Differential and Integral Equations Volume 19, Number 4 (2006), 361–368

REGULARITY OF WEAK SOLUTION TO A $p-curl$-SYSTEM

Hong-Ming Yin
Department of Mathematics, Washington State University
Pullman, WA 99164

(Submitted by: DiBenedetto Emmanuele)

Abstract. In this note we study the regularity of weak solutions to a nonlinear steady-state Maxwell's equation in conductive media: $\nabla \times \left[|\nabla \times \mathbf{H}|^{p-2}\nabla \times \mathbf{H}\right] = \mathbf{F}(x), p > 1$, where $\mathbf{H}(x)$ represents the magnetic field while $\mathbf{F}(x)$ is the internal magnetic current. It is shown that the weak solution to the above system is of class $C^{1+\alpha}$, which is optimal. The basic idea is to introduce a suitable potential and then to transform the system into a $p-$Laplacian equation subject to a Neumann type of boundary condition. The desired regularity is established by using the known theory for the scalar p-Laplacian equation.

1. Introduction

Let $p > 1$ and $\mathbf{F}(x)$ be a three-dimensional vector function defined in a bounded domain Ω in R^3 with $div\mathbf{F}(x) = 0$ on Ω. In this note a bold letter represents a three-dimensional vector or vector function. We are interested in finding the minimum over the Banach space $H_0^p(curl, div0, \Omega)$ for the following functional:

$$I[\mathbf{H}] = \frac{1}{p}\int_\Omega |\nabla \times \mathbf{H}|^p dx + \int_\Omega \mathbf{F}\cdot\mathbf{H}dx,$$

where

$$H_0^p(curl, div0, \Omega) := \{\mathbf{H} \in L^p(\Omega) : \nabla \times \mathbf{H} \in L^p(\Omega), \nabla\cdot\mathbf{H} = 0, x \in \Omega, \mathbf{N}\times\mathbf{H} = 0, x \in \partial\Omega\}.$$

By using the Dirichlet principle, we see that the minimum of $I[\mathbf{H}]$ is the solution of the following $p-curl$-system:

$$\nabla \times \left[|\nabla \times \mathbf{H}|^{p-2}\nabla \times \mathbf{H}\right] = \mathbf{F}(x), \qquad x \in \Omega, \tag{1.1}$$

$$\nabla\cdot\mathbf{H} = 0, \qquad x \in \Omega, \tag{1.2}$$

Accepted for publication: December 2005.
AMS Subject Classifications: 35Q60.

$$\mathbf{N} \times \mathbf{H}(x) = 0, \qquad\qquad x \in \partial\Omega, \tag{1.3}$$

where $\mathbf{N}$ is the outward unit normal on $\partial\Omega$.

Note that if $\mathbf{H}(x) = \{0, 0, h(x_1, x_2)\}$, it is not difficult to see that the $p - curl$-system (1.1) becomes the well-known $p-$Laplacian equation which has been studied extensively by many authors (see Choe [4],DiBenedetto [6], Evans [7], Lieberman [8, 9] etc., as well as the references therein). On the other hand, Equation (1.1) is the steady-state approximation for Bean's critical-state model for type-II superconductors ([3, 10, 13, 14]) where the magnetic field $\mathbf{H}$ is approximated by the solution of the $p - curl$-evolution system:

$$\mathbf{H}_t + \nabla \times \left[|\nabla \times \mathbf{H}|^{p-2} \nabla \times \mathbf{H} \right] = \mathbf{F}(x, t). \tag{1.4}$$

The interested reader may consult [1, 3, 13] for further physical background. The existence of a unique weak solution for the time-dependent $p - curl$-system (1.4) as well as the steady-state case (1.1)-(1.3) is established in [13, 14]. Moreover, some regularity of weak solution is also discussed in [14]. In this note we use the idea of [12] to show the optimal regularity for the weak solution of the steady-state system (1.1)-(1.3). The results of [6] and [8] (Theorem 2) play a crucial rule in this paper. Throughout this paper the following hypothesis is assumed to hold:

H(1.1):

(a) Let Ω be a bounded and simply-connected domain in R^3 with boundary $\partial\Omega \in C^2$.

(b) Let $\mathbf{F}(x) \in H(div0, \Omega) \bigcap C^{\alpha}(\bar{\Omega})$.

The main result in this note is the following theorem.

Theorem. *Under the assumption* H(1.1) *the weak solution* $\mathbf{H}(x)$ *of the problem* (1.1)–(1.3) *belongs to* $C^{1+\alpha}(\bar{\Omega})$ *for some* $\alpha \in (0, 1)$. *Moreover, there exists a constant* C *such that*

$$||\mathbf{H}||_{C^{1+\alpha}(\bar{\Omega})} \leq C,$$

where C *depends only on* $p, \Omega, \partial\Omega$ *and* C^{α}*-norm of* $\mathbf{F}(x)$.

Remark 1.1. The $C^{1+\alpha}$-regularity of weak solution to the time-dependent $p - curl$-system (1.4) is still open.

2. PROOF

We recall some function spaces associated with *curl* and *div*-operators ([5]). Other Sobolev spaces such as $H^1(\Omega)$ are the same as usual. Let $p > 1$.

$$H^p(curl, \Omega) \quad := \quad \{\mathbf{U} \in L^p(\Omega) : \nabla \times \mathbf{U} \in L^p(\Omega)\};$$

$$\begin{aligned} H^p(div,\Omega) &:= \{\mathbf{U}\in L^p(\Omega):\nabla\cdot\mathbf{U}\in L^p(\Omega)\};\\ H^p(curl0,\Omega) &:= \{\mathbf{U}\in H^p(curl,\Omega):\nabla\times\mathbf{U}(x)=0, x\in\Omega\};\\ H^p(div0,\Omega) &:= \{\mathbf{U}\in H^p(div,\Omega):\nabla\cdot\mathbf{U}=0, x\in\Omega\}. \end{aligned}$$

The norm of $H^p(curl,\Omega)$ is defined as

$$||\mathbf{U}||_{H^p(curl,\Omega)} = \Big[\int_\Omega [|\mathbf{U}|^p + |\nabla\times\mathbf{U}|^p]dx\Big]^{\frac{1}{p}}.$$

The norm of $H^p(div,\Omega)$ is defined similarly. Under these norms, $H^p(curl,\Omega)$ and $H^p(div,\Omega)$ are Banach spaces. Moreover, when $p=2$ we simply denote them by $H(curl,\Omega)$ and $H(div,\Omega)$. They are Hilbert spaces with the following inner products, respectively,

$$\begin{aligned} (\mathbf{U},\mathbf{K}) &= \int_\Omega [\mathbf{U}\cdot\mathbf{K} + (\nabla\times\mathbf{U})\cdot(\nabla\times\mathbf{K})]dx\\ (\mathbf{U},\mathbf{K}) &= \int_\Omega [\mathbf{U}\cdot\mathbf{K} + (\nabla\cdot\mathbf{U})\cdot(\nabla\cdot\mathbf{K})]dx. \end{aligned}$$

We first show some elementary properties.

Lemma 2.1. *For a vector function $\mathbf{F}(x)\in H(div0,\Omega)$ there exists a vector function $\mathbf{G}(x)\in H(curl,div0,\Omega)\bigcap C^\alpha(\bar\Omega)$ such that*

$$\mathbf{F}(x) = \nabla\times\mathbf{G}(x), \qquad x\in\Omega.$$

Moreover, each component of $\mathbf{G}(x)$ are strictly positive on $\bar\Omega$.

Proof. We begin with the classical case by assuming $\mathbf{F}(x)\in C^1(\Omega)$ with $\nabla\cdot\mathbf{F}(x)=0$ on Ω. Since Ω is a simply-connected domain, we know from elementary calculus that there exists a vector function, denoted by $\mathbf{G}_0(x)\in C^1(\bar\Omega)$, such that

$$\nabla\times\mathbf{G}_0(x) = \mathbf{F}(x), \qquad x\in\Omega.$$

It is clear that for any $\psi(x)\in C^2(\bar\Omega)$ the vector function $\mathbf{G}_0(x)+\nabla\psi$ also satisfies the above system (called Gauge invariance in the electromagnetic theory). From the theory of elliptic equations we see that there exists at least one solution for the following equation:

$$\Delta\psi + \nabla\cdot\mathbf{G}_0(x) = 0, x\in\Omega.$$

By choosing $\psi_0(x)$ to be the solution of the above equation, we find that $\mathbf{G}(x)=\mathbf{G}_0(x)+\nabla\psi_0(x)$ satisfies the div-condition. From the result of [2] we further know that $\mathbf{G}(x)\in C^\alpha(\bar\Omega)$. Moreover, by adding a large constant if necessary we can choose $\mathbf{G}(x)$ such that each component is positive on $\bar\Omega$.

Finally, since $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$ by using the standard approximation, we see the conclusion of Lemma 2.1 holds for any $\mathbf{F}(x) \in H(div0, \Omega)$. □

Lemma 2.2. *For a vector function* $\mathbf{K}(x) \in H(curl0, \Omega)$ *there exists a scalar function* $\psi(x) \in H^1(\Omega)$ *such that*

$$\mathbf{K}(x) = \nabla\psi(x), \qquad x \in \Omega.$$

Proof. Again we may assume that $\mathbf{K}(x) \in C^1(\bar{\Omega})$. Since the domain Ω is simply connected and $\nabla\times\mathbf{K}(x) = 0, x \in \Omega$, then $\mathbf{K}(x)$ must be a conservative field. Hence there exists a potential function $\psi(x) \in C^1(\bar{\Omega})$ such that

$$\mathbf{K}(x) = \nabla\psi(x), \qquad x \in \bar{\Omega}.$$

Since $C^1(\bar{\Omega})$ is dense in $H^1(\Omega)$, we see the conclusion of Lemma 2.2 holds for any $\mathbf{K}(x) \in H(curl0, \Omega)$. □

Lemma 2.3. *Suppose* $\mathbf{H}(x) \in H^p(curl, \Omega)$ *with* $\mathbf{N} \times \mathbf{H}(x) = 0$ *on* $\partial\Omega$ *in the sense of trace. Then*

$$\mathbf{N} \cdot (\nabla \times \mathbf{H}(x)) = 0, \qquad x \in \partial\Omega$$

in the sense of distribution.

Proof*: Assume that* $\mathbf{H}(x)$ *is smooth. For any* $\psi(x) \in H^1(\Omega)$,

$$\int_{\partial\Omega} [\mathbf{N} \cdot (\nabla \times \mathbf{H})\psi] ds = \int_{\Omega} \nabla[(\nabla \times \mathbf{H})\psi] dx$$
$$= \int_{\Omega} [(\nabla \times \mathbf{H}) \cdot \nabla\psi] dx = \int_{\partial\Omega} [(\mathbf{N} \times \mathbf{H}) \cdot (\nabla\psi)] ds = 0.$$

It follows that $\mathbf{N} \cdot (\nabla \times \mathbf{H}) = 0$ *in the sense of distribution.*

Proof of the Theorem. From Lemma 2.1, we choose

$$\mathbf{G}(x) \in H(curl, div0, \Omega) \cap C^{\alpha}(\bar{\Omega})$$

such that

$$\nabla \times \mathbf{G} = \mathbf{F}(x), \nabla \cdot \mathbf{G}(x) = 0, \qquad x \in \Omega.$$

Now we can rewrite Equation (1.1) as follows:

$$\nabla \times [|\nabla \times \mathbf{H}|^{p-2}\nabla \times \mathbf{H} - \mathbf{G}] = 0, \qquad x \in \Omega. \tag{2.1}$$

On the other hand, Lemma 2.2 implies that there exists a scalar function $\psi(x) \in H^1(\Omega)$ such that

$$|\nabla \times \mathbf{H}|^{p-2}\nabla \times \mathbf{H} - \mathbf{G} = \nabla\psi, \qquad x \in \Omega. \tag{2.2}$$

It follows that

$$|\nabla \times \mathbf{H}| = |\nabla\psi + \mathbf{G}|^{\frac{1}{p-1}}, \qquad x \in \Omega. \tag{2.3}$$

From Equations (2.2)-(2.3) we find

$$\begin{aligned}\nabla\times\mathbf{H} &= |\nabla\times\mathbf{H}|^{-(p-2)}[\nabla\psi+\mathbf{G}]\\ &= |\nabla\psi+\mathbf{G}|^{-\frac{p-2}{p-1}}[\nabla\psi+\mathbf{G}], \qquad x\in\Omega.\end{aligned} \tag{2.4}$$

Using the identity $div(curl\mathbf{K})=0$ in the weak sense for any vector $\mathbf{K}\in H(curl,\Omega)$ with $\mathbf{N}\times\mathbf{K}=0$ on $\partial\Omega$ we have

$$\nabla\big[|\nabla\psi+\mathbf{G}|^{-\frac{p-2}{p-1}}(\nabla\psi+\mathbf{G})\big]=0, \qquad x\in\Omega. \tag{2.5}$$

On the boundary $\partial\Omega$, we use Lemma 2.3 to obtain

$$\mathbf{N}\cdot[|\nabla\psi+\mathbf{G}|^{-\frac{p-2}{p-1}}(\nabla\psi+\mathbf{G})]=0, \qquad x\in\partial\Omega,$$

which is equivalent to

$$\nabla_n\psi=-\mathbf{N}\cdot\mathbf{G}, \qquad x\in\partial\Omega. \tag{2.6}$$

Note that

$$-\frac{p-2}{p-1}=\frac{p}{p-1}-2:=q-2,$$

where $q=\frac{p}{p-1}>1$ for any $p>1$.

To apply the regularity results for the scalar $p-$Laplacian equation, we have to verify that Equation (2.5) satisfies the same structure conditions as those in [6]. Let

$$\mathbf{A}(x,t,\nabla\psi):=|\nabla\psi+\mathbf{G}|^{q-2}[\nabla\psi+\mathbf{G}].$$

Then

$$\begin{aligned}\mathbf{A}(x,t,\mathbf{V})\cdot\mathbf{V} &= |\mathbf{V}+\mathbf{G}|^{q-2}[\mathbf{V}+\mathbf{G}]\cdot\mathbf{V}\\ &= |\mathbf{V}+\mathbf{G}|^{q}-|\mathbf{V}+\mathbf{G}|^{q-2}[\mathbf{V}+\mathbf{G}]\cdot\mathbf{G}\\ &\geq |\mathbf{V}+\mathbf{G}|^{q}-\varepsilon|\mathbf{V}+\mathbf{G}|^{q}-C(\varepsilon)|\mathbf{G}|^{q}\\ &\geq \frac{1-\varepsilon}{2^q}|\mathbf{V}|^{q}-C(\varepsilon)|\mathbf{G}|^{q},\end{aligned}$$

where at the final step we have used the following inequality:

$$|\mathbf{B}_1+\mathbf{B}_2|^q\leq 2^q[|\mathbf{B}_1|^q+|\mathbf{B}_2|^q],$$

where $\mathbf{B}_1$ and $\mathbf{B}_2$ are two vectors and $q>1$.

By choosing $\varepsilon=\frac{1}{2}$, we see

$$\mathbf{A}(x,t,\mathbf{V})\cdot\mathbf{V}\geq a_0|\mathbf{V}|^q-C,$$

where $a_0>0$ and C depends only on $|\mathbf{G}|^q$.

Moreover, for any vectors $\mathbf{B}_1, \mathbf{B}_2$ and $\beta \in (0,1)$ the following inequality also holds:

$$|\mathbf{B}_1 + \mathbf{B}_2|^\beta \leq |\mathbf{B}_1|^\beta + |\mathbf{B}_2|^\beta.$$

It follows that

$$|\mathbf{A}(x,t,\mathbf{V})| \leq C|\mathbf{V}|^{q-1} + C|\mathbf{G}|^{q-1},$$

where $C = max\{1, 2^{q-1}\}$.

It follows that the structure conditions (A_1) to (A_5) of [6] are satisfied. We see that the weak solution $\psi(x)$ is Hölder continuous for any $q > 1$ and there exists a constant C and an exponent $\alpha \in (0,1)$ such that

$$||\psi||_{C^\alpha(\bar{\Omega})} \leq C,$$

where C depends only on known data.

To obtain $C^{1+\alpha}$-regularity, we can use the idea of Lieberman [8] to prove the desired result since the structure conditions of [8] or [4] (see Theorem 2 of [8], page 1204 or Theorem 4.1 in [4]) are not satisfied here unless $\mathbf{G}(x)$ is a positive constant. Indeed, when $\mathbf{G}(x)$ is a positive constant, then the structure conditions in Theorem 1 of [8] with $k = \mathbf{G} > 0$ clearly hold. One can use the result of Theorem 2 of [8] to conclude the desired result. When $\mathbf{G}(x)$ is not a constant, we can use the perturbation argument to establish the regularity. Since the proof is quite similar to that of [9], we only give the idea of the proof here.

Indeed, for the homogeneous p-Laplacian equation, one has the $C^{1+\alpha}$-estimate (see Lemma 4 of [8], page 1210). Next one can use the perturbation argument to prove the general case. For any fixed $x_0 \in \Omega$, we consider the perturbed homogeneous equation in a ball $B_r(x_0) \subset \Omega$:

$$\begin{aligned} &\nabla[|\mathbf{G}(x_0) + \nabla v|^{q-2}\nabla v] = 0, \qquad x \in B_r(x_0), \\ &v(x) = \psi(x), \qquad \partial B_r(x_0). \end{aligned}$$

Since $\psi(x)$ is Hölder continuous, Lemma 5 of [8] holds. By using the same method of [9] (also see the proof of Theorem 4.5 [4], pages 36-38), we see $\psi(x)$ is of class $C^{1+\alpha}(\bar{\Omega})$ and there exists a constant C such that

$$||\psi||_{C^{1+\alpha}(\bar{\Omega}')} \leq C,$$

where Ω' is a compact sub-domain of Ω and C depends only on known data and $d := dist\{\Omega', \partial\Omega\}$.

The proof of $C^{1+\alpha}$-regularity for ψ near the boundary is also similar to [8] (Lemma 6, page 1213) since the boundary condition (2.6) and Lemma 2.1 imply that $\mathbf{N} \cdot \mathbf{G}(x)$ is Hölder continuous (see Theorem 2, condition (0.7)). We omit the details here.

From Equation (2.4), we know that $\mathbf{H}(x)$ satisfies

$$\begin{aligned} &\nabla \times \mathbf{H} = |\nabla\psi + \mathbf{G}|^{q-2}[\nabla\psi + \mathbf{G}], && x \in \Omega, \\ &\nabla \cdot \mathbf{H}(x) = 0, && x \in \Omega, \\ &\mathbf{N} \times \mathbf{H}(x) = 0, && x \in \partial\Omega. \end{aligned}$$

Ω is simply connected and hence the second Betti number of Ω is zero. From the result of [2] (Theorem 2.1) we find that $\mathbf{H}(x) \in C^{1+\alpha}(\bar{\Omega})$ and

$$||\mathbf{H}||_{C^{1+\alpha}(\bar{\Omega})} \leq C[||\nabla \times \mathbf{H}||_{C^{\alpha}(\Omega)} \leq C_2,$$

where C_2 depends only on $\Omega, \partial\Omega$ and C_1. □

Remark 2.1. The fact that the domain Ω is simply connected is necessary not only for Lemma 2.1 and Lemma 2.2 but also for the result of [2] since the $C^{1+\alpha}$-estimate of $\mathbf{U}$ depends on $div\mathbf{H}, curl\mathbf{H}, \mathbf{N} \times \mathbf{H}$ and the topology of the domain (the first and second Betti numbers, see [2] for details).

Remark 2.2. Unlike the steady-state case, for the time-dependent $p-curl$-system (1.4) there must be a restriction on the exponent p in order to derive $C^{1+\alpha}$-regularity (see [6]).

Acknowledgement. The author would like to thank Professor G. Lieberman for helpful discussions.

REFERENCES

[1] C.P. Bean, *Magnetization of high-field superconductors,* Rev. Mod. Phys., 36 (1964), 31–39.

[2] J. Bolik and W. von Wahl, *Estimating* ∇u *in terms of* $div\mathbf{U}, curl\mathbf{U}$*, either* $(\nu, \mathbf{U})$ *or* $\nu \times \mathbf{U}$ *and the topology,* Mathematical Methods in the Applied Sciences, 20 (1997), 737–744.

[3] S.J. Chapman, *A hierarchy of models for type-II superconductors,* SIAM Review, 42 (2000), 555–598.

[4] H.J. Choe, "Degenerate Elliptic and Parabolic Equations and Variational Inequality," Lectures Notes Series, Number 16, Seoul National University, Korean, 1993.

[5] P. Ciarlet, Jr. and E. Sonnedrücker, *A decomposition of the electromagnetic field-Application to the Darwin Model,* Mathematical Models and Methods in Applied Sciences, 7 (1997), 1085–1120.

[6] E. DiBenedetto, "Degenerate Parabolic Equations," Springer-Verlag, New York, 1993.

[7] L.C. Evans, *A new proof of local* $C^{1,\alpha}$ *regularity for solutions of certain degenerate elliptic P.D.E.,* J. of Diff. Eqs., 45 (1982), 356–373.

[8] G.M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations,* Nonlinear Analysis 12 (1988), 1203–1219.

[9] G.M. Lieberman, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural'tseva for elliptic equations,* Comm. P.D.E., 16 (1991), 311–361.

[10] L. Prigozhin, *On the Bean critical-state model in superconductivity,* European J. Appl. Math., 7 (1996), 237–247.

[11] W. von Wahl, *Estimating* ∇u *by* div$\mathbf{u}$ *and* curl$\mathbf{u}$*,* Math. Method. Appl. Sci., 20 (1997), 737–744.

[12] H.M. Yin, *Optimal regularity of solution to a degenerate elliptic system arising in electromagnetic fields,* Comm. on Pure and Applied Analysis, 1 (2002), 127–134.

[13] H.M. Yin, *On a* $p-$*Laplacian type of evolution system and applications to the Bean model in the type-II superconductivity theory,* Quarterly of Applied Mathematics, LIX (2001), 47–66.

[14] H.M. Yin, B.Q. Li, and J. Zou, *A degenerate evolution system modeling Bean's critical-state type-II superconductors,* Discrete and Continuous Dynamical Systems, 8 (2002), 781–794.