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EXISTENCE AND STABILITY IN THE $\alpha-$NORM FOR SOME PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS WITH INFINITE DELAY

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Abstract. In this work, we discuss the existence, regularity and stability of solutions for some partial functional differential equations with infinite delay. We assume that the linear part generates an analytic semigroup on a Banach space X and the nonlinear part is a Lipschitz continuous function with respect to the fractional power norm of the linear part.

1. Introduction

The aim here is to study the existence, the regularity and the stability of solutions of some class of partial functional differential equations with infinite delay and deviating arguments in terms involving spatial derivatives. The following model provides an example of such a situation:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}u(t,\xi) + \displaystyle\int_{-\infty}^{0} \dfrac{\partial}{\partial \xi}u(t+\theta,\xi)d\mu_1(\theta) + \displaystyle\int_{-\infty}^{0} u(t+\theta,\xi)d\mu_2(\theta) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \geq 0 \text{ and } 0 \leq \xi \leq \pi, \\ u(t,0) = u(t,\pi) = 0 \text{ for } t \geq 0, \\ u(\theta,\xi) = u_0(\theta,\xi) \text{ for } \theta \leq 0 \text{ and } 0 \leq \xi \leq \pi, \end{cases} \tag{1.1}$$

where the initial data u_0 is a given function from $(-\infty,0]\times[0,\pi]$ to $\mathbb{R}$, μ_1 and μ_2 are positive measures on $(-\infty,0]$. Equation (1.1) can be written in

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the following abstract form for partial functional differential equations with infinite delay:

$$\begin{cases} \dfrac{d}{dt}x(t) = -Ax(t) + F(x_t) \text{ for } t \geq 0 \\ x_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.2}$$

where $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space X, $\mathcal{B}$ is a Banach space of functions mapping $(-\infty, 0]$ to X and satisfying some axioms that will be introduced later. For $0 < \alpha < 1$, A^α denotes the fractional power of A; we assume that F is defined on a subspace $\mathcal{B}_\alpha$ with values in X, where $\mathcal{B}_\alpha$ is defined by

$$\mathcal{B}_\alpha = \{\varphi \in \mathcal{B} : \varphi(\theta) \in D(A^\alpha) \text{ for } \theta \leq 0 \text{ and } A^\alpha \varphi \in \mathcal{B}\},$$

the function $A^\alpha \varphi$ is defined by

$$(A^\alpha \varphi)(\theta) = A^\alpha (\varphi(\theta)) \text{ for } \theta \leq 0.$$

We suppose that F is Lipschitz continuous with respect to the fractional power norm of A^α. For every $t \geq 0$, the history function $x_t \in \mathcal{B}_\alpha$ is defined by

$$x_t(\theta) = x(t+\theta) \text{ for } \theta \leq 0.$$

We will discuss the existence, the regularity of solutions and the stability of equilibria in the α-norm for equation (1.2). Recall that when F is Lipschitz continuous in $\mathcal{B}$ with respect to the X-norm, equation (1.2) has been extensively studied by several authors; for more details we refer to [6], [7], [9], [10] and the references therein.

This work is motivated by the papers of Travis and Webb [21] and [22], where the authors studied the existence and stability in the $\alpha-$norm for partial functional differential equations with finite delay; they assumed that $F : C_\alpha = C([-r, 0]; D(A^\alpha)) \to X$ is continuous, where C_α is the Banach space of continuous functions from $[-r, 0]$ to $D(A^\alpha)$, endowed with the following norm

$$|\varphi|_\alpha = \sup_{-r \leq \theta \leq 0} |A^\alpha \varphi(\theta)|.$$

The authors investigated several results regarding the existence, the regularity and the stability of solutions in C_α. Recently, in [1], the authors established several results about the existence and the stability in the $\alpha-$norm for neutral partial functional differential equations.

The organization of this work is as follows. In Section 2, we study the existence and the regularity of the mild solutions in the $\alpha-$norm for equation (1.2). In Section 3, we establish a result about the linearized stability near an equilibrium of (1.2). In Section 4, we establish some fundamental smoothness results on the solution semigroup. In Section 5, we discuss the stability of solutions when F is linear. To illustrate our approach, we propose to study the stability of equation (1.1).

2. Existence and regularity of solutions

Let us recall some results that will be used throughout this work. Assume that,

($\mathbf{H_1}$) $-A$ is the infinitesimal generator of an analytic semigroup $(T(t))_{t\geq 0}$ on a Banach space X and $0 \in \rho(A)$, where $\rho(A)$ is the resolvent set of A.

Then, there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $|T(t)| \leq Me^{\omega t}$ for $t \geq 0$. Without loss of generality, we assume that $\omega > 0$. If the assumption $0 \in \rho(A)$ is not satisfied, one can substitute the operator A for the operator $(A - \sigma I)$ with σ large enough so that $0 \in \rho(A - \sigma I)$ and so we can always assume that $0 \in \rho(A)$.

For the fractional power $(A^\alpha, D(A^\alpha))$, for $0 < \alpha < 1$, and its inverse $A^{-\alpha}$, one has the following known result.

Theorem 2.1. ([19], pages 69-75) *Let $0 < \alpha < 1$ and assume that* ($\mathbf{H_1}$) *holds. Then*

(i) $D(A^\alpha)$ is a Banach space with the norm $|x|_\alpha = |A^\alpha x|$ for $x \in D(A^\alpha)$,

(ii) $T(t) : X \to D(A^\alpha)$ for $t > 0$,

(iii) $A^\alpha T(t)x = T(t)A^\alpha x$ for $x \in D(A^\alpha)$ and $t \geq 0$,

(iv) for every $t > 0$, $A^\alpha T(t)$ is bounded on X and there exists $M_\alpha > 0$ such that

$$|A^\alpha T(t)| \leq M_\alpha \frac{e^{\omega t}}{t^\alpha} \quad \text{for } t > 0, \tag{2.1}$$

(v) $A^{-\alpha}$ is a bounded linear operator on X with $D(A^\alpha) = \mathrm{Im}(A^{-\alpha})$,

(vi) if $0 < \alpha < \beta < 1$, then $D(A^\beta) \hookrightarrow D(A^\alpha)$,

(v) there exists $N_\alpha > 0$ such that

$$\left|(T(t) - I)A^{-\alpha}\right| \leq N_\alpha t^\alpha \text{ for } t > 0.$$

In the sequel, we denote by X_α the Banach space $(D(A^\alpha), |\cdot|_\alpha)$. Recall that $A^{-\alpha}$ is given by the following formulas

$$A^{-\alpha} = \frac{\sin(\pi\alpha)}{\alpha} \int_0^\infty t^{-\alpha}(t + A)^{-1} dt \tag{2.2}$$

or

$$A^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} T(t) dt. \tag{2.3}$$

Both integrals converge in the uniform operator topology. Consequently, if $T(t)$ is compact for every $t > 0$, then $A^{-\alpha}$ is compact for every $0 < \alpha < 1$. Moreover, if $0 < \alpha < \beta < 1$, then $A^{-\beta} : X \to X_\alpha$ is also compact.

From now on, we use an axiomatic definition of the phase space $\mathcal{B}$ which was first introduced by Hale and Kato in [15]. We assume that $\mathcal{B}$ is the normed space of functions mapping $(-\infty, 0]$ into X and satisfying the following fundamental axioms:

(**A**) there exist a positive constant N, a locally bounded function $M(\cdot)$ on $[0, +\infty)$ and a continuous function $K(\cdot)$ on $[0, +\infty)$, such that if $x : (-\infty, a] \to X$ is continuous on $[\sigma, a]$ with $x_\sigma \in \mathcal{B}$, for some $\sigma < a$, then for all $t \in [\sigma, a]$,

(i) $x_t \in \mathcal{B}$,

(ii) $t \to x_t$ is continuous with respect to $|\cdot|$ on $[\sigma, a]$,

(iii) $N|x(t)| \leq |x_t|_{\mathcal{B}} \leq K(t-\sigma) \sup_{\sigma \leq s \leq t} |x(s)| + M(t-\sigma)|x_\sigma|_{\mathcal{B}}$.

(**B**) $\mathcal{B}$ is a Banach space.

Lemma 2.2. [17] *Let C_{00} be the space of continuous functions mapping $(-\infty, 0]$ into X with compact supports and C_{00}^a be the subspace of functions with supports included in $[-a, 0]$ endowed with the uniform norm topology. Then $C_{00}^a \hookrightarrow \mathcal{B}$.*

Let $\mathcal{B}_\alpha = \{\varphi \in \mathcal{B} : \varphi(\theta) \in D(A^\alpha)$ for $\theta \leq 0$ and $A^\alpha \varphi \in \mathcal{B}\}$and provide $\mathcal{B}_\alpha$ with the following norm

$$|\varphi|_{\mathcal{B}_\alpha} = |A^\alpha \varphi|_{\mathcal{B}} \text{ for } \varphi \in \mathcal{B}_\alpha.$$

Assume that

($\mathbf{H}_2$) $F : \mathcal{B}_\alpha \to X$ is a Lipschitz continuous function for some $0 < \alpha < 1$. Let $k > 0$ be such that

$$|F(\varphi_1) - F(\varphi_2)| \leq k|\varphi_1 - \varphi_2|_{\mathcal{B}_\alpha} \text{ for } \varphi_1, \varphi_2 \in \mathcal{B}_\alpha.$$

The rest of this section will be devoted to the existence of the so-called mild and strict solutions.

Definition 2.3. A function $x : (-\infty, \infty) \to X_\alpha$ is called a mild solution of equation (1.2) if

(i) $x(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(x_s)ds$ for $t \geq 0$,

(ii) $x_0 = \varphi$.

Definition 2.4. A function $x : (-\infty, \infty) \to X_\alpha$ is called a strict solution of equation (1.2) if

(i) $t \to x(t)$ is continuously differentiable on $[0,\infty)$,
(ii) $x(t) \in D(A)$ for $t \geq 0$,
(iii) $x_0 = \varphi$,
(iv) x satisfies equation (1.2) for $t \geq 0$.
Suppose that
$(\mathbf{H}_3)$ $A^{-\alpha}\varphi \in \mathcal{B}$ for $\varphi \in \mathcal{B}$, where the function $A^{-\alpha}\varphi$ is defined by

$$\left(A^{-\alpha}\varphi\right)(\theta) = A^{-\alpha}\left(\varphi(\theta)\right) \text{ for } \theta \leq 0.$$

Lemma 2.5. *Assume that* $(\mathbf{H}_1)$ *and* $(\mathbf{H}_3)$ *hold. Then,* $\mathcal{B}_\alpha$ *is a Banach space.*

Proof. Let $(\varphi_n)_n$ be a Cauchy sequence in $\mathcal{B}_\alpha$. Then, $(A^\alpha\varphi_n)_n$ is a Cauchy sequence in $\mathcal{B}$. Since $\mathcal{B}$ is a Banach space, there exists a function $\varphi \in \mathcal{B}$ such that

$$|A^\alpha\varphi_n - \varphi|_{\mathcal{B}} \to 0 \text{ as } n \to \infty.$$

Let $\widetilde{\varphi}$ be defined by

$$\widetilde{\varphi} = A^{-\alpha}\varphi.$$

By assumption $(\mathbf{H}_3)$, the function $\widetilde{\varphi}$ belongs to $\mathcal{B}$ and

$$|\varphi_n - \widetilde{\varphi}|_{\mathcal{B}_\alpha} \to 0 \text{ as } n \to \infty.$$

Consequently, $\mathcal{B}_\alpha$ is a Banach space.

Theorem 2.6. *Assume that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$ *and* $(\mathbf{H}_3)$ *hold. Then, for* $\varphi \in \mathcal{B}_\alpha$, *equation* (1.2) *has a unique mild solution which is defined for all* $t \geq 0$.

Proof. Let $a > 0$ and $C([0,a]; X_\alpha)$ be the space of continuous functions from $[0,a]$ to X_α provided with the uniform norm topology. For $\varphi \in \mathcal{B}_\alpha$, we define the set

$$\Lambda = \{y \in C([0,a]; X_\alpha) : y(0) = \varphi(0)\}.$$

Let $y \in \Lambda$. We introduce the extension $\widetilde{y}$ of y on $(-\infty, a]$ by

$$\widetilde{y}(t) = \begin{cases} y(t) \text{ for } t \in [0,a] \\ \varphi(t) \text{ for } t \leq 0. \end{cases}$$

Let $\mathcal{J}$ be the operator defined on Λ by

$$\mathcal{J}(y)(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(\widetilde{y}_s)ds \text{ for } t \in [0,a].$$

We claim that $\mathcal{J}(\Lambda) \subset \Lambda$. In fact, let $y \in \Lambda$, $t_0 \in [0,a]$ and $t_0 < t < a$. Then

$$A^\alpha(\mathcal{J}(y)(t) - \mathcal{J}(y)(t_0)) = T(t)A^\alpha\varphi(0) - T(t_0)A^\alpha\varphi(0)$$

$$+\int_0^{t_0} A^\alpha \left(T(t-s)-T(t_0-s)\right) F(\widetilde{y}_s)ds + \int_{t_0}^t A^\alpha T(t-s)F(\widetilde{y}_s)ds.$$

Since

$$T(t)A^\alpha\varphi(0) - T(t_0)A^\alpha\varphi(0) \to 0 \text{ as } t \to t_0$$

and

$$\int_0^{t_0} A^\alpha \left(T(t-s)-T(t_0-s)\right) F(\widetilde{y}_s)ds = (T(t-t_0)-I)\int_0^{t_0} A^\alpha T(t_0-s)F(\widetilde{y}_s)ds,$$

it follows that

$$\int_0^{t_0} A^\alpha \left(T(t-s)-T(t_0-s)\right) F(\widetilde{y}_s)ds \to 0 \text{ as } t \to t_0.$$

Moreover,

$$\int_{t_0}^t \left|A^\alpha T(t-s)F(\widetilde{y}_s)\right| ds \to 0 \text{ as } t \to t_0.$$

Consequently,

$$A^\alpha(\mathcal{J}(y)(t) - \mathcal{J}(y)(t_0)) \to 0 \text{ as } t \to t_0 \text{ and } t > t_0.$$

Arguing as above, one can show that if $t_0 > 0$, then,

$$A^\alpha(\mathcal{J}(y)(t) - \mathcal{J}(y)(t_0)) \to 0 \text{ as } t \to t_0 \text{ and } t < t_0.$$

This implies that $\mathcal{J}(y) \in \Lambda$ for all $y \in \Lambda$. In order to show that $\mathcal{J}$ has a unique fixed point in Λ, we use the strict contraction principle. In fact, let $y, z \in \Lambda$ and $t \in [0,a]$. Then,

$$(\mathcal{J}(y) - \mathcal{J}(z))(t) = \int_0^t T(t-s)(F(\widetilde{y}_s) - F(\widetilde{z}_s))ds.$$

Taking the $\alpha-$norm, we obtain

$$\left|(\mathcal{J}(y) - \mathcal{J}(z))(t)\right|_\alpha \le M_\alpha k \int_0^t \frac{e^{\omega(t-s)}}{(t-s)^\alpha} \left|\widetilde{y}_s - \widetilde{z}_s\right|_{\mathcal{B}_\alpha} ds.$$

By axiom (**A**), one has

$$\left|\widetilde{y}_s - \widetilde{z}_s\right|_{\mathcal{B}_\alpha} \le K(s) \sup_{0\le\tau\le s} \left|y(\tau) - z(\tau)\right|_\alpha.$$

Let $K_a = \sup_{t\in[0,a]} K(t)$. Then,

$$\left|\mathcal{J}(y) - \mathcal{J}(z)\right|_\alpha \le \left(M_\alpha k K_a \int_0^a \frac{e^{\omega s}}{s^\alpha} ds\right) \left|y - z\right|_\alpha,$$

where $|y-z|_{\alpha}$ denotes the supremum norm in $C([0,a];X_{\alpha})$. If we choose a such that

$$M_{\alpha}kK_a\int_0^a \frac{e^{\omega s}}{s^{\alpha}}ds<1,$$

then, $\mathcal{J}$ is a strict contraction on Λ and it has a unique fixed point x which is the unique mild solution of equation (1.2) on $[0,a]$. To extend the solution x to $[a,2a]$, we consider the following equation

$$\begin{cases} \dfrac{d}{dt}y(t)=-Ay(t)+F(y_t) \text{ for } t\in[a,2a] \\ y_a=x_a\in\mathcal{B}_{\alpha}. \end{cases} \tag{2.4}$$

To show that equation (2.4) has a unique mild solution, we consider the operator $\mathcal{J}_a$ defined on $\Lambda_a=\{y\in C([a,2a];X_{\alpha}):y(a)=x(a)\}$ by

$$\mathcal{J}_a(y)(t)=T(t-a)x(a)+\int_a^t T(t-s)F(\widetilde{y}_s)ds \text{ for } t\in[a,2a],$$

where the function $\widetilde{y}$ is defined by

$$\widetilde{y}(t)=\begin{cases} y(t) \text{ for } t\in[a,2a] \\ x(t) \text{ for } t\le a. \end{cases}$$

Then $\mathcal{J}_a(\Lambda_a)\subset\Lambda_a$. Moreover, for $y,z\in\Lambda_a$, one has

$$|(\mathcal{J}_a(y)-\mathcal{J}_a(z))(t)|_{\alpha}\le \quad M_{\alpha}k\int_a^t\frac{e^{\omega(t-s)}}{(t-s)^{\alpha}}\,|\widetilde{y}_s-\widetilde{z}_s|_{\mathcal{B}_{\alpha}}\,ds.$$

On the other hand,

$$|\widetilde{y}_s-\widetilde{z}_s|_{\mathcal{B}_{\alpha}}\le K(s-a)\sup_{a\le\tau\le s}|\widetilde{y}(\tau)-\widetilde{z}(\tau)|_{\alpha}.$$

It follows that

$$|(\mathcal{J}_a(y)-\mathcal{J}_a(z))(t)|_{\alpha}\le\left(M_{\alpha}kK_a\int_a^t\frac{e^{\omega(t-s)}}{(t-s)^{\alpha}}ds\right)|y-z|_{\alpha,a}$$

where $|y-z|_{\alpha,a}$ denotes the supremum norm in $C([a,2a];X_{\alpha})$. Finally we arrive at

$$|(\mathcal{J}_a(y)-\mathcal{J}_a(z))(t)|_{\alpha}\le\left(M_{\alpha}kK_a\int_0^a\frac{e^{\omega s}}{s^{\alpha}}ds\right)|y-z|_{\alpha,a} \text{ for } t\in[a,2a]$$

and

$$|\mathcal{J}_a(y)-\mathcal{J}_a(z)|_{\alpha,a}\le\left(M_{\alpha}kK_a\int_0^a\frac{e^{\omega s}}{s^{\alpha}}ds\right)|y-z|_{\alpha,a}.$$

Consequently, the mapping $\mathcal{J}_a$ has a unique fixed point in Λ_a which is an extension of the solution x to $[a, 2a]$. Proceeding inductively, the solution x is uniquely and continuously extended to $[na, (n+1)a]$ for all $n \geq 1$.

For the regularity of the mild solution, we suppose that $\mathcal{B}$ satisfies the following axiom:

(**C**) if $(\varphi_n)_n$ is a Cauchy sequence in $\mathcal{B}$ and converges compactly to φ in $(-\infty, 0]$, then $\varphi \in \mathcal{B}$ and $|\varphi_n - \varphi|_{\mathcal{B}} \longrightarrow 0$ as $n \to \infty$.

Theorem 2.7. *Assume that* ($\mathbf{H}_1$), ($\mathbf{H}_2$), ($\mathbf{H}_3$) *hold and* $\mathcal{B}$ *satisfies axiom* (**C**). *Furthermore, assume that* $F : \mathcal{B}_\alpha \to X$ *is continuously differentiable and* F' *is Lipschitz continuous. Let* $\varphi \in \mathcal{B}_\alpha$ *be such that*

$$\varphi' \in \mathcal{B}_\alpha, \ \varphi(0) \in D(A) \ \textit{and} \ \varphi'(0) = -A\varphi(0) + F(\varphi).$$

Then, the corresponding mild solution x *is a strict solution of equation (1.2).*

Proof. The proof will be done through several steps. Let $a > 0$ and x denote the mild solution of equation (1.2). Consider the equation

$$\begin{cases} y(t) = T(t)\varphi'(0) + \int_0^t T(t-s) F'(x_s)(y_s) ds \text{ for } t \in [0, a] \\ y_0 = \varphi'. \end{cases} \tag{2.5}$$

Using the strict contraction principle, one can show that equation (2.5) has a unique continuous mild solution y on $(-\infty, a]$. Let $z \in C((-\infty, a]; X_\alpha)$ be defined by

$$z(t) = \begin{cases} \varphi(0) + \int_0^t y(s)\, ds \text{ for } t \in [0, a] \\ \varphi(t) \text{ for } t \in (-\infty, 0]. \end{cases}$$

We claim that

Lemma 2.8.

$$z_t = \varphi + \int_0^t y_s ds \ \textit{for} \ t \geq 0.$$

The proof of Lemma 2.8 is based on the following lemma which gives sufficient conditions to compute the integrals in $\mathcal{B}$ in term of the integrals in X.

Lemma 2.9. [18] *Assume that* $\mathcal{B}$ *satisfies axiom* (**C**). *Let* $\vartheta : [a, b] \to \mathcal{B}$ *be a continuous function such that* $\vartheta(t)(\theta)$ *is continuous for* $(t, \theta) \in [a, b] \times (-\infty, 0]$. *Then,*

$$\Big(\int_a^b \vartheta(s) ds\Big)(\theta) = \int_a^b \vartheta(s)(\theta) ds \ \textit{for} \ \theta \leq 0.$$

Proof of Lemma 2.8. Since the function $(s,\theta)\to y(s+\theta)$ is continuous from $[0,t]\times(-\infty,0]$ to X_α and $X_\alpha \hookrightarrow X$, it follows that $(s,\theta)\to y(s+\theta)$ is continuous from $[0,t]\times(-\infty,0]$ to X and, by Lemma 2.9, we get

$$\Big(\int_0^t y_s ds\Big)(\theta) = \int_0^t y(s+\theta)ds = \int_\theta^{t+\theta} y(s)ds \text{ for } \theta \le 0.$$

If $t+\theta \ge 0$, then

$$\int_\theta^{t+\theta} y(s)ds = \int_\theta^0 y(s)ds + \int_0^{t+\theta} y(s)ds = \varphi(0)-\varphi(\theta) + \int_0^{t+\theta} y(s)ds.$$

Since

$$\int_0^{t+\theta} y(s)ds = z(t+\theta)-\varphi(0),$$

it follows that

$$\Big(\int_0^t y_s ds\Big)(\theta) = z(t+\theta)-\varphi(\theta). \tag{2.6}$$

Next, if $t+\theta \le 0$, then

$$\int_\theta^{t+\theta} y(s)ds = \varphi(t+\theta)-\varphi(\theta). \tag{2.7}$$

Combining (2.6) and (2.7), we deduce that

$$z_t = \varphi + \int_0^t y_s ds \text{ for } t \ge 0.$$

Now, we will show that $x=z$ on $[0,a]$. To see this, notice that equation (2.5) yields

$$\int_0^t y(s)ds = \int_0^t T(s)\varphi'(0)ds + \int_0^t\int_0^s T(s-\sigma)F'(x_\sigma)(y_\sigma)d\sigma ds. \tag{2.8}$$

Since, by Lemma 2.8 the function $t\to F(z_t)$ is continuously differentiable on $[0,a]$ and for $0\le t\le a$,

$$\frac{d}{dt}\int_0^t T(t-s)F(z_s)ds = T(t)F(\varphi) + \int_0^t T(t-s)F'(z_s)(y_s)ds,$$

we have

$$\int_0^t T(s)F(\varphi)ds = \int_0^t T(t-s)F(z_s)ds - \int_0^t\int_0^s T(s-\sigma)F'(z_\sigma)(y_\sigma)d\sigma ds. \tag{2.9}$$

Hence

$$z(t) = \varphi(0) + \int_0^t T(s)(-A\varphi(0) + F(\varphi))ds + \int_0^t \int_0^s T(s-\sigma)F'(x_\sigma)(y_\sigma)d\sigma ds$$
$$= T(t)\varphi(0) + \int_0^t T(s)F(\varphi)ds + \int_0^t \int_0^s T(s-\sigma)F'(x_\sigma)(y_\sigma)d\sigma ds.$$

Moreover, by (2.9) we get that

$$z(t) = T(t)\varphi(0) + \int_0^t T(t-s)F(z_s)ds + \int_0^t \int_0^s T(s-\sigma)(F'(x_\sigma) - F'(z_\sigma))(y_\sigma)d\sigma ds.$$

Consequently,

$$x(t) - z(t) = \int_0^t T(t-s)(F(x_s) - F(z_s))ds + \int_0^t \int_0^s T(s-\sigma)(F'(x_\sigma) - F'(z_\sigma))(y_\sigma)d\sigma ds.$$

Since F and F' are Lipschitz continuous on $\mathcal{B}_\alpha$, let ϱ be the Lipschitz constant of F and F'. Taking the X_α-norm and using Axiom $(\mathbf{A}) - (iii)$, we deduce that

$$|x(t) - z(t)|_\alpha \leq \left(\varrho M_\alpha K_a(1 + a \sup_{0\leq\sigma\leq a} |y_\sigma|_{\mathcal{B}_\alpha}) \int_0^a \frac{e^{\omega s}}{s^\alpha} ds\right) \sup_{0\leq\tau\leq a} |x(\tau) - z(\tau)|_\alpha$$

for $t \in [0, a]$. If a is chosen such that

$$\varrho M_\alpha K_a(1 + a \sup_{0\leq\sigma\leq a} |y_\sigma|_{\mathcal{B}_\alpha}) \int_0^a \frac{e^{\omega s}}{s^\alpha} ds < 1,$$

then $x(t) = z(t)$ for $t \in [0, a]$. We claim that $x(t) = z(t)$ for $t \geq 0$. We proceed by contradiction and assume that there exists $t_1 > 0$ such that $x(t_1) \neq z(t_1)$. Let t^* be the smallest number such that $x(t) \neq z(t)$. Then

$$t^* = \inf \{t > 0 : |x(t) - z(t)| > 0\}.$$

By continuity, one has $x(t) = z(t)$ for $t \leq t^*$ and there exists $\varepsilon > 0$ such that

$$|x(t) - z(t)| > 0 \text{ for } t \in (t^*, t^* + \varepsilon).$$

It follows that for $t \in [t^*, t^* + \varepsilon]$

$$x(t) - z(t) = \int_{t^*}^t T(t-s)(F(x_s) - F(z_s))ds$$

$$+\int_{t^*}^{t}\int_{t^*}^{s} T(s-\sigma)(F'(x_\sigma)-F'(z_\sigma))(y_\sigma)d\sigma ds$$

and

$$\begin{aligned}&|x(t)-z(t)|_\alpha \\ &\leq \Big(\varrho M_\alpha K_{t^*}(1+\varepsilon \sup_{t^*\leq\sigma\leq t^*+\varepsilon}|y_\sigma|_{\mathcal{B}_\alpha})\int_0^\varepsilon \frac{e^{\omega s}}{s^\alpha}ds\Big)\sup_{t^*\leq\tau\leq t^*+\varepsilon}|x(\tau)-z(\tau)|_\alpha .\end{aligned}$$

If we choose ε such that

$$\varrho M_\alpha K_{t^*}(1+\varepsilon \sup_{t^*\leq\sigma\leq t^*+\varepsilon}|y_\sigma|_{\mathcal{B}_\alpha})\int_0^\varepsilon \frac{e^{\omega s}}{s^\alpha}ds<1,$$

then $x(t)=z(t)$ for $t\in[t^*,t^*+\varepsilon]$ which gives a contradiction. Consequently $x(t)=z(t)$ for $t\geq 0$. We conclude that $t\to x_t$ is continuously differentiable from $\mathbb{R}^+$ to $\mathcal{B}_\alpha$ and $F(x.)\in C^1(\mathbb{R}^+,X)$. To end the proof, we use the following lemma.

Lemma 2.10. ([19], Corollary 2.5, page 107) *Let $h:\mathbb{R}^+\to X$ be continuously differentiable and q satisfy*

$$q(t)=T(t)q_0+\int_0^t T(t-s)h(s)ds \text{ for } t\in\mathbb{R}^+.$$

If $q_0\in D(A)$, then q is continuously differentiable on $\mathbb{R}^+$ and

$$q'(t)=-Aq(t)+h(t) \text{ for } t\in\mathbb{R}^+.$$

As a consequence of Lemma 2.10, we deduce that x is a strict solution of equation (1.2) on $\mathbb{R}^+$.

3. The solution semigroup and the linearized stability

For $t\geq 0$, we define the operator $U(t)$ on $\mathcal{B}_\alpha$ by

$$U(t)(\varphi)=x_t(.,\varphi),$$

where $x(.,\varphi)$ is the mild solution of equation (1.2).

Proposition 3.1. $(U(t))_{t\geq 0}$ *is a strongly continuous semigroup on $\mathcal{B}_\alpha$; that is,*

(i) $U(0)=I$,

(ii) $U(t+s)=U(t)U(s)$ *for* $t,s\geq 0$,

(iii) for all $\varphi\in\mathcal{B}_\alpha$, $U(t)(\varphi)$ is a continuous function of $t\geq 0$ with values in $\mathcal{B}_\alpha$,

(iv) $(U(t))_{t\geq 0}$ *satisfies the translation property; that is, for* $t \geq 0$ *and* $\theta \leq 0$*, one has*

$$(U(t)(\varphi))(\theta) = \begin{cases} (U(t+\theta)(\varphi))(0) \text{ for } t+\theta \geq 0 \\ \varphi(t+\theta) \text{ for } t+\theta \leq 0, \end{cases}$$

(v) for all $t \geq 0$*,* $U(t)$ *is Lipschitz continuous from* $\mathcal{B}_\alpha$ *to* $\mathcal{B}_\alpha$*. Moreover, for all* $a > 0$*, there exists a positive constant* $l(a)$ *such that*

$$|U(t)\phi - U(t)\psi|_{\mathcal{B}_\alpha} \leq l(a)\, |\phi - \psi|_{\mathcal{B}_\alpha} \text{ for } t \in [0, a]\,.$$

Proof. Parts $(i) - (iv)$ are just consequences of the definition of the mild solutions. To show part (v), let $\varphi, \psi \in \mathcal{B}_\alpha$; Then

$$|U(t)\phi - U(t)\psi|_{\mathcal{B}_\alpha} = |A^\alpha x_t(.,\phi) - A^\alpha x_t(.,\psi)|_{\mathcal{B}}\,,$$

which implies

$$\begin{aligned} &|U(t)\phi - U(t)\psi|_{\mathcal{B}_\alpha} \\ &\leq K(t) \sup_{0\leq s\leq t} |A^\alpha x(s,\phi) - A^\alpha x(s,\psi)| + M(t)\, |A^\alpha \phi - A^\alpha \psi|_{\mathcal{B}} \end{aligned}$$

and

$$\begin{aligned} &|U(t)\phi - U(t)\psi|_{\mathcal{B}_\alpha} \leq K(t) \sup_{0\leq s\leq t} |A^\alpha T(s)(\phi(0) - \psi(0))| \\ &+ K(t) \sup_{0\leq s\leq t} \left| \int_0^s A^\alpha T(s-\sigma)\Big(F(x_\sigma(.,\phi)) - F(x_\sigma(.,\psi))\Big) d\sigma \right| \\ &\quad + M(t)\, |A^\alpha \phi - A^\alpha \psi|_{\mathcal{B}}\,. \end{aligned}$$

Using axiom $(\mathbf{A}) - (iii)$, we deduce that, for $a > 0$, there exist positive constants $m(a)$ and $n(a)$ such that for $t \in [0, a]$,

$$\begin{aligned} &|U(t)\phi - U(t)\psi|_{\mathcal{B}_\alpha} \\ &\leq m(a)\, |\phi - \psi|_{\mathcal{B}_\alpha} + n(a) \sup_{0\leq s\leq t} \int_0^s \frac{e^{\omega(s-\sigma)}}{(s-\sigma)^\alpha}\, |U(\sigma)\phi - U(\sigma)\psi|_{\mathcal{B}_\alpha}\, d\sigma \end{aligned}$$

and

$$|U(t)\phi - U(t)\psi|_{\mathcal{B}_\alpha} \leq m(a)|\phi - \psi|_{\mathcal{B}_\alpha} + n(a) \int_0^a \frac{e^{\omega s}}{s^\alpha} ds \sup_{0\leq \tau\leq a} |U(\tau)\phi - U(\tau)\psi|_{\mathcal{B}_\alpha}\,.$$

If we choose a small enough such that $n(a)\int_0^a \frac{e^{\omega s}}{s^\alpha} ds < 1$, then

$$|U(t)\phi - U(t)\psi|_{\mathcal{B}_\alpha} \leq \frac{m(a)}{1 - n(a)\int_0^a \frac{e^{\omega s}}{s^\alpha} ds}\, |\phi - \psi|_{\mathcal{B}_\alpha} \text{ for } t \in [0, a]\,. \tag{3.1}$$

If we proceed by steps, we can prove that the estimate (3.1) can be extended to all $a \geq 0$ and (v) follows.

In the sequel, we are concerned with the stability of equilibria. Recall that an equilibrium is a constant mild solution x^* of equation (1.2). Without loss of generality, we assume that $x^* = 0$ and $F(0) = 0$. To establish a linearized stability principle, we assume that:

($\mathbf{H}_4$) F is differentiable at zero.

Then the linearized equation at zero is given by

$$\begin{cases} \dfrac{d}{dt}y(t) = -Ay(t) + L(y_t) \text{ for } t \geq 0, \\ y_0 = \varphi \in \mathcal{B}_\alpha, \end{cases} \tag{3.2}$$

where $L = F'(0)$. Let $(S(t))_{t\geq 0}$ denote the solution semigroup associated with equation (3.2) and defined by

$$S(t)\varphi = y_t(.,\varphi) \text{ for } \varphi \in \mathcal{B}_\alpha \text{ and } t \geq 0,$$

where y is the mild solution of equation (3.2) .

Theorem 3.2. *Assume that* ($\mathbf{H}_1$), ($\mathbf{H}_2$), ($\mathbf{H}_3$) *and* ($\mathbf{H}_4$) *hold. Then, for* $t > 0$, *the derivative of* $U(t)$ *with respect to* φ *at zero is* $S(t)$.

Proof. Let $\varphi \in \mathcal{B}_\alpha$ and $t \geq 0$. Then,

$$|U(t)\varphi - S(t)\varphi|_{\mathcal{B}_\alpha} \leq K(t) \sup_{0\leq s\leq t} |x(s) - y(s)|_\alpha \,,$$

which implies that, for each $a > 0$, there exists a positive constant $\beta(a)$ such that, for $t \in [0, a]$,

$$|U(t)\varphi - S(t)\varphi|_{\mathcal{B}_\alpha} \leq \beta(a) \left\{ \sup_{0\leq s\leq t} \int_0^s \frac{e^{\omega(s-\sigma)}}{(s-\sigma)^\alpha} |U(\sigma)\varphi - S(\sigma)\varphi|_{\mathcal{B}_\alpha} d\sigma \right.$$
$$\left. + \sup_{0\leq s\leq t} \int_0^s \frac{e^{\omega(s-\sigma)}}{(s-\sigma)^\alpha} \left| F(S(\sigma)(\varphi)) - F'(0)(S(\sigma)\varphi) \right| d\sigma \right\}.$$

Let $\varepsilon > 0$. By assumption ($\mathbf{H}_4$), there exists $\delta > 0$ such that

$$\left| F(S(s)\varphi) - F'(0)(S(s)\varphi) \right| \leq \varepsilon \, |\varphi|_{\mathcal{B}_\alpha} \text{ for } |\varphi|_{\mathcal{B}_\alpha} \leq \delta \text{ and } s \in [0, a] \,.$$

Hence, for $t \in [0, a]$,

$$|U(t)\varphi - S(t)\varphi|_{\mathcal{B}_\alpha}$$
$$\leq \left(\beta(a)\varepsilon \int_0^a \frac{e^{\omega s}}{s^\alpha} ds \right) |\varphi|_{\mathcal{B}_\alpha} + \left(\beta(a) \int_0^a \frac{e^{\omega s}}{s^\alpha} ds \right) \sup_{0\leq s\leq a} |U(s)\varphi - S(s)\varphi|_{\mathcal{B}_\alpha} \,.$$

If we choose a such that $\beta(a)\int_0^a \frac{e^{\omega s}}{s^\alpha} ds < 1$, then

$$|U(t)\varphi - S(t)\varphi|_{\mathcal{B}_\alpha} \leq \varepsilon \frac{\beta(a)\int_0^a \frac{e^{\omega s}}{s^\alpha} ds}{1-\beta(a)\int_0^a \frac{e^{\omega s}}{s^\alpha} ds} |\varphi|_{\mathcal{B}_\alpha} \text{ for } |\varphi|_{\mathcal{B}_\alpha} \leq \delta \text{ and } t \in [0,a],$$

which implies that $U(t)$ is differentiable at 0 and $D_\varphi U(t)(0) = S(t)$, for t small enough. One can proceed by steps to show that the relationship $D_\varphi U(t)(0) = S(t)$ holds, for every $t > 0$.

Theorem 3.3. *Assume that conditions* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$ *and* $(\mathbf{H}_4)$ *hold. If* $(S(t))_{t\geq 0}$ *is exponentially stable in the sense that there exist* $M_0 > 0$ *and* $\epsilon > 0$ *such that*

$$|S(t)|_{\mathcal{B}_\alpha} \leq M_0 e^{-\epsilon t} \text{ for } t \geq 0,$$

then, the zero equilibrium of $(U(t))_{t\geq 0}$ *is locally exponentially stable in the sense that there exist* $\delta > 0$, $\mu > 0$ *and* $k_0 \geq 1$ *such that*

$$|U(t)\varphi|_{\mathcal{B}_\alpha} \leq k_0 e^{-\mu t} |\varphi|_{\mathcal{B}_\alpha} \text{ for } \varphi \in \mathcal{B}_\alpha \text{ with } |\varphi|_{\mathcal{B}_\alpha} \leq \delta \text{ and } t \geq 0.$$

Moreover, if $\mathcal{B}_\alpha$ *can be decomposed as* $\mathcal{B}_\alpha = \mathcal{B}_{\alpha,1} \oplus \mathcal{B}_{\alpha,2}$, *where* $\mathcal{B}_{\alpha,i}$ *are* S*-invariant subspaces of* $\mathcal{B}_\alpha$, $\mathcal{B}_{\alpha,1}$ *is finite dimensional and with*

$$\omega_0 = \lim_{h\to\infty} \frac{1}{h} \log |S(h)/\mathcal{B}_{\alpha,2}|_{\mathcal{B}_\alpha},$$

we have

$$\inf\{|\lambda| : \lambda \in \sigma(S(t)/\mathcal{B}_{\alpha,1})\} > e^{\omega_0 t},$$

then, zero is not stable in the sense that there exist $\varepsilon > 0$ *and sequences* $(\varphi_n)_n$ *converging to* 0 *and* $(t_n)_n$ *of positive real numbers such that*

$$|U(t_n)\varphi_n|_{\mathcal{B}_\alpha} > \varepsilon.$$

The above theorem is a consequence of Theorem 3.2 and of the following result.

Theorem 3.4. [11] *Let* $(V(t))_{t\geq 0}$ *be a nonlinear strongly continuous semigroup on a subset* Ω *of a Banach space* Z *and assume that* $x_0 \in \Omega$ *is an equilibrium of* $(V(t))_{t\geq 0}$ *such that* $V(t)$ *is differentiable at* x_0 *for each* $t \geq 0$, *with* $W(t)$ *the derivative at* x_0 *of* $V(t)$, $t \geq 0$. *Then,* $(W(t))_{t\geq 0}$ *is a strongly continuous semigroup of bounded linear operators on* Z. *If* $(W(t))_{t\geq 0}$ *is exponentially stable, then* x_0 *is a locally exponentially stable equilibrium of* $(V(t))_{t\geq 0}$. *Moreover, if* Z *can be decomposed as* $Z = Z_1 \oplus Z_2$ *where* Z_i *are* W*-invariant subspaces of* Z, Z_1 *is finite dimensional and with* $\omega_1 = \lim_{h\to\infty} \frac{1}{h} \log |W(h)/Z_2|$ *we have*

$$\inf\{|\lambda| : \lambda \in \sigma(W(t)/Z_1)\} > e^{\omega_1 t},$$

then, the equilibrium x_0 *is not stable in the sense that there exist* $\varepsilon > 0$ *and sequences* $(x_n)_n$ *converging to* x_0 *and* $(t_n)_n$ *of positive real numbers such that* $|V(t_n)x_n - x_0| > \varepsilon$.

4. Smoothness of the solution semigroup

Assume that

($\mathbf{H}_5$) The semigroup $(T(t))_{t\geq 0}$ is compact for $t > 0$.

Now, we state the following fundamental result which will play a crucial role in studying the asymptotic behavior of solutions in the linear cases.

Theorem 4.1. *Assume that* ($\mathbf{H}_1$), ($\mathbf{H}_2$), ($\mathbf{H}_3$) *and* ($\mathbf{H}_5$) *hold. Then, the semigroup* $(U(t))_{t\geq 0}$ *is decomposed as follows:* $U(t) = U_1(t) + U_2(t)$, *for* $t \geq 0$, *where* $U_2(t)$ *is compact on* $\mathcal{B}_\alpha$ *for* $t > 0$ *and* $U_1(t)$ *is the semigroup solution of the following equation*

$$\begin{cases} x'(t) = -Ax(t) \text{ for } t \geq 0, \\ x_0 = \varphi \in \mathcal{B}_\alpha. \end{cases} \tag{4.1}$$

Proof. The solution semigroup $U_1(t)$ associated with equation (4.1) satisfies

$$(U_1(t)\varphi)(\theta) = \begin{cases} T(t+\theta)\varphi(0) \text{ for } t+\theta \geq 0 \\ \varphi(t+\theta) \text{ for } t+\theta \leq 0. \end{cases}$$

Let $U_2(t) = U(t) - U_1(t)$. Then,

$$(U_2(t)\varphi)(\theta) = \begin{cases} 0 \text{ for } t+\theta \leq 0 \\ \displaystyle\int_0^{t+\theta} T(t+\theta-s)F(U(s)\varphi)ds \text{ for } t+\theta \geq 0. \end{cases}$$

Notice that the support of each function $U_2(t)\varphi$ is included in $[-t, 0]$. Let D be a bounded set in $\mathcal{B}_\alpha$. We claim that $\{U_2(t)\varphi(\theta) : \varphi \in D\}$ is relatively compact for each $\theta \geq -t$ and $\{U_2(t)\varphi : \varphi \in D\}$ is equicontinuous in $C([-t, 0]; X_\alpha)$. In fact, let $0 < \alpha < \beta < 1$. Since by assumption ($\mathbf{H}_5$) and Theorem 2.1, we know that $A^{-\beta} : X \to X_\alpha$ is compact, then it is enough to prove that

$$\left\{ A^\beta \int_0^{t+\theta} T(t+\theta-s)F(U(s)\varphi)ds : \varphi \in D \right\}$$

is bounded in X. Since the semigroup $(U(t))_{t\geq 0}$ is locally bounded in t and φ, then there exists a positive constant δ_1 such that

$$\left| A^\beta \int_0^{t+\theta} T(t+\theta-s)F(U(s)\varphi)ds \right| = M_\beta \delta_1 \int_0^{t+\theta} \frac{e^{\omega s}}{s^\beta} ds \text{ for } \varphi \in D.$$

Consequently, $\{U_2(t)\varphi(\theta) : \varphi \in D\}$ is bounded in X_β and is relatively compact in X_α, for each $\theta \geq -t$. To prove the equicontinuity property in the $\alpha-$norm, we take $\theta > \theta_0 \geq -t$, then,

$$\begin{aligned} &A^\alpha U_2(t)\varphi(\theta) - A^\alpha U_2(t)\varphi(\theta_0) \\ &= A^\alpha \int_0^{t+\theta_0} (T(t+\theta-s) - T(t+\theta_0-s))F(U(s)\varphi)ds \\ &\quad + A^\alpha \int_{t+\theta_0}^{t+\theta} T(t+\theta-s)F(U(s)\varphi)ds. \end{aligned}$$

From the local boundedness of $U(t)$ in t and φ, we can see that there exists a positive constant δ_2 such that

$$\left| \int_{t+\theta_0}^{t+\theta} A^\alpha T(t+\theta-s)F(U(s)\varphi)ds \right| \leq M_\alpha \delta_2 \int_{t+\theta_0}^{t+\theta} \frac{e^{\omega s}}{s^\alpha} ds \text{ for } \varphi \in D,$$

which implies that

$$\left| \int_{t+\theta_0}^{t+\theta} A^\alpha T(t+\theta-s)F(U(s)\varphi)ds \right| \to 0 \,\text{as}\; \theta \to \theta_0 \text{ uniformly in } \varphi \in D.$$

Moreover,

$$\begin{aligned} &\int_0^{t+\theta_0} A^\alpha (T(t+\theta-s) - T(t+\theta_0-s))F(U(s)\varphi)ds \\ &= (T(\theta-\theta_0) - I)\int_0^{t+\theta_0} A^\alpha T(t+\theta_0-s)F(U(s)\varphi)ds. \end{aligned}$$

Now, there is a compact set K_1 in X such that

$$\int_0^{t+\theta_0} A^\alpha T(t+\theta_0-s)F(U(s)\varphi)ds \in K_1 \text{ for } \varphi \in D.$$

By Banach-Steinhaus's theorem, we know that

$$\lim_{\theta\to\theta_0} \sup_{\varsigma\in K_1} |(T(\theta-\theta_0) - I)\varsigma| = 0.$$

Hence,

$$\lim_{\theta\to\theta_0^+} |U_2(t)\varphi(\theta) - U_2(t)\varphi(\theta_0)|_\alpha = 0 \text{ uniformly in } \varphi \in D.$$

The proof is similar if $\theta < \theta_0$. Consequently, by Ascoli-Arzela's theorem, we deduce that $\{U_2(t)\varphi : \varphi \in D\}$ is relatively compact in $C([-t, 0]\,; X_\alpha)$ and by Lemma 2.2, $\{U_2(t)\varphi : \varphi \in D\}$ is relatively compact in $\mathcal{B}_\alpha$.

5. Spectral analysis of the linear equation

In this section, we are concerned with the stability of the linear equation (3.2). We assume that $\mathcal{B}$ satisfies the following axiom:

(**D**) there exists a constant $\nu \in \mathbb{R}$ such that for every $x \in X$ and $\lambda \in \mathbb{C}$ with $\mathcal{R}e(\lambda) > \nu$, one has

$$\varepsilon_\lambda \otimes x \in \mathcal{B} \text{ and } \sup_{|x|\leq 1} |\varepsilon_\lambda \otimes x| < \infty,$$

where $(\varepsilon_\lambda \otimes x)(\theta) = e^{\lambda\theta}x$ for $\theta \leq 0$.

For $\lambda \in \mathbb{C}$ such that $\mathcal{R}e(\lambda) > \nu$, we define the linear operator $\Delta(\lambda)$ by

$$\begin{cases} D(\Delta(\lambda)) = \{x \in X_\alpha : x \in D(A) \text{ and } -Ax + L(e^{\lambda\cdot}I) \in X_\alpha\} \\ \Delta(\lambda) = \lambda I + A - L(e^{\lambda\cdot}I), \end{cases}$$

where $L(e^{\lambda\cdot}I)$ is the bounded linear operator on X defined by

$$L(e^{\lambda\cdot}I)(x) = L(\varepsilon_\lambda \otimes x) \text{ for } x \in X.$$

Let $(A_S, D(A_S))$ be the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$ and $\sigma_p(A_S)$ be the point spectrum of A_S.

Theorem 5.1. *Suppose that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ *hold and axioms* $(\mathbf{C})$ *and* $(\mathbf{D})$ *are satisfied. Let* $\lambda \in \mathbb{C}$ *be such that* $\mathcal{R}e(\lambda) > \nu$. *Then, the following are equivalent:*

(i) $\lambda \in \sigma_p(A_S)$,

(ii) $\ker \Delta(\lambda) \neq \{0\}$.

Proof. Let $\lambda \in \sigma_p(A_S)$ with $\mathcal{R}e(\lambda) > \nu$. Then, there exists $\varphi \in D(A_S)$, $\varphi \neq 0$, such that $A_S\varphi = \lambda\varphi$, which implies that

$$\lim_{t\to 0^+} \frac{S(t)\varphi - \varphi}{t} = \lambda\varphi.$$

In particular, by axiom $(\mathbf{A}) - (iii)$, one has,

$$\lim_{t\to 0^+} \Big(\frac{S(t)\varphi - \varphi}{t}\Big)(0) = \lambda\varphi(0).$$

Since

$$\Big(\frac{S(t)\varphi - \varphi}{t}\Big)(0) = \frac{T(t)\varphi(0) - \varphi(0)}{t} + \frac{1}{t}\int_0^t T(t-s)L(S(s)\varphi)ds \text{ for } t > 0$$

and letting $t \to 0$, we get that

$$\varphi(0) \in D(A) \text{ and } -A\varphi(0) + L(\varphi) = \lambda\varphi(0). \tag{5.1}$$

On the other hand, by the spectral mapping theorem 2.4, page 46, [19], we obtain that

$$e^{\lambda t} \in \sigma_p(S(t)) \text{ and } S(t)\varphi = e^{\lambda t}\varphi \text{ for all } t > 0.$$

Let $t > 0$ and $\theta \leq 0$ be such that $t+\theta \geq 0$. Then, by the translation property of the solution semigroup, we get that

$$(S(t)\varphi)(\theta) = (S(t+\theta)\varphi)(0) = e^{\lambda t}\varphi(\theta) = e^{\lambda(t+\theta)}\varphi(0),$$

which implies that

$$\varphi(\theta) = e^{\lambda\theta}\varphi(0) \text{ for } \theta \leq 0.$$

Since $\varphi \neq 0$, which is equivalent to saying that $\varphi(0) \neq 0$, and by (5.1), one obtains that $\varphi(0) \in \ker \Delta(\lambda)$. Conversely, first one can observe that if φ satisfies all the conditions of Theorem 2.7, then Lemma 2.8 shows that $A_S\varphi = \varphi'$. Now, if we take $a \in D(A)$ such that $a \neq 0$ and $\Delta(\lambda)a = 0$, then the function $\varepsilon_\lambda \otimes a$ satisfies all the conditions of Theorem 2.7 and, by Lemma 2.8, we deduce that

$$A_S(\varepsilon_\lambda \otimes a) = \lambda(\varepsilon_\lambda \otimes a),$$

which implies that $\lambda \in \sigma_p(A_S)$.

For $\varphi \in \mathcal{B}$, $t \geq 0$ and $\theta \leq 0$, we define

$$\left[\widetilde{W}(t)\varphi\right](\theta) = \begin{cases} \varphi(0) \text{ for } t+\theta \geq 0 \\ \varphi(t+\theta) \text{ for } t+\theta < 0. \end{cases}$$

Then, $(\widetilde{W}(t))_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{B}$. We set

$$\widetilde{W}_0(t) = \widetilde{W}(t)_{/\mathcal{B}_0}, \text{ where } \mathcal{B}_0 = \{\varphi \in \mathcal{B} : \varphi(0) = 0\}.$$

Definition 5.2. We say that $\mathcal{B}$ is a uniform fading memory space if the following conditions hold:

(i) if a uniformly bounded sequence $(\varphi_n)_n$ in C_{00} converges to a function φ compactly on $(-\infty, 0]$, then φ is in $\mathcal{B}$ and $|\varphi^n - \varphi|_{\mathcal{B}} \to 0$ as $n \to +\infty$,

(ii) $\left|\widetilde{W}_0(t)\right|_{\mathcal{B}} \to 0$ as $t \to +\infty$.

Let $\nu_0 = \inf\{\ \nu \in \mathbb{R}$ such that $(\mathbf{D})$ is satisfied $\}$.

Lemma 5.3. [18] *If $\mathcal{B}$ is a uniform fading memory space, then $\nu_0 < 0$.*

Lemma 5.4. [17] *If $\mathcal{B}$ is a uniform fading memory space, then K and M can be chosen such that K is bounded on $\mathbb{R}^+$ and $M(t) \to 0$ as $t \to 0$.*

Definition 5.5. We say that $\lambda \in \mathbb{C}$ is a characteristic value of equation (3.2) if $\mathrm{Re}(\lambda) > \nu_0$ and $\ker \Delta(\lambda) \neq \{0\}$.

The computation of each characteristic value is completely obtained by the following characteristic equation:

$$\lambda \in \mathbb{C},\ \mathrm{Re}(\lambda) > \nu_0 \text{ and } \ker \Delta(\lambda) \neq \{0\}.$$

Let Z be a Banach space. We introduce the Kuratowskii measure of noncompactness $\chi(\Omega)$ of a set $\Omega \subset Z$ by

$$\chi(\Omega) = \inf\left\{d > 0 : \Omega \text{ has a finite cover of diameter } < d\right\}.$$

For a bounded linear operator $\mathcal{H}$, the Kuratowskii measure of noncompactness $\alpha(\mathcal{H})$ of $\mathcal{H}$ is defined by

$$\chi(\mathcal{H}) = \inf\{\eta \in \mathbb{R}^+ : \chi(\mathcal{H}(D)) \leq \eta\chi(D) \text{ for every bounded subset } D \text{ of } Z\}.$$

Lemma 5.6. *Suppose that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ *and* $(\mathbf{H}_5)$ *hold. Then, for all* $\varepsilon > 0$, *there exists* $C_\varepsilon > 0$ *such that*

$$\chi(U_1(t)) \leq C_\varepsilon M(t-\varepsilon) \text{ for } t > \varepsilon.$$

Proof. Let Ω be a bounded set in $\mathcal{B}_\alpha$ such that $\chi(\Omega) = \beta$. For $\delta > 0$, there exists a finite cover $(\Omega_i)_{1\leq i\leq n}$ of Ω such that $diam(\Omega_i) < \beta + \delta$. Using axiom $(\mathbf{A}) - (iii)$, we have for $t > \varepsilon$ and $\varphi, \phi \in \Omega$,

$$\begin{aligned}|U_1(t)\varphi - U_1(t)\phi|_{\mathcal{B}_\alpha} \leq{}& K(t-\varepsilon) \sup_{\varepsilon\leq s\leq t} |T(s)\varphi(0) - T(s)\phi(0)|_\alpha \\ &+ M(t-\varepsilon)\,|U_1(\varepsilon)\varphi - U_1(\varepsilon)\phi|_{\mathcal{B}_\alpha}.\end{aligned} \tag{5.2}$$

On the other hand

$$|U_1(\varepsilon)\varphi - U_1(\varepsilon)\phi|_{\mathcal{B}_\alpha} \leq K(\varepsilon) \sup_{0\leq s\leq \varepsilon} |T(s)\varphi(0) - T(s)\phi(0)|_\alpha + M(\varepsilon)|\varphi - \phi|_{\mathcal{B}_\alpha},$$

which implies that

$$|U_1(\varepsilon)\varphi - U_1(\varepsilon)\phi|_{\mathcal{B}_\alpha} \leq C_\varepsilon\, |\varphi - \phi|_{\mathcal{B}_\alpha}, \tag{5.3}$$

where $C_\varepsilon = K(\varepsilon)H \sup\limits_{0\leq s\leq \varepsilon} |T(s)| + M(\varepsilon)$. Consequently,

$$\begin{aligned}&|U_1(t)\varphi - U_1(t)\phi|_{\mathcal{B}_\alpha} \\ &\leq K(t-\varepsilon) \sup_{\varepsilon\leq s\leq t} |T(s)\varphi(0) - T(s)\phi(0)|_\alpha + M(t-\varepsilon)C_\varepsilon\, |\varphi - \phi|_{\mathcal{B}_\alpha}.\end{aligned} \tag{5.4}$$

Ω is bounded in $\mathcal{B}_\alpha$ and by axiom $(\mathbf{A}) - (iii)$, $\{A^\alpha\psi(0) : \psi \in \Omega\}$ is bounded in X. Using the compactness of the semigroup $(T(t))_{t\geq 0}$, we can see that $\{T(.)A^\alpha\psi(0) : \psi \in \Omega\}$ is relatively compact in $C([\varepsilon, t]\,; X)$. Then, there exists a finite cover $(\Gamma_i)_{1\leq i\leq m}$ of $\{T(.)A^\alpha\psi(0) : \psi \in \Omega\}$ in $C([\varepsilon, t]\,; X)$ such that $diam(\Gamma_i) < \delta$.

Let $\Omega_{i,j} = \{\psi \in \Omega_i : T(.)A^{\alpha}\psi(0) \in \Gamma j\}$. Then, $\Omega \subset \bigcup_{i,j} \Omega_{i,j}$. By (5.4), for $\varphi, \phi \in \Omega_{i,j}$, we have

$$|U_1(t)\varphi - U_1(t)\phi|_{\mathcal{B}_\alpha} \leq K(t-\varepsilon)\delta + M(t-\varepsilon)C_\varepsilon(\beta+\delta)\,|\varphi - \phi|_{\mathcal{B}_\alpha}.$$

Letting δ go to zero, one obtains

$$\chi(U_1(t)) \leq C_\varepsilon M(t-\varepsilon) \text{ for } t > \varepsilon.$$

Recall that the essential growth bound $\omega_{ess}(S)$ of $(S(t))_{t\geq 0}$ is defined by

$$\omega_{ess}(S) = \lim_{t\to\infty}\frac{1}{t}\log\chi(S(t)) = \inf_{t>0}\frac{1}{t}\log\chi(S(t)).$$

Lemma 5.7. *Suppose that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ *and* $(\mathbf{H}_5)$ *hold. If* $\mathcal{B}$ *is a uniform fading memory space, then* $\omega_{ess}(S) < 0$.

Proof. By Theorem 4.1, we know that $S(t)$ is decomposed as follows:

$$S(t) = S_1(t) + S_2(t) \text{ for } t \geq 0,$$

where $S_2(t)$ is compact for all $t \geq 0$, which implies that $\chi(S(t)) = \chi(S_1(t))$, for $t > 0$. From Lemma 5.6, we know that for each $\varepsilon > 0$, there exists a positive constant C_ε such that

$$\chi(S_1(t)) \leq C_\varepsilon M(t-\varepsilon) \text{ for } t > \varepsilon.$$

Since $\mathcal{B}$ is a uniform fading memory space, then

$$M(t) \to 0 \text{ as } t \to \infty.$$

Consequently, $\omega_{ess}(S) < 0$.

Definition 5.8. Let $\mathcal{H} : Z \to Z$ be a closed linear operator with a dense domain $D(\mathcal{H})$ in Z. We denote by $\sigma(\mathcal{H})$ the spectrum of $\mathcal{H}$. The essential spectrum $\sigma_{ess}(\mathcal{H})$ of $\mathcal{H}$ is the set of λ in $\sigma(\mathcal{H})$ such that at least one of the following holds:

(*i*) $\operatorname{Im}(\lambda I - \mathcal{H}) = \{(\lambda I - \mathcal{H})z : z \in Z\}$ is not closed,

(*ii*) the generalized eigenspace $M_\lambda(\mathcal{H}) = \bigcup_{k\geq 1} \ker(\lambda I - \mathcal{H})^k$ of λ is infinite dimensional,

(*iii*) $\lambda \in \overline{\sigma(\mathcal{H})\backslash\lambda}$.

The asymptotic behavior of the semigroup $(S(t))_{t\geq 0}$ is completely obtained by the so-called growth bound $\omega_0(S)$ which is defined by

$$\omega_0(S) = \inf\left\{\kappa > 0 : \sup_{t\geq 0} e^{-\kappa t}\,|S(t)| < \infty\right\}.$$

We know from [23] that $\omega_0 = \max\{\omega_{ess}, s'(A_S)\}$, where

$$s'(A_S) = \sup\{\operatorname{Re}\lambda : \lambda \in \sigma(A_S) - \sigma_{ess}(A_S)\}.$$

Recall that $\sigma(A_S) - \sigma_{ess}(A_S)$ contains a finite number of eigenvalues of A_S. Consequently, the stability of $(S(t))_{t\geq 0}$ is completely determined by $s'(A_S)$ and by Lemma 5.7 we get the following result.

Theorem 5.9. *Suppose that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ *and* $(\mathbf{H}_5)$ *hold. If* $\mathcal{B}$ *is a uniform fading memory space, then the following properties hold:*

(i) if $s'(A_S) < 0$, *then* $(S(t))_{t\geq 0}$ *is exponentially stable,*

(ii) if $s'(A_S) = 0$, *then there exists* $\varphi \in \mathcal{B}_\alpha$ *such that* $|S(t)\varphi|_{\mathcal{B}_\alpha} = |\varphi|_{\mathcal{B}_\alpha}$ *for* $t \geq 0$,

(iii) if $s'(A_S) > 0$, *then there exists* $\varphi \in \mathcal{B}_\alpha$ *such that* $|S(t)\varphi|_{\mathcal{B}_\alpha} \to \infty$ *as* $t \to \infty$.

From Theorem 3.3, we deduce the following stability result in the nonlinear case.

Theorem 5.10. *Suppose that* $(\mathbf{H}_1)$, $(\mathbf{H}_2)$, $(\mathbf{H}_3)$, $(\mathbf{H}_4)$ *and* $(\mathbf{H}_5)$ *hold. If* $\mathcal{B}$ *is a uniform fading memory space, then the following properties hold:*

(i) if $s'(A_S) < 0$, *then the zero equilibrium of* $(U(t))_{t\geq 0}$ *is locally exponentially stable,*

(ii) if $s'(A_S) > 0$, *then zero is unstable of* $(U(t))_{t\geq 0}$.

In the sequel, we give some sufficient conditions in order to determine $(A_S, D(A_S))$. We assume that $\mathcal{B}$ satisfies the following axiom:

$(\mathbf{E})$ let $(\varphi_n)_n$ be a sequence in $\mathcal{B}$ such that $\varphi_n \to 0$, as $n \to \infty$ in $\mathcal{B}$; then for all $\theta \leq 0$, $\varphi_n(\theta) \to 0$, as $n \to \infty$.

First, one can observe that if axiom $(\mathbf{E})$ holds, then axiom $(\mathbf{C})$ follows.

Theorem 5.11. *Assume that* $\mathcal{B}$ *satisfies* $(\mathbf{A})$, $(\mathbf{B})$ *and* $(\mathbf{E})$. *If* $\mathcal{B}$ *is a subspace of the space of continuous functions from* $(-\infty, 0]$ *to* X, *then*

$$\begin{cases} D(A_S) = \{\varphi \in \mathcal{B}_\alpha : \varphi' \in \mathcal{B}_\alpha, \varphi(0) \in D(A) \text{ and } \varphi'(0) = -A\varphi(0) + L(\varphi)\} \\ A_S\varphi = \varphi'. \end{cases}$$

Proof. Let B be the infinitesimal generator of the semigroup $(S(t))_{t\geq 0}$ and $\varphi \in D(B)$. Then,

$$\begin{cases} \lim\limits_{t\to 0^+} \dfrac{S(t)\varphi - \varphi}{t} = \psi \text{ exists in } \mathcal{B}_\alpha \\ B\varphi = \psi. \end{cases}$$

Using axiom (**E**), we get that

$$\lim_{t\to 0^+}\frac{\varphi(t+\theta)-\varphi(\theta)}{t}=\psi(\theta) \text{ for } \theta\in(-\infty,0),$$

which means that the right derivative $D^+\varphi$ exists on $(-\infty,0)$ and equals ψ. Since each function in $\mathcal{B}$ is continuous on $(-\infty,0]$, it follows that $D^+\varphi$ is continuous on $(-\infty,0)$. The following lemma is needed.

Lemma 5.12. ([19], Corollary 1.2, page 43) *Let Θ be a continuous and right differentiable function on $[a,b)$. If the right derivative function $D^+\Theta$ is continuous on $[a,b)$, then Θ is continuously differentiable on $[a,b)$.*

By Lemma 5.12, we deduce that the function φ is continuously differentiable and $\varphi'=\psi$ on $(-\infty,0)$. Moreover, $\lim_{\theta\to 0}D^+\varphi(\theta)=\psi(0)$, which implies that the function φ is continuously differentiable from $(-\infty,0]$ to X_α and $\varphi'=\psi$ on $(-\infty,0]$. Now, if we take $\theta=0$, then

$$\Big(\frac{S(t)\varphi-\varphi}{t}\Big)(0)=\frac{T(t)\varphi(0)-\varphi(0)}{t}+\frac{1}{t}\int_0^t T(t-s)L(S(s)\varphi)ds.$$

Since $X_\alpha\hookrightarrow X$, we obtain

$$\Big(\frac{S(t)\varphi-\varphi}{t}\Big)(0)\to\varphi'(0) \text{ as } t\to 0 \text{ with the } X\text{-norm},$$

therefore,

$$\frac{1}{t}\int_0^t T(t-s)L(S(s)\varphi)ds\to L(\varphi) \text{ as } t\to 0.$$

Hence,

$$\varphi(0)\in D(A) \text{ and } \varphi'(0)=-A\varphi(0)+L(\varphi).$$

Consequently,

$$\begin{cases} D(B)\subseteq\{\varphi\in\mathcal{B}_\alpha:\varphi'\in\mathcal{B}_\alpha,\varphi(0)\in D(A) \text{ and } \varphi'(0)=-A\varphi(0)+L(\varphi)\} \\ B\varphi=\varphi' \end{cases}$$

Conversely, if we take $\varphi\in\mathcal{B}_\alpha$ such that

$$\varphi'\in\mathcal{B}_\alpha,\ \varphi(0)\in D(A) \text{ and } \varphi'(0)=-A\varphi(0)+L(\varphi).$$

Then, Theorem 2.6 implies that $t\to S(t)\varphi$ is continuously differentiable from $[0,+\infty)$ to $\mathcal{B}_\alpha$, which gives $\varphi\in D(B)$.

6. Application

In order to apply the abstract previous results, we propose to study the following model:

$$\begin{cases} \dfrac{\partial}{\partial t}u(t,\xi) = \dfrac{\partial^2}{\partial \xi^2}u(t,\xi) + a_1\dfrac{\partial}{\partial \xi}u(t-r,\xi) + \displaystyle\int_{-\infty}^{0} g(\theta)u(t+\theta,\xi)d\theta \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{for } t \geq 0 \text{ and } 0 \leq \xi \leq \pi, \\ \\ u(t,0) = u(t,\pi) = 0 \text{ for } t \geq 0, \\ u(\theta,\xi) = u_0(\theta,\xi) \text{ for } -\infty < \theta \leq 0 \text{ and } 0 \leq \xi \leq \pi, \end{cases} \tag{6.1}$$

where a_1 and r are positive constants, $g : (-\infty, 0] \to I\!R$ is a positive function, and $u_0 : (-\infty, 0] \times [0, \pi] \to I\!R$ is an appropriate function which will be described below.

In order to rewrite equation (6.1) in the abstract form, we consider equation (6.1) in $X = L^2([0,\pi];\mathbb{R})$ and we define the operator A on X by

$$\begin{cases} D(A) = H^2(0,\pi) \cap H_0^1(0,\pi) \\ Am = -m''. \end{cases}$$

Then, by Theorem 2.7, page 211 in [19], we get that $-A$ generates an analytic semigroup $(T(t))_{t\geq 0}$ on X. Moreover the imbedding $H^2(0,\pi) \hookrightarrow L^2([0,\pi];\mathbb{R})$ is compact, which implies that $T(t)$ is compact for $t > 0$. The spectrum $\sigma(-A)$ is the point spectrum and $\sigma(-A) = \{-n^2 : n \geq 1\}$. The associated eigenfunctions $(e_n)_{n\geq 1}$ are given by $e_n(\xi) = \sqrt{\frac{2}{\pi}}\sin(n\xi)$, for $\xi \in [0,\pi]$ and $n \geq 1$. From [22], the semigroup $T(t)$ is explicitly given by

$$T(t)y = \sum_{n=1}^{\infty} e^{-n^2 t} < y, e_n > e_n \text{ for } y \in X.$$

If we choose $\alpha = \frac{1}{2}$, then

$$\begin{cases} A^{\frac{1}{2}}T(t)y = \sum_{n=1}^{\infty} n e^{-n^2 t} < y, e_n > e_n \text{ for } y \in X. \\ A^{-\frac{1}{2}}y = \sum_{n=1}^{\infty} \frac{1}{n} < y, e_n > e_n \text{ for } y \in X. \\ A^{\frac{1}{2}}y = \sum_{n=1}^{\infty} n < y, e_n > e_n \text{ for } y \in D(A^{\frac{1}{2}}). \end{cases} \tag{6.2}$$

Lemma 6.1. [22] *If $m \in D(A^{\frac{1}{2}})$, then m is absolutely continuous, $m' \in X$ and*

$$|m'| = |A^{\frac{1}{2}}m|.$$

For $\gamma > 0$, we choose the space

$$\mathcal{B} = C_\gamma = \left\{ \varphi \in C((-\infty, 0]; X) : \lim_{\theta \to -\infty} e^{\gamma\theta}\varphi(\theta) \text{ exists in } X \right\}$$

provided with the following norm

$$|\varphi|_\gamma = \sup_{\theta \le 0} e^{\gamma\theta} |\varphi(\theta)| \text{ for } \varphi \in C_\gamma.$$

Then, $\mathcal{B}$ satisfies axioms (**A**), (**B**), (**E**) with $\nu_0 = -\gamma$ and C_γ is a uniform fading memory space. The norm in $\mathcal{B}_{\frac{1}{2}}$ is given by

$$|\varphi|_{\mathcal{B}_{\frac{1}{2}}} = \sup_{\theta \le 0} e^{\gamma\theta} \left| A^{\frac{1}{2}}\varphi(\theta) \right| = \sup_{\theta \le 0} e^{\gamma\theta} \sqrt{\int_0^\pi \left(\frac{\partial}{\partial \xi}(\varphi)(\theta)(\xi) \right)^2 d\xi}.$$

Assume that,

($\mathbf{H}_6$) $e^{-2\gamma \cdot} g \in L^2(\mathbb{R}^-)$.

Let $L : \mathcal{B}_{\frac{1}{2}} \to X$ be defined by

$$L(\varphi)(\xi) = a_1 \frac{\partial}{\partial \xi}(\varphi)(-r)(\xi) + \int_{-\infty}^0 g(\theta)\varphi(\theta)(\xi) d\theta \text{ for a.e. } \xi \in [0, \pi].$$

We claim that L is a bounded linear operator from $\mathcal{B}_{\frac{1}{2}}$ to X. In fact, L can be decomposed as follows: $L = L_1 + L_2$, where

$$L_1(\varphi)(\xi) = a_1 \frac{\partial}{\partial \xi}(\varphi)(-r)(\xi)$$

and

$$L_2(\varphi)(\xi) = \int_{-\infty}^0 g(\theta)\varphi(\theta)(\xi) d\theta \text{ for a.e. } \xi \in [0, \pi].$$

Hence,

$$\begin{aligned} \int_0^\pi L_1(\varphi)(\xi)^2 d\xi &= a_1^2 \int_0^\pi \left(\frac{\partial}{\partial \xi}(\varphi)(-r)(\xi) \right)^2 d\xi \\ &= a_1^2 \left| A^{\frac{1}{2}}\varphi(-r) \right|^2 \le a_1^2 e^{2\gamma r} \sup_{\theta \le 0} e^{2\gamma\theta} \left| A^{\frac{1}{2}}\varphi(\theta) \right|^2, \end{aligned}$$

which implies that L_1 is a bounded linear operator from $\mathcal{B}_{\frac{1}{2}}$ to X. For L_2, one has

$$L_2(\varphi)(\xi)^2 \le \int_{-\infty}^0 g(\theta)^2 e^{-4\gamma\theta} d\theta \int_{-\infty}^0 e^{4\gamma\theta}\varphi(\theta)(\xi)^2 d\theta;$$

it follows that

$$\int_0^\pi L_2(\varphi)(\xi)^2 d\xi \le \int_{-\infty}^0 g(\theta)^2 e^{-4\gamma\theta} d\theta \int_{-\infty}^0 e^{4\gamma\theta}\Big(\int_0^\pi \varphi(\theta)(\xi)^2 d\xi\Big) d\theta.$$

Hence,

$$\int_0^\pi L_2(\varphi)(\xi)^2 d\xi \le \frac{1}{2\gamma}\Big(\int_{-\infty}^0 g(\theta)^2 e^{-4\gamma\theta} d\theta\Big) \sup_{\theta\le 0}\Big(e^{2\gamma\theta}\int_0^\pi \varphi(\theta)(\xi)^2 d\xi\Big).$$

Using formulas (6.2), one has

$$\int_0^\pi \varphi(\theta)(\xi)^2 d\xi \le \int_0^\pi \Big(\frac{\partial}{\partial\xi}\varphi(\theta)(\xi)\Big)^2 d\xi = \Big|A^{\frac{1}{2}}\varphi(\theta)\Big|^2.$$

Consequently, L_2 is a bounded linear operator from $\mathcal{B}_{\frac{1}{2}}$ to X. Let $v(t) = u(t,.)$, for $t \ge 0$ and $\psi(\theta)(\xi) = u_0(\theta,\xi)$ for $\theta \le 0$ and $\xi \in [0,\pi]$. Assume that $\psi(\theta) \in D(A^{\frac{1}{2}})$ for $\theta \le 0$, with

$$\sup_{\theta\le 0} e^{\gamma\theta}\sqrt{\int_0^\pi \Big(\frac{\partial}{\partial\xi}u_0(\theta,\xi)\Big)^2 d\xi} < \infty \tag{6.3}$$

and

$$\lim_{\theta\to\theta_0}\int_0^\pi \Big(\frac{\partial}{\partial\xi}u_0(\theta,\xi) - \frac{\partial}{\partial\xi}u_0(\theta_0,\xi)\Big)^2 d\xi = 0 \text{ for all } \theta_0 \le 0. \tag{6.4}$$

Then, (6.3) and (6.4) imply that $\psi \in \mathcal{B}_{\frac{1}{2}}$. Equation (6.1) takes the following abstract form

$$\begin{cases} \dfrac{d}{dt}v(t) = -Av(t) + L(v_t) \text{ for } t \ge 0 \\ v_0 = \psi \in \mathcal{B}_{\frac{1}{2}}. \end{cases} \tag{6.5}$$

Moreover, for each function $\psi \in \mathcal{B}_{\frac{1}{2}}, A^{-\frac{1}{2}}\psi \in \mathcal{B}_{\frac{1}{2}}$. Consequently, we obtain the following result.

Proposition 6.2. *Assume that* $(\mathbf{H}_6)$ *holds; then equation* (6.5) *has a unique mild solution which is defined for* $t \ge 0$ *and the solutions define a strongly continuous semigroup on* $\mathcal{B}_{\frac{1}{2}}$.

We state the main result of this part on the stability of the solutions.

Theorem 6.3. *Assume that* $(\mathbf{H}_6)$ *holds and*

$$0 < \int_{-\infty}^0 g(\theta)d\theta < 1 - \frac{a_1^2}{4}.$$

Then, the solution semigroup of equation (6.5) *is exponentially stable.*

Proof. Let G be the infinitesimal generator of the solution semigroup of equation (6.5). According to Corollary 5.9, it is enough to show that $s'(G) < 0$, which is true only if

$$\sup\{\operatorname{Re}\lambda : \lambda \in \sigma(G) - \sigma_{ess}(G) \text{ and } \operatorname{Re}\lambda > -\gamma\} < 0.$$

Since, by Theorem 5.1, the characteristic equation is given by

$$\begin{cases} \operatorname{Re}\lambda > -\gamma, \ f \in D(A) \text{ with } f \neq 0 \\ \lambda f - f'' - a_1 e^{-\lambda r} f' - \left(\int_{-\infty}^{0} g(\theta) e^{\lambda\theta} d\theta\right) f = 0, \end{cases} \tag{6.6}$$

it follows that

$$\lambda - \int_{-\infty}^{0} g(\theta) e^{\lambda\theta} d\theta \in \sigma_p\left(\frac{d^2}{d\xi^2} + a_1 e^{-\lambda r}\frac{d}{d\xi}\right),$$

where $\sigma_p(\frac{d^2}{d\xi^2} + a_1 e^{-\lambda r}\frac{d}{d\xi})$ denotes the point spectrum of $\frac{d^2}{d\xi^2} + a_1 e^{-\lambda r}\frac{d}{d\xi}$ with the domain $D(\frac{d^2}{d\xi^2} + a_1 e^{-\lambda r}\frac{d}{d\xi}) = H^2(0,\pi) \cap H_0^1(0,\pi)$. On the other hand, from [22], we know that

$$\sigma_p\left(\frac{d^2}{d\xi^2} + a_1 e^{-\lambda r}\frac{d}{d\xi}\right) = \left\{-n^2 - \frac{a_1^2 e^{-2\lambda r}}{4} : n \in \mathbb{N}^*\right\}.$$

The characteristic equation (6.6) becomes

$$\begin{cases} \operatorname{Re}\lambda > -\gamma, \\ \lambda = \int_{-\infty}^{0} g(\theta) e^{\lambda\theta} d\theta - n^2 - \frac{a_1^2 e^{-2\lambda r}}{4} \text{ for } n \in \mathbb{N}^*. \end{cases}$$

We proceed by contradiction and assume that there exists a characteristic value λ such that $\operatorname{Re}\lambda \geq 0$. Then, for $n \in \mathbb{N}^*$,

$$\operatorname{Re}\lambda = \int_{-\infty}^{0} g(\theta) e^{\operatorname{Re}(\lambda\theta)} \cos(\operatorname{Im}(\lambda\theta)) d\theta - n^2 - \frac{a_1^2 e^{-2\operatorname{Re}(\lambda r)} \cos(\operatorname{Im}(-2\lambda r))}{4},$$

and

$$\operatorname{Re}\lambda \leq \int_{-\infty}^{0} g(\theta) d\theta - 1 + \frac{a_1^2}{4} < 0,$$

which gives a contradiction. Consequently, each characteristic value has negative real part and by Corollary 5.9, we conclude that the solution semigroup of equation (6.5) is exponentially stable.

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REFERENCES

[1] M. Adimy and K. Ezzinbi, *Existence and stability in the α-norm for some partial functional differential equations of neutral type,* Annali di Matematica Pura ed Applicata, Springer-Verlag, (to appear 2005).

[2] M. Adimy and K. Ezzinbi, *A class of linear partial neutral functional differential equations with non-dense domain,* Journal of Differential Equations, 147 (1998), 285–332.

[3] M. Adimy and K. Ezzinbi, *Existence and linearized stability for partial neutral functional differential with nondense domains,* Differential Equations and Dynamical Systems, 7 (1999), 371–417.

[4] M. Adimy and K. Ezzinbi, *Strict solutions of nonlinear hyperbolic neutral differential equations,* Applied Mathematics Letters, 12 (1999), 107–112.

[5] M. Adimy and K. Ezzinbi, *Local existence and linearized stability for partial functional differential equations,* Dynamic Systems and Applications, 7 (1998), 389–404.

[6] M. Adimy, H. Bouzahir, and K. Ezzinbi, *Existence for a class of partial functional differential equations with infinite delay,* Nonlinear Analysis Theory, Methods and Applications, 46 (2001), 91–112.

[7] M. Adimy, H. Bouzahir, and K. Ezzinbi, *Local existence and stability for some partial functional differential equations with infinite delay,* Nonlinear Analysis, Theory, Methods and Applications, 48 (2002), 323–348.

[8] M. Adimy, H. Bouzahir, and K. Ezzinbi, *Existence and stability for some partial neutral functional differential equations with infinite delay,* Journal of Mathematical Analysis and Applications, 294 (2004), 438–461.

[9] R. Benkhalti and K. Ezzinbi, *Existence and stability for some partial neutral functional differential equations,* Differential and Integral Equations, 17 (2004), 1423–1436.

[10] R. Benkhalti, H. Bouzahir, and K. Ezzinbi, *Existence of a periodic solution for some partial functional differential equations with infinite delay,* Journal of Mathematical Analysis and Applications, 256 (2001), 257–280.

[11] W. Desch and W. Schappacher, *Linearized stability for nonlinear semigroups, Differential Equations in Banach Spaces,* A.Favini and E.Obrecht, (eds.), 61–73, Lecture Notes, Springer-Verlag, Vol. 1223, (1986).

[12] K. Engel and R. Nagel, "One-parameter Semigroups for Linear Evolution Equations," Graduate Texts in Mathematics, Springer-Verlag, Vol. 194, (2001).

[13] K. Ezzinbi, *Existance and stability for some partial functional differential equations with infinite delay,* Electronic Journal of Differential Equations, Vol. 2003, No.116, 1–13, (2003).

[14] K. Ezzinbi and H. Tamou, *Abstract seminlinear functional differential equations of retarded type, Dynamics of Continuous, Discrete and Impulsive Systems,* Serie A, Mathematical Analysis, 8 (2001), 281.

[15] J. Hale and J. Kato, *Phase space for retarded equations with unbounded delay,* Funkcial Ekvac, 21 (1978), 11–41.

[16] J.K. Hale and S. Lunel, "Introduction to Functional Differential Equations," Springer-Verlag, New York, (1993).

[17] Y. Hino, S. Murakami, and T. Naito, "Functional Differential Equations with Unbounded Delay," Lectures Notes, Springer-Verlag, Vol. 1473 (1991).

[18] T. Naito, J. S. Shin, and S. Murakami, *The generator of the solution semigroup for the general linear functional differential equation,* Bull. Univ.Electro-Communications, Vol. 11, No.1, 29–38, (1998).

[19] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer-Verlag, Berlin, (1983).

[20] C.C. Travis and G.F. Webb, *Existence and stability for partial functional differential equations,* Transactions of the American Mathematical Society, 200 (1974), 395–418.

[21] C.C. Travis and G.F. Webb, *Partial differential equations with deviating arguments in the time variable,* Journal of Mathematical Analysis and Applications, 56 (1976), 397–409.

[22] C.C. Travis and G F. Webb, *Existence, stability and compactness in the $\alpha-$norm for partial functional differential equations,* Transactions of the American Mathematical Society, 240 (1978), 129–143.

[23] G.F. Webb, "Theory of Nonlinear Age-dependent Population Dynamics," Marcel-Dekker Inc., New York (1985).

[24] J. Wu, "Theory and Applications of Partial Functional Differential Equations," Applied Mathematical Sciences, Springer-Verlag, Vol. 119 (1996).