldots < t_n < b, n \in \mathbb{N}$. As a consequence of the uniform boundedness theorem, for any closed subspace Y of X,

$$\begin{aligned} &\mathbf{V}_Y^\bullet\big(V_0; [r,t)\big) \\ &:= \sup\Big\{\mathbf{v}^\bullet\big(V_0(\cdot,s)x; [s,t)\big);\ r \le s < t, x \in Y, \|x\| \le 1\Big\} < \infty. \end{aligned} \tag{1.4}$$

We also consider the following hypothesis:

(H.1.2) There exists a closed linear subspace Y of X such that $V_0(t,r)\ X \subseteq Y$ and

$$\mathbf{V}_Y^\bullet\big(V_0; [r,t)\big) \to 0, \qquad t \downarrow r, \text{ uniformly in } r \in [0,\tau).$$

Differently from [9] we do not assume that the operators $V_0(t,r)$ are dual operators. This creates quite a few technical difficulties which force us to make additional assumptions the formulation of which we postpone to the next section because they are hard to understand without longer explanations involving the evolution semigroup associated with U_0. At this point we only mention that the assumptions (H.2.1) ... (H.2.3) in the subsequent theorem involve a closed linear subspace $X^\sharp$ of X^* which contains the embedding of Z into X^* and is mapped into itself by all operators $U_0^*(r,t)$ and $V_0^*(r,t)$.

Theorem 1.1. *Let U_0 be a dual evolutionary system satisfying (1.1) and V_0 be a cumulative output for U_0 such that (H.1.1) and (H.1.2) are satisfied. Then, if (H.2.1) ... (H.2.3) also hold, there exists an evolutionary system $U(t,r)$, $0 \le r \le t < \tau$, on X such that, for every $z \in Z$, $U^*(r,t)z$ is a continuous function of (r,t) with values in $X^* = Z^{**}$, and*

$$\langle z, U(t,r)x\rangle = \langle z, U_0(t,r)x\rangle + \int_r^t \big\langle V_0(ds,r)x, U^*(s,t)z\big\rangle. \tag{1.5}$$

The integral is to be interpreted in a weak* Stieltjes sense. While this is the shortest way of formulating the result in a meaningful way, valuable additional information can be gathered (see Section 3):

Remark 1.2. Associated with V_0, there is a uniquely determined Volterra Stieltjes kernel $V(t,r) : X \to Y$, $0 \le r \le t < \tau$, which satisfies (H.1.1) and (H.1.2) and is characterized by the Stieltjes product formulas

$$V(t,r) = V_0(t,r) + \int_r^t V_0(t,s)V(ds,r) = V_0(t,r) + \int_r^t V(t,s)V_0(ds,r). \tag{1.6}$$

V has a series representation in terms of multiple Stieltjes products of V_0 and is called the *resolvent kernel* associated with V_0. V is a cumulative output for U. U can also be represented as

$$\langle z, U(t,r)x\rangle = \langle z, U_0(t,r)x\rangle + \int_r^t \big\langle U_\diamond(s,t)z, V(ds,r)x\big\rangle, \quad z \in Z. \tag{1.7}$$

If U_0 and V_0 are given with $\tau = \infty$, we obtain U and V with $\tau = \infty$ as well, provided that (1.2) holds with $\tau = \infty$ and (H.1.1) and (H.1.2) hold for every finite τ. This follows from the local character of (1.5) and (1.6) and the fact that the resolvent kernel is uniquely determined by (1.6).

The assumption that U_0 is a dual evolutionary system can be replaced by the assumption that $U_0^*(s,t)z$ is continuous in (s,t) for each $z \in Z$ at the expense of an additional assumption which is too technical for an introduction (Theorem 3.1).

An important special case is X being an abstract L space and $V_0(t,r)x$ being positive and non-decreasing in $t \geq r$ for every $x \in X_+$. Then $V_0(\cdot,r)x$ automatically is of bounded variation, and (H.1.1) (d) follows from the other assumptions because, for $x \in X_+$,

$$\mathbf{v}^\bullet\big(V_0(\cdot,s)x;[a,b)\big) = \|V_0(b-,s)x - V_0(a,s)x\|, \qquad s \leq a \leq b < \tau.$$

If the evolutionary system U_0 is positive, so is the perturbed system U.

1.2. **Perspectives and applications.** Theorem 1.1 has been proved in [9] for the special case that the operators $V_0(t,r)$ are dual operators. Then the perturbed system U inherits the property of being a dual evolutionary system, because the perturbation procedure can be performed on the space Z leading to a perturbed backwards evolutionary system. While the construction of the resolvent kernel V can be done on X as well and U can be defined by (1.7), we have not succeeded in directly showing that the resolvent kernel V is a cumulative output for U and U an evolutionary system, for the weak* integral in (1.7) does not commute with bounded linear operators unless they are dual operators.

Here we work around this difficulty by using evolution semigroups and the associated integrated semigroups. Evolution semigroups associated with forward evolutionary systems have recently been used for many purposes (see [6, 14] and the references therein); since we start from a dual evolutionary system, we go back to the space-time approach suggested by Paquet [13] for backwards evolutionary system. This provides a C_0 semigroup $T_\diamond$ on $C_0([0,\tau),Z)$ induced by the evolutionary system $U_\diamond$. The dual semigroup $T = T_\diamond^*$ can be associated with the dual evolutionary system U_0 (see Section 2). We avoid the problems with weak* integration by using the l.L.c. (locally Lipschitz continuous) integrated semigroup, Ψ, obtained by integrating the dual evolution semigroup T [2, 3]. Ψ is perturbed by a resolvent output, induced by the cumulative output V_0 [15], leading to a new locally Lipschitz continuous integrated semigroup, Φ. It remains to be shown that Φ is associated with a (not necessarily dual) semigroup S and that S is an evolution semigroup associated with a forward evolutionary system on X, namely U in Theorem 1.1.

As an application, we considered in [9, Section 9] a size-structured population model which involves per capita birth rates $\beta(t, y)$ that depend on time t and body size y. The population state space X is $M([0, \infty))$, the space of finite Borel measures on $[0, \infty)$, identified with the dual space of $Z = C_0[0, \infty)$, the space of continuous functions vanishing at infinity. The approach in [9] requires the operators $V_0(t, s)$ to be dual operators, in particular $V_0^*(s, t)$ to map Z into Z. This in turn enforces a condition for the birth rate,

$$\lim_{y \to \infty} \beta(t, y) = 0 \quad \text{uniformly in } t \in [0, \tau),$$

i.e., the birth rate tends to 0 for large body sizes. This assumption which may be unrealistic in certain applications can be dropped with the approach presented here.

In this paper we apply our theory to a general class of evolutionary systems operating on the space $X = [C_0(\Omega)]^*$ of Borel measures on a locally compact Hausdorff space Ω where all open subsets of Ω are σ-compact (Section 4). This class, which is associated with transition functions, has already been considered in [8] where a measure-theoretic rather than functional analytic construction is made. This more concrete approach makes it possible to drop the topological assumptions for Ω and relax the various continuity assumptions for $V_0(t, s)$ in (t, s), which are required here as a technicality of Stieltjes integration. As a trade-off, the approach presented here provides some regularity information, as the evolutionary system U on X has $U^*(s, t)z$ being continuous in (s, t) for every $z \in C_0(\Omega)$.

1.3. **Outlook on nonlinear evolution equations.** This perturbation theory shall also provide an access to nonlinear evolution problems, parallel to [7]. As preparation, we also perturb output families.

A two-parameter family of bounded linear operators $P_0(t, r) : X \to X_1$, $0 \le r \le t < \tau$, from X into a Banach space X_1 is called an *output family* for U_0, if

$$P_0(t, s)U_0(s, r) = P_0(t, r), \qquad 0 \le r \le s \le t < \tau. \tag{1.8}$$

Obviously, if P_0 is a strongly continuous output family for U_0, then $V_0(t, r) = \int_r^t P_0(s, r)ds$ is a cumulative output for U_0 and vice-versa.

We formulate the following assumptions for P_0:

(H.1.3) (a) $P_0(t, s)$ is operator-norm continuous in (t, s).
(b) $\sup_{0 \le r \le t < \tau} \|P_0(t, r)\| < \infty$.

Theorem 1.3. *Let the assumptions of Theorem* 1.1 *be satisfied and* $P_0(t,r)$, $0 \le r \le t < \tau$, *an output family for* U_0 *which satisfies* $(H.1.3)$ *and* $(H.2.4)$. *Then there exists an output family* P *for the perturbed evolutionary system* U *which satisfies* $(H.1.3)$ *and*

$$P(t,r) = P_0(t,r) + \int_r^t P(t,s)V_0(ds,r), \qquad 0 \le r \le t < \tau.$$

(H.2.4) is related to the technical conditions in Theorem 1.1 and assumes the existence of a total subspace Z_1 of X_1^* which is mapped into $X^\sharp$ by all operators $P_0^*(r,t)$. $X^\sharp$ is the subspace of X^* mentioned prior to Theorem 1.1. Recall that Z_1 is a total subspace of X_1^* if and only if $x_1 = 0$ is the only element in X_1 such that $\langle x_1, x^* \rangle = 0$ for all $x^* \in Z_1$. P is given by

$$P(t,r) = P_0(t,r) + \int_r^t P_0(t,s)V(ds,r), \qquad 0 \le r \le t < \tau, \tag{1.8}$$

where V is the resolvent kernel associated with V_0 (Remark 1.2).

The connection between output families and nonlinear evolution problems is the following:

For ease of exposition let $\tau = \infty$ and $\mathcal{O}$ a set of functions $E : [0,\infty) \to X_1$. We consider families $U^E(t,r)$ and $P^E(t,r)$ of forward evolutionary systems U^E and their outputs P^E, $E \in \mathcal{O}$, and assume

$$U^E(t+r, s+r) = U^{E_r}(t,s), \qquad P^E(t+r,s+r) = P^{E_r}(t,s),$$

where E_r is the shift of E, $E_r(s) = E(r+s)$. Now assume that for each $x \in X$ there exists a unique $E \in \mathcal{O}$ such that

$$E(t) = P^E(t,0)x \qquad \forall t \ge 0, \tag{1.9}$$

and set $\Theta(t,x) = U^E(t,0)x$. Then Θ is a semiflow on X. This can be seen as follows:

$$\begin{aligned} E_s(t) =& P^E(t+s,0)x = P^E(t+s,s)U^E(s,0)x \\ =& P^{E_s}(t,0)U^E(s,0)x = P^{E_s}(t,0)\Theta(s,x). \end{aligned}$$

This means that E_s solves (1.9) with $\Theta(s,x)$ replacing x. By definition of Θ,

$$\begin{aligned} \Theta\left(t, \Theta(s,x)\right) =& U^{E_s}(t,0)\Theta(s,x) = U^{E_s}(t,0)U^E(s,0)x \\ =& U^E(t+s,s)U^E(s,0)x = U^E(t+s,0)x = \Theta(t+s,x). \end{aligned}$$

The key is solving equation (1.9) which is a fixed-point problem for E. To this end, the dependence of P^E on E must be studied. This can be done through formulas (1.6) and (1.8) which allow us to formulate appropriate assumptions in terms of elementary evolutionary systems U_0^E and associated output families P_0^E and cumulative outputs V_0^E.

2. Evolution semigroups on dual spaces and their (cumulative) outputs

Let Z be a Banach space, $X = Z^*$ its dual space, and $X^\sharp$ a closed subspace of X^* such that $X^\sharp$ contains the canonical embedding of Z into its bi-dual X^*. In the following we identify Z with is canonical embedding. Given on X is a two parameter system $U_0(t,s) : X \to X$, $0 \le s \le t < \tau < \infty$.

Let $U_0^*(s,t)$ denote the dual operator of $U_0(t,s)$. We make the following assumption:

(H.2.0)
- (a) $U_0^*(s,t)z$ is a continuous function of (s,t) with values in $X^\sharp$ for every $z \in Z$, $0 \le s \le t < \tau$.
- (b) $\sup_{0 \le s \le t < \tau} \|U_0(t,s)\| < \infty$.

Let $\mathcal{Z} = C_0([0,\tau), Z)$, the Banach space of continuous functions on $[0,\tau)$ with values in Z and limit 0 at τ, endowed with the supremum norm. $\mathcal{Z}^*$ can be identified with the space of countably additive weakly* regular Borel measures on $[0,\tau)$ with values in X (e.g., [DiU], page 182), and this way $\mathrm{BM}([0,\tau), X^\sharp)$, the space of bounded Borel measurable functions with values in $X^\sharp$ and relatively compact range, can be considered a closed subspace of $\mathcal{Z}^{**}$ which contains $\mathcal{Z}$.

2.1. **The evolution semigroup.** We define an operator family $T_\diamond(t)$ from $\mathcal{Z}$ to $C_0([0,\tau), X^\sharp)$ as follows,

$$(T_\diamond(t)f)(s) = \begin{cases} U_0^*(s, s+t)f(s+t); & s+t < \tau, \\ 0; & s+t \ge \tau > s, \end{cases} \qquad f \in \mathcal{Z}. \tag{2.1}$$

Since $T_\diamond(t)f$ can be considered an element of $\mathcal{Z}^{**}$ by (H.2.0),

$$\langle f, T(t)f^* \rangle = \langle f^*, T_\diamond(t)f \rangle, \qquad f \in \mathcal{Z}, f^* \in \mathcal{Z}^*, \tag{2.2}$$

defines an operator family $T = \{T(t), t \ge 0\}$ on $\mathcal{Z}^*$. Let $T^*(t)$ denote the dual operator of $T(t)$. Then

$$[T^*(t)f](s) = U_0^*(s, s+t)f(s+t), \qquad f \in \mathcal{Z},$$

where the right-hand side is interpreted as 0 if $s + t \ge 0$. We make the following assumption which states that this relation extends under certain circumstances.

(H.2.1) If $f \in C_0\big([0,\tau), X^\sharp\big)$ and $U_0^*(s,t+s)f(t+s)$ is a continuous function of (s,t) with values in $X^\sharp$, $t,s \ge 0, t+s < \tau$, then $T^*(t)f \in C_0([0,\tau), X^\sharp)$ and $[T^*(t)f](s) = U_0^*(s,t+s)f(t+s)$.

The last expression is interpreted as 0 if $t+s \ge \tau$. For simplicity we identify $C_0\big([0,\tau), Z\big)$ and $C_0\big([0,\tau), X^\sharp\big)$ with the respective spaces of continuous functions on $[0,\infty)$ which are 0 on $[\tau,\infty)$. We extend U_0 by

$$U_0(t,r)x = \begin{cases} x, & t \ge r \ge \tau, \\ 0, & t \ge \tau > r, \end{cases} \qquad x \in X.$$

This extension preserves the property of being an evolutionary system, though there may be a discontinuity at $t=\tau$. This does not affect whether or not $U_0^*(s,t+s)f(t+s)$ is a continuous function of $(t,s) \in [0,\infty)^2$ because this expression is 0 if $t+s \ge \tau$, no matter how the evolutionary system is extended beyond τ.

We define an operator family $\Psi_\diamond(t)$ from $\mathcal{Z}$ to $C_0([0,\tau), X^\sharp)$ as follows,

$$[\Psi_\diamond(t)f](s) = \int_0^t U_0^*(s,s+r)f(s+r)dr, \qquad f \in \mathcal{Z}, s \in [0,\tau]. \tag{2.3}$$

Again we obtain an operator family on $\mathcal{Z}^*$ by

$$\langle f, \Psi(t)f^*\rangle = \langle f^*, \Psi_\diamond(t)f\rangle. \tag{2.4}$$

Let $\Psi^*(t)$ denote the dual operator of $\Psi(t)$ on $\mathcal{Z}^{**}$. We make the following assumption.

(H.2.2) If $f \in C_0\big([0,\tau), X^\sharp\big)$ and $U_0^*(s,t+s)f(t+s)$ is a continuous function of (s,t) with values in $X^\sharp$, $t,s \ge 0$, $t+s \le \tau$, then $\Psi^*(t)f \in C_0\big([0,\tau), X^\sharp\big)$ and

$$\left[\Psi^*(t)f\right](s) = \int_0^t U_0^*(s,s+r)f(s+r)dr \qquad \forall t \ge 0, s \in [0,\tau).$$

We introduce a closed linear subspace $\mathcal{Z}^\sharp$ of $C_0([0,\tau], X^\sharp)$ as follows:

$$f \in \mathcal{Z}^\sharp \iff \begin{cases} f \in C_0\big([0,\tau], X^\sharp\big) \text{ and} \\ U_0^*(s,s+t)f(t+s) \text{ is a continuous function} \\ \text{of } (s,t) \text{ with values in } X^\sharp, t,s \ge 0, t+s < \tau. \end{cases} \tag{2.5}$$

The hypotheses (H.2.0), (H.2.1), (H.2.2) can be translated as

Lemma 2.1. (a) $\mathcal{Z} \subseteq \mathcal{Z}^\sharp$.

(b) *If $f \in \mathcal{Z}^\sharp$, then $(T^*(t)f)(s) = U_0^*(s,s+t)f(s+t)$ for all $t,s \ge 0$ with $t+s<\tau$, $T^*(t)f \in C_0([0,\tau), X^\sharp)$, and $T^*(t)f$ is continuous in t.*

(c) $\Psi^*(t)f = \int_0^t T^*(r)f dr$ *for all* $f \in \mathcal{Z}^\sharp$, $\Psi^*(t)f \in C_0([0,\tau), X^\sharp)$, *and* $\Psi^*(t)f$ *is locally Lipschitz continuous in* t.

Proposition 2.2. *Let* U_0 *be an evolutionary system on* X *and* $(H.2.0)$, $(H.2.1)$, *and* $(H.2.2)$ *hold. Then the following also hold:*

(a) $T^*(t)\mathcal{Z}^\sharp \subseteq \mathcal{Z}^\sharp$ *and the restrictions* $T^\sharp(t)$ *of* $T^*(t)$ *to* $\mathcal{Z}^\sharp$ *form a strongly continuous semigroup on* $\mathcal{Z}^\sharp$.

(b) $\Psi^*(r)\mathcal{Z}^\sharp \subseteq \mathcal{Z}^\sharp$ *for all* $r \geq 0$, $T^*(t)\Psi^*(r) = \Psi^*(t+r) - \Psi^*(t)$, *and the restrictions* $\Psi^\sharp(t)$ *of* $\Psi^*(t)$ *to* $\mathcal{Z}^\sharp$ *form a l.L.c. integrated semigroup on* $\mathcal{Z}^\sharp$.

Proof. (a) Let $t, r \geq 0, f \in \mathcal{Z}^\sharp$. Set $g = T^*(r)f$. Then $g(s) = U_0^*(s, s + r)f(s+r)$ and $g \in C_0([0,\tau), X^\sharp)$ by Lemma 2.1 (b). Further,

$$\begin{aligned} U_0^*(s, s+t)g(s+t) &= U_0^*(s, t+s)U_0^*(s+t, s+t+r)f(s+t+r) \\ &= U_0^*(s, s+t+r)f(s+t+r). \end{aligned}$$

So $U_0^*(s, t+s)g(s+t)$ is continuous in (t,s) and an element in $X^\sharp$ by definition of $Z^\sharp$. So $g \in \mathcal{Z}^\sharp$ and, by (H.2.1),

$$[T^*(t)g](s) = U_0^*(s, s+t+r)f(s+r+t) = [T^*(t+r)f](s).$$

Recalling that $g = T^*(r)f$, $T^*(t)T^*(r)f = T^*(t+r)f$. The strong continuity of T^* follows from Lemma 2.1 (b).

(b) Let $t, r \geq 0, f \in \mathcal{Z}$ and set $g = \Psi^*(r)f$. Then

$$g(s) = \int_0^r U_0^*(s, s+u)f(s+u)du$$

and $g \in C_0([0,\tau), X^\sharp)$ by Lemma 2.1 (c). Further,

$$\begin{aligned} U_0^*(s, s+t)g(s+t) &= U_0^*(s, s+t)\int_0^r U_0^*(s+t, s+t+u)f(s+t+u)du \\ &= \int_0^r U_0^*(s, s+t+u)f(s+t+u)du = \int_t^{t+r} U_0^*(s, s+v)f(s+v)dv \end{aligned}$$

is continuous in (t,s) and an element in $X^\sharp$. So $g \in \mathcal{Z}^\sharp$. The rest now easily follows from Lemma 2.1. □

Corollary 2.3. *Let* U_0 *be an evolutionary system and* $(H.2.0)$, $(H.2.1)$, *and* $(H.2.2)$ *hold. Then* T *is a weakly** *continuous semigroup on* $\mathcal{Z}^*$, Ψ *is a non-degenerate l.L.c. integrated semigroup on* $\mathcal{Z}^*$, *and* $\Psi(r)T(t) = \Psi(r+t) - \Psi(r)$ *for all* $t, r \geq 0$.

We recall that a one-parameter family Ψ of bounded linear operators is called *non-degenerate* if, for any $x \in X$, $\Psi(t)x = 0$ for all $t > 0$ implies that $x = 0$. T is called the *evolution semigroup* associated with U_0 and Ψ the associated *integrated evolution semigroup*. We will need a converse of sorts of Corollary 2.3.

Let $M[0,\tau)$ be the space of all real-valued Borel measures on $[0,\tau)$ with finite total variation and $M[0,\tau)\otimes X$ the subset of $\mathcal{Z}^*$ consisting of functionals $\mu \otimes x$ with $\mu \in M[0,\tau)$ and $x \in X$,

$$\langle f, \mu \otimes x\rangle = \int_{[0,\tau)} \langle f(s), x\rangle \mu(ds), \qquad \forall f \in \mathcal{Z}.$$

Lemma 2.4. *If a one-parameter system S on $\mathcal{Z}^*$ is related to a two-parameter system U on X by $[S^*(t)f](s) = U^*(s, s+t)f(s+t)$ for all $f \in \mathcal{Z}$ and $t, s \geq 0$, $t+s < \tau$, and if S is a semigroup, then U is a forward evolutionary system.*

Proof. It follows from our assumption that

$$S(t)(\delta_s \otimes x) = \delta_{t+s} \otimes U(t+s, s)x, \qquad 0 \leq s+t < \tau,$$

where δ_s is the Dirac measure concentrated at s. Let $t, r, s \geq 0$ and $t+r+s < \tau$,

$$\begin{aligned} &\delta_{t+r+s} \otimes [U(t+r+s, s)x] = S(t+r)(\delta_s \otimes x) = S(t)S(r)(\delta_s \otimes x) \\ &= S(t)\big(\delta_{r+s} \otimes U(r+s, s)x\big) = \delta_{t+r+s} \otimes [U(t+r+s, r+s)U(r+s, s)x]. \end{aligned}$$

Now let $z \in Z$. Choose a function $\phi \in C_0([0,\tau), \mathbb{R})$ such that $\phi(t+s+r) = 1$. Set $f(s) = \phi(s)z$. Then $f \in C_0([0,\tau), Z)$ and $\langle f, \delta_{t+r+s}\tilde{x}\rangle = \langle z, \tilde{x}\rangle$ for all $\tilde{x} \in X$. So

$$\big\langle z, U(t+r+s, s)x\big\rangle = \big\langle z, U(t+r+s, r+s)U(r+s, s)x\big\rangle. \qquad \square$$

2.2. **Cumulative Outputs.** Let $V(t, s)$, $0 \leq s \leq t < \tau$, be a Volterra Stieltjes kernel. In addition to (H.1.1) we assume

(H.2.3) $V^*(s, t)$ maps $X^\sharp$ into $X^\sharp$, whenever $0 \leq s \leq t < \tau$.

By (H.1.1)(d), $V^*(s, \cdot)$ with $V^*(s, t) = (V(t, s))^*$ is of bounded semi-variation on $[s, \tau)$, i.e.,

$$\mathbf{v}\big(V^*(s,\cdot); [s,\tau)\big) := \sup \left\{ \Big\| \sum_{j=1}^{n} \big(V^*(s, t_j) - V^*(s, t_{j-1})\big)x_j^* \Big\| \right\} < \infty,$$

where the supremum is taken over all partitions $s = t_0 < \ldots < t_n < \tau, n \in \mathbb{N}$ and all $x_j^* \in X^*, \|x_j^*\| \leq 1$. This follows from the duality relation between the variation of V and the semi-variation of V^*,

$$\mathbf{v}\big(V^*(s,\cdot);[s,\tau)\big) = \sup\Big\{\mathbf{v}\big(\big(V(\cdot,s)x;[s,\tau)\big);x \in X, \|x\| \leq 1\Big\}.$$

See [9, Proposition 3.11]. In particular, by (1.4),

$$\sup_{0\leq s<\tau} \mathbf{v}\big(V^*(s,\cdot);[s,\tau)\big) = \mathbf{V}_X^{\bullet}\big(V,[0,\tau)\big) < \infty. \tag{2.6}$$

We define an operator family on $\mathcal{Z}^*$ by first defining an operator family $W^*(t)$ on $C_0\big([0,\tau),X^\sharp\big)$,

$$[W^*(t)f](s) = \int_s^{(t+s)\wedge\tau} V^*(s,d\sigma)f(\sigma), \qquad 0 \leq s < \tau, \tag{2.7}$$

for all $f \in C_0\big([0,\tau),X^\sharp\big)$. By (H.1.1) and (H.2.3), $W^*(t)f$ is defined and continuous on $[0,\tau)$, and takes values in $X^\sharp$. See [9, Section 3.2].

Recall that $C_0\big([0,\tau),X^\sharp\big)$ can be considered a closed subspace of $\mathcal{Z}^{**}$ which contains $\mathcal{Z}$. The operator family W on $\mathcal{Z}^*$ is defined by

$$\langle f, W(t)f^*\rangle = \langle f^*, W^*(t)f\rangle, \qquad t \geq 0,\ f \in \mathcal{Z},\ f^* \in \mathcal{Z}^*. \tag{2.8}$$

The notation W^* for the operator family in (2.7) is justified because

$$\langle W(t)f^*, f\rangle = \langle f^*, W^*(t)f\rangle, \qquad f \in C_0([0,\tau),X^*), f^* \in \mathcal{Z}^*.$$

This follows from the assumption that $V(t+s,s)$ is norm-continuous in s.

Lemma 2.5. *For $f \in \mathcal{Z}$, $W^*(t)f$ is a continuous function of $t \geq 0$, $W^*(0)f = 0$.*

Proof. Let $0 \leq r \leq t \leq u < \tau$, $f \in \mathcal{Z}$. For convenience we extend $f(s) = 0$ and $V(r,s) = 0$, $s \geq \tau$. By (H.1.1), $V^*(r,s)f(s)$ is continuous in $(r,s) \in [0,\tau) \times [0,\infty)$ and $V^*(r,\cdot)$ is of bounded semi-variation on $[0,\infty)$ with $\mathbf{v}(V^*(r,\cdot);[r,\infty)) = \mathbf{v}(V^*(r,\cdot);[r,\tau))$. We first assume that f has compact support in $[0,\eta] \subseteq [0,\tau)$. Then $[W^*(t)f](s) = 0$ for $s > \eta$. We show: $\|[W^*(u)f](s) - [W^*(r)f](s)\| \to 0$ as $u,r \to t$, uniformly in $s \in [0,\eta]$. To this end, we observe that

$$\begin{aligned}[W^*(u)f](s) - [W^*(r)f](s) &= \int_{r+s}^{u+s} V^*(s,d\sigma)f(\sigma)\\ =&V^*(s,u+s)f(t+s) - V^*(s,r+s)f(t+s)\\ &+ \int_{r+s}^{u+s} V^*(s,d\sigma)[f(\sigma) - f(t+s)].\end{aligned}$$

Then

$$\big\|[W^*(u)f](s) - [W^*(r)f](s)\big\|$$

$$\leq \big\|V^*(s,u+s)f(t+s) - V^*(s,r+s)f(t+s)\big\|$$
$$+ \mathbf{v}\big(V^*(s,\cdot),[s,\eta]\big)\sup\big\{\|f(\sigma)-f(\rho)\|;\sigma,\rho\in[0,\eta],|\rho-\sigma|\leq u-r\big\}.$$

By the triangle inequality, we can continue our estimate,

$$\begin{aligned}
\leq &\big\|V^*(s,u+s)f(u+s) - V^*(s,u+s)f(t+s)\big\| \\
&+ \big\|V^*(s,u+s)f(u+s) - V^*(s,r+s)f(r+s)\big\| \\
&+ \big\|V^*(r,u+s)f(r+s) - V^*(s,r+s)f(t+s)\big\| \\
&+ \sup_{0\leq s<\tau}\mathbf{v}\big(V^*(s,\cdot),[s,\eta]\big)\sup\big\{\|f(\sigma)-f(\rho)\|;\sigma,\rho\in[0,\eta],|\rho-\sigma|\leq u-r\big\} \\
\leq &\big\|V^*(s,u+s)f(u+s) - V^*(s,r+s)f(r+s)\big\| \\
&+ \sup_{0\leq s<\tau}\mathbf{v}\big(V^*(s,\cdot),[s,\eta]\big) \\
&\qquad 3\sup\big\{\|f(\sigma)-f(\rho)\|;\sigma,\rho\in[0,\eta],|\rho-\sigma|\leq u-r\big\}.
\end{aligned}$$

Recall that $\sup_{0\leq s<\tau}\mathbf{v}\big(V^*(s,\cdot),[s,\eta]\big) < \infty$ by (2.6). Since f is uniformly continuous on $[0,\eta]$ and $V^*(s,t+s)f(t+s)$ is a uniformly continuous function of $(s,t)\in[0,\eta]\times[0,\infty)$, the last expression converges to 0 as $u,r\to t$, uniformly over $s\in[0,\eta]$ and hence over $[0,\tau)$. This implies that $W^*(u)f - W^*(r)f \to 0$ as $u,r\to t$. If $f\in C_0([0,\tau),Z)$, then f can be uniformly approximated by functions f_n with compact support in $[0,\tau)$. Then

$$\big\|[W^*(t)f](s) - [W^*(t)f_n](s)\big\| \leq \mathbf{v}(V(s,\cdot);[0,\tau))\|f-f_n\|$$

and $W^*(t)f$ is continuous in $t\geq 0$ as a uniform limit of continuous functions. Similarly we show that $W^*(t)f\to 0$ as $t\to 0$. □

Lemma 2.6. *$W(t):\mathcal{Z}^*\to\mathcal{Z}^*$ is of strong bounded variation in $t\geq 0$.*

Proof. Equivalently we show that $W^*(t):\mathcal{Z}\to C_0([0,\tau),X^*)$ is of bounded semi-variation. Let $0=t_0<\cdots t_{n+1}<\infty$ and $f_1,\dots f_n\in\mathcal{Z}$, $\|f_j\|\leq 1$. Then

$$\Big\|\sum_{j=0}^{n}(W^*(t_{j+1})-W^*(t_j))f_j\Big\| = \sup_{0\leq s<\tau}\Big\|\sum_{j=0}^{n}\int_{(t_j+s)\wedge\tau}^{(t_{j+1}+s)\wedge\tau}V^*(s,d\sigma)f_j(\sigma)\Big\|.$$

For every $s\in[0,\tau)$ we define a piecewise continuous bounded function f_s on $[s,\tau)$ by

$$f_s(\sigma) = \begin{cases} f_j(\sigma); & t_j+s\leq\sigma<t_{j+1}+s \\ 0; & \sigma\geq t_{n+1}+s\end{cases};\quad \sigma\in[s,\tau).$$

In particular f_s is regulated. Since $V(t,s) : X \to X$ is of strong bounded variation in $t \in [s,\tau)$ for every $s \in [0,\tau)$ by (H.1.1), $V^*(s,t)$ is of bounded semi-variation in $t \in [s,\tau)$ for every $s \in [0,\tau)$. So

$$\Big\|\sum_{j=0}^{n}(W^*(t_{j+1}) - W^*(t_j))f_j\Big\| = \sup_{0\le s<\tau}\Big\|\int_s^\tau V^*(s,d\sigma)f_s(\sigma)\Big\|$$
$$\le \sup_{0\le s<\tau}\Big(\mathbf{v}\big(V^*(s,\cdot);[s,\tau)\big)\sup_{s\le\sigma<\tau}\|f_s(\sigma)\|\Big) \le \mathbf{V}_X^\bullet\big(V;[0,\tau)\big).$$

The last but one inequality follows from [9, Theorem 3.3]. The last inequality is a consequence of the duality between semi-variation $\mathbf{v}$ and the variation $\mathbf{V}^\bullet$ in (1.3) ([9, Proposition 3.11] and of the construction of f_s with $\|f_s(\sigma)\| \le 1$ for all $\sigma \in [s,\tau)$. □

Lemma 2.7. *For every $f^* \in \mathcal{Z}^*$, $W(t)f^*$ is continuous in $t \ge 0$.*

Proof. Let $f \in \mathcal{Z}^*$. By Lemma 2.6, $W(t)f^*$ is of bounded variation in $t \ge 0$. Then the strong right and left limits $W(t+)f^*$ exist for $t \ge 0$ and $t > 0$ respectively. Since $W^*(t)f$ is continuous in $t \ge 0$ for every $f \in Z$, we have

$$\langle f, W(t+)f^*\rangle = \langle W^*(t)f, f^*\rangle = \langle f, W(t)f\rangle$$

for every $f \in \mathcal{Z}$, and so $W(t+)f^* = W(t)f^*$. So $W(t)f$ is right continuous in $t \ge 0$. Similarly it follows that $W(t)f$ is left continuous in $t > 0$. □

We introduce two Stieltjes convolutions:

$$(V_0 \star V)(t,r)x = \int_r^t V_0(t,s)V(ds,r)x = \int_0^\tau V_0(t,s)V(ds,r)x, \tag{2.9}$$

$$(W_0 \star W)(t)f^* = \int_0^t W_0(t-s)W(ds)f^*, \tag{2.10}$$

for two- or one-parameter Volterra Stieltjes kernels.

Lemma 2.8. *If V and V_0 satisfy (H.1.1) and (H.1.2), then $(V_0 \star V)(t,r)$ is defined and operator-norm continuous in (t,r).*

Proof. We first notice that (H.1.2) implies that $V_0(s+h,s) \to 0$ as $h \to 0$ in operator norm on Y, locally uniformly in $s \in [0,\tau)$. Together with (H.1.1) (b) this implies that $V_0(t,s)$ is continuous in (t,s) with respect to the operator-norm on Y. Since all operators $V(s,r)$ map into Y, (2.9) is well defined and

$$(V_0 \star V)(t+r,r)x = \int_0^t V_0(t+r,s+r)V(r+ds,r)x.$$

Since both V_0 and V satisfy (H.1.2) (b), $(V_0 \star V)(t+r,r)$ is continuous in r with respect to the operator norm on X. Since $V(r,s)$ maps X into Y and $V_0(t,s)$ is operator-norm continuous in (t,s) on Y, $(V_0 \star V)(t+r,r)$ is operator-norm continuous in t on X, locally uniformly in r. This implies our assertion. □

Lemma 2.9. *Let $g:[0,\infty) \to C_0\big([0,\tau),X^*\big)$ be continuous. If W is associated with a Volterra-Stieltjes kernel V via* (2.7), *then*

$$\Big(\int_0^t W^*(dr)g(t-r)\Big)(s) = \int_0^{(t+s)\wedge\tau} V^*(s,d\sigma)\big[g(t+s-\sigma)\big](\sigma), \quad 0 \le s < \tau.$$

Proof. Without restriction of generality we can assume that g is differentiable. Integrating by parts, using (2.7), and changing the order of integration we obtain

$$\begin{aligned}
&\Big[\int_0^t W^*(dr)g(t-r)\Big](s) = \int_0^t \big[W^*(r)g'(t-r)\big](s)dr - \big[W^*(t)g(0)\big](s)\\
&= \int_0^t \Big(\int_s^{(r+s)\wedge\tau} V^*(s,d\sigma)[g'(t-r)](\sigma)\Big)dr - \int_s^{(r+s)\wedge\tau} V^*(s,d\sigma)[g(0)](\sigma)\\
&= \int_s^{(t+s)\wedge\tau} V^*(s,d\sigma)\Big(\int_{\sigma-s}^t [g'(t-r)](\sigma)dr - [g(0)](\sigma)\Big)\\
&= \int_s^{(t+s)\wedge\tau} V^*(s,d\sigma)\big[g(t+s-\sigma)\big](\sigma).
\end{aligned}$$

□

Lemma 2.10. *Let the kernels V_0 and V satisfy* $(H.1.1)$ *and* $(H.1.2)$. *Let the kernels W_0, W on $\mathcal{Z}^*$ be associated with the kernels V_0 and V on X via* (2.7). *Then $W_0 \star W$ is associated with $V_0 \star V$ via* (2.7).

Proof. Let $f \in C_0^1([0,\tau),Z)$. We can assume that f has compact support in $[0,\tau)$. We extend f to $[\tau,\infty)$ by 0. Let $t,s \ge 0$. Assume that W_0 is associated with V_0 via (2.7). Then

$$\begin{aligned}
(W_0^*(t)f)(s) &= \int_0^{(t+s)\wedge\tau} V_0^*(s,d\sigma)f(\sigma)\\
&= V_0^*(s,(t+s)\wedge\tau)f(t+s) - \int_0^{(t+s)\wedge\tau} V_0^*(s,\sigma)f'(\sigma)d\sigma.
\end{aligned}$$

Since $V_0^*(s,\sigma)$ is strongly continuous on Z, $g(t) = W_0^*(t)f$ provides a function $g \in C_0([0,\tau),X^*)$. It is easy to see that

$$(W_0 \star W)^*(t)f = \int_0^t W^*(dr)W_0^*(t-r)f.$$

By Lemma 2.9, for the convolution (2.10),

$$[(W_0 \star W)^*(t)f](s) = \int_0^{(t+s)\wedge\tau} V^*(s,d\sigma)[W_0^*(t+s-\sigma)f](\sigma)$$

$$= \int_0^{(t+s)\wedge\tau} V^*(s,d\sigma)\Big[V_0^*(\sigma,(t+s)\wedge\tau)f(t+s) - \int_0^{(t+s)\wedge\tau} V_0^*(\sigma,u)f'(u)du\Big]$$

$$=(V^* \star V_0^*)(s,(t+s)\wedge\tau)f(t+s) - \int_0^{(t+s)\wedge\tau} (V_0^* \star V^*)(s,u)f'(u)du$$

$$= \int_0^{(t+s)\wedge\tau} (V_0 \star V)^*(s,du)f(u). \qquad \square$$

If $U(t,s)$ is a two-parameter system on X such that $U^*(s,t)z$ is continuous in (s,t) for every $z \in Z$, then we can define the following convolution which again is denoted by $\star$,

$$\langle z, (U \star V)(t,r)x\rangle = \int_r^t \langle V(ds,r)x, U^*(s,t)z\rangle, \qquad z \in Z, x \in X = Z^*.$$

The integral is taken in a weak* Stieltjes sense. The $\star$ notation is justified because we have

$$[(U \star V)^*(r,t)]z = \int_r^t V^*(r,ds)U^*(s,t)z, \qquad z \in Z.$$

This is a strong Stieltjes integral, because $U^*(s,t)z$ is strongly continuous and $V^*(r,\cdot)$ of bounded semi-variation. If $T(t)$ is a one-parameter system on $\mathcal{Z}^*$ such that $T^*(t)f$ is continuous in $t \geq 0$ for every $f \in \mathcal{Z}$, we define $T \star W$ by

$$\langle f, (T \star W)(t)f^*\rangle = \int_0^t \langle W(ds)f^*, T^*(t-s)f\rangle$$

and again obtain

$$(T \star W)^*(t)f = \int_0^t W^*(dr)T^*(t-r)f, \qquad f \in \mathcal{Z}.$$

Lemma 2.11. *Let U_0 be a forward two-parameter system on X satisfying (H.2.0), (H.2.1), and (H.2.3) and let the operator family T on $\mathcal{Z}^*$ be associated with U_0 via $[T^*(t)f](s) = U_0^*(s,s+t)f(s+t)$ for $f \in \mathcal{Z}$. Let the kernels V and W be associated by (2.7). Then, if $s+t \leq \tau$,*

$$[(T \star W)^*(t)f](s) = [(U_0 \star V)^*(s,s+t)]f(t+s) \qquad \forall f \in \mathcal{Z}.$$

Proof. By Lemma 2.9,

$$((T \star W)^*(t)f)(s) = \int_s^{t+s} V^*(s, d\sigma)(T^*(t+s-\sigma)f)(\sigma)$$

$$= \int_s^{t+s} V^*(s, d\sigma) U_0^*(\sigma, t+s) f(t+s) = [(U_0 \star V)^*(s, s+t)] f(t+s). \qquad \square$$

A two-parameter operator family $V(t, r)$ is a *cumulative output for a two-parameter system* $U(t, r)$, as defined in the Introduction, if and only if $U^*(r, s)V^*(s, t) = V^*(r, t) - V^*(r, s)$.

Analogously, a one-parameter family $W(t)$ is a *cumulative output for a one-parameter semigroup* $T(t)$, if $W(t)T(r) = W(t+r) - W(r)$, equivalently if $T^*(r)W^*(r) = W^*(t+r) - W^*(r)$. $W(t)$ is a cumulative output for an integrated semigroup Ψ, if

$$W(t)\Psi(r) = \int_0^r [W(t+u) - W(u)]du \qquad \forall t, r \geq 0.$$

Lemma 2.12. *If* $V_0(t, r)$ *is a cumulative output for the evolutionary system* $U_0(r, t)$ *satisfying* $(H.2.0)$, $(H.2.1)$, *and* $(H.2.3)$, *then* W_0 *is a cumulative output for* T *and* Ψ*.*

Proof. We use the dual characterization. Let $f \in \mathcal{Z}$ and $t, s \geq 0$, $t+s < \tau$, $g = W^*(r)f$. By (2.7),

$$U^*(s, s+t)g(t+s) = U^*(s, s+t)\Big(\int_{t+s}^{(t+s+r)\wedge\tau} V^*(t+s, d\sigma)f(\sigma)\Big).$$

Since this is a strong Stieltjes integral and V a cumulative output for U,

$$U^*(s, s+t)g(t+s) = \int_{t+s}^{(t+s+r)\wedge\tau} U^*(s, s+t)V^*(t+s, d\sigma)f(\sigma)$$

$$= \int_{t+s}^{(t+s+r)\wedge\tau} V^*(s, d\sigma)f(\sigma) = [W^*(t+r)f](s) - [W^*(t)f](s).$$

So $g \in \mathcal{Z}^\sharp$ by (H.1.1). By Lemma 2.1 (b), $[T^*(t)g](s) = [W^*(t+r)f](s) - [W^*(t)f](s)$. If $s+t \geq \tau$, both sides of the equation are 0. Since $g = W^*(r)f$, $T^*(t)W^*(r)f = W^*(t+r)f - W^*(t)f$. Integrating over t and using Lemma 2.1 (c),

$$\Psi^*(t)W^*(r)f = \int_0^t [W^*(s+r) - W^*(s)]f ds.$$

By duality, W is a cumulative output for T and Ψ. $\square$

Lemma 2.13. *Let* U *satisfy* $(H.2.0)$ *and* V *satisfy* $(H.1.1)$*. Then* $(U \star V)^*(r, t)z$ *is continuous in* (r, t) *for every* $z \in Z$*.*

Proof. Now let $0 \le t < \tau$ and $0 \le r < \tau - t$. Then

$$(U \star V)^*(r, r+t)z = \int_0^t V^*(r, r+ds)U^*(s+r, r+t)z$$

is continuous in $r \in [0, \tau - t)$ because $V^*(r, r+s)$ is operator-norm continuous in r and $U^*(s,t)z$ is continuous in (s,t). Let $r \le t < v < \tau$. Then

$$\begin{aligned}
&(U \star V)^*(r, r+v)z - (U \star V)^*(r, r+t)z \\
=&\int_{r+t}^{r+v} V^*(r, ds)U^*(s, r+v)z \\
&+ \int_r^{r+t} V^*(r, ds)[U^*(s, r+v)z - U^*(s, r+t)z] \\
=&V^*(r, r+v)z - V^*(r, r+t)z + \int_{r+t}^{r+v} V^*(r, ds)[U^*(s, r+v)z - z] \\
&+ \int_r^{r+t} V^*(r, ds)[U^*(s, r+v)z - U^*(s, r+t)z].
\end{aligned}$$

Since $U^*(r,t)z$ and $V^*(r,t)z$ are continuous in (r,t), we have

$$(U \star V)^*(r, r+v)z - (U \star V)^*(r, r+t)z \to 0, \qquad t, v \to u \ge r,$$

locally uniformly in r. This shows that $(U \star V)^*(r, r+t)z$ is continuous in $t \in [0, \tau - r)$ locally uniformly in $r \in [0, \tau)$. Combining this with our previous continuity result, our assertion follows. $\square$

2.3. **Output families.** Let the bounded linear operators $P_0(t,s) : X \to X_1$ form a two-parameter family satisfying the assumptions (H.1.3). We associate with the family P_0 on X a one-parameter family Q_0 of bounded linear operators from $\mathcal{Z}^* = M([0,\tau), X)$ to X_1,

$$Q_0(t)\mu = \int_{[0,\tau)} P_0(t+s, s)\mu(ds), \tag{2.11}$$

for every measure $\mu \in \mathcal{Z}^*$. Choosing $\mu = \delta_s \otimes x$, we see that

$$Q_0^*(t)x^* = \begin{cases} P_0^*(s, s+t)x^*; & s+t < \tau, \\ 0; & s+t \ge \tau > s \end{cases}, \qquad x^* \in X_1^*.$$

We make the following assumption.

(H.2.4) There exists a total subspace Z_1 of X_1^* such that $P_0^*(s,t)$ maps Z_1 into $X^\sharp$.

Similarly as in Lemma 2.12 and Lemma 2.4, we prove the following result.

Proposition 2.14. (a) *If P_0 is an output family for U_0, then Q_0 is an output family for T and Ψ, i.e.,*

$$Q_0(t)T(s) = Q_0(t+s), \qquad Q_0(t)\Psi(r) = \int_0^r Q(t+s)ds.$$

(b) *Let the one-parameter systems S on $\mathcal{Z}^*$ and Q from $\mathcal{Z}^*$ to X_1 be related to the two-parameter systems U and P on $X = Z^*$ by*

$$[S^*(t)f](s) = U^*(s, t+s)f(t+s), \qquad f \in \mathcal{Z},$$

with the right-hand side being interpreted as 0 if $t + s \geq \tau$, and

$$[Q^*(t)x^*](s) = P^*(s, s+t)x^*, \qquad x^* \in Z_1.$$

Then P is an output family for U, whenever Q is an output family for S.

3. Perturbation

We prove extensions of Theorem 1.1 and Theorem 1.3 which relax the assumption that the evolutionary system U_0 consists of dual operators.

We recall that the Favard class of a linear operator $B : D(B) \to \mathcal{X}_2$, with $D(B)$ being a linear subspace of $\mathcal{X}_1$ and $\mathcal{X}_1, \mathcal{X}_2$ being Banach spaces, is defined as follows:

An element $x \in \mathcal{X}_1$ is contained in the *Favard class* of B if and only if there exists a sequence of elements $x_n \in D(B)$ such that $x_n \to x$ as $n \to \infty$ and $\sup_{n\in\mathbb{N}} \|Bx_n\| < \infty$.

Obviously $D(B)$ is contained in the Favard class of B. B is called a *Favard operator* if $D(B)$ coincides with the Favard class of B.

Dual operators associated with densely defined linear operators are Favard operators. Operators forming a l.L.c. integrated semigroup map the space into the Favard class of the generator (Lemma 3.2 b).

Theorem 3.1. *Let U_0 be an evolutionary system on the dual space $X = Z^*$ which satisfies $(H.2.0) \cdots (H.2.2)$. Assume that the generator B of the integrated evolution semigroup associated with U_0 is a Favard operator. Further let V_0 be a cumulative output for the evolutionary system U_0 such that $(H.1.1)$, $(H.1.2)$ and $(H.2.3)$ hold. Then there exists an evolutionary system $U(t,r)$, $0 \leq r \leq t < \tau$, on X such that, for every $z \in Z$, $U^*(r,t)z$ is a continuous function of (r,t) with values in $X^\sharp$ and*

$$\langle z, U(t,r)x\rangle = \langle z, U_0(t,r)x\rangle + \int_r^t \langle V_0(ds,r)x, U^*(s,t)z\rangle.$$

The next paragraphs are devoted to the proof of Theorem 3.1. Definition (2.1) provides the evolution semigroup T associated with U_0.

Let the Volterra Stieltjes kernel $V_0(t,s) : X \to X, 0 \le s \le t < \tau$, be a cumulative output for U_0. Recall the closed linear subspace Y of X introduced in (H.1.2). We consider the restrictions of $V_0(t,s)$ from Y to Y, which are operator-norm continuous in (t,s) by (H.1.1) (b) and (H.1.2), and let $V_0^\bullet(s,t)$ denote the dual operators on Y^*. In the terminology of [9], they form a regular Volterra Stieltjes operator kernel on Y^*, in particular the family is of bounded semi-variation. By [9, Theorem 4.8], $V_0^\bullet$ has a unique regular resolvent kernel $V^\bullet(s,t) : Y^* \to Y^*$, $0 \le s \le t < \tau$, which is a regular Volterra Stieltjes kernel on Y^* as well,

$$V^\bullet(r,t) = \int_r^t V^\bullet(r,ds)V_0^\bullet(s,t) + V_0^\bullet(r,t) = \int_r^t V_0^\bullet(r,ds)V^\bullet(s,t) + V_0^\bullet(r,t),$$

which is given by a series expansion

$$V^\bullet(r,t) = \sum_{j=0}^{\infty} V_j^\bullet(r,t), \qquad V_{j+1}^\bullet(r,t) = \int_r^t V_0^\bullet(r,ds)V_j^\bullet(s,t).$$

The convergence holds in the uniform operator norm, uniformly in (r,t), and in bounded semi-variation. We extend $V^\bullet(r,t)$ to an operator from Y^* to X^* by setting

$$V^\bullet(r,t) = \int_r^t V_0^*(r,ds)V^\bullet(s,t) + V_0^*(r,t)$$

and retain the series representation (converging in operator norm) with

$$V_{j+1}^\bullet(r,t) = \int_r^t V_0^*(r,ds)V_j^\bullet(s,t).$$

$V^\bullet$ is now a regular Volterra Stieltjes kernel from Y^* to X^*. Let $V^{\bullet *}(t,s) : X^{**} \to Y^{**}$ be the dual operators of $V^\bullet(s,t)$. They satisfy an analogous series representation which implies that the restriction $V(r,t)$ to X maps into Y. By duality, $V(\cdot,r)x$ is of bounded variation on $[s,\tau)$ for every $s \in [0,\tau)$ and every $x \in X$; actually V is a regular Volterra Stieltjes kernel on X, satisfies (1.6) and is given by the series expansion (converging in operator norm)

$$V(t,r) = \sum_{j=0}^{\infty} V_j(t,r), \qquad V_{j+1}(t,r)x = \int_r^t V_j(t,s)V_0(ds,r)x.$$

The Stieljes integral exists because $V_0(s,r)x \in Y$ and the restriction of $V_j(t,s)$ to Y is operator-norm-continuous in (t,s). The dual operators

$V^*(t,r) : X^* \to X^*$ associated with $V(t,r)$ map $X^\sharp$ into $X^\sharp$. This follows from (H.2.3) which has been assumed for V_0 and the recursive formulas involving strong Stieltjes integrals,

$$V_{j+1}^*(r,t)x^* = \int_r^t V_0^*(r,ds)V_j^*(s,t)x^*,$$

and the series representation (converging in operator norm)

$$V^*(r,t) = \sum_{j=0}^\infty V^*(r,t).$$

Let W_0 and W be the families on $\mathcal{Z}^*$ associated with V_0 and W by (2.7). Then W_0 and W are of strong bounded variation by Lemma 2.6, W_0 is a cumulative output for T and Ψ by Lemma 2.12, and, by Lemma 2.10 and (1.6),

$$W(t) = W_0(t) + \int_0^t W_0(t-s)W(ds) = W_0(t) + \int_0^t W(t-s)W_0(ds),$$

with the series expansion

$$W(t) = \sum_{j=0}^\infty W_j(t), \qquad W_{j+1}(t) = \int_0^t W_j(t-s)W_0(ds).$$

The Laplace Stieltjes transform

$$F(\lambda) = \int_0^\infty e^{-\lambda t}W_0(dt) = \lambda\int_0^\infty e^{-\lambda t}W_0(t)dt$$

exists, because $W_0(t)$ is constant for $t > \tau$. Further $F(\lambda) \to 0$ strongly on $\mathcal{Z}^*$ as $\lambda \to \infty$, because $W_0(t)$ is strongly continuous and $W_0(0) = 0$ by Lemma 2.7.

The operator family $F(\lambda)$ is a resolvent output for the generator B of T which is also a generator of Ψ [15, Section 4.3],

$$F(\lambda)(\mu - B)^{-1} = \frac{1}{\mu - \lambda}(F(\lambda) - F(\mu)).$$

Let

$$G(\lambda) = (I - F(\lambda))^{-1}F(\lambda) = \sum_{j=1}^\infty (F(\lambda))^j.$$

$G(\lambda)$ is the Laplace Stieltjes transform of W. From here on, the procedure is analogous to the one in [15, Sections 4 to 6], to which we refer for some of the details. The resolvent output relation shows that the definition

$$C = F(\lambda)(\lambda - B), \qquad D(C) = D(B),$$

is independent of large λ and defines a perturbator $C : D(B) \to X$ of B [15, Section 4.1]. The definition

$$\Phi(t)x = \Psi(t)x + \int_0^t \Psi(t-s)W(ds)x, \tag{3.1}$$

provides a l.L.c. operator family, because Ψ is l.L.c. and $W(t)x$ is of bounded variation. The property that $\Psi(t)$ and $W(t)$ are constant for $t > \tau$ is inherited by Φ. Taking Laplace-Stieltjes transforms shows that

$$\int_0^\infty e^{-\lambda t}\Phi(dt) = (\lambda - A)^{-1}, \qquad A = B + C \tag{3.2}$$

and

$$\Phi(t)x = \Psi(t)x + \int_0^t \Phi(t-s)W_0(ds)x. \tag{3.3}$$

By (3.2), Φ is an integrated semigroup with A being its generator [15, Theorem 3.3].

Lemma 3.2. *(a) A is a Favard operator.*
(b) $\Phi(t)$ maps $\mathcal{Z}^*$ into $D(A)$.

Remark 3.3. For the proof of Theorem 3.1, we only need part (b) of Lemma 3.2. Part (a) only serves to prove part (b).

Proof. (a) (H.1.2) implies that the spectral radius of $F(\lambda)$ converges to 0 as $\lambda \to \infty$. So we can choose some $\lambda > 0$ such that $F(\lambda)$ is invertible. Consider a sequence $(f_n^*) \in D(A) = D(B)$ such that $f_n^* \to f^*$ with $f^* \in \mathcal{Z}^*$ and the sequence (Af_n^*) is bounded. Then the sequence $(\lambda - A)f_n^*$ is bounded. Now

$$(\lambda - A)f_n^* = (\lambda - B - C)f_n^* = (I - F(\lambda))(\lambda - B)f_n^*.$$

Hence the elements

$$(\lambda - B)f_n^* = (I - F(\lambda))^{-1}(\lambda - A)f_n^*$$

form a bounded sequence and so do the elements Bf_n^*. Since B is a Favard operator by assumption, $f^* \in D(B) = D(A)$.

(b) Let $f^* \in \mathcal{Z}^*, t \geq 0$. Set

$$g_n^* = \frac{1}{h_n}\int_t^{t+h_n} \Phi(s)f^* ds$$

with some sequence $0 < h_n \to 0$ as $n \to \infty$. Since A is the generator of the integrated semigroup Φ, $g_n^* \in D(A)$ and

$$Ag_n^* = \frac{1}{h_n}\big(\Phi(t+h_n) - \Phi(t)\big)f^* - f^*.$$

Since Φ is locally Lipschitz continuous, $g_n^* \to \Phi(t)f^*$ as $n \to \infty$ and the elements Ag_n^* form a bounded sequence. By (a), $\Phi(t)f^* \in D(A)$. □

By Lemma 3.2 (b), we can define

$$S(t)f^* = f^* + A\Phi(t)f^* \tag{3.4}$$

and obtain a semigroup S on $\mathcal{Z}^*$ which satisfies $\Phi(r)S(t) = \Phi(t+r) - \Phi(t)$ for all $t, r \geq 0$. S is called the *integral semigroup* associated with Φ [17, Section 2.4].

Proposition 3.4. *The semigroup S in* (3.4) *has the dual representation*

$$S^*(t)f = T^*(t)f + \int_0^t W^*(ds)T^*(t-s)f \qquad \forall f \in \mathcal{Z}^\sharp.$$

Proof. Since $F(\lambda) \to 0$ strongly as $\lambda \to \infty$, we have

$$Cf^* = \lim_{\lambda\to\infty} \lambda F(\lambda)f^* \qquad \forall f^* \in D(B).$$

Taking Laplace transforms one sees that

$$F(\lambda)\Psi(t) = W_0(t)(\lambda - B)^{-1}.$$

By [15, Theorem 4.13], $\lim_{\lambda\to\infty} \lambda F(\lambda)\Psi(t) = W_0(t)$. It is not difficult to see that $G(\lambda)$ is a resolvent output for $A = B + C$,

$$G(\lambda)(\mu - A)^{-1} = \frac{1}{\lambda - \mu}\big(G(\mu) - G(\lambda)\big), \qquad G(\lambda)(\lambda - A) = B.$$

The uniqueness properties of the Laplace transform imply that

$$G(\lambda)\Phi(t) = W(t)(\lambda - A)^{-1}.$$

Since also $Cx = \lim_{\lambda\to\infty} \lambda G(\lambda)x$ for $x \in D(B)$, the same arguments as before imply that $C\Phi(t)x = W(t)x$.

We use the duality relation between B and the operator $B^\odot$ in $\mathcal{X}^\odot$ with $\mathcal{X} = \mathcal{Z}^*$

$$\mathcal{X}_B^\odot = \{x^* \in \mathcal{X}^*; \|\lambda(\lambda - B)^{-1*}x^* - x^*\| \to 0, \lambda \to \infty\}.$$

See [15, Section 3.3] and [16, (2.7), (2.8)]. The following holds for $x^* \in \mathcal{X}^*$:

$$x^* \in \mathcal{X}_B^\odot \iff \Psi^*(t)x^* \text{ is strongly differentiable in } t \geq 0.$$

Further, for $x^\odot \in \mathcal{X}_B^\odot$,

$$T^\odot(t)x^\odot := \frac{d}{dt}[\Psi^*(t)x^\odot], \qquad x^\odot \in X_B^\odot,$$

defines a C_0-semigroup $T^\odot$ on $X_B^\odot$. The infinitesimal generators of $T^\odot$ is denoted by $B^\odot$. We let $x^\odot \in D(B^\odot)$. Then

$$\langle x^\odot, B\Phi(t)f^*\rangle = \langle \Phi^*(t)B^\odot x^\odot, f^*\rangle$$

and

$$\begin{aligned}\Phi^*(t)B^\odot x^\odot =&\Psi^*(t)B^\odot x^\odot + \int_0^t W^*(ds)\Psi^*(t-s)B^\odot x^\odot \\ =&T^\odot(t)x^\odot - x^\odot + \int_0^t W^*(ds)[T^\odot(t-s)x^\odot - x^\odot].\end{aligned}$$

So, for all $x^\odot \in D(B^\odot)$,

$$\langle x^\odot, B\Phi(t)f^*\rangle = \Big\langle T^\odot(t)x^\odot - x^\odot + \int_0^t W^*(ds)T^\odot(t-s)x^\odot - W^*(t)x^\odot,\ f^*\Big\rangle.$$

Since $D(B^\odot)$ is dense in $\mathcal{X}_B^\odot$, this equality holds for all $x^\odot \in \mathcal{X}_B^\odot$. By Lemma 2.1, $\mathcal{Z}^\sharp$ is a subspace of $\mathcal{X}_B^\odot$ and $T^*(t)f = T^\odot(t)f$ for all $f \in \mathcal{Z}^\sharp$. Using $W(t)f^* = C\Phi(t)f^*$ and $S(t)f^* = f^* + (B+C)\Phi(t)f^*$,

$$S^*(t)f = T^*(t)f + \int_0^t W^*(ds)T^*(t-s)f \qquad \forall f \in \mathcal{Z}^\sharp. \qquad \square$$

The formula in Proposition 3.4 can be rewritten as $S = T + T \star W$. To finish the proof of Theorem 3.1, we define U by the analog of (1.7),

$$\langle z, U(t,r)x\rangle = \langle z, U_0(t,r)x\rangle + \int_r^t \big\langle V(ds,r)x, U_0^*(s,t)z\big\rangle, \quad z \in Z, x \in X = Z^*. \tag{3.5}$$

By Lemma 2.11, $[S^*(t)f](s) = U^*(s,s+t)f(s+t)$ for all $f \in \mathcal{Z}$. U is an evolutionary system by Lemma 2.4. By duality

$$U^*(r,t)z = U_0^*(r,t)z + \int_r^t V^*(r,ds)U_0^*(s,t)z, \qquad z \in Z. \tag{3.6}$$

Since this is a strong Stieltjes integral and $U_0^*(s,t)z \in X^\sharp$ by (H.2.0) and $V^*(r,x)$ maps $X^\sharp$ into itself, $U^*(r,t)z \in X^\sharp$ for all $z \in Z$. Since $V - V_0 = V_0 \star V$, $V(t,s) - V_0(t,s)$ is operator-norm continuous on X in (t,s) by Lemma 2.8. Now Lemma 2.13 implies that $U^*(s,t)z$ is continuous in (s,t) for $z \in Z$. This shows that $S^*(t)f$ is continuous in $t \geq 0$ for $f \in \mathcal{Z}$. By a Laplace transform argument,

$$S^*(t)f = T_\diamond(t)f + \int_0^t W_0^*(ds)S^*(t-s)f \qquad \forall f \in \mathcal{Z}.$$

Lemma 2.11 implies the formula in Theorem 3.1.

We prove a perturbation result for cumulative outputs.

Proposition 3.5. *Let W_0 be a cumulative output for the integrated semigroup Ψ and W the resolvent kernel for W_0 and Φ the perturbed integrated semigroup. Let Ξ_0 be also a cumulative output for Ψ, $\Xi_0(t) : \mathcal{X} \to \mathcal{X}_1$ operator-norm continuous, where $\mathcal{X}_1$ is also a Banach space. Then*

$$\Xi = \Xi_0 + \Xi_0 \star W$$

defines a cumulative output of Φ which satisfies

$$\Xi = \Xi_0 + \Xi \star W_0.$$

Proof. Since cumulative outputs of l.L.c. integrated semigroups are exponentially bounded [15, Lemma 4.14], we can define

$$\check{\Xi}(\lambda) = \lambda \int_0^\infty e^{-\lambda t} \Xi(t) dt.$$

Then

$$\check{\Xi}(\lambda) = \check{\Xi}_0(\lambda)(I + \check{W}(\lambda)) = \check{\Xi}_0(\lambda)(I - \check{W}_0(\lambda))^{-1}.$$

Hence

$$\check{\Xi}(\lambda) - \check{\Xi}(\lambda)\check{W}_0(\lambda) = \check{\Xi}_0(\lambda).$$

The uniqueness properties of the Laplace transform provide that $\Xi = \Xi_0 + \Xi \star W_0$. Recall that W is a cumulative output of Φ if and only if

$$\check{W}(\lambda)\check{\Phi}(\mu) = \frac{1}{\mu - \lambda}(\check{W}(\lambda - \check{W}(\mu)).$$

By definition of Ξ,

$$\check{\Xi}(\lambda)\check{\Phi}(\mu) = \check{\Xi}_0(\lambda)\check{\Phi}(\mu) + \check{\Xi}_0(\lambda)\check{W}(\lambda)\check{\Phi}(\mu).$$

Since W is a cumulative output for Φ,

$$\check{\Xi}(\lambda)\check{\Phi}(\mu) = \check{\Xi}_0(\lambda)\check{\Psi}(\mu)(I + \check{W}(\mu)) + \check{\Xi}_0(\lambda)\frac{1}{\lambda - \mu}\left(\check{W}(\mu) - \check{W}(\lambda)\right).$$

Since Ξ_0 is a cumulative output for Ψ,

$$\begin{aligned}\check{\Xi}(\lambda)\check{\Phi}(\mu) &= \frac{1}{\lambda - \mu}\Big(\big[\check{\Xi}_0(\mu) - \check{\Xi}_0(\lambda)\big]\left(I + \check{W}(\mu)\right) + \check{\Xi}_0(\lambda)\big[\check{W}(\mu) - \check{W}(\lambda)\big]\Big) \\ &= \frac{1}{\lambda - \mu}\left(\check{\Xi}(\mu) - \check{\Xi}(\lambda)\right).\end{aligned}$$

Hence Ξ is a cumulative output for Φ. □

It follows from the definition of a cumulative output that $\Xi^*(s)$ maps $\mathcal{X}^*$ into $\mathcal{X}_A^\odot$ and

$$S^\odot(t)\Xi^*(s) = \Xi^*(t+s) - \Xi^*(t).$$

In our particular situation, by duality

$$\Xi(s)S(t)f = \Xi(t+s) - \Xi(t),$$

and Ξ is also a cumulative output of the integral semigroup S. If Ξ is formed by integrals of a family P, we obtain that P is an output family for S.

Theorem 3.6. *Let U_0 be an evolutionary system on the dual space $X = Z^*$ which satisfies $(H.2.0)\cdots(H.2.2)$. Assume that the generator B of the integrated semigroup associated with U_0 is a Favard operator. Further let V_0 be a cumulative output for the evolutionary system U_0 such that $(H.1.1)$, $(H.1.2)$ and $(H.2.3)$ hold. Finally consider an output family for U_0 of bounded linear operators $P_0(r,t) : X \to X_1$, $0 \le r \le t < \tau$, which satisfies $(H.1.3)$ and $(H.2.4)$. Then there exists an output family for the perturbed evolutionary system U which satisfies $(H.1.3)$ and*

$$P(t,r) = P_0(t,r) + \int_r^t P(t,s)V_0(ds,r), \qquad 0 \le r \le t < \tau,$$

and is given by

$$P(t,r) = P_0(t,r) + \int_r^t P_0(t,s)V(ds,r), \qquad 0 \le r \le t < \tau.$$

Proof. We define

$$P(t,r) = P_0(t,r) + \int_r^t P_0(s,t)V(ds,r), \qquad 0 \le r \le t < \tau.$$

A similar proof as the one for Lemma 2.8 shows that $P(t,r)$ is operator-norm continuous in (t,r). Further $P^*(r,t)$ maps Z_1 into $X^\sharp$. We set

$$Q(t)\mu = \int_{[0,\tau)} P(t+r,r)\mu(dr), \qquad \mu \in \mathcal{Z}^*,$$

with $P(t+r,r)$ being interpreted as 0 if $t+s \ge \tau$.

We establish the analogous relation between P_0 and Q_0, i.e., (2.14). Then Q_0 is an output family for the semigroup T and the associated integrated semigroup Ψ. Further

$$Q^*(t)x^* = Q_0^*(t)x^* + \int_0^t W^*(ds)Q_0^*(t-s)x^*, \qquad x^* \in X_1^*.$$

Taking Laplace transforms, we see that

$$Q^*(t)x^* = Q_0^*(t)x^* + \int_0^t W_0^*(ds)Q^*(t-s)x^*,$$

which implies the second equation in our theorem. We set

$$\Xi_0^*(t)f = \int_0^t Q_0^*(s)f\,ds, \quad \text{and} \quad \langle f, \Xi_0(t)f^*\rangle = \langle f^*, \Xi_0^*(t)f\rangle.$$

Similarly we relate an operator family $\Xi(t)$ with Q. Then Ξ_0 is a cumulative output for T and Ψ. Further

$$\Xi^*(t)f = \Xi_0^*(t)f + \int_0^t W^*(ds)\Xi_0^*(t-s).$$

Since $\Xi_0(t)$ is operator-norm continuous, by duality,

$$\Xi(t) = \Xi_0(t) + \int_0^t \Xi_0(t-s)W(ds).$$

By Proposition 3.5, Ξ is a cumulative output of the perturbed integrated semigroup Φ and the associated integral semigroup S. This implies that Q is an output family for S and P an output family for U. □

4. Evolutionary systems on spaces of measures

A Hausdorff space Ω is called *locally compact* if for every $\omega \in \Omega$ there exists an open subset D of Ω such that $x \in D$ and $\bar{D}$ is compact [12, XI.6].

The subset of a Hausdorff space is called *σ-bounded*, if it is the countable union of compact sets [5, Section 57].

A Hausdoff space is called *σ-compact* if it is locally compact and σ-bounded [12, XI.7]. Such a space is also called *countable at infinity* [4, Definition 7.4.4].

We call a Hausdorff space *completely σ-compact* if if is locally compact and every open subset is σ-bounded.

In order to illustrate that completely σ-compact Hausdorff spaces form a sufficiently large class we prove the following result.

Proposition 4.1. (a) *A locally compact space with countable base is completely σ-compact.*

(b) *A subspace of a completely σ-compact space Ω is completely σ-compact, if and only if it is of the form $D \cap C$ where D is an open and C is a closed subset of Ω.*

(c) *The topological product of finitely many completely σ-compact spaces is completely σ-compact.*

(d) *The topological product of infinitely many completely σ-compact spaces is completely σ-compact, provided that all but finitely many of the spaces are compact.*

Proof. (a) Let Ω be a locally compact Hausdorff space with countable base. By [4, Section 7.4, Expression 2(c)], Ω is σ-compact. By [4, Theorem 7.6.1], Ω is a Polish space, i.e., its topology is induced by a complete metric. Then every open set is the countable union of closed sets. Since Ω is σ-compact, every closed set in Ω is the countable union of compact sets. Hence every open set is the countable union of compact sets.

(b) Recall that an analogous statement is true for locally compact spaces [12, Chapter XI, Theorem 6.5]. This shows the necessity of the representation. We need to show that $D \cap C$ is σ-bounded. Let $\tilde{D}$ be open in the topological space $D \cap C$. Then $\tilde{D} = D \cap C \cap \hat{D}$ with some open subset $\hat{D}$ of Ω. Since $D \cap \hat{D}$ is open and Ω is completely σ-bounded, there exists a sequence (C_n) of compact subsets C_n of Ω such that $D \cap \hat{D} = \bigcup_{n\in\mathbb{N}} C_n$. Then the sets $C_n \cap C$ are compact and $\tilde{D} = \bigcup_{n\in\mathbb{N}}(C_n \cap C)$.

(c) and (d) easily follow from analogous statements for locally compact spaces [12, Chapter XI, Theorem 6.5]. □

Corollary 4.2. *A topological subspace of $\mathbb{R}^n$ is completely σ-compact if and only if it is the intersection of an open and a closed subset of $\mathbb{R}^n$.*

Throughout this section, let Ω be a completely σ-compact space. Then every Borel measure of finite variation is regular [1, Theorem 38.4]. $M(\Omega)$ denotes the Banach space of Borel measures on Ω endowed with the total variation as norm. $M(\Omega)$ is an abstract L space. Let $\mathrm{BC}(\Omega)$ denote the Banach space of bounded continuous real-valued functions on Ω endowed with the supremum norm, $C_{00}(\Omega)$ the subspace of functions with compact support, and $C_0(\Omega)$ its closure. Then $M(\Omega)$ can be identified with the dual space of $C_0(\Omega)$ by the Riesz representation theorem. In the language of Section 2, we have $Z = C_0(\Omega)$ and $X = Z^* = M(\Omega)$. Further we choose $X^\sharp = \mathrm{BM}(\Omega)$, the Banach space of bounded Borel measurable functions on Ω, with supremum norm. In a natural way, $X^\sharp$ can be considered a closed subspace of X^* containing Z by $\langle f, x\rangle = \int_\Omega f(\omega)x(d\omega)$.

4.1. **Measure kernels.** Let $\mathcal{B}$ denote the σ-algebra of Borel sets in Ω. A *measure kernel* (*transition function*) is a function $K : \mathcal{B} \times \Omega \times \Delta_\tau \to [0, \infty)$ where $\Delta_\tau = \{(t, s) : 0 \le s \le t < \tau\}$ and

- $K(\cdot, \omega, t, s)$ is a measure on $\mathcal{B}$ for every $\omega \in \Omega$, $(t, s) \in \Delta_\tau$,
- $K(D, \cdot, t, s)$ is Borel measurable on Ω for every $D \in \mathcal{B}$, $(t, s) \in \Delta_\tau$.

- $\sup\{K(\Omega, \omega, t, s);\ \omega \in \Omega,\ 0 \le s \le t < \tau\} < \infty$.

We can associate a two-parameter operator family $U_0(t,s)$ on X with a measure kernel K_0 by setting

$$[U_0(t,s)x](D) = \int_\Omega K_0(D, \omega, t, s) x(d\omega). \tag{4.1}$$

For $f \in X^\sharp$ =BM(Ω), by changing the order of integration,

$$\langle f, U_0(t,s)x\rangle = \int_\Omega \Big(\int_\Omega f(\xi) K_0(d\xi, \omega, t, s)\Big) x(d\omega)$$

which shows that $U_0^*(s,t)$ maps $X^\sharp$ into $X^\sharp$ and

$$[U_0^*(s,t)f](\omega) = \int_\Omega f(\xi) K_0(d\xi, \omega, t, s) \qquad \forall \omega \in \Omega, f \in X^\sharp. \tag{4.2}$$

4.2. **Evolutionary systems and the Chapman-Kolmogorov equations.** If U_0 is related to a measure kernel K_0 by formula (4.1), then U_0 is an evolutionary system if and only if K_0 satisfies the *Chapman-Kolmogorov* equations

(H.4.1) $K_0(D, \omega, t, r) = \int_\Omega K_0(D, \xi, t, s) K_0(d\xi, \omega, s, r)$ for all $D \in \mathcal{B}$, $0 \le r \le t < \tau$.

To satisfy (H.2.0) we make the following assumption:

(H.4.2) For every $z \in C_0(\Omega)$, $\int_\Omega z(\xi) K_0(d\xi, \omega, t, s)$ is continuous in $(t,s) \in \Delta_\tau$, uniformly for $\omega \in \Omega$, and $\int_\Omega z(\xi) K_0(d\xi, \omega, t, s) \to z(\omega)$ as $t \downarrow s$, $\omega \in \Omega$.

$\mathcal{Z} = C_0([0,\tau), Z)$ can be identified with $C_0(\Omega_\tau)$, where $\Omega_\tau = [0,\tau) \times \Omega$. Since Ω_τ is completely σ-compact by Proposition 4.1 (c), $\mathcal{Z}^*$ can be identified with $M(\Omega_\tau)$, the Banach space of measures on the Borel sets in Ω_τ, while $C_0([0,\tau), X^\sharp)$ can be embedded into BM(Ω_τ). BM(Ω_τ) can be identified with a closed subspace of $\mathcal{Z}^{**}$. By (2.1),

$$[T_\diamond(t)f](s,\omega) = \int_\Omega f(t+s, \xi) K_0(d\xi, \omega, t+s, s) \qquad f \in C_0(\Omega_\tau).$$

This implies that, for every Borel set $\tilde{D}$ in Ω_τ and every measure $\mu \in M(\Omega_\tau)$,

$$[T(t)\mu](\tilde{D}) = \int_\Omega \Big(\int_\Omega \chi_{\tilde{D}}(t+s, \xi) K_0(d\xi, \omega, t+s, s)\Big) \mu(d(s,\omega)).$$

By duality we see that $T^*(t)$ maps BM(Ω_τ) into itself with

$$[T^*(t)f](s,\omega) = \int_\Omega f(t+s, \xi) K_0(d\xi, \omega, t+s, s) \qquad \forall f \in \mathrm{BM}(\Omega_\tau).$$

This implies (H.2.1). By (H.4.2), $\int_\Omega z(\xi)K_0(d\xi,\omega,t,s)$ is Borel measurable in (ω,t,s) for $z \in C_0(\Omega)$. It follows ([Ber], Sections 54 and 55) that $K_0(D,\omega,t,s)$ is Borel measurable in (ω,t,s) for all compact sets D that are countable intersections of open sets and all open sets that are countable unions of compact sets, for the characteristic functions of both type of sets can be pointwise approximated by functions in $C_0(\Omega)$. Since, by assumption, all open subsets of Ω are σ-bounded, $K(D,\cdot)$ is measurable for all open sets and so for all Borel sets D in Ω. Since every $f \in$BM(Ω) can be approximated by linear combinations of characteristic functions of Borel sets, $\int_\Omega f(\xi)K_0(d\xi,\omega,t,s)$ is Borel measurable in (ω,t,s) for $f \in$BM(Ω). We combine similar arguments as we have used before with Fubini's theorem and obtain (H.2.2); in particular

$$[\Psi^*(t)f](s) = \int_0^t \Big(\int_\Omega f(r+s,\xi)K_0(d\xi,\omega,r+s,s)\Big)dr \quad \forall f \in \mathrm{BM}(\Omega_\tau).$$

We notice that $\Psi^*(t)$ leaves $\mathcal{X}^\sharp =$BM(Ω_τ) invariant. So the restrictions $\Psi^\sharp(t)$ of $\Psi^*(t)$ to $\mathcal{X}^\sharp$ form a (possibly degenerate) integrated semigroup on $\mathcal{X}^\sharp$. The dual space of $\mathcal{X}^\sharp$ can be identified with the finitely additive set functions on Ω_τ.

Proposition 4.3. *Let K_0 be a measure kernel satisfying* (H.4.1) *and* (H.4.2)*. Then formula* (4.1) *defines an evolutionary system U_0 on $X = M(\Omega)$ such that* (H.2.0), (H.2.1) *and* (H.2.2) *are satisfied.*

4.3. **Cumulative outputs.** Let L be another measure-kernel. We assume that

(H.4.3) For all Borel sets D in Ω, $\tilde\omega \in \Omega$, $0 \le r \le s \le t < \tau$,

$$\int_\Omega L(D,\omega,t,s)K_0(d\omega,\tilde\omega,s,r) = L(D,\tilde\omega,t,r) - L(D,\tilde\omega,s,r).$$

(H.4.4) If D is a Borel set in Ω and $\omega \in \Omega$, $s \in [0,\tau)$, then $L(D,\omega,\cdot,s)$ is non-decreasing on $[s,\tau)$ and $L(\Omega,\omega,s,s) = 0$ for all $\omega \in \Omega$, $s \in [0,\tau)$.

(H.4.5) For every Borel set D and every $\omega \in \Omega$, $L(D,\omega,t,s)$ is continuous in $(t,s) \in \Delta_\tau$. Whenever $0 \le r \le t < \tau$, then $L(D,\omega,t+s,r+s)$ is continuous in $s \in [0,\tau-t)$, uniformly in all $\omega \in \Omega$ and all Borel sets $D \subseteq \Omega$. For every $z \in C_0(\Omega)$,

$$\int_\Omega z(\tilde\omega)L(d\tilde\omega,\omega,t,s)$$

is continuous in $(t,s) \in \Delta_\tau$, uniformly for all $\omega \in \Omega$.

For the next assumption we define the *support of* L, $\Omega_\flat$, as $\Omega_\flat = \Omega \setminus \mathcal{O}$ where $\mathcal{O}$ is the largest open set in Ω such that

$$L(\mathcal{O}, \omega, t, s) = 0 \qquad \forall \omega \in \Omega, 0 \le s \le t < \tau.$$

(H.4.6.) $L(\Omega_\flat, \omega, s+h, s) \to 0$ as $0 < h \to 0$, uniformly in all $\omega \in \Omega_\flat$, $s \in [0, \tau)$.

We associate a linear bounded operator $V_0(t,r) : X \to X$ with the measure kernel L by a formula analogous to (4.1). If $f = \chi_D$ is an indicator function, then the analog of (4.2) implies that

$$[V_0^*(r,t)(\chi_D)](\omega) = L(D, \omega, t, r),$$

and, by (H.4.3),

$$[U_0^*(r,s)V_0^*(s,t)(\chi_D)](\omega) = \int_\Omega L(D, \tilde{\omega}, t, s) K_0(d\tilde{\omega}, \omega, r, r)$$
$$= L(D, \omega, t, r) - L(D, \omega, s, r) = [V_0^*(r,t)\chi_D](\omega) - [V_0^*(r,s)\chi_D](\omega).$$

Since these operators are linear and bounded, the identity $U_0^*(r,s)\, V_0^*(s,t)f = V_0^*(r,t)f - V_0^*(r,s)f$ also holds for linear combinations f of indicator functions and, since the latter form a dense subspace of $X^\sharp$, also for all $f \in X^\sharp$. It follows from (H.4.4), that $V_0(\cdot, s)x$ is of bounded variation on $[s, \tau)$ and $v^\bullet\big(V_0(\cdot, s)x; [s,t)\big) = \|V_0(t,s)x\|$. By choosing Y to be the subspace of Borel measures μ with $\mu(D) = 0$ for all Borel sets D with $D \cap \Omega_\flat = \emptyset$, (H.4.6) implies (H.1.2). The rest of (H.1.1) follows from (H.4.5). We summarize.

Proposition 4.4. *Let the measure kernel* K_0 *satisfy* (H.4.1) *and* (H.4.2) *and the measure kernel* L *satisfy* (H.4.3) *to* (H.4.6)*. Let the operator families* U_0 *and* V_0 *be associated with* K_0 *and* L*, respectively, via* (4.1)*. Then* V_0 *is a cumulative output for* U_0 *in such a way that the assumptions* (H.1.1), (H.1.2), (H.2.0), ..., (H.2.3) *are satisfied.*

4.4. **The case of a dual evolutionary system.**

Theorem 4.5. *Let* Ω *be completely* σ*-compact. Let the measure kernel* K_0 *satisfy (H.4.1) and (H.4.2) and the measure kernel* L *satisfy (H.4.3) to (H.4.6). Further assume the following.*

- *For every* $z \in C_0(\Omega)$ *and* $(t,s) \in \Delta_\tau$, $\int_\Omega z(\xi) K_0(d\xi, \omega, t, s)$ *is continuous in* $\omega \in \Omega$.
- *For every compact subset* D_1 *of* Ω, *every* $(t,s) \in \Delta_\tau$, *and every* $\epsilon > 0$ *there exists a compact subset* D_2 *of* Ω *such that*

$$K_0(D_1, \omega, t, s) \le \epsilon \qquad \forall \omega \in \Omega \setminus D_2.$$

Let the operator families U_0 and V_0 be associated with K_0 and L, respectively, via (4.1). Then there exists a measure kernel K which satisfies (H.4.1) and (H.4.2) such that the associated evolutionary system U solves the equation

$$U^*(r,t)z = U_0^*(r,z) + \int_0^t V_0^*(r,ds)U^*(s,t)z, \qquad 0 \le r \le t < \tau, z \in Z.$$

Proof. The additional assumptions imply that $U_0^*(t,s)$ maps $Z = C_0(\Omega)$ into itself such that U_0 is a dual evolutionary system. The existence of the evolutionary system U follows from Theorem 1.1. We show that U is associated with a measure kernel K. Taking the Banach lattice structure of X into account, one can show that the operators $U(t,r)$ preserve non-negative measures. Since $U^*(s,t)z \in X^\sharp$ =BM(Ω), for each $\omega \in \Omega$, the mapping $z \mapsto [U^*(s,t)z](\omega)$ is a continuous linear functional on $Z = C_0(\Omega)$. Hence there exists a unique Borel measure $K(\cdot,\omega,t,s)$ such that

$$[U^*(s,t)z](\omega) = \int_\Omega z(\xi)K(d\xi,\omega,t,s),$$

this expression is Borel measurable in ω and continuous in (t,s), and

$$\sup_{\omega\in\Omega} K(\Omega,\omega,t,s) \le \|U(t,s)\|.$$

Since $U^*(s,t)$ is strongly continuous from Z to $X^\sharp$, K satisfies (H.4.2). As we have argued before, this implies that $K(D,\omega,t,s)$ is Borel measurable in (ω,t,s) for all Borel sets D in Ω and K is a measure kernel. K also satisfies the Chapman-Kolomogorov equation (H.4.1) because U is an evolutionary system. □

4.5. **The case of second countable Ω.** The additional assumptions in Theorem 4.5 can be omitted if we make stronger assumptions for Ω, namely that Ω is second countable, i.e., the topology of Ω has a countable base. Then Ω is completely σ-compact by Proposition 4.1 (a). By [4, Theorem 7.6.3], $Z = C_0(\Omega)$ is separable. Ω_τ is a locally compact space with countable base as well and $\mathcal{Z} = C_0(\Omega_\tau)$ is separable. Since we have not been able to show that the generator B of the integrated semigroup Ψ is a Favard operator, we show directly that $\Phi(t)$ maps $\mathcal{X}$ into $D(A)$. See Remark 3.3. We need some preparations.

Lemma 4.6. *$\mathcal{X}^\sharp \cap \mathcal{X}^\odot$ separates points in $\mathcal{X} = \mathcal{Z}^\star$.*

Proof. Let $x \in \mathcal{X}$ be such that $\langle x, x^* \rangle = 0$ for all $x^* \in \mathcal{X}^\sharp \cap \mathcal{X}^\odot$. Let $y^* \in \mathcal{X}^\sharp$. Then $(\lambda - B)^{-1*} y^* \in \mathcal{X}^\sharp \cap \mathcal{X}^\odot$. By assumption,

$$0 = \langle x, (\lambda - B)^{-1*} y^* \rangle = \langle (\lambda - B)^{-1} x, y^* \rangle.$$

Since $y^* \in \mathcal{X}^\sharp$ has been arbitrary and $\mathcal{X}^\sharp = BM(\Omega_\tau)$ separates points in $\mathcal{X} = \mathrm{M}(\Omega_\tau)$, $(\lambda - B)^{-1} x = 0$. This implies $x = 0$. □

Since $T^*(t)$ extends $T^\odot(t)$ and maps $\mathcal{X}^\sharp = \mathrm{BM}(\Omega_\tau)$ into itself, $T^\odot(t)$ maps $\mathcal{X}^\sharp \cap \mathcal{X}^\odot$ into itself and its restrictions to this Banach space form a C_0-semigroup which is generated by the part of $B^\odot$ in $\mathcal{X}^\sharp \cap \mathcal{X}^\odot$ which is denoted by $B^\sharp$. The resolvents of $B^\sharp$ are the restrictions of $(\lambda - B)^{-1*}$ on $\mathcal{X}^\sharp$. We show that B behaves like a dual operator of $B^\sharp$.

Lemma 4.7. *Let $x, y \in \mathcal{X}$ and $\langle x, B^\sharp x^* \rangle = \langle y, x^* \rangle$ $\forall x^* \in D(B^\sharp)$. Then $x \in D(B)$ and $y = Bx$.*

Proof. Assume that $x, y \in \mathcal{X}$ satisfy the assumptions in this lemma. Let $x^* \in \mathcal{X}^\sharp \cap \mathcal{X}^\odot$. Then $(\lambda - B)^{-1*} x^* \in D(B^\sharp)$. By assumption,

$$\langle x, B^\sharp (\lambda - B)^{-1*} x^* \rangle = \langle y, (\lambda - B)^{-1*} x^* \rangle.$$

By duality,

$$\langle -x + \lambda(\lambda - B)^{-1} x, x^* \rangle = \langle (\lambda - B)^{-1} y, x^* \rangle.$$

Since $x^* \in \mathcal{X}^\sharp \cap \mathcal{X}^\odot$ has been arbitrary and $\mathcal{X}^\sharp \cap \mathcal{X}^\odot$ separates points in $\mathcal{X}$,

$$-x + \lambda(\lambda - B)^{-1} x = (\lambda - B)^{-1} y.$$

This implies that $x \in D(B)$ and $Bx = y$. □

Let X be a normed vector space and X^* its dual space. A subset $\tilde{X}$ of X^* is called *sequentially weakly* closed* if the following holds: If (x_j^*) is a sequence in $\tilde{X}$, $x^* \in X^*$ and $\langle x, x_j^* \rangle \to \langle x, x^* \rangle$ as $j \to \infty$ for all $x \in X$, then $x^* \in \tilde{X}$.

Lemma 4.8. *Let Ω be a completely σ-compact Hausdorff space, $Z = C_0(\Omega)$ and $X = Z^* = \mathrm{M}(\Omega)$. The space of bounded Borel measurable functions $X^\sharp = \mathrm{BM}(\Omega)$ is a sequentially weakly* closed subspace of X^* and equals every sequentially weakly* closed subspace of* $\mathrm{BM}(\Omega)$ *that contains Z.*

Proof. The first statement is left to the reader. Let $\tilde{X}$ be a sequentially weakly* closed subspace of $X^\sharp$ that contains Z. Let $\mathcal{B} = \{D \subseteq \Omega; \chi_D \in \tilde{X}\}$. Let D be an open subset of Ω. Since Ω is completely σ-compact, D is the countable union of compact sets. Since Ω is locally compact, χ_D is the pointwise limit of an increasing sequence (z_j) of functions in Z. Since $Z \subseteq \tilde{X}$ and $\langle \chi_D, x \rangle = \lim_{j \to \infty} \langle z_j, x \rangle$ for all $x \in \mathrm{M}(\Omega)$ by the theorem of monotone convergence, $\chi_D \in \tilde{X}$ and $D \in \mathcal{B}$. Since $\tilde{X}$ is a vector space, $\mathcal{B}$ is an algebra. Since $\tilde{X}$ is sequentially weakly* closed, $\mathcal{B}$ is a monotone class. This implies that $\mathcal{B}$ contains every Borel set. By the same token, $\tilde{X}$ contains all finitely-valued Borel measurable functions and their bounded pointwise limits, i.e., all bounded Borel measurable functions. □

Lemma 4.9. *Assume that the topology of* Ω *has a countable base,* $Z = C_0(\Omega)$, $X = Z^*$. *Let* $w : [a,b] \to X$ *be of bounded variation. Then there exist a finite non-negative Borel measure* μ_w *on* $[a,b]$ *and a function* $\psi_w : [a,b] \to X$ *such that, for all* $x^* \in X^\sharp$, $\langle \psi_w(s), x^* \rangle$ *is Borel measurable in* s, $\|\psi_w(s)\| \le 1$ *for* μ_w*-a.a.* $s \in [a,b]$, *and*

$$\int_a^b \langle dw(s), g(s) \rangle = \int_{[a,b]} \langle \psi_w(s), g(s) \rangle \mu_w(ds)$$

for all continuous $g : [a,b] \to X^\sharp$.

Proof. Fix w. Since w is of bounded variation, the variation of w, $v(t) = \mathbf{v}(w,[a,t])$, is of bounded variation, too. So there exists a unique non-negative Borel measure $\mu = \mu_w$ on $[a,b]$ such that $v(t+) - v(r+) = \mu((r,t])$. Since the topology of Ω has a countable base, $Z = C_0(\Omega)$ is separable. Now let $z \in Z$. Then $\langle z, w(t) \rangle$ is of bounded variation and

$$\mathbf{v}(\langle z, w(\sqcup) \rangle; [r,t]) \le \|z\| \mathbf{v}(w; [r,t]).$$

There exists a signed measure ν_z which is uniquely determined by z and w such that

$$\langle z, w(t+) - w(r+) \rangle = \nu_z([r,t]).$$

Further, $|\nu_z((r,t])| \le \|z\| \mu((r,t])$. This estimate also holds for all Borel sets. By the Radon-Nikodym theorem, there exists $\psi_z : [a,b] \to \mathbb{R}$ such that

$$\nu_z(B) = \int_B \psi_z(s) \mu(ds)$$

and

$$\int_B |\psi_z(s)| \mu(ds) = |\nu_z|(B) \le \|z\| \mu(B).$$

ψ_z is uniquely determined up to a set of μ-measure 0 which depends on z. The last inequality implies that

$$\psi_z(s) \le \|z\| \text{ for } \mu\text{-a.a. } s \in [a,b].$$

The proof now continues exactly as the one of Theorem 34 in [10, Chapter 1]. Since Z is separable, the definition $\langle z, \psi(s) \rangle = \psi_z(s)$ provides a function

$\psi = \psi_w : [a, b) \to X = Z^*$, $\|\psi(s)\| \le 1$ for μ-a.a. $s \in [a, b)$, and

$$\langle w(t+) - w(r+), x^* \rangle = \int_{(r,t]} \langle \psi_w(s), x^* \rangle \mu_w(ds) \qquad \forall x^* = Jz, z \in Z,$$

where J is the canonical embedding of Z into $Z^{**} = X^*$. The set of $x^* \in X^\sharp$ for which this equality holds is sequentially weakly* closed, and so the equality holds for all $x^* \in X^\sharp$ by Lemma 4.8. The equality of the Stieltjes and the measure-theoretic integral follows for all piecewise constant left-continuous functions. Since every continuous functions can be uniformly approximated by piecewise constant left-continuous functions, the equality of the two integrals follows for every continuous function. □

We define

$$\langle f, \tilde{S}(t)x \rangle := \langle f, T(t)x \rangle + \int_0^t \langle W(ds)x, T^*(t-s)f \rangle \qquad \forall f \in \mathcal{Z}, x \in \mathcal{X}.$$

Fix $t > 0$. We apply Lemma 4.9 for Ω_τ rather than Ω with $w(s) = W(s)x$ and $g(s) = T^*(t-s)f$ and write ψ_x for ψ_w and $\mu_x = \mu_w$,

$$\langle f, \tilde{S}(t)x \rangle = \langle f, T(t)x \rangle + \int_0^t \big\langle \psi_x(s), T^*(t-s)f \big\rangle \mu_x(ds).$$

For all $f \in \mathcal{X}^\sharp$, the map $(s, r) \mapsto \langle \psi_x(s), T_0^*(r)f \rangle$ is Borel measurable and

$$\langle \tilde{S}(t)x, f \rangle = \langle T(t)x, f \rangle + \int_0^t \big\langle \psi_x(s), T^*(t-s)f \big\rangle \mu_x(ds).$$

This follows from Lemma 4.8 and the fact that the set of $f \in \mathcal{X}^\sharp$ for which the last two statements hold contains $J\mathcal{Z}$ and is sequentially weakly*-closed. For all $f \in \mathcal{X}^\sharp \cap \mathcal{X}^\odot$, $T^*(r)f$ is a continuous function of r with values in $\mathcal{X}^\sharp$ and, by Lemma 4.9,

$$\langle \tilde{S}(t)x, f \rangle = \langle T(t)x, f \rangle + \int_0^t \big\langle W(ds)x, T^*(t-s)f \big\rangle.$$

For all $f \in D(B^\sharp)$,

$$\begin{aligned}\langle \tilde{S}(t)x, f \rangle =& \langle \Psi(t)x, B^\sharp f \rangle + \langle x, f \rangle + \int_0^t \big\langle W(ds)x, \Psi^*(t-s)B^\sharp f + f \big\rangle \\ =& \langle \Phi(t)x, B^\sharp f \rangle + \langle W(t)x + x, f \rangle.\end{aligned}$$

By Lemma 4.7, $\Phi(t)x \in D(B) = D(A)$ which we still needed to prove to have the following result.

Theorem 4.10. *Let Ω be locally compact and the topology of Ω have a countable base. Let the measure kernel K_0 satisfy $(H.4.1)$ and $(H.4.2)$ and the measure kernel L satisfy $(H.4.3)$ to $(H.4.6)$. Let the operator families U_0 and V_0 be associated with K_0 and L, respectively, via (4.1). Then there exists a measure kernel K which satisfies $(H.4.1)$ and $(H.4.2)$ such that the associated evolutionary system U solves the equation*

$$U^*(r,t)z = U_0^*(r,z) + \int_0^t V_0^*(r,ds)U^*(s,t)z, \qquad 0 \le r \le t < \tau, z \in Z.$$

4.6. **Output families.** Let $H : \Omega \times \Delta_\tau \to X_1$. X_1 could be any Banach space, but we consider the following assumptions for the special case that X_1 is the dual of a Banach space Z_1.

(H.4.7) (a) For all $(t,s) \in \Delta_\tau$, $z \in Z_1$, $\langle z, H(\cdot,t,s)\rangle$ is Borel measurable on Ω.

(b) $H(\omega,\cdot)$ is continuous on Δ_τ, uniformly in $\omega \in \Omega$.

(c) $\sup\big\{H(\omega,t,s);\ \omega \in \Omega, 0 \le s \le t < \tau\big\} < \infty$.

(d) For all $\omega \in \Omega$, $0 \le r \le s \le t \le \tau$, $z \in Z_1$,

$$\int_\Omega \big\langle z, H(\xi,t,s)\big\rangle K_0(d\xi,\omega,s,r) = \big\langle z, H(\omega,t,r)\big\rangle.$$

(e) If X_1 is not a dual space, we assume that $H(\cdot,t,s)$ is Borel measurable on Ω and Z_1 is a total subspace of X_1^*.

We associate a two-parameter operator family P_0 of operators from $M(\Omega)$ to X_1 with H by

$$\big\langle z, P_0(t,s)x\big\rangle = \int_\Omega \big\langle z, H(\omega,t,s)\big\rangle x(d\omega). \tag{4.3}$$

If H satisfies (H.4.7), then P_0 is an output family for U_0 which satisfies (H.1.3) and (H.2.4). Notice that $P_0^*(s,t)z = \langle z, H(\cdot,t,s)\rangle \in \mathrm{BM}(\Omega) = X^\sharp$.

Proposition 4.11. *Let the measure kernel K_0 satisfy* (H.4.1) *and* (H.4.2)*, the measure kernel L satisfy* (H.4.3) *to* (H.4.6)*, and the measure kernel H satisfy* (H.4.7)*. Let the operator families U_0, V_0, and P_0 be associated with K_0, L, and H, respectively, via* (4.1) *and* (4.3)*. Let Ω have a countable base or let the extra assumptions of Theorem* 4.5 *be satisfied. Then the assumptions of Theorem* 3.5*are satisfied.*

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