

THE NAVIER-STOKES EQUATIONS WITH INITIAL DATA IN UNIFORMLY LOCAL L^p SPACES

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Abstract. In this paper we will construct local mild solutions of the Cauchy problem for the incompressible homogeneous Navier-Stokes equations in d -dimensional Euclidian space with initial data in uniformly local L^p (L^p_{uloc}) spaces where p is greater than or equal to d . For the proof, we shall establish $L^p_{uloc} - L^q_{uloc}$ estimates for some convolution operators. We will also show that the mild solution associated with L^d_{uloc} almost periodic initial data at time zero becomes uniformly local almost periodic (L^∞ -almost periodic) in any positive time.

1. INTRODUCTION

In this paper, we consider the Cauchy problem for the incompressible homogeneous Navier-Stokes equations with viscosity 1 in \mathbb{R}^d where $d \geq 2$. The system of equations is of the form

$$\begin{cases} u_t - \Delta u + (u, \nabla)u + \nabla p = 0, & t > 0, \quad x \in \mathbb{R}^d, \\ \nabla \cdot u = 0, & x \in \mathbb{R}^d, \\ u(0, x) = u_0(x), & x \in \mathbb{R}^d, \end{cases} \quad (\text{NS})$$

where $u = (u^1, u^2, \dots, u^d)$ is the unknown velocity vector field, p is the unknown pressure scalar field and $u_0 = (u_0^1, u_0^2, \dots, u_0^d)$ is the given initial velocity satisfying $\nabla \cdot u_0 = 0$.

Our main purpose here is to solve (NS) for initial data which may not decay at space infinity but may not necessarily be locally bounded. There are many works which construct mild solutions of the Navier-Stokes equations on various function spaces (e.g. [17], [3], [9], [16], [12], [13], [19], [6], [5], [7], [10], [18], [24], [4]). E.B. Fabes, B.F. Jones and N.M. Rivière [9], T. Kato [16], Y. Giga and T. Miyakawa [12] constructed mild solutions of (NS) with initial data in L^p spaces where p is larger than the space dimension d .

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Moreover, in [16] and [12], the case $p = d$ is discussed. However, all functions in L^p spaces decay at space infinity when p is finite. When one considers nondecaying flows at space infinity as we would like to do, the function space for initial data should be a space of functions which do not decay at space infinity. The L^∞ space, considered by J.R. Cannon and G.H. Knightly [3], M. Cannone [5], Y. Giga, K. Inui and S. Matsui [10] is of course such a kind of function space, and both the Besov spaces with negative regularity considered by M. Cannone and Y. Meyer [6], M. Cannone [5], H. Kozono and M. Yamazaki [19], etc. and the function space considered by H. Koch and D. Tataru [18] are such kinds of function spaces too. However there is no work constructing mild solutions with initial data in uniformly local type spaces, which naturally contain functions which may not decay at space infinity. (For the definition of uniformly local Triebel-Lizorkin spaces and Besov Spaces, see, e.g., [28].) In this paper, we shall construct mild solutions of (NS) with initial data in uniformly local L^p spaces where p is greater than or equal to the space dimension d . The method is quite similar to that of E.B. Fabes, B.F. Jones and N.M. Rivière [9], T. Kato [16], Y. Giga and T. Miyakawa [12] except that we use our newly obtained convolution type estimate instead of Young's inequality for convolutions. Uniformly local L^p spaces consist of functions which are locally in L^p and whose L^p norm in any Euclidean ball with radius 1 is uniformly bounded. When p is finite, they obviously contain functions which do not decay at space infinity but are not necessarily locally bounded. As an application, we shall prove that the initial data at time zero of L^d_{uloc} -almost periodic functions become uniformly almost periodic ($=L^\infty$ -almost periodic) functions in any positive time (the notion of L^p_{uloc} -almost periodic function where $p \in [1, \infty)$ was originally defined by A. S. Besicovitch [2] in \mathbb{R} . Ours is a generalization of that notion in \mathbb{R}^d .) This is an extension of a recent result of Y. Giga, A. Mahalov and B. Nicolaenko [11] where they proved L^∞ -almost periodic initial data stay L^∞ -almost periodic under the Navier-Stokes flow.

Uniformly local L^p spaces were used by J. Ginibre and G. Velo [14] for complex Ginzburg-Landau equations and used by P. G. Lemarié-Rieusset [20], [21] and Y. Taniuchi [26] for equations of fluid mechanics. In his work [20] and [21], Lemarié-Rieusset constructed in three-dimensional Euclidean space a suitable weak solution which is *local in time* with arbitrary initial data in uniformly local L^2 space. Furthermore, he constructed a suitable weak solution which is *global in time* with arbitrary initial data in the closure of compactly supported smooth functions in uniformly local L^2 spaces. Y. Taniuchi [26] obtained uniformly local L^p estimates of vorticity equations.

However, he only considers L^p_{uloc} - L^p_{uloc} type estimates of convolution type operators, while we also treat L^p_{uloc} - L^q_{uloc} type estimates of convolution type operators in which the indices p and q may be different. Let us be more precise. We consider the equations (NS) with initial data in $L^p_{uloc,\rho}$ space for any positive number ρ and any $p \in [d, \infty]$. When p is finite, the space $L^p_{uloc,\rho}(\mathbb{R}^d)$ is defined as follows:

$$L^p_{uloc,\rho}(\mathbb{R}^d) := \{f \in L^1_{loc}(\mathbb{R}^d); \|f\|_{L^p_{uloc,\rho}} := \sup_{x \in \mathbb{R}^d} \left(\int_{|x-y|<\rho} |f(y)|^p dy \right)^{\frac{1}{p}} < \infty\}. \tag{1.1}$$

For simplicity of notation we set $L^\infty_{uloc,\rho}(\mathbb{R}^d) := L^\infty(\mathbb{R}^d)$. When p is finite, the space $L^p_{uloc,\rho}$ naturally contains both the space L^p and the space L^∞ . The space $L^p_{uloc,\rho}$ also contains all L^p -periodic functions, i.e., periodic functions which are locally p -th integrable in \mathbb{R}^d . We include the parameter ρ here, since the existence time estimate of the mild solutions can be different if ρ is different. Moreover, varying ρ , we can reproduce T. Kato's global existence result for small initial data; more precisely, one can construct a unique mild solution globally in time if L^d norms of the initial data are sufficiently small.

To solve (NS) we convert the equations to the integral equation of the form

$$u(t) = e^{t\Delta}u_0 - \int_0^t e^{(t-s)\Delta} \mathbf{P} \nabla \cdot (u \otimes u) ds. \tag{1.2}$$

Here, $e^{t\Delta}$ is the heat semigroup, \mathbf{P} is the Helmholtz projection, and $f \otimes g := (f_i g_j)_{1 \leq i, j \leq d}$ is a tensor product of $f = (f_1, f_2, \dots, f_d)$ and $g = (g_1, g_2, \dots, g_d)$. The solution of the integral equation (1.2) is called the mild solution of (NS) with initial data u_0 . The precise meaning of each term, and the fact that they are well defined, follows from the L^p_{uloc} - L^q_{uloc} type estimates of convolution type operators which we shall obtain. Now we would like to state our main results.

Theorem 1.1. (Existence and uniqueness) *(i) Let $p \in (d, \infty]$ and $\rho > 0$. Then, for all $u_0 \in (L^p_{uloc,\rho}(\mathbb{R}^d))^d$ such that $\nabla \cdot u_0 = 0$, there exist a positive T and a unique mild solution $u \in L^\infty((0, T); (L^p_{uloc,\rho})^d) \cap C((0, T); (L^p_{uloc,\rho})^d)$ of (NS) with initial data u_0 on $(0, T) \times \mathbb{R}^d$. The existence time T can be taken to satisfy*

$$T^{\frac{1}{2} + \frac{d}{2p}} \rho^{-\frac{2d}{p}} + T^{\frac{1}{2} - \frac{d}{2p}} \geq \frac{\gamma}{\|u_0\|_{L^p_{uloc,\rho}}},$$

where γ is a positive constant depending only on d and p .

(ii) Let $\rho > 0$. For all $u_0 \in \overline{(\cup_{p>d} L^p_{uloc,\rho}(\mathbb{R}^d))}^{\|\cdot\|_{L^d_{uloc,\rho}}}$ such that $\nabla \cdot u_0 = 0$, there exist a positive T and a mild solution $u \in L^\infty((0, T); (L^d_{uloc,\rho})^d) \cap C((0, T); (L^d_{uloc,\rho})^d)$ of (NS) with initial data u_0 on $(0, T) \times \mathbb{R}^d$. This solution may be chosen so that for all $T' \in (0, T)$ we have

$$\sup_{0 < t < T'} t^{\frac{1}{2}} \|u(t, \cdot)\|_{L^\infty} < \infty \quad \text{and} \quad \lim_{t \rightarrow 0} t^{\frac{1}{2}} \|u(t, \cdot)\|_{L^\infty} = 0.$$

With this extra condition on the L^∞ norm, such a solution is unique. The existence time T can be taken to satisfy $T \geq \min\{\rho^2, \alpha\}$, where α is a positive number satisfying

$$\sup_{0 < t < \alpha} t^{\frac{1}{4}} \|e^{t\Delta} u_0\|_{L^{2d}_{uloc,\rho}} \leq \gamma,$$

where γ is a positive constant depending only on d .

(iii) Let $\rho > 0$. There exists $\epsilon > 0$ such that for all $u_0 \in (L^d_{uloc,\rho})^d$ with $\|u_0\|_{L^d_{uloc,\rho}} \leq \epsilon$, there exist a positive T and a unique mild solution $u \in L^\infty((0, T); (L^d_{uloc,\rho})^d) \cap C((0, T); (L^d_{uloc,\rho})^d)$ of (NS) with initial data u_0 on $(0, T) \times \mathbb{R}^d$ such that $u(0, \cdot) = u_0$. The existence time T can be taken to satisfy

$$T^{\frac{1}{4}} \rho^{-\frac{1}{2}} + 1 \geq \frac{\gamma}{\|u_0\|_{L^d_{uloc,\rho}}},$$

where γ is a positive constant depending only on d .

H. Koch and D. Tataru [18] showed that for any given $T > 0$, one can construct a local mild solution of (NS) which exists at least until time T if the bmo^{-1} norm of the initial data is sufficiently small. Especially, they constructed local mild solutions for any initial data in vmO^{-1} . They also showed that one can construct global mild solutions for small initial data in BMO^{-1} . The definitions of bmo^{-1} , vmO^{-1} and BMO^{-1} are the following.

For $f \in \mathcal{S}'(\mathbb{R}^n)$ (i.e., f is a tempered distribution), we set

$$\|f\|_{BMO_T^{-1}} := \sup_{x \in \mathbb{R}^n, 0 < R^2 < T} \left(\frac{1}{|B(x, R)|} \int_{B(x, R)} \int_0^{R^2} |e^{t\Delta} f(y)|^2 dt dy \right)^{\frac{1}{2}},$$

where $B(x, R)$ is a ball centered at x with radius R and $|B(x, R)|$ is the Lebesgue measure of the ball $B(x, R)$. Then

$$\begin{aligned} BMO^{-1} &:= \{f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{BMO^{-1}} := \|f\|_{BMO_\infty^{-1}} < \infty\}, \\ bmo^{-1} &:= \{f \in \mathcal{S}'(\mathbb{R}^n); \|f\|_{bmo^{-1}} := \|f\|_{BMO_1^{-1}} < \infty\}, \end{aligned}$$

$$vmo^{-1} := \{f \in bmo^{-1}; \lim_{T \rightarrow 0} \|f\|_{BMO_T^{-1}} = 0\}.$$

The inclusion relations of $L^p_{uloc,\rho}$, bmo^{-1} , vmo^{-1} and BMO^{-1} are as follows:

$$\begin{aligned} L^p_{uloc,\rho} &\subset vmo^{-1} \text{ if } p > d, \\ L^p_{uloc,\rho} &\subset bmo^{-1} \text{ if } p \geq d, \\ L^d &\subset BMO^{-1}. \end{aligned}$$

These relations can be proved using the result of M. E. Taylor [27], but we will reproduce them in a different way in Section 2. Related to mild solutions constructed by H. Koch and D. Tataru [18], H. Miura [22] showed some uniqueness theorems of mild solutions of (NS).

Although our function spaces $L^p_{uloc,\rho}$ when $d \leq p \leq +\infty$, are contained in bmo^{-1} , our results are not included in H. Koch and D. Tataru [18], since they impose conditions such as

$$\sup_{0 < t < T} t^{\frac{1}{2}} \|u(t, \cdot)\|_{L^\infty} < \infty$$

for solutions. We shall show that the uniqueness of the solution holds without this condition, if the initial data belongs to $L^p_{uloc}(\mathbb{R}^n)$ for some $p \in (n, \infty]$. Another advantage is that the definition of $L^p_{uloc,\rho}$ is very simple and it obviously contains some functions which may have singularities and may not decay at space infinity. Moreover, the convergences of mild solutions to initial data when time goes to zero are relatively simple in our case. For describing the convergence of mild solutions to initial data, we define the subspace $\mathcal{L}^p_{uloc,\rho}$ as the closure of the space of bounded uniformly continuous functions $BUC(\mathbb{R}^d)$ in the space $L^p_{uloc,\rho}$, i.e.,

$$\mathcal{L}^p_{uloc,\rho} := \overline{BUC(\mathbb{R}^d)}^{\|\cdot\|_{L^p_{uloc,\rho}}}. \tag{1.3}$$

Recall that the subspace $\mathcal{L}^\infty_{uloc,\rho}(\mathbb{R}^d)$ is the space $BUC(\mathbb{R}^d)$. The space $\mathcal{L}^p_{uloc,\rho}$ is useful since we can show that the solutions converge to the initial data in the $L^p_{uloc,\rho}$ norm if the initial data belong to $\mathcal{L}^p_{uloc,\rho}$. In fact, we shall prove the following theorem.

Theorem 1.2. (Convergence to initial data) *(i) Let $p \in (d, \infty]$ and $\rho > 0$. Let $u \in L^\infty((0, T); (L^p_{uloc,\rho})^d)$ be a unique mild solution with initial data $u_0 \in L^p_{uloc,\rho}$. Then, for any compact set $K \subset \mathbb{R}^d$, we have*

$$\lim_{t \rightarrow 0} \|u(t) - u_0\|_{L^p(K)} = 0. \tag{1.4}$$

Moreover,

$$\lim_{t \rightarrow 0} \|u(t) - u_0\|_{L^p_{uloc,\rho}} = 0 \quad (1.5)$$

holds if and only if $u_0 \in \mathcal{L}^p_{uloc,\rho}$.

(ii) Let $\rho > 0$. Let u be a unique mild solution in $L^\infty((0, T); (L^d_{uloc,\rho})^d)$ which satisfies $t^{\frac{1}{2}}u(t) \in L^\infty((0, T); (L^\infty)^d)$, $\lim_{t \rightarrow 0} t^{\frac{1}{2}}\|u(t)\|_{L^\infty} = 0$. Then,

$$\lim_{t \rightarrow 0} \|u(t) - u_0\|_{L^d(K)} = 0. \quad (1.6)$$

Moreover, under the above condition,

$$\lim_{t \rightarrow 0} \|u(t) - u_0\|_{L^d_{uloc,\rho}} = 0 \quad (1.7)$$

holds if and only if $u_0 \in \mathcal{L}^d_{uloc,\rho}$.

One of the keys to our results is new $L^p_{uloc,\rho}$ - $L^q_{uloc,\rho}$ estimates we obtain, which are the following.

Let $1 \leq q \leq p \leq \infty$. Then for any $f \in L^p_{uloc,\rho}$, we have

$$\|e^{t\Delta}f\|_{L^p_{uloc,\rho}} \leq \left(\frac{C_1}{\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}} \right) \|f\|_{L^q_{uloc,\rho}}, \quad (1.8)$$

$$\|\nabla e^{t\Delta}f\|_{L^p_{uloc,\rho}} \leq \left(\frac{C_3}{t^{\frac{1}{2}}\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_4}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}} \right) \|f\|_{L^q_{uloc,\rho}}. \quad (1.9)$$

For $F \in (L^p_{uloc,\rho})^{d \times d}$,

$$\|e^{t\Delta}\mathbf{P}\nabla \cdot F\|_{L^p_{uloc,\rho}} \leq \left(\frac{C_5}{t^{\frac{1}{2}}\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_6}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}} \right) \|F\|_{L^q_{uloc,\rho}}. \quad (1.10)$$

Here $e^{t\Delta}$ is the heat semigroup, \mathbf{P} is the Helmholtz projection, and C_1, C_3, C_5 are positive constants depending only on d , and C_2, C_4, C_6 are positive constants depending only on d, p and q .

Let us state the outline of the proof. By rescaling, we may assume that $\rho = 1$. To obtain this estimate, we decompose \mathbb{R}^d into countable cubes whose centers are lattice points, i.e., $\mathbb{R}^d = \cup_{k \in \mathbb{Z}^d} S(k, \frac{1}{2})$, where $S(x, \theta) := \{y; \max_{1 \leq i \leq d} |y_i - x_i| \leq \theta\}$. We can decompose any measurable function f in \mathbb{R}^d into

$$f(x) = \sum_{k \in \mathbb{Z}^d} \chi_{S(k, \frac{1}{2})}(x) f(x), \quad a.e. \ x \in \mathbb{R}^d,$$

where $\chi_A(x)$ is the characteristic function of a subset A in \mathbb{R}^d . We decompose both a convolution kernel and a convoluted function in this way. Using Young's inequality for convolutions and the relation $\text{supp } f_1 * f_2 \subset \text{supp } f_1 +$

supp f_2 when $f_1 * f_2$ is well defined, we can obtain the desired estimates. In the proof, we also use the technique of estimating certain sums by certain integrals. After our work was completed, we were informed of the work by J.M. Arriera, A. Rodriguez-Bernal, J.W. Cholewa and T. Dlotko [1]. They treated linear parabolic equations in uniformly local L^p spaces. In particular, they also established $L^p_{uloc} - L^q_{uloc}$ estimates for the heat semigroup and their proof is similar to ours. However, we shall obtain $L^p_{uloc} - L^q_{uloc}$ estimates for more general convolution kernels including $e^{t\Delta}$, $\nabla e^{t\Delta}$ and $e^{t\Delta} \mathbf{P}\nabla \cdot$. For details, see Section 3.2.

This paper is organized as follows. In Section 2, we will state some properties of $L^p_{uloc,\rho}$ spaces and $\mathcal{L}^p_{uloc,\rho}$ spaces. In Section 3, we will prove our key $L^p_{uloc,\rho} - L^q_{uloc,\rho}$ estimates of convolution operators with integrable functions satisfying some conditions. By using these estimates, we will construct mild solutions of (NS) which are smooth in time and space. We also discuss the convergence of mild solutions to the initial data as time goes to zero. We will also obtain the existence time estimates of mild solutions which may depend on the parameter ρ . In Section 4, we will define L^p_{uloc} -almost periodic functions in all dimensions and we will show that the initial data at time zero of L^p_{uloc} -almost periodic functions become uniformly almost periodic (= L^∞ -almost periodic) functions in any positive time.

2. $L^p_{uloc,\rho}$ SPACE AND $\mathcal{L}^p_{uloc,\rho}$ SPACE

In this section, we state several properties of the function spaces $L^p_{uloc,\rho}$ and $\mathcal{L}^p_{uloc,\rho}$. When $\rho = 1$, we write L^p_{uloc} , \mathcal{L}^p_{uloc} instead of $L^p_{uloc,1}$, $\mathcal{L}^p_{uloc,1}$, respectively. We will often use the estimates for the operator $e^{t\Delta}$ in Section 3.2 (Recall that the estimates obtained in Section 3.2 are independent of the results in this section.) The inclusion relations for $L^p_{uloc,\rho}$ are as follows.

Proposition 2.1. (i) For any $\rho_1, \rho_2 > 0$, we have $L^p_{uloc,\rho_1} = L^p_{uloc,\rho_2}$ with equivalent norms.

(ii) For $1 \leq p_1 \leq p_2 \leq \infty$ and $\rho > 0$, we have $L^{p_2}_{uloc,\rho} \subset L^{p_1}_{uloc,\rho}$.

(iii) For any $\rho > 0$ and $1 \leq p \leq \infty$, we have $L^p \subset L^p_{uloc,\rho}$, $L^\infty \subset L^p_{uloc,\rho}$.

(iv) Let $\rho > 0$ and $p > d$. Then, we have $L^p_{uloc,\rho} \subset vmo^{-1} \subset bmo^{-1}$.

(v) Let $\rho > 0$. Then, we have $L^d_{uloc,\rho} \subset bmo^{-1}$.

Proof. The proofs of (i), (ii) and (iii) are easy, so we omit them. We use $L^p_{uloc,\rho} - L^q_{uloc,\rho}$ estimates for $e^{t\Delta}$ in Section 3.2 to prove the assertions (iv)

and (v). By Hölder’s inequality,

$$\begin{aligned} \int_{|x-x_0|\leq\sqrt{s}} |e^{s\Delta} f|^2 dx &\leq \left(\int_{|x-x_0|\leq\sqrt{s}} dx \right)^{1-\frac{2}{p}} \|e^{s\Delta} f\|_{L^p_{uloc,\sqrt{s}}}^2 \\ &\leq C s^{\frac{d}{2}(1-\frac{2}{p})} \|f\|_{L^p_{uloc,\sqrt{s}}}^2, \end{aligned}$$

where C is a positive constant depending only on d and p . So we have

$$\begin{aligned} \|f\|_{BMO_R^{-1}}^2 &\leq \sup_{0<t<R} \sup_{x_0\in\mathbb{R}^d} \frac{1}{t^{\frac{d}{2}}} \int_0^t \int_{|x-x_0|\leq\sqrt{s}} |e^{s\Delta} f|^2 dx ds \\ &\leq C \sup_{0<t<R} \sup_{x_0\in\mathbb{R}^d} \frac{1}{t^{\frac{d}{2}}} \int_0^t s^{\frac{d}{2}(1-\frac{2}{p})} \|f\|_{L^p_{uloc,\sqrt{s}}}^2 ds \\ &\leq C \sup_{0<t<R} t^{1-\frac{d}{p}} \|f\|_{L^p_{uloc,R}}^2 \leq CR^{1-\frac{d}{p}} \|f\|_{L^p_{uloc,R}}^2. \end{aligned}$$

This implies that $L^p_{uloc,\rho} \subset bmo^{-1}$ if $p \geq d$ and $L^p_{uloc,\rho} \subset vmo^{-1}$ if $p > d$. So (iv) and (v) hold.

Remark 2.1. The space L^∞ is not dense in L^p_{uloc} , hence not in $L^p_{uloc,\rho}$. More strongly, we have that the subset $\cup_{q>p} L^q_{uloc}$ is not dense in L^p_{uloc} . For example, let $\phi \in C^\infty_0(\mathbb{R}^d)$ be a function such that $\text{supp } \phi \subset B(0, 1)$, $\int |\phi|^p = 1$ and $\phi \geq 0$ where $B(0, 1)$ is a unit ball whose center is the origin. Choose any countable points $\{x_n\}_{n\geq 1}$ such that $B(x_n, 1) \cap B(x_m, 1) = \emptyset$. Set $\phi_n(x) = n^{\frac{d}{p}} \phi(n(x - x_n))$. Then $\text{supp } \phi_n \subset B(x_n, \frac{1}{n})$. So if we set

$$\tilde{\phi}(x) = \begin{cases} \phi_n(x), & \text{for } x \in B(x_n, 1) \\ 0, & \text{otherwise,} \end{cases}$$

then we have

$$\|\tilde{\phi}\|_{L^p_{uloc}} = \sup_n \|\phi_n\|_{L^p(B(x_n, 1))} = 1.$$

Let $q > p$. For any $g \in L^q_{uloc}(\mathbb{R}^d)$, we have $\|\tilde{\phi} - g\|_{L^p_{uloc}} \geq 1$. Indeed,

$$\begin{aligned} \|\tilde{\phi} - g\|_{L^p_{uloc}} &\geq \|\phi_n - g\|_{L^p(B(x_n, 1))} \geq \|\phi_n - g\|_{L^p(B(x_n, \frac{1}{n}))} \\ &\geq \left| \|\phi_n\|_{L^p(B(x_n, \frac{1}{n}))} - \|g\|_{L^p(B(x_n, \frac{1}{n}))} \right| \geq 1 - |B(x_n, \frac{1}{n})|^{\frac{1}{p}(1-\frac{p}{q})} \|g\|_{L^q_{uloc}} \\ &\geq 1 - \frac{C}{n^{d(\frac{1}{p}-\frac{1}{q})}} \|g\|_{L^q_{uloc}} \rightarrow 1 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

This implies that $\tilde{\phi}$ does not belong to $\overline{\cup_{q>p} L^q_{uloc}} \| \cdot \|_{L^p_{uloc}}$.

We have the following characterizations of $\mathcal{L}_{uloc,\rho}^p$.

Proposition 2.2. *For any $\rho > 0$, the following three statements are equivalent:*

- (i) $f \in \mathcal{L}_{uloc,\rho}^p$.
- (ii) $\lim_{|y| \rightarrow 0} \|f(\cdot + y) - f(\cdot)\|_{L_{uloc,\rho}^p} = 0$.
- (iii) $\lim_{t \rightarrow 0^+} \|e^{t\Delta} f - f\|_{L_{uloc,\rho}^p} = 0$.

Proof. Without loss of generality, we may assume that $\rho = 1$. For $p = \infty$, the above assertion is well known so we omit it.

Let $p \in [1, \infty)$. From the estimates (3.18) and (3.19) in Section 3.2, $e^{t\Delta} f$ is well defined and belongs to BUC when $t > 0$ for any $f \in L_{uloc}^p$. This implies (iii) \Rightarrow (i).

If $f \in \mathcal{L}_{uloc}^p$, there exists a sequence $\{f_n\}_{n \geq 1} \subset BUC$ such that $f_n \rightarrow f$ in L_{uloc}^p . So we have

$$\begin{aligned} & \|f(\cdot + y) - f(\cdot)\|_{L_{uloc}^p} \\ & \leq \|f(\cdot + y) - f_n(\cdot + y)\|_{L_{uloc}^p} + \|f_n(\cdot + y) - f_n(\cdot)\|_{L_{uloc}^p} + \|f_n(\cdot) - f(\cdot)\|_{L_{uloc}^p} \\ & \leq 2\|f(\cdot) - f_n(\cdot)\|_{L_{uloc}^p} + |B|^{\frac{1}{p}} \|f_n(\cdot + y) - f_n(\cdot)\|_{L^\infty}, \end{aligned}$$

where $|B|$ is Lebesgue measure of the unit ball in \mathbb{R}^d . The above inequality shows that (i) \Rightarrow (ii) holds. Let us show (ii) \Rightarrow (iii). Since

$$e^{t\Delta} f - f = \int_{\mathbb{R}^d} \frac{1}{(4\pi)^{\frac{d}{2}}} e^{-\frac{|z|^2}{4}} (f(x - \sqrt{t}z) - f(x)) dz,$$

we have

$$\|e^{t\Delta} f - f\|_{L_{uloc}^p} \leq \int_{\mathbb{R}^d} \frac{1}{(4\pi)^{\frac{n}{2}}} e^{-\frac{|z|^2}{4}} \|f(\cdot - \sqrt{t}z) - f(\cdot)\|_{L_{uloc}^p} dz.$$

So (iii) follows from (ii) and Lebesgue's convergence theorem.

Remark 2.2. This characterization was also obtained in [1].

Remark 2.3. The characterization (ii) in the above proposition is convenient to check whether a function f belongs to $\mathcal{L}_{uloc,\rho}^p$ or not. We will know that the characterization (iii) is important and useful when we consider the convergence of a mild solution to the initial data.

3. MILD SOLUTIONS IN $L^p_{uloc,\rho}$ AND $\mathcal{L}^p_{uloc,\rho}$

We will state the main results of this paper in this section. In Section 3.1, we will define mild solutions, i.e., solutions of the integral equations for the Navier-Stokes equations.

3.1. Definition of mild solutions. First, we define the convolution operator $e^{t\Delta}\mathbf{P}\nabla\cdot$ which appears in the nonlinear term of the integral equations of the Navier-Stokes equations. Notice that we regard $e^{t\Delta}\mathbf{P}\nabla\cdot$ as one operator. So we do not treat the operators $e^{t\Delta}\nabla\cdot$ and \mathbf{P} separately.

Definition 3.1. We define the operator $e^{t\Delta}\mathbf{P}\nabla\cdot$ (where \mathbf{P} is the Helmholtz projection) as follows.

For any $F \in (L^p_{uloc}(\mathbb{R}^d))^{d \times d}$, $1 \leq p \leq \infty$, we define

$$(e^{t\Delta}\mathbf{P}\nabla\cdot)F := \sum_{i=1}^d (e^{t\Delta}\mathbf{P} \frac{\partial}{\partial x_i})F \mathbf{e}_i, \quad (3.1)$$

where $\{\mathbf{e}_i\}_{i=1}^d$ is the standard bases in \mathbb{R}^d and the matrix-valued convolution kernel of the operator $(e^{t\Delta}\mathbf{P} \frac{\partial}{\partial x_i})$ is defined as

$$\left(\frac{1}{t^{\frac{d}{2}}}\partial_i(K_{j,k}\left(\frac{x}{\sqrt{t}}\right)) + \partial_i G_t(x)\delta_{j,k}\right)_{1 \leq j,k \leq d},$$

where $G_t(x)$ is the Gauss kernel

$$G_t(x) := \frac{1}{(4\pi t)^{\frac{d}{2}}} \exp\left(-\frac{|x|^2}{4t}\right)$$

and

$$K_{j,k} := -\mathcal{F}^{-1}\left(\frac{\xi_j \xi_k}{|\xi|^2} \exp(-|\xi|^2)\right).$$

Here \mathcal{F}^{-1} is the inverse Fourier transform

$$\mathcal{F}^{-1}g(x) := (2\pi)^{-\frac{d}{2}} \int_{\mathbb{R}^d} g(\xi) e^{ix \cdot \xi} d\xi.$$

The following lemma is the pointwise estimate of the function $K_{j,k}(x)$. This pointwise estimate was originally obtained by C. W. Oseen [23] (see also P. G. Lemarié-Rieusset [21], and Y. Shibata and S. Shimizu [25]). But we shall give the proof of this estimate for the sake of completeness.

Lemma 3.1. *Let $K_{j,k}(x)$ be the function defined as above. Then, for any non-negative multi-index $\alpha \in \mathbb{N}^d$, there exists $C_\alpha > 0$ such that*

$$|\partial_x^\alpha K_{j,k}(x)| \leq \frac{C_\alpha}{(1 + |x|)^{d+|\alpha|}}, \tag{3.2}$$

where C_α depends only on d and α .

Proof. By definition of $K_{j,k}(x)$, we have

$$\partial_x^\alpha K_{j,k}(x) = (2\pi)^{-\frac{d}{2}}(i)^{|\alpha|} \int_{\mathbb{R}^d} \frac{\xi_j \xi_k \xi^\alpha}{|\xi|^2} e^{-|\xi|^2} e^{ix \cdot \xi} d\xi, \tag{3.3}$$

where $\xi^\alpha = \xi^{\alpha_1} \dots \xi^{\alpha_d}$. Let $r \in (0, 1)$ and let $a_1(\xi)$ be a smooth radial symmetric function such that $a_1(\xi) = 1$ for $|\xi| < r$, $a_1(\xi) = 0$ for $|\xi| > 2r$, $0 \leq a_1(\xi) \leq 1$ and $|\partial_\xi^\alpha a_1(\xi)| \leq \frac{C_1}{|\xi|^{|\alpha|}}$. Let $a_2(\xi) = 1 - a_1(\xi)$. Then we have

$$\begin{aligned} \partial_x^\alpha K_{j,k}(x) &= (2\pi)^{-\frac{d}{2}}(i)^{|\alpha|} \int_{\mathbb{R}^d} \frac{\xi_j \xi_k \xi^\alpha}{|\xi|^2} a_1(\xi) e^{-|\xi|^2} e^{ix \cdot \xi} d\xi \\ &+ (2\pi)^{-\frac{d}{2}}(i)^{|\alpha|} \int_{\mathbb{R}^d} \frac{\xi_j \xi_k \xi^\alpha}{|\xi|^2} a_2(\xi) e^{-|\xi|^2} e^{ix \cdot \xi} d\xi \\ &=: I_1(x) + I_2(x). \end{aligned}$$

For I_1 we have the estimate

$$|I_1(x)| \leq (2\pi)^{-\frac{d}{2}} \int_{|\xi| \leq 2r} |\xi|^{|\alpha|} d\xi \leq A_1 r^{d+|\alpha|}, \tag{3.4}$$

where A_1 is a positive constant depending only on α and d .

Let us estimate I_2 . For any $x \in \mathbb{R}^d$ and multi-index β , we have by integration by parts,

$$\begin{aligned} x^\beta I_2(x) &= (2\pi)^{-\frac{d}{2}}(i)^{|\alpha|} \int_{\mathbb{R}^d} \frac{\xi_j \xi_k \xi^\alpha}{|\xi|^2} a_2(\xi) e^{-|\xi|^2} x^\beta e^{ix \cdot \xi} d\xi \\ &= (2\pi)^{-\frac{d}{2}}(i)^{|\alpha|-|\beta|} \int_{\mathbb{R}^d} \frac{\xi_j \xi_k \xi^\alpha}{|\xi|^2} a_2(\xi) e^{-|\xi|^2} \partial_\xi^\beta e^{ix \cdot \xi} d\xi \\ &= (2\pi)^{-\frac{d}{2}}(i)^{|\alpha|-|\beta|} (-1)^{|\beta|} \int_{\mathbb{R}^d} \partial_\xi^\beta \left(\frac{\xi_j \xi_k \xi^\alpha}{|\xi|^2} a_2(\xi) e^{-|\xi|^2} \right) e^{ix \cdot \xi} d\xi. \end{aligned}$$

From the inequality

$$|\partial_\xi^\theta a_2(\xi)| \leq \frac{C_1}{|\xi|^{|\theta|}}, \text{ for } |\theta| \geq 1 \quad \text{and} \quad |\xi|^\theta e^{-\frac{|\xi|^2}{2}} \leq C,$$

we easily see that

$$|\partial_\xi^\beta \left(\frac{\xi_j \xi_k \xi^\alpha}{|\xi|^2} a_2(\xi) e^{-|\xi|^2} \right)| \leq \frac{C_\beta}{|\xi|^{|\beta| - |\alpha|}} e^{-\frac{|\xi|^2}{2}}.$$

Hence,

$$\begin{aligned} |x^\beta I_2(x)| &\leq (2\pi)^{-\frac{d}{2}} \int_{|\xi| \geq r} \frac{C_\beta}{|\xi|^{|\beta| - |\alpha|}} e^{-\frac{|\xi|^2}{2}} d\xi \\ &= (2\pi)^{-\frac{d}{2}} C_\beta |S^{d-1}| \int_r^\infty \frac{1}{h^{|\beta| - |\alpha| - d + 1}} e^{-\frac{h^2}{2}} dh \\ &= \frac{C'_\beta}{r^{|\beta| - |\alpha| - d}}, \text{ if } |\beta| \neq |\alpha| + d, \end{aligned}$$

where C'_β is a positive constant depending only on β , α and d .

We also have the estimate $|I_2(x)| \leq C$. Thus, we have for $|\beta| > |\alpha| + d$,

$$|I_2(x)| \leq \frac{A_2}{(1 + |x|)^{|\beta| - |\alpha| - d}}, \quad (3.5)$$

where A_2 is a positive constant depending only on β , α and d .

If we take $r = (1 + |x|)^{-1}$, then we have from (3.4) and (3.5),

$$|I_1(x)| \leq \frac{A_1}{(1 + |x|)^{d + |\alpha|}} \quad \text{and} \quad |I_2(x)| \leq \frac{A_2}{(1 + |x|)^{d + |\alpha|}}.$$

The proof is now completed.

Next we define the mild solutions of (NS) as the solutions of the integral equations associated with (NS).

Definition 3.2. Let $p \geq d$. The function $u \in L^\infty((0, T); (L^p_{uloc, \rho})^d)$ is called a mild solution of the Navier-Stokes equations on $(0, T) \times \mathbb{R}^d$ if there exists $u_0 \in L^p_{uloc, \rho}$ with $\operatorname{div} u_0 = 0$ such that

$$u(t) = e^{t\Delta} u_0 - \int_0^t e^{(t-s)\Delta} \mathbf{P} \nabla \cdot (u \otimes u) ds \quad (3.6)$$

holds.

Remark 3.1. By the $L^p_{uloc, \rho} - L^q_{uloc, \rho}$ estimates in the following section, the terms $e^{t\Delta} u_0$ and $e^{(t-s)\Delta} \mathbf{P} \nabla \cdot (u \otimes u)$ can be defined pointwise if $u_0 \in (L^p_{uloc, \rho})^d$ and $u \in (L^p_{uloc, \rho})^d$, $p \geq 1$.

3.2. $L^p_{uloc,\rho} - L^q_{uloc,\rho}$ **estimates.** In this section, we shall prove the key estimates for the convolution operators, $e^{t\Delta}$, $\nabla e^{t\Delta}$, and $e^{t\Delta}\mathbf{P}\nabla\cdot$. In fact, we shall give the proof for a certain class of convolution operators, which includes these three operators. We say that a function is radial decreasing if it is radial symmetric and nonincreasing.

Theorem 3.1. *Let $1 \leq q \leq p \leq \infty$. Let $F(x), H(x)$ be two real-valued functions in \mathbb{R}^d and let $|F(x)| \leq H(x)$ hold. Furthermore, assume that H is a bounded, integrable and radial decreasing function in \mathbb{R}^d .*

We set $F_{t,m}(x) = t^{-\frac{d}{2}-m}F(x/t^{\frac{1}{2}})$ for $t > 0, m \geq 0$. Then, for any function $g \in L^q_{uloc,\rho}(\mathbb{R}^d)$, we can define pointwise

$$F_{t,m} * g(x) = \int_{\mathbb{R}^d} F_{t,m}(x - y)g(y)dy.$$

Furthermore, we have the estimate

$$\|F_{t,m} * g\|_{L^p_{uloc,\rho}} \leq \left(\frac{C_1\|H\|_{L^1}}{t^m\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2\|H\|_{L^r}}{t^{m+\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}} \right) \|g\|_{L^q_{uloc,\rho}}, \tag{3.7}$$

where r is the number satisfying $\frac{1}{p} = \frac{1}{r} + \frac{1}{q} - 1$, and C_1, C_2 are positive constants depending only on d .

Proof. It suffices to prove that

$$\|H_{t,0} * g\|_{L^p_{uloc,\rho}} \leq \left(\frac{C_1\|H\|_{L^1}}{\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2\|H\|_{L^r}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}} \right) \|g\|_{L^q_{uloc,\rho}}. \tag{3.8}$$

Moreover, by rescaling we may assume that $\rho = 1$. Indeed, if we set $f_\rho(x) = \rho f(\rho x)$, then we easily see the relations

$$\|f_\rho\|_{L^p} = \rho^{1-\frac{d}{p}}\|f\|_{L^p}, \tag{3.9}$$

$$\|f_\rho\|_{L^p_{uloc}} = \rho^{1-\frac{d}{p}}\|f\|_{L^p_{uloc,\rho}}, \tag{3.10}$$

$$\|f * g\|_{L^p_{uloc,\rho}} = \rho^{\frac{d}{p}-1}\|(f * g)_\rho\|_{L^p_{uloc}} = \rho^{\frac{d}{p}+d-2}\|f_\rho * g_\rho\|_{L^p_{uloc}}. \tag{3.11}$$

So if we have the estimate (3.8) for $\rho = 1$, then

$$\begin{aligned} \|H_{t,0} * g\|_{L^p_{uloc,\rho}} &= \rho^{\frac{d}{p}+d-2}\|(H_{t,0})_\rho * g_\rho\|_{L^p_{uloc}} \\ &\leq \rho^{\frac{d}{p}+d-2}\left(C_1\|H_\rho\|_{L^1} + \frac{C_2\|H_\rho\|_{L^r}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}\right)\|g_\rho\|_{L^q_{uloc}} \end{aligned}$$

$$\begin{aligned}
&= \rho^{\frac{d}{p}+d-2} (C_1 \rho^{1-d} \|H\|_{L^1} + \rho^{1-\frac{d}{r}} \frac{C_2 \|H\|_{L^r}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}}) \rho^{1-\frac{d}{q}} \|g\|_{L^q_{uloc,\rho}} \\
&= \left(\frac{C_1 \|H\|_{L^1}}{\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2 \|H\|_{L^r}}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}} \right) \|g\|_{L^q_{uloc,\rho}}.
\end{aligned}$$

First we decompose \mathbb{R}^d into countable cubes whose centers are lattice points, sides are of length one and parallel to the axis, i.e., $\mathbb{R}^d = \cup_{k \in \mathbb{Z}^d} S(k, \frac{1}{2})$, where $S(x, \theta) := \{y ; \max_{1 \leq i \leq d} |y_i - x_i| \leq \theta\}$. Recall that $|S(k, \frac{1}{2}) \cap S(l, \frac{1}{2})| = 0$ if $k \neq l$, where $|A|$ is Lebesgue measure of any Euclidean subset A .

One can easily check that there exist positive constants C and C' depending only on d such that

$$C \sup_{x \in \mathbb{R}^d} \|f\|_{L^p(B(x,1))} \leq \sup_{k \in \mathbb{Z}^d} \|f\|_{L^p(S(k, \frac{1}{2}))} \leq C' \sup_{x \in \mathbb{R}^d} \|f\|_{L^p(B(x,1))}, \quad (3.12)$$

where $B(x, \theta) := \{y ; |y - x| < \theta\}$. We now decompose the function $H_{t,0}(x)$ as

$$H_{t,0}(x) = \sum_{k \in \mathbb{Z}^d} \chi_{S(k, \frac{1}{2})}(x) H_{t,0}(x), \quad a.e. \ x,$$

and

$$g(x) = \sum_{k \in \mathbb{Z}^d} \chi_{S(k, \frac{1}{2})}(x) g(x), \quad a.e. \ x.$$

Let the number r satisfy $\frac{1}{p} = \frac{1}{r} + \frac{1}{q} - 1$.

Note that $\text{supp } f * g \subset \text{supp } f + \text{supp } g$ when $f * g$ is well defined. For any $k \in \mathbb{Z}^d$, we have

$$\begin{aligned}
&\|H_{t,0} * g\|_{L^p(S(k, \frac{1}{2}))} = \|(\sum_{k' \in \mathbb{Z}^d} \chi_{S(k', \frac{1}{2})} H_{t,0}) * (\sum_{k'' \in \mathbb{Z}^d} \chi_{S(k'', \frac{1}{2})} g)\|_{L^p(S(k, \frac{1}{2}))} \\
&= \|\sum_{k', k'' \in \mathbb{Z}^d} (\chi_{S(k', \frac{1}{2})} H_{t,0}) * (\chi_{S(k'', \frac{1}{2})} g)\|_{L^p(S(k, \frac{1}{2}))}. \quad (3.13)
\end{aligned}$$

Since $(\chi_{S(k', \frac{1}{2})} H_{t,0}) * (\chi_{S(k'', \frac{1}{2})} g)$ is well defined and

$$\text{supp}(\chi_{S(k', \frac{1}{2})} H_{t,0}) * (\chi_{S(k'', \frac{1}{2})} g) \subset S(k' + k'', 1)$$

holds, we have, by Young's inequality,

$$\begin{aligned}
(R.H.S.) \text{ of (3.13)} &\leq \sum_{\substack{k', k'' \in \mathbb{Z}^d \\ \max |k'_i + k''_i - k_i| \leq 1}} \|\chi_{S(k', \frac{1}{2})} H_{t,0}\|_{L^r} \|\chi_{S(k'', \frac{1}{2})} g\|_{L^q} \quad (3.14) \\
&\leq \sum_{k' \in \mathbb{Z}^d} 3^d \|\chi_{S(k', \frac{1}{2})} H_{t,0}\|_{L^r} \sup_{k'' \in \mathbb{Z}^d} \|g\|_{L^q(S(k'', \frac{1}{2}))}.
\end{aligned}$$

So we have to estimate the quantity $\sum_{k' \in \mathbb{Z}^d} \|\chi_{S(k', \frac{1}{2})} H_{t,0}\|_{L^r}$.

$$\begin{aligned} \sum_{k \in \mathbb{Z}^d} \|\chi_{S(k, \frac{1}{2})} \frac{1}{t^{\frac{d}{2}}} H\left(\frac{x}{\sqrt{t}}\right)\|_{L^r} &= \sum_{k \in \mathbb{Z}^d} \left(\int_{S(k, \frac{1}{2})} \left(\frac{1}{t^{\frac{d}{2}}} H\left(\frac{x}{\sqrt{t}}\right)\right)^r dx \right)^{\frac{1}{r}} \\ &= \sum_{k \in \mathbb{Z}^d} \left(\int_{S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}})} \left(\frac{1}{t^{\frac{d}{2}}} H(y)\right)^r t^{\frac{d}{2}} dy \right)^{\frac{1}{r}}. \end{aligned} \tag{3.15}$$

We set $y_{k,t} \in \mathbb{R}^d$ as the closest point to the origin in $S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}})$. In the right-hand side of (3.15), since $H(y)$ is radial decreasing, if k satisfies $\max_{1 \leq i \leq d} |k_i| \geq 2$, we can estimate

$$\left(\int_{S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}})} H(y)^r dy \right)^{\frac{1}{r}} \leq \left(\frac{1}{\sqrt{t}}\right)^{\frac{d}{r}} H(y_{k,t}).$$

The sum of the other terms (that is, k satisfies $\max_{1 \leq i \leq d} |k_i| \leq 1$) can be bounded from above by $3^d \|\frac{1}{t^{\frac{d}{2}}} H(\frac{x}{\sqrt{t}})\|_{L^r}$. So the right-hand side of (3.15) can be estimated as

$$\begin{aligned} (R.H.S.) \text{ of (3.15)} &\leq \frac{1}{t^{\frac{d}{2} - \frac{d}{2r}}} \sum_{\substack{k \in \mathbb{Z}^d \\ \max |k_i| \geq 2}} \left(\frac{1}{\sqrt{t}}\right)^{\frac{d}{r}} H(y_{k,t}) + 3^d \|\frac{1}{t^{\frac{d}{2}}} H\left(\frac{\cdot}{\sqrt{t}}\right)\|_{L^r} \\ &\leq \sum_{\substack{k \in \mathbb{Z}^d \\ \max |k_i| \geq 2}} \left(\frac{1}{\sqrt{t}}\right)^d H(y_{k,t}) + \frac{3^d}{t^{\frac{d}{2}(1-\frac{1}{r})}} \|H\|_{L^r}. \end{aligned} \tag{3.16}$$

We now consider the estimate of the term

$$\sum_{\substack{k \in \mathbb{Z}^d \\ \max |k_i| \geq 2}} \left(\frac{1}{\sqrt{t}}\right)^d H(y_{k,t}).$$

We draw a line from the origin to the point $y_{k,t}$. This line intersects several cubes of the form $(S(\frac{l}{\sqrt{t}}, \frac{1}{2\sqrt{t}}))^i$ (inside of the cube) where $l \in \mathbb{Z}^d$. We order these cubes as follows. The first cube is the cube $S(0, \frac{1}{2\sqrt{t}})$. The second cube is the cube which the line meets when it goes out of the first cube, and so on. We correspond the cube $S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}})$ to the second-to-last cube in this order. We denote this correspondence I_t , i.e., $I_t(S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}}))$ is the second-to-last cube in this order.

Then all points in the cube $I_t(S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}}))$ are closer to the origin than the point $y_{k,t}$. Hence,

$$\left(\frac{1}{\sqrt{t}}\right)^d H(y_{k,t}) \leq \int_{I_t(S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}}))} H(y) dy,$$

since H is radial decreasing. It is easy to see that

$$\#\{k' \in \mathbb{Z}^d ; I_t(S(\frac{k'}{\sqrt{t}}, \frac{1}{2\sqrt{t}})) = S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}})\} \leq 5^d$$

for any $k \in \mathbb{Z}^d$ with $\max_{1 \leq i \leq d} |k_i| \geq 2$. So

$$\begin{aligned} (R.H.S.) \text{ of (3.16)} &\leq \sum_{\substack{k \in \mathbb{Z}^d \\ \max |k_i| \geq 2}} \int_{I_t(S(\frac{k}{\sqrt{t}}, \frac{1}{2\sqrt{t}}))} H(y) dy + \frac{3^d}{t^{\frac{d}{2}(1-\frac{1}{r})}} \|H\|_{L^r} \\ &\leq 5^d \int_{\mathbb{R}^d} H(y) dy + \frac{3^d}{t^{\frac{d}{2}(1-\frac{1}{r})}} \|H\|_{L^r} \\ &\leq 5^d \|H\|_{L^1} + \frac{3^d}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}} \|H\|_{L^r}. \end{aligned} \quad (3.17)$$

This completes the proof.

Proposition 3.1. *For $F_1(x) := \exp(-|x|^2/4)$, $F_2(x) := |x| \exp(-|x|^2/4)$, and $F_3(x) := \frac{1}{(1+|x|)^{d+1}}$, there exist positive, bounded, integrable and radial decreasing functions H_1, H_2, H_3 satisfying $F_i \leq H_i$ ($i = 1, 2, 3$).*

Proof. Clearly, we can take $H_1 = F_1$ and $H_3 = F_3$. For F_2 , if we compute the derivative of $|x| \exp(-|x|^2/4)$, we can see that it suffices to set $H_2(x) = \sqrt{\frac{2}{e}}$ if $|x| < \sqrt{2}$, $H_2(x) = |x| \exp(-|x|^2/4)$ if $|x| \geq \sqrt{2}$.

From Theorem 3.1 and the proposition above, we can establish $L^p_{uloc,\rho} - L^q_{uloc,\rho}$ estimates for the linear term and the nonlinear term in the integral equations (3.6).

Corollary 3.1. *Let $1 \leq q \leq p \leq \infty$. Then for any $f \in L^p_{uloc,\rho}$, we have*

$$\|e^{t\Delta} f\|_{L^p_{uloc,\rho}} \leq \left(\frac{C_1}{\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_2}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}} \right) \|f\|_{L^q_{uloc,\rho}}, \quad (3.18)$$

$$\|\nabla e^{t\Delta} f\|_{L^p_{uloc,\rho}} \leq \left(\frac{C_3}{t^{\frac{1}{2}} \rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_4}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}} \right) \|f\|_{L^q_{uloc,\rho}}. \quad (3.19)$$

For $F \in (L^p_{uloc,\rho})^{d \times d}$,

$$\|e^{t\Delta} \mathbf{P}\nabla \cdot F\|_{L^p_{uloc,\rho}} \leq \left(\frac{C_5}{t^{\frac{1}{2}} \rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{C_6}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})+\frac{1}{2}}} \right) \|F\|_{L^q_{uloc,\rho}}, \tag{3.20}$$

holds. Here, C_1, C_3, C_5 are positive constants depending only on d , and C_2, C_4, C_6 are positive constants depending only on d, p and q .

Proof. Combining Theorem 3.1, Proposition 3.1, and Lemma 3.1, we can easily get the above estimates. We omit the details.

Remark 3.2. From Lemma 3.1 and pointwise estimates for derivatives of the Gauss kernel, we also obtain $L^p_{uloc,\rho} - L^q_{uloc,\rho}$ estimates for derivatives of the convolution operators; that is, it follows that

$$\|\partial_t^\alpha \partial_x^\beta e^{t\Delta} f\|_{L^p_{uloc,\rho}} \leq \frac{C}{t^{|\alpha|+\frac{|\beta|}{2}}} \left(\frac{1}{\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{1}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}} \right) \|f\|_{L^q_{uloc,\rho}}, \tag{3.21}$$

$$\|\partial_t^\alpha \partial_x^\beta e^{t\Delta} \mathbf{P}\nabla \cdot F\|_{L^p_{uloc,\rho}} \leq \frac{C'}{t^{|\alpha|+\frac{|\beta|}{2}+\frac{1}{2}}} \left(\frac{1}{\rho^{d(\frac{1}{q}-\frac{1}{p})}} + \frac{1}{t^{\frac{d}{2}(\frac{1}{q}-\frac{1}{p})}} \right) \|F\|_{L^q_{uloc,\rho}}, \tag{3.22}$$

where C, C' are positive constants depending only on d, p, q, α and β .

Here $\alpha \in \mathbb{N}$ and $\beta \in \mathbb{N}^d$ is any multi-index.

Remark 3.3. The $L^p_{uloc} - L^q_{uloc}$ estimates for the heat semigroup (3.21) are obtained in [1]. When $p = q = \infty$, the estimate (3.22) is obtained in [10].

3.3. Existence and uniqueness of mild solutions with initial data in $L^p_{uloc,\rho}$ and $\mathcal{L}^p_{uloc,\rho}$. In this section we prove the main results in this paper. In order to prove the existence and uniqueness of mild solutions, we use the following Picard contraction theorem.

Theorem 3.2. (The Picard contraction theorem) *Let E be a Banach space and let B be a bounded bilinear transform from $E \times E$ to E satisfying*

$$\|B(e, f)\|_E \leq C_B \|e\|_E \|f\|_E.$$

Then, if $0 < \delta < \frac{1}{4C_B}$ and if $e_0 \in E$ is such that $\|e_0\|_E \leq \delta$, the equation $e = e_0 - B(e, e)$ has a solution with $\|e\|_E \leq 2\delta$. This solution is unique in the ball $\overline{B}(0, 2\delta)$.

Proof. This result is well known, so we omit it. (See, e.g. [21].)

We define the Picard iteration sequence $\{e_n\}_{n=1}^\infty \subset E$ as

$$e_{n+1} = e_0 - B(e_n, e_n),$$

where B is a bounded bilinear transform in the above theorem.

Proof of Theorem 1.1. (i) Let \mathcal{E}_T be a Banach space defined as

$$\mathcal{E}_T = \{f \in L^\infty((0, T); (L^p_{uloc, \rho})^d) : t^{\frac{d}{2p}} f(t, \cdot) \in L^\infty((0, T); (L^\infty)^d)\}$$

with norm $\|f\|_{\mathcal{E}_T} := \sup_{0 < t < T} \|f(t, \cdot)\|_{L^p_{uloc, \rho}} + \sup_{0 < t < T} t^{\frac{d}{2p}} \|f(t, \cdot)\|_{L^\infty}$.

Let us estimate the bilinear form

$$B(f, g) = \int_0^t e^{(t-\tau)\Delta} \mathbf{P}\nabla \cdot (f \otimes g) d\tau,$$

where f and g belong to \mathcal{E}_T . By the estimate of the kernel of the convolution operator $e^{(t-\tau)\Delta} \mathbf{P}\nabla \cdot$ and Hölder's inequality

$$\|fg\|_{L^{\frac{p}{2}}_{uloc, \rho}} \leq \|f\|_{L^p_{uloc, \rho}} \|g\|_{L^p_{uloc, \rho}},$$

we have

$$\begin{aligned} \|B(f, g)\|_{L^p_{uloc, \rho}}(t) &\leq \int_0^t \|e^{(t-\tau)\Delta} \mathbf{P}\nabla \cdot (f \otimes g)\|_{L^p_{uloc, \rho}} d\tau \\ &\leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}} \rho^{\frac{d}{p}}} + \frac{C_6}{(t-\tau)^{\frac{1}{2} + \frac{d}{2p}}} \right) \|f \otimes g\|_{L^{\frac{p}{2}}_{uloc, \rho}} d\tau \\ &\leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}} \rho^{\frac{d}{p}}} + \frac{C_6}{(t-\tau)^{\frac{1}{2} + \frac{d}{2p}}} \right) d\tau \|f\|_{\mathcal{E}_T} \|g\|_{\mathcal{E}_T} \\ &= C(t^{\frac{1}{2}} \rho^{-\frac{d}{p}} + t^{\frac{1}{2} - \frac{d}{2p}}) \|f\|_{\mathcal{E}_T} \|g\|_{\mathcal{E}_T}. \end{aligned}$$

And we have

$$\begin{aligned} \|B(f, g)\|_{L^\infty}(t) &\leq \int_0^t \|e^{(t-\tau)\Delta} \mathbf{P}\nabla \cdot (f \otimes g)\|_{L^\infty} d\tau \\ &\leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}} \rho^{\frac{d}{p}}} + \frac{C_6}{(t-\tau)^{\frac{1}{2} + \frac{d}{2p}}} \right) \|f \otimes g\|_{L^p_{uloc, \rho}} d\tau \\ &\leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}} \rho^{\frac{d}{p}}} + \frac{C_6}{(t-\tau)^{\frac{1}{2} + \frac{d}{2p}}} \right) \frac{1}{\tau^{\frac{d}{2p}}} \tau^{\frac{d}{2p}} \|f\|_{L^\infty} \|g\|_{L^p_{uloc, \rho}} d\tau \\ &\leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}} \rho^{\frac{d}{p}}} + \frac{C_6}{(t-\tau)^{\frac{1}{2} + \frac{d}{2p}}} \right) \frac{1}{\tau^{\frac{d}{2p}}} d\tau \|f\|_{\mathcal{E}_T} \|g\|_{\mathcal{E}_T} \\ &\leq C(t^{\frac{1}{2} - \frac{d}{2p}} \rho^{-\frac{d}{p}} + t^{\frac{1}{2} - \frac{d}{p}}) \|f\|_{\mathcal{E}_T} \|g\|_{\mathcal{E}_T}. \end{aligned}$$

Thus,

$$\sup_{0 < t < T} t^{\frac{d}{2p}} \|B(f, g)\|_{L^\infty}(t) \leq C(T^{\frac{1}{2}} \rho^{-\frac{d}{p}} + T^{\frac{1}{2} - \frac{d}{2p}}) \|f\|_{\mathcal{E}_T} \|g\|_{\mathcal{E}_T}.$$

From the above estimates we have

$$\|B(f, g)\|_{\mathcal{E}_T} \leq C(T^{\frac{1}{2}} \rho^{-\frac{d}{p}} + T^{\frac{1}{2} - \frac{d}{2p}}) \|f\|_{\mathcal{E}_T} \|g\|_{\mathcal{E}_T}. \tag{3.23}$$

From the estimate (3.18) we easily see that $\|e^{t\Delta} u_0\|_{\mathcal{E}_T} \leq C(T^{\frac{d}{2p}} \rho^{-\frac{d}{p}} + 1) \|u_0\|_{L^p_{uloc,\rho}}$. So we find that we have a mild solution in \mathcal{E}_T for T small enough, by the Picard contraction principle. Moreover it is easy to see that the existence time T can be taken to satisfy

$$T^{\frac{1}{2} + \frac{d}{2p}} \rho^{-\frac{2d}{p}} + T^{\frac{1}{2} - \frac{d}{2p}} \geq \frac{\gamma}{\|u_0\|_{L^p_{uloc,\rho}}}, \tag{3.24}$$

where γ is a positive constant depending only on d and p .

Next we show the uniqueness of the mild solution in $L^p_{uloc,\rho}$. Let $u_1, u_2 \in L^\infty((0, T); (L^p_{uloc,\rho})^d)$ be two mild solutions with initial data $u_0 \in (L^p_{uloc,\rho}(\mathbb{R}^d))^d$, $p > d$. We set

$$K = \max\left\{ \sup_{0 < t < T} \|u_1\|_{L^p_{uloc,\rho}}(t), \sup_{0 < t < T} \|u_2\|_{L^p_{uloc,\rho}}(t) \right\}.$$

Then

$$\begin{aligned} & \|u_1 - u_2\|_{L^p_{uloc,\rho}}(t) \\ & \leq \int_0^t \|e^{(t-\tau)\Delta} \mathbf{P}\nabla \cdot ((u_1 - u_2) \otimes u_1) + u_2 \otimes (u_1 - u_2)\|_{L^p_{uloc,\rho}} d\tau \\ & \leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}}} + \frac{C_6}{(t-\tau)^{\frac{1}{2} + \frac{d}{2p}}} \right) \|((u_1 - u_2) \otimes u_1) + u_2 \otimes (u_1 - u_2)\|_{L^{\frac{p}{2}}_{uloc,\rho}} d\tau \\ & \leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}}} + \frac{C_6}{(t-\tau)^{\frac{1}{2} + \frac{d}{2p}}} \right) \|u_1 - u_2\|_{L^p_{uloc,\rho}} (\|u_1\|_{L^p_{uloc,\rho}} + \|u_2\|_{L^p_{uloc,\rho}}) d\tau \\ & \leq 2KC(T'^{\frac{1}{2}} \rho^{-\frac{d}{p}} + T'^{\frac{1}{2} - \frac{d}{2p}}) \sup_{0 < t < T'} \|u_1 - u_2\|_{L^p_{uloc,\rho}}(t), \end{aligned}$$

so we have

$$\sup_{0 < t < T'} \|u_1 - u_2\|_{L^p_{uloc,\rho}}(t) \leq 2KC(T'^{\frac{1}{2}} \rho^{-\frac{d}{p}} + T'^{\frac{1}{2} - \frac{d}{2p}}) \sup_{0 < t < T'} \|u_1 - u_2\|_{L^p_{uloc,\rho}}(t).$$

Thus, for sufficiently small $T' > 0$ it follows that $u_1 = u_2$ in $0 < t < T'$. Repeating this argument, we see that $u_1 = u_2$ in $0 < t < T$.

We shall show $u \in C((0, T); (L_{uloc, \rho}^p)^d)$. Let $0 < t < t + h < T$. Since $u(t) = e^{t\Delta}u_0 - B(u, u)(t)$, we have

$$\begin{aligned} & \|u(t+h) - u(t)\|_{L_{uloc, \rho}^p} \\ & \leq \|e^{(t+h)\Delta}u_0 - e^{t\Delta}u_0\|_{L_{uloc, \rho}^p} + \|B(u, u)(t+h) - B(u, u)(t)\|_{L_{uloc, \rho}^p}. \end{aligned}$$

From Corollary 3.1 we know $e^{t\Delta}u_0 \in \mathcal{L}_{uloc, \rho}^p$ for any $t > 0$. So by Proposition 2.2,

$$\|e^{(t+h)\Delta}u_0 - e^{t\Delta}u_0\|_{L_{uloc, \rho}^p} = \|e^{h\Delta}e^{t\Delta}u_0 - e^{t\Delta}u_0\|_{L_{uloc, \rho}^p} \rightarrow 0, \text{ as } h \rightarrow 0.$$

Next,

$$\begin{aligned} & \|B(u, u)(t+h) - B(u, u)(t)\|_{L_{uloc, \rho}^p} \\ & \leq \int_0^t \|e^{(h+t-s)\Delta}\mathbf{P}\nabla \cdot u \otimes u - e^{(t-s)\Delta}\mathbf{P}\nabla \cdot u \otimes u\|_{L_{uloc, \rho}^p} ds \\ & \quad + \int_t^{t+h} \|e^{(t+h-s)\Delta}\mathbf{P}\nabla \cdot u \otimes u\|_{L_{uloc, \rho}^p} ds = I_1 + I_2. \end{aligned}$$

Again, from Corollary 3.1 and Remark 3.2, we know $e^{(t-s)\Delta}\mathbf{P}\nabla \cdot u \otimes u$ belongs to $\mathcal{L}_{uloc, \rho}^p$ for $0 \leq s < t$, so we have from Proposition 2.2,

$$\|e^{(h+t-s)\Delta}\mathbf{P}\nabla \cdot u \otimes u - e^{(t-s)\Delta}\mathbf{P}\nabla \cdot u \otimes u\|_{L_{uloc, \rho}^p} \rightarrow 0 \text{ as } h \rightarrow 0,$$

for $0 \leq s < t$.

On the other hand,

$$\begin{aligned} & \|e^{(h+t-s)\Delta}\mathbf{P}\nabla \cdot u \otimes u - e^{(t-s)\Delta}\mathbf{P}\nabla \cdot u \otimes u\|_{L_{uloc, \rho}^p} \\ & \leq \left(\frac{C_5}{(t+h-s)^{\frac{1}{2}}\rho^{\frac{d}{p}}} + \frac{C_6}{(t+h-s)^{\frac{1}{2}+\frac{d}{2p}}} \right) \|u\|_{L_{uloc, \rho}^p} \|u\|_{L_{uloc, \rho}^p} \\ & \quad + \left(\frac{C_5}{(t-s)^{\frac{1}{2}}\rho^{\frac{d}{p}}} + \frac{C_6}{(t-s)^{\frac{1}{2}+\frac{d}{2p}}} \right) \|u\|_{L_{uloc, \rho}^p} \|u\|_{L_{uloc, \rho}^p} \\ & \leq 2 \left(\frac{C_5}{(t-s)^{\frac{1}{2}}\rho^{\frac{d}{p}}} + \frac{C_6}{(t-s)^{\frac{1}{2}+\frac{d}{2p}}} \right) \|u\|_{L_{uloc, \rho}^p} \|u\|_{L_{uloc, \rho}^p}. \end{aligned}$$

So by Lebesgue's convergence theorem, we have $I_1 \rightarrow 0$ as $h \rightarrow 0$.

$$I_2 \leq \int_t^{t+h} \left(\frac{C_5}{(t+h-s)^{\frac{1}{2}}\rho^{\frac{d}{p}}} + \frac{C_6}{(t+h-s)^{\frac{1}{2}+\frac{d}{2p}}} \right) \|u \otimes u\|_{L_{uloc, \rho}^p} ds$$

$$\leq C(h^{\frac{1}{2}}\rho^{-\frac{d}{p}} + h^{\frac{1}{2}-\frac{d}{2p}}) \sup_{0 < t < T} \|u(t)\|_{L^p_{uloc,\rho}} \rightarrow 0 \text{ as } h \rightarrow 0.$$

This implies that $u \in C((0, T); (L^p_{uloc,\rho})^d)$.

(ii) Let $\mathcal{E}_T, \mathcal{F}_T$ be Banach spaces defined as

$$\begin{aligned} \mathcal{E}_T &= \{f \in L^\infty((0, T); (L^d_{uloc,\rho})^d) : \\ &\quad t^{\frac{1}{2}}f(t, \cdot) \in L^\infty((0, T); (L^\infty)^d), \lim_{t \rightarrow 0} t^{\frac{1}{2}}\|f(t, \cdot)\|_{L^\infty} = 0\}, \\ \mathcal{F}_T &= \{f : t^{\frac{1}{4}}f(t, \cdot) \in L^\infty((0, T); (L^{2d}_{uloc,\rho})^d), \lim_{t \rightarrow 0} t^{\frac{1}{4}}\|f(t, \cdot)\|_{L^{2d}_{uloc,\rho}} = 0\}, \end{aligned}$$

with norm $\|f\|_{\mathcal{E}_T} = \sup_{0 < t < T} \|f(t, \cdot)\|_{L^d_{uloc,\rho}} + \sup_{0 < t < T} t^{\frac{1}{2}}\|f(t, \cdot)\|_{L^\infty}$ and $\|f\|_{\mathcal{F}_T} = \sup_{0 < t < T} t^{\frac{1}{4}}\|f(t, \cdot)\|_{L^{2d}_{uloc,\rho}}$, respectively. We can easily check that $\mathcal{E}_T \subset \mathcal{F}_T$. Indeed, for any $f \in \mathcal{E}_T$ we have

$$\begin{aligned} &t^{\frac{1}{4}} \left(\int_{|y-x| < \rho} |f(t, y)|^{2d} dy \right)^{\frac{1}{2d}} \\ &\leq t^{\frac{1}{4}} \|f(t, \cdot)\|_{L^\infty}^{\frac{1}{2}} \left(\int_{|y-x| < \rho} |f(t, y)|^d dy \right)^{\frac{1}{2d}} \\ &\leq \left(\sup_{0 < t < T} t^{\frac{1}{2}} \|f(t, \cdot)\|_{L^\infty} \right)^{\frac{1}{2}} \left(\sup_{0 < t < T} \|f(t, \cdot)\|_{L^d_{uloc,\rho}} \right)^{\frac{1}{2}}, \end{aligned}$$

hence $f \in \mathcal{F}_T$. For f and g in \mathcal{F}_T , applying the estimate of the kernel of the convolution operator $e^{(t-\tau)\Delta} \mathbf{P} \nabla \cdot$, we obtain

$$\begin{aligned} \|B(f, g)\|_{L^{2d}_{uloc,\rho}}(t) &\leq \int_0^t \|e^{(t-\tau)\Delta} \mathbf{P} \nabla \cdot (f \otimes g)\|_{L^{2d}_{uloc,\rho}} d\tau \\ &\leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}}\rho^{\frac{1}{2}}} + \frac{C_6}{(t-\tau)^{\frac{3}{4}}} \right) \|f \otimes g\|_{L^d_{uloc,\rho}} d\tau \\ &\leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}}\rho^{\frac{1}{2}}} + \frac{C_6}{(t-\tau)^{\frac{3}{4}}} \right) \tau^{-\frac{1}{2}} \tau^{\frac{1}{4}} \|f\|_{L^{2d}_{uloc,\rho}} \tau^{\frac{1}{4}} \|g\|_{L^{2d}_{uloc,\rho}} d\tau \\ &\leq C(\rho^{-\frac{1}{2}} + t^{-\frac{1}{4}}) \|f\|_{\mathcal{F}_T} \|g\|_{\mathcal{F}_T}, \end{aligned} \tag{3.25}$$

thus $B(f, g) \in \mathcal{F}_T$. For f and g in \mathcal{E}_T we have

$$\begin{aligned} &\|B(f, g)\|_{L^d_{uloc,\rho}} + \|B(g, f)\|_{L^d_{uloc,\rho}} \\ &\leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}}\rho^{\frac{1}{2}}} + \frac{C_6}{(t-\tau)^{\frac{3}{4}}} \right) (\|f \otimes g\|_{L^{\frac{2d}{3}}_{uloc,\rho}} + \|g \otimes f\|_{L^{\frac{2d}{3}}_{uloc,\rho}}) d\tau \end{aligned}$$

$$\begin{aligned}
&\leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}}\rho^{\frac{1}{2}}} + \frac{C_6}{(t-\tau)^{\frac{3}{4}}} \right) \|f\|_{L_{uloc,\rho}^{2d}} \|g\|_{L_{uloc,\rho}^d} d\tau \\
&\leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}}\rho^{\frac{1}{2}}} + \frac{C_6}{(t-\tau)^{\frac{3}{4}}} \right) \tau^{-\frac{1}{4}} d\tau \|f\|_{\mathcal{F}_T} \|g\|_{\mathcal{E}_T} \\
&= C(1 + t^{\frac{1}{4}}\rho^{-\frac{1}{2}}) \|f\|_{\mathcal{F}_T} \|g\|_{\mathcal{E}_T},
\end{aligned}$$

and

$$\begin{aligned}
&\|B(f, g)\|_{L^\infty} + \|B(g, f)\|_{L^\infty} \\
&\leq \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}}\rho^{\frac{1}{2}}} + \frac{C_6}{(t-\tau)^{\frac{3}{4}}} \right) (\|f \otimes g\|_{L_{uloc,\rho}^{2d}} + \|g \otimes f\|_{L_{uloc,\rho}^{2d}}) d\tau \\
&\leq 2 \int_0^t \left(\frac{C_5}{(t-\tau)^{\frac{1}{2}}\rho^{\frac{1}{2}}} + \frac{C_6}{(t-\tau)^{\frac{3}{4}}} \right) \tau^{-\frac{3}{4}} \tau^{\frac{1}{4}} \|f\|_{L_{uloc,\rho}^{2d}} \tau^{\frac{1}{2}} \|g\|_{L^\infty} d\tau \\
&\leq C(t^{-\frac{1}{2}} + t^{-\frac{1}{4}}\rho^{-\frac{1}{2}}) \|f\|_{\mathcal{F}_T} \|g\|_{\mathcal{E}_T},
\end{aligned}$$

so $B(f, g) \in \mathcal{E}_T$ and

$$\|B(f, g)\|_{\mathcal{E}_T} + \|B(g, f)\|_{\mathcal{E}_T} \leq C(1 + T^{\frac{1}{4}}\rho^{-\frac{1}{2}}) \|f\|_{\mathcal{F}_T} \|g\|_{\mathcal{E}_T}. \quad (3.26)$$

The estimate (3.25) proves that the Picard contraction scheme will provide a solution to the Navier-Stokes equations $u \in \mathcal{F}_T$ provided that the norm of $(e^{t\Delta}u_0)_{0 < t < T}$ in \mathcal{F}_T is small enough.

For $u_0 \in \overline{(\cup_{p>d} L_{uloc,\rho}^p)}^{\|\cdot\|_{L_{uloc,\rho}^d}}$, $(e^{t\Delta}u_0)_{0 < t < T}$ satisfies this condition if we take T sufficiently small, since it follows that

$$\lim_{T \rightarrow 0} \sup_{0 < t < T} t^{\frac{1}{4}} \|e^{t\Delta}u_0\|_{L_{uloc,\rho}^{2d}} = 0. \quad (3.27)$$

Indeed, for $u_0 \in (L_{uloc,\rho}^p)^d$ with $p > d$, we have

$$\|e^{t\Delta}u_0\|_{L_{uloc,\rho}^{2d}} \leq \left(\frac{C_1}{\rho^{d(\frac{1}{p}-\frac{1}{2d})}} + \frac{C_2}{t^{\frac{d}{2}(\frac{1}{p}-\frac{1}{2d})}} \right) \|u_0\|_{L_{uloc,\rho}^p},$$

so

$$\lim_{T \rightarrow 0} \sup_{0 < t < T} t^{\frac{1}{4}} \|e^{t\Delta}u_0\|_{L_{uloc,\rho}^{2d}} = 0.$$

Since the family of operators $(t^{\frac{1}{4}}e^{t\Delta})_{\mathcal{L}(L_{uloc,\rho}^d, L_{uloc,\rho}^{2d})}$ is equicontinuous, any

$u_0 \in \overline{(\cup_{p>d} L_{uloc,\rho}^p)}^{\|\cdot\|_{L_{uloc,\rho}^d}}$ also satisfies the above condition.

Especially, the existence time T can be taken to satisfy

$$T^{\frac{1}{4}}\rho^{-\frac{1}{2}} + 1 \geq \frac{\gamma}{\sup_{0 < t < T} t^{\frac{1}{4}} \|e^{t\Delta} u_0\|_{L^2_{uloc,\rho}}{}^d}.$$

Here γ is a positive constant depending only on d . From this estimate, one can deduce the following existence time estimate

$$T \geq \min\{\rho^2, \alpha\}, \tag{3.28}$$

where α is a positive number such that

$$\sup_{0 < t < \alpha} t^{\frac{1}{4}} \|e^{t\Delta} u_0\|_{L^2_{uloc,\rho}}{}^d \leq \frac{\gamma}{2}.$$

Next we show that a solution $u \in \mathcal{F}_T$ with initial value $u_0 \in (L^d_{uloc,\rho})^d$ also belongs to \mathcal{E}_T . For the Picard iteration sequence $\{u^{(n)}\}_{n \geq 1}$ in \mathcal{F}_T , we obtain from the property (3.27)

$$\limsup_{T \rightarrow 0} \sup_n \|u^{(n)}\|_{\mathcal{F}_T} = 0.$$

Furthermore, by the estimate (3.26), we have that the sequence $\{u^{(n)}\}_{n \geq 1}$ is bounded in \mathcal{E}_T if T is sufficiently small. Then, again by the estimate (3.26)

$$\begin{aligned} \|u^{(n+1)} - u^{(n)}\|_{\mathcal{E}_T} &= \|B(u^{(n)}, u^{(n)}) - B(u^{(n-1)}, u^{(n-1)})\|_{\mathcal{E}_T} \\ &\leq \|B(u^{(n)} - u^{(n-1)}, u^{(n)})\|_{\mathcal{E}_T} + \|B(u^{(n-1)}, u^{(n)} - u^{(n-1)})\|_{\mathcal{E}_T} \\ &\leq C \|u^{(n)} - u^{(n-1)}\|_{\mathcal{F}_T} (\|u^{(n)}\|_{\mathcal{E}_T} + \|u^{(n-1)}\|_{\mathcal{E}_T}). \end{aligned}$$

This implies that $\{u^{(n)}\}_{n \geq 1}$ is a Cauchy sequence in \mathcal{E}_T . Hence the limit function u belongs to \mathcal{E}_T . This proves the theorem (ii).

It remains to prove that $u \in C((0, T); (L^d_{uloc,\rho})^d)$. Let $0 < t < t + h < T$.

$$\begin{aligned} &\|u(t+h) - u(t)\|_{L^d_{uloc,\rho}} \leq \|e^{(t+h)\Delta} u_0 - e^{t\Delta} u_0\|_{L^d_{uloc,\rho}} \\ &+ \int_t^{t+h} \|e^{(t+h-s)\Delta} \mathbf{P}\nabla \cdot (u \otimes u)\|_{L^d_{uloc,\rho}} ds \\ &+ \int_0^t \|e^{(t+h-s)\Delta} \mathbf{P}\nabla \cdot (u \otimes u) - e^{(t-s)\Delta} \mathbf{P}\nabla \cdot (u \otimes u)\|_{L^d_{uloc,\rho}} ds \\ &=: J_1 + J_2 + J_3. \end{aligned}$$

Since $e^{t\Delta}u_0 \in \mathcal{L}_{uloc,\rho}^d$ for any $t > 0$, by Proposition 2.2, the term J_1 converges to zero as h goes to zero. From Corollary 3.1 we have

$$\begin{aligned} J_2 &\leq C \int_t^{t+h} \frac{1}{(t+h-s)^{\frac{1}{2}}} \|u\|_{L_{uloc,\rho}^d} \|u\|_{L^\infty} ds \\ &\leq C \frac{h^{\frac{1}{2}}}{t^{\frac{1}{2}}} \sup_{0 < t < T} \|u(t)\|_{L_{uloc,\rho}^d} \sup_{0 < t < T} t^{\frac{1}{2}} \|u(t)\|_{L^\infty} \rightarrow 0 \text{ as } h \rightarrow 0. \end{aligned}$$

Again from Corollary 3.1 and Remark 3.2, we know the function $e^{(t-s)\Delta}\mathbf{P}\nabla \cdot (u \otimes u)$ belongs to $\mathcal{L}_{uloc,\rho}^d$ for $0 \leq s < t$, so we have

$$\|e^{(h+t-s)\Delta}\mathbf{P}\nabla \cdot (u \otimes u) - e^{(t-s)\Delta}\mathbf{P}\nabla \cdot (u \otimes u)\|_{L_{uloc,\rho}^d} \rightarrow 0 \text{ as } h \rightarrow 0,$$

for $0 \leq s < t$. On the other hand, we have

$$\begin{aligned} &\|e^{(h+t-s)\Delta}\mathbf{P}\nabla \cdot (u \otimes u) - e^{(t-s)\Delta}\mathbf{P}\nabla \cdot (u \otimes u)\|_{L_{uloc,\rho}^d} \\ &\leq \left(\frac{C}{(t+h-s)^{\frac{1}{2}}} + \frac{C'}{(t-s)^{\frac{1}{2}}} \right) \|u\|_{L_{uloc,\rho}^d} \|u\|_{L^\infty} \\ &\leq \frac{C+C'}{(t-s)^{\frac{1}{2}}s^{\frac{1}{2}}} \left(\sup_{0 < t < T} \|u(t)\|_{L_{uloc,\rho}^d} \right) \left(\sup_{0 < t < T} t^{\frac{1}{2}} \|u(t)\|_{L^\infty} \right). \end{aligned}$$

By Lebesgue's convergence theorem, we have $J_3 \rightarrow 0$ as $h \rightarrow 0$.

(iii) The proof is similar to the proof of (ii). In this case, let \mathcal{E}_T and \mathcal{F}_T be defined as

$$\begin{aligned} \mathcal{E}_T &= \{f \in L^\infty((0, T); (L_{uloc,\rho}^d)^d) : t^{\frac{1}{2}}f(t, \cdot) \in L^\infty((0, T); (L^\infty)^d)\}, \\ \mathcal{F}_T &= \{f : t^{\frac{1}{4}}f(t, \cdot) \in L^\infty((0, T); (L_{uloc,\rho}^{2d})^d)\}. \end{aligned}$$

The norms of \mathcal{E}_T and \mathcal{F}_T are the same as in (ii). Then we have $\mathcal{E}_T \subset \mathcal{F}_T$ and by arguing similarly as in (ii),

$$\|B(f, g)\|_{\mathcal{F}_T} \leq C(1 + T^{\frac{1}{4}}\rho^{-\frac{1}{2}}) \|f\|_{\mathcal{F}_T} \|g\|_{\mathcal{F}_T} \quad (3.29)$$

$$\|B(f, g)\|_{\mathcal{E}_T} + \|B(g, f)\|_{\mathcal{E}_T} \leq C(1 + T^{\frac{1}{4}}\rho^{-\frac{1}{2}}) \|f\|_{\mathcal{F}_T} \|g\|_{\mathcal{E}_T}. \quad (3.30)$$

$$\|e^{\Delta t}u_0\|_{\mathcal{F}_T} \leq C(1 + T^{\frac{1}{4}}\rho^{-\frac{1}{2}}) \|u_0\|_{L_{uloc,\rho}^d}. \quad (3.31)$$

$$\|e^{\Delta t}u_0\|_{\mathcal{E}_T} \leq C(1 + T^{\frac{1}{2}}\rho^{-1}) \|u_0\|_{L_{uloc,\rho}^d}. \quad (3.32)$$

So as in the case as (ii), it suffices to show that if $\epsilon > 0$ is sufficiently small and $\|u_0\|_{L_{uloc,\rho}^d} \leq \epsilon$, then there exists a unique mild solution in \mathcal{F}_T for small

T . By considering the Picard iteration scheme, we can easily prove this assertion. The existence time T^* can be taken to satisfy

$$T^{\frac{1}{4}}\rho^{-\frac{1}{2}} + 1 \geq \frac{\gamma}{\|u_0\|_{L^d_{uloc,\rho}}}, \tag{3.33}$$

where γ is a positive constant depending only on d . Also arguing the same way as in (ii), we obtain $u \in C((0, T); (L^d_{uloc,\rho})^d)$. We omit the details. The proof of the theorem is now complete.

Remark 3.4. The existence time estimate for arbitrary $\rho > 0$ can also be deduced directly from that estimate for $\rho = 1$ if we consider an appropriate scaling of the solution of (NS) and use the relation (17).

Remark 3.5. We do not know the local existence of solutions for general initial data in $L^d_{uloc,\rho}$. One of the main difficulties is that the space L^∞ or $L^p_{uloc,\rho}$ ($p > d$) is not dense in $L^d_{uloc,\rho}$.

Remark 3.6. The existence of local mild solutions with initial data in L^∞ space is proved in Y. Giga, S. Matsui and K. Inui [10]. They also proved that the mild solutions are smooth with respect to the space variables and the time variable when the time is positive. We see that any mild solution $u(t)$ constructed in the proof of Theorem 1.1 belongs to $(L^\infty(\mathbb{R}^d))^d$ for each $t > 0$. So we know from the results of [10] that this mild solution is also smooth with respect to x and t for $t > 0$.

Remark 3.7. If $u_0 \in (\mathcal{L}^d_{uloc}(\mathbb{R}^d))^d$, we know from the result of H. Miura [22] that the associated mild solution $u(t) \in L^\infty((0, T); (L^d_{uloc})^d)$ satisfying $t^{\frac{1}{2}}u(t) \in L^\infty((0, T); (L^\infty)^d)$ is unique. So this mild solution has to satisfy $\lim_{T \rightarrow 0} \sup_{0 < t < T} t^{\frac{1}{2}}\|u(t)\|_{L^\infty} = 0$. This fact will be used in Theorem 4.1.

Remark 3.8. When $u_0 \in (L^p_{uloc,\rho})^d$, $p > d$, the inequality $\|u_0\|_{L^p_{uloc,\rho}} \leq \|u_0\|_{L^p}$ holds for any $\rho > 0$. So if we let $\rho > 0$ go to ∞ in (3.24), we have the estimate

$$T^{\frac{1}{2} - \frac{d}{2p}} \geq \frac{\gamma}{\|u_0\|_{L^p}}.$$

Hence, we can reproduce T. Kato's results [16].

Next we discuss the critical case $p = d$. From the existence time estimate (3.33), for sufficiently small data $u_0 \in (L^d(\mathbb{R}^d))^d$ we have

$$T^{\frac{1}{4}}\rho^{-\frac{1}{2}} + 1 \geq \frac{\gamma}{\|u_0\|_{L^d}}.$$

Thus, we can find that there exists $\epsilon > 0$ such that (NS) can be solved globally in time for any initial data $u_0 \in (L^d)^d$ such that $\|u_0\|_{L^d} < \epsilon$ (we can take ϵ as $\frac{\gamma}{2}$). Hence we can reproduce T. Kato's result [16] in this case too. So in a certain sense, our results may be regarded as a generalization of T. Kato's results about mild solutions of (NS) to uniformly local settings.

3.4. Convergence to initial data. In this section we consider the convergence of mild solutions to initial data as t goes to zero. The characterization of $\mathcal{L}_{uloc,\rho}^p$ in Proposition 2.2 is essentially used. We shall show that if u_0 belongs to $\mathcal{L}_{uloc,\rho}^p$ then $u(t)$ converges to u_0 in $L_{uloc,\rho}^p$ -norm as t goes to zero. Moreover, from the estimates of the nonlinear term, we shall show that the converse also holds if $p > d$.

Proof of Theorem 1.2. (i) It is easy to check that

$$\|B(u, u)(t)\|_{L_{uloc,\rho}^p} \leq Ct^{\frac{1}{2} - \frac{d}{2p}} \left(\sup_{0 < t < T} \|u(t)\|_{L_{uloc}^p} \right)^2 \rightarrow 0, \text{ as } t \rightarrow 0. \quad (3.34)$$

Moreover, for any compact subset K in \mathbb{R}^d , we have

$$\|e^{t\Delta}u_0 - u_0\|_{L^p(K)} \leq \int_{\mathbb{R}^d} \frac{1}{(4\pi)^{\frac{d}{2}}} e^{-\frac{|z|^2}{4}} \|u_0(\cdot - \sqrt{t}z) - u_0(\cdot)\|_{L^p(K)} dz.$$

By Lebesgue's convergence theorem, we have

$$\lim_{t \rightarrow 0} \|e^{t\Delta}u_0 - u_0\|_{L^p(K)} = 0.$$

Thus,

$$\lim_{t \rightarrow 0} \|u(t) - u_0\|_{L^p(K)} = 0$$

holds. For any $u_0 \in L_{uloc,\rho}^p$, we have from (3.34) and Proposition 2.2,

$$\begin{aligned} & \lim_{t \rightarrow 0} \|u(t) - u_0\|_{L_{uloc,\rho}^p} = 0 \\ \Leftrightarrow & \lim_{t \rightarrow 0} \|e^{t\Delta}u_0 - u_0\|_{L_{uloc,\rho}^p} = 0 \\ \Leftrightarrow & u_0 \in \mathcal{L}_{uloc,\rho}^p. \end{aligned}$$

(ii) Since

$$\|B(u, u)(t)\|_{L_{uloc,\rho}^d} \leq C \left(\sup_{0 < t < T} \|u(t)\|_{L_{uloc,\rho}^d} \right) \left(\sup_{0 < t < T} t^{\frac{1}{2}} \|u(t)\|_{L^\infty} \right),$$

we have

$$\lim_{t \rightarrow 0} \|B(u, u)(t)\|_{L_{uloc,\rho}^d} = 0$$

from the assumption for u . Therefore, arguing as in (i), we know that the assertions hold.

4. L^p_{uloc} -ALMOST PERIODIC INITIAL DATA

In this section we define the class of L^p_{uloc} -almost periodic functions in \mathbb{R}^d . We shall show that for any L^p_{uloc} -almost periodic initial data, the associated mild solutions become uniformly almost periodic ($=L^\infty$ -almost periodic) for any positive time. This result is an extension of the result of Y. Giga, A. Mahalov and B. Nicolaenko [11] which is concerned with L^∞ -almost periodic initial data.

4.1. **L^p_{uloc} -almost periodic functions.** A.S. Besicovitch [2] defined the class of L^p_{uloc} -almost periodic functions in \mathbb{R} . In this section we shall extend that notion to multi-dimensional cases. A set $E \subset \mathbb{R}^d$ is said to be relatively dense if there exists a number $r > 0$ such that any Euclidian ball with radius r has some intersection with E . Let f be in $L^p_{uloc}(\mathbb{R}^d)$ and $\epsilon \geq 0$. A point $\tau \in \mathbb{R}^d$ is called a translation point of f belonging to ϵ if $\|f(\cdot + \tau) - f(\cdot)\|_{L^p_{uloc}} \leq \epsilon$ holds. Let $E_p\{\epsilon, f\}$ denote the set of all translation points of f belonging to ϵ .

Definition 4.1. Let $p \in [1, \infty]$ and let f be in $L^p_{uloc}(\mathbb{R}^d)$. Then f is called L^p_{uloc} -almost periodic if $f \in \overline{BUC(\mathbb{R}^d)}^{L^p_{uloc}}$ ($= \mathcal{L}^p_{uloc}(\mathbb{R}^d)$) and for any $\epsilon > 0$, $E_p\{\epsilon, f\}$ is relatively dense.

The next characterization of L^p_{uloc} -almost periodicity is useful.

Lemma 4.1. Let $p \in [1, \infty]$. A function $f \in L^p_{uloc}(\mathbb{R}^d)$ is L^p_{uloc} -almost periodic if and only if

$$A_f := \{f(\cdot + \tau); \tau \in \mathbb{R}^d\}$$

is precompact in $L^p_{uloc}(\mathbb{R}^d)$.

In order to prove this lemma, we need the following proposition which is a consequence of Proposition 2.2.

Proposition 4.1. Let $p \in [1, \infty]$. Let $f \in L^p_{uloc}(\mathbb{R}^d)$. Assume that A_f defined above is precompact in $L^p_{uloc}(\mathbb{R}^d)$. Then f belongs to \mathcal{L}^p_{uloc} .

Proof. From Proposition 2.2, it suffices to show that

$$\lim_{|y| \rightarrow 0} \|f(\cdot + y) - f(\cdot)\|_{L^p_{uloc}} = 0. \tag{4.1}$$

Let $\{y_n\}_{n \geq 1}$ be an arbitrary sequence such that $y_n \rightarrow 0$ as $n \rightarrow \infty$. Since A_f is relatively compact, there exists a subsequence $\{y_{n(l)}\}_{l \geq 1}$ and a function $g \in L^p_{uloc}$ such that $f(\cdot + y_{n(l)}) \rightarrow g(\cdot)$ in L^p_{uloc} as $l \rightarrow \infty$. On the other hand,

for any compact set $K \subset \mathbb{R}^d$, we easily see that $f(\cdot + y_{n(l)}) \rightarrow f(\cdot)$ in $L^1(K)$ as $l \rightarrow \infty$. Thus $f = g$. This implies that (4.1) holds.

Proof of Lemma 4.1. Assume that f is an L^p_{uloc} -almost periodic function. Let ϵ be a given positive number. From the definition of almost periodicity, there exists $r(\epsilon/3) > 0$ such that for any $y \in \mathbb{R}^d$ there exist $x \in B(0, r(\epsilon/3))$ (a ball with radius $r(\epsilon/3)$ centered at the origin) and $\tau \in E\{\epsilon/3, f\}$ satisfying $y = \tau + x$. Let $\{h_i\}_{i=1}^\infty \subset \mathbb{R}^d$ be a given sequence. Then for each i , there exist $x_i \in B(0, r(\epsilon/3))$ and $\tau_i \in E\{\epsilon/3, f\}$ such that $h_i = \tau_i + x_i$. Since $\{x_i\}$ is bounded, there exists a subsequence $\{x_{i(l)}\}$ which converges to some $x_0 \in \mathbb{R}^d$. Since $f \in \mathcal{L}^p_{uloc}$, from Proposition 2.2 there exists $\delta > 0$ such that $\|f(\cdot + z) - f(\cdot)\|_{L^p_{uloc}} < \epsilon/3$ for z with $|z| < \delta$. We have $l_0 \in \mathbb{N}$ such that $\{x_{i(l)}\}_{l=l_0}^\infty \subset B(x_0, \delta/2)$. Collecting these facts, we obtain

$$\begin{aligned}
& \|f(\cdot + h_{i(l)}) - f(\cdot + h_{i(k)})\|_{L^p_{uloc}} \\
&= \|f(\cdot + \tau_{i(l)} + x_{i(l)}) - f(\cdot + \tau_{i(k)} + x_{i(k)})\|_{L^p_{uloc}} \\
&\leq \|f(\cdot + \tau_{i(l)} + x_{i(l)}) - f(\cdot + \tau_{i(k)} + x_{i(l)})\|_{L^p_{uloc}} \\
&\quad + \|f(\cdot + \tau_{i(k)} + x_{i(l)}) - f(\cdot + \tau_{i(k)} + x_{i(k)})\|_{L^p_{uloc}} \\
&= \|f(\cdot + \tau_{i(l)}) - f(\cdot + \tau_{i(k)})\|_{L^p_{uloc}} + \|f(\cdot) - f(\cdot + x_{i(k)})\|_{L^p_{uloc}} \\
&\leq \|f(\cdot + \tau_{i(l)}) - f(\cdot)\|_{L^p_{uloc}} + \|f(\cdot + \tau_{i(k)}) - f(\cdot)\|_{L^p_{uloc}} \\
&\quad + \|f(\cdot) - f(\cdot + x_{i(k)} - x_{i(l)})\|_{L^p_{uloc}} \\
&< \epsilon.
\end{aligned}$$

This implies that A_f is precompact in L^p_{uloc} .

Next we assume that $f \in L^p_{uloc}(\mathbb{R}^d)$ is not L^p_{uloc} -almost periodic. If $f \notin \mathcal{L}^p_{uloc}$, then from Proposition 4.1 we have that A_f is not relatively compact. So we may assume that there exists $\epsilon > 0$ such that $E\{\epsilon, f\}$ is not relatively dense. Then for any $h_1 \in \mathbb{R}^d$ there exist $r_1 > 2|h_1|$ and $h_2 \in \mathbb{R}^d$ such that

$$B(h_2, r_1) \cap E\{\epsilon, f\} = \emptyset.$$

Then $h_2 - h_1 \notin E\{\epsilon, f\}$ since $h_2 - h_1 \in B(h_2, r_1)$. Next we have $r_2 > 2(|h_1| + |h_2|)$ and $h_3 \in \mathbb{R}^d$ such that

$$B(h_3, r_2) \cap E\{\epsilon, f\} = \emptyset.$$

Then $h_3 - h_1, h_3 - h_2 \notin E\{\epsilon, f\}$ since $h_3 - h_1, h_3 - h_2 \in B(h_3, r_2)$. Repeating this, we obtain a sequence $\{h_i\} \subset \mathbb{R}^d$ such that $h_i - h_j \notin E\{\epsilon, f\}$ for $i \neq j$, or equivalently,

$$\|f(\cdot + h_i) - f(\cdot + h_j)\|_{L^p_{uloc}} = \|f(\cdot + h_i - h_j) - f(\cdot)\|_{L^p_{uloc}} > \epsilon.$$

This shows that A_f is not precompact in $L^p_{uloc}(\mathbb{R}^d)$.

4.2. L^p_{uloc} -almost periodic initial data. We shall show that for any L^p_{uloc} -almost periodic initial data which is divergence free, the associated mild solution becomes uniformly almost periodic ($=L^\infty$ -almost periodic) for any positive time. It suffices to consider the case $p = d$, since any L^p_{uloc} -almost periodic function where $p > d$ is also L^d_{uloc} -almost periodic. Note that any L^d_{uloc} -almost periodic function is by definition in \mathcal{L}^d_{uloc} . Thus the associated mild solution $u(t)$ satisfying $u(t) \in L^\infty((0, T); (L^d_{uloc})^d)$ and $t^{\frac{1}{2}}u(t) \in L^\infty((0, T); (L^\infty)^d)$ uniquely exists. In the proof of the theorem below, we essentially use Lemma 4.1. The key of the proof is to show the continuity of the mild solutions with respect to initial data. We shall use the following notation. For $f \in L^1_{loc}(\mathbb{R}^d)$, we write $f^\tau(x) := f(x + \tau)$.

Theorem 4.1. *Assume that u_0 is L^d_{uloc} almost periodic and satisfies $\operatorname{div} u_0 = 0$. Let $u(t) \in L^\infty((0, T); (L^d_{uloc})^d)$ be the associated mild solution of (NS) satisfying $t^{\frac{1}{2}}u(t) \in L^\infty((0, T); (L^\infty)^d)$. Then, for $0 < t < T$, $u(t)$ is L^∞ -almost periodic.*

Proof. Recall that by uniqueness the mild solution $u(t)$ satisfies

$$\sup_{0 < t < T} \|u(t)\|_{L^d_{uloc}} \leq K, \quad \lim_{T \rightarrow 0} \sup_{0 < t < T} t^{\frac{1}{2}} \|u(t)\|_{L^\infty} = 0,$$

where K is a positive constant depending only on $\|u_0\|_{L^d_{uloc}}$, T and d (see Remark 3.7). Interpolating these, we have

$$\lim_{T \rightarrow 0} \sup_{0 < t < T} t^{\frac{1}{4}} \|u(t)\|_{L^{2d}_{uloc}} = 0.$$

From Lemma 4.1, it suffices to show that $A_{u(t)} := \{u(t, \cdot + \tau); \tau \in \mathbb{R}^d\}$ is precompact in $(L^\infty(\mathbb{R}^d))^d$. For this, we shall show the continuous dependence of the mild solutions on initial data.

Let τ, τ' be any points in \mathbb{R}^d . Since $e^{t\Delta} \mathbf{P}\nabla \cdot$ is a convolution operator, we have

$$u^\tau(t) = e^{t\Delta} u_0^\tau - \int_0^t e^{(t-s)\Delta} \mathbf{P}\nabla \cdot (u^\tau \otimes u^\tau) ds.$$

From Corollary 3.1, we have

$$\begin{aligned} \|u^\tau - u^{\tau'}\|_{L^\infty}(t) &\leq \|e^{t\Delta}(u_0^\tau - u_0^{\tau'})\|_{L^\infty} \\ &+ \int_0^t \|e^{(t-s)\Delta} \mathbf{P}\nabla \cdot ((u^\tau - u^{\tau'}) \otimes u^\tau + u^{\tau'} \otimes (u^{\tau'} - u^\tau))\|_{L^\infty} ds \end{aligned}$$

$$\begin{aligned}
&\leq C(1+t^{-\frac{1}{2}})\|u_0^\tau - u_0^{\tau'}\|_{L_{uloc}^d} \\
&+ C \int_0^t \left(\frac{1}{(t-s)^{\frac{1}{2}}} + \frac{1}{(t-s)^{\frac{3}{4}}} \right) \|(u^\tau - u^{\tau'}) \otimes u^\tau + u^{\tau'} \otimes (u^{\tau'} - u^\tau)\|_{L_{uloc}^{2d}} ds \\
&\leq C(1+t^{-\frac{1}{2}})\|u_0^\tau - u_0^{\tau'}\|_{L_{uloc}^d} \\
&+ C \int_0^t \left(\frac{1}{(t-s)^{\frac{1}{2}}} + \frac{1}{(t-s)^{\frac{3}{4}}} \right) \frac{1}{s^{\frac{3}{4}}} ds \sup_{0 < t < T'} t^{\frac{1}{4}} \|u\|_{L_{uloc}^{2d}}(t) \sup_{0 < t < T'} t^{\frac{1}{2}} \|u^\tau - u^{\tau'}\|_{L^\infty}(t) \\
&\leq C(1+t^{-\frac{1}{2}})\|u_0^\tau - u_0^{\tau'}\|_{L_{uloc}^d} \\
&+ C'(t^{-\frac{1}{4}} + t^{-\frac{1}{2}}) \sup_{0 < t < T'} t^{\frac{1}{4}} \|u\|_{L_{uloc}^{2d}}(t) \sup_{0 < t < T'} t^{\frac{1}{2}} \|u^\tau - u^{\tau'}\|_{L^\infty}(t).
\end{aligned}$$

Thus,

$$\begin{aligned}
\sup_{0 < t < T'} t^{\frac{1}{2}} \|u^\tau - u^{\tau'}\|_{L^\infty}(t) &\leq C(1+t^{\frac{1}{2}})\|u_0^\tau - u_0^{\tau'}\|_{L_{uloc}^d} \\
&+ C'(1+t^{\frac{1}{4}}) \sup_{0 < t < T'} t^{\frac{1}{4}} \|u\|_{L_{uloc}^{2d}}(t) \sup_{0 < t < T'} t^{\frac{1}{2}} \|u^\tau - u^{\tau'}\|_{L^\infty}(t).
\end{aligned}$$

Since $\lim_{T \rightarrow 0} \sup_{0 < t < T} t^{\frac{1}{4}} \|u(t)\|_{L_{uloc}^{2d}} = 0$, for sufficiently small T' , we have

$$\sup_{0 < t < T'} t^{\frac{1}{2}} \|u^\tau - u^{\tau'}\|_{L^\infty}(t) \leq 2C(1+T^{\frac{1}{2}})\|u_0^\tau - u_0^{\tau'}\|_{L_{uloc}^d}.$$

This implies that $u(t)$ is L^∞ -almost periodic for $0 < t < T'$. As for $T' < t < T$, recall that $u(t)$ coincides with $v(t - \frac{T'}{2})$ which is the mild solution of (NS) with initial data $u(\frac{T'}{2})$. By the result of Y. Giga, A. Mahalov and B. Nicolaenko [11], we see that the mild solution $u(t)$ is L^∞ -almost periodic because $u(\frac{T'}{2})$ is L^∞ -almost periodic. This completes the proof.

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