Differential and Integral Equations Volume 19, Number 5 (2006), 537–544

ON THE BLOW-UP RATE BY REGULAR VARIATION FUNCTIONS

Amin Boumenir
Department of Mathematics, University of West Georgia
Carrollton, GA 30118

(Submitted by: Y. Giga)

Abstract. We develop a direct method for obtaining the blow-up rate of positive solutions of semilinear parabolic equations by using the theory of regularly varying functions. The method is applicable any time the blow-up set is a single point.

1. Introduction

There are several methods for obtaining asymptotics of the blow-up rate of positive solutions of reaction diffusion equations. They include rescaling, stability techniques and maximum principle, [4, 7]. The purpose of this note is to develop a direct method by using the theory of regular varying functions, which has been successfully used in nonlinear ordinary differential equations, [5, 9]. For the sake of simplicity we shall deal first with one-dimensional equations of the type, [1, 3]

$$u_t = (u_x^\sigma)_x + f(u) \tag{1.1}$$

and the classical particular case when $\sigma = 1$ and $f(u) = |u|^{p-1}u$

$$u_t = u_{xx} + \lambda |u|^{p-1} u. \tag{1.2}$$

These examples are treated at the end and agree with the results obtained by [8] and [3] by different methods. We would like to point out that under some mild conditions our method is applicable anytime we have a single point blow-up, see [2, 3]. The main idea is based on the observation that if the blow-up set is a point, then the equation holds in the classical sense in its neighborhood.

Accepted for publication: March 2006.
AMS Subject Classifications: 35B40.

2. Functions of regular variation

For the sake of simplicity consider first the following one-dimensional reaction diffusion equation

$$u_t = u_{xx} + f(u). \tag{2.1}$$

It is known that a solution can blow up in finite time, i.e., $\lim_{(x,t)\to(a,T)} u(x,t) = \infty$ if $\int^{\infty} \frac{1}{f(u)} du < \infty$. To find its behavior close to the singularity we first proceed to a change of variables; set $u = \varphi(w)$ such that $w \to 0$ if $u \to \infty$. If φ is one to one in the neighborhood of $w = 0^+$, then $w(x,t)$ is also a solution of

$$w_t = w_{xx} + w_x^2 \frac{\varphi''(w)}{\varphi'(w)} + \frac{f(\varphi(w))}{\varphi'(w)}. \tag{2.2}$$

The relation $u = \varphi(w)$ establishes a well-known connection between blow-up and dead core, quenching and extinction. The main advantage of working with w instead of u is that the former solution decreases to zero. To proceed further we need the limits

$$\lim_{w\to 0} \frac{f(\varphi(w))}{\varphi'(w)} \quad \text{and} \quad \lim_{w\to 0} \frac{w\varphi''(w)}{\varphi'(w)} \tag{2.3}$$

to exist and be finite. We recall that $F \in L^{1,loc}(0,\infty)$ is of regular variation at infinity and index α, which we denote by RV_α^∞, if $\lim_{x\to\infty} \frac{F(\lambda x)}{F(x)} = \lambda^\alpha$. For example if $F \in RV_\alpha^\infty$ we have $\lim_{x\to\infty} \frac{xF(x)}{\int_0^x F(t)dt} = \alpha + 1$. It is known that if f is of regular variation at infinity then the function φ and the limits in (2.3) also exist. The space RV_α^0 is defined by using the change of variable $F\left(\frac{1}{x}\right)$ and a calculus on limits of RV functions can be found in [6, 9].

The next step is to replace w_{xx} and w_x^2 in (2.2) by equivalent expressions involving w instead of its derivatives. Since two variables, x and t are involved, it is more convenient to use a path, so as to reduce the number of variables. Recall that the blow-up of the solution $u(x,t)$ by the transformation is equivalent to $w(a,T) = 0$. Without loss of generality we can take $a = T = 0$ and choose the following path

$$(x,t) \to (0,0), \text{ where } \quad t = -\frac{x^2}{2}.$$

In this case by L'Hospital's rule we have

$$\lim_{x\to 0} \frac{w(x, \frac{-x^2}{2})}{\frac{x^2}{2}} = \lim_{x\to 0} \left(\frac{w_x(x, \frac{-x^2}{2})}{x} - w_t\Big(x, \frac{-x^2}{2}\Big)\right).$$

If we assume that $w_x(0,0)=0$, then we can repeat the previous step to obtain

$$\lim_{x\to 0}\frac{w_x(x,-\frac{x^2}{2})}{x}=\lim_{x\to 0}\,(w_{xx}-xw_{xt})$$

and so

$$\lim_{x\to 0}2\frac{w(x,-\frac{x^2}{2})}{x^2}=\lim_{x\to 0}\left(w_{xx}-x\,w_{xt}-w_t\right). \tag{2.4}$$

To replace w_x in (2.2) observe that $\lim_{x\to 0}\frac{w}{x}=\lim_{x\to 0}\left(w_x-x\,w_t\right)$ and if $\lim_{x\to 0}x\,w_t$ is finite then

$$\lim_{x\to 0}w_x=\lim_{x\to 0}\left(xw_t+\frac{w}{x}\right). \tag{2.5}$$

Therefore if $\lim_{w\to 0}\frac{w\varphi''(w)}{\varphi'(w)}$ is finite then equation (2.2), as $x\to 0$, becomes

$$\frac{\varphi''(w)}{\varphi'(w)}\left(\frac{w}{x}+xw_t\right)^2+\frac{f(\varphi(w))}{\varphi'(w)}+2\frac{w}{x^2}+xw_{xt}=o(1). \tag{2.6}$$

If we assume that

$$\lim_{w\to 0}\frac{f(\varphi(w))}{\varphi'(w)}=-b\neq 0,\qquad \lim_{x\to 0}\frac{x^2w_t}{w}=0,\qquad \lim_{w\to 0}\frac{w\varphi''(w)}{\varphi'(w)}=d,$$

then as $x\to 0$

$$\frac{w}{x^2}\left[\frac{w\varphi''(w)}{\varphi'(w)}\right]+\frac{f(\varphi(w))}{\varphi'(w)}+\frac{2w}{x^2}=o(1)$$

yields

$$w\Big(x,-\frac{x^2}{2}\Big)\Big(\frac{2+d}{x^2}\Big)\approx b\neq 0,$$

$$w\Big(x,-\frac{x^2}{2}\Big)\approx\frac{b}{2+d}\,x^2\qquad\text{as}\quad x\to 0,$$

where we need $\frac{b}{2+d}>0$ for w to be positive. We can then return to the original function $u(x,t)$.

$$u\Big(x,-\frac{x^2}{2}\Big)=\varphi(w)\approx\varphi\Big(\frac{b}{2+d}\,x^2\Big)\ \text{ as }\quad x\to 0.$$

So we have just proved the following:

Proposition 1. *Assume that the solution of* (2.1) *has a single point blow-up at* $(0,0)$ *and let* $\varphi(w)>0$ *be such that* $\lim_{w\to 0^+}\varphi(w)=\infty$,

$$\lim_{w\to 0}\frac{f(\varphi(w))}{\varphi'(w)}=-b\neq 0,\ \ \lim_{w\to 0}\frac{w\varphi''(w)}{\varphi'(w)}=d,\ \ \frac{b}{2+d}>0,$$

$\lim_{x\to 0} w_x(x, -\frac{1}{2}x^2) = 0$, $\lim_{x\to 0} \frac{x^2 w_t}{w} = 0$ *and* $\lim_{x\to 0} x w_{xt} \to 0$ *as* $x \to 0$ *on the path* $(x, \frac{-x^2}{2})$, *then*

$$u\Big(x, -\frac{x^2}{2}\Big) \approx \varphi\Big(\frac{b}{2+d}x^2\Big) \quad as \quad x \to 0.$$

3. Examples

A) Let us apply the above proposition to find the rate of blow-up for the classical

$$u_t = u_{xx} + e^u.$$

Because of the exponential growth $f(u) = e^u$ we shall choose $\varphi(w) = Ln\left(\frac{1}{w}\right)$. Clearly

$$\frac{f(\varphi(w))}{\varphi'(w)} = \frac{e^{Ln\left(\frac{1}{w}\right)}}{\frac{-1}{w}} = -1$$

and so $d = -1$ and $-b = -1 \neq 0$. Under the assumptions of the above proposition, the rate of the blow-up is

$$u\Big(x, -\frac{x^2}{2}\Big) \approx -Ln(x^2) = -2Ln(|x|).$$

B) We now treat an example worked out in [3], where

$$u_t = \frac{\partial}{\partial x}(u_x^{\sigma+1}) + u^\beta \qquad \beta > 1.$$

Set $u = w^\alpha$, where $\alpha < 0$ is to be chosen later. Then

$$w_t = A w^{(\alpha-1)(\sigma+1)+1-\alpha}\, w_x^\sigma\, w_{xx} + B w_x^{\sigma+2} w^{(\alpha-1)(\sigma+1)-1+1-\alpha} + \frac{1}{\alpha} w^{(\beta-1)\alpha+1}, \tag{3.1}$$

where $A = (\sigma+1)\alpha^\sigma$ and $B = \alpha^\sigma(\alpha-1)(\alpha+1)$. Now we only need to choose $\alpha(\beta-1)+1 = 0$, i.e., $\alpha = \frac{-1}{\beta-1} < 0$, so that

$$-w_t + A w^{(\alpha-1)\sigma} w_x^\sigma\, w_{xx} + B w_x^{\sigma+2} w^{(\alpha-1)\sigma-1} + \frac{1}{\alpha} = 0.$$

Therefore,

$$\lim_{x\to 0}\Big(-w_t + A w^{(\alpha-1)\sigma} w_x^\sigma w_{xx} + B w_x^{\sigma+2} w^{(\alpha-1)\sigma-1}\Big) = -\frac{1}{\alpha}.$$

Thus, by the above proposition, we have

$$\lim_{x\to 0}\Big(A w^{(\alpha-1)\sigma} \frac{w^\sigma}{x^\sigma}\frac{2w}{x^2} + B\frac{w^{\sigma+2}}{x^{\sigma+2}} w^{(\alpha-1)\sigma-1}\Big) = -\frac{1}{\alpha},$$

$$\lim_{x\to 0}\Big(2A\frac{w^{\alpha\sigma+1}}{x^{\sigma+2}}+B\frac{w^{\sigma\alpha+1}}{x^{\sigma+2}}\Big)=-\frac{1}{\alpha},$$

$$w\approx\Big(\frac{1}{2A+B}\frac{-1}{\alpha}\Big)^{\frac{1}{\alpha\sigma+1}}x^{\frac{(\sigma+2)}{\sigma\alpha+1}}\qquad \text{as}\quad x\to 0,$$

$$u=w^{\alpha}\approx\Big(\frac{1}{2A+B}\frac{-1}{\alpha}\Big)^{\frac{\alpha}{\alpha\sigma+1}}x^{\frac{(\sigma+2)\alpha}{\sigma\alpha+1}}\approx\Big(\frac{1}{2A+B}\frac{-1}{\alpha}\Big)^{\frac{\alpha}{\alpha\sigma+1}}x^{\frac{\sigma+2}{\sigma+1-\beta}}$$

as $x\to 0$ and $\sigma+1<\beta$. This was the asymptotic found in [3].

4. The n-dimensional case

We now turn to the following equation $u_t=\Delta u+f(u)$ where $x\in\mathbb{R}^n$. We shall see that the method also extends over to n-dimensional cases in a similar fashion. We start by a change of variables $u=\varphi(w)$ where φ is a certain smooth scalar function to be determined later. In this case the w-equation is

$$w_t=\Delta w+\frac{|\nabla w|^2}{w}\frac{w\varphi''(w)}{\varphi'(w)}+\frac{f\left(\varphi(w)\right)}{\varphi'(w)}. \tag{4.1}$$

As previously done, let us assume that the blow-up occurs at one point, at the origin say, and choose the path $t=-\frac{|x|^2}{2n}$, i.e., such that for $x=(x_1, x_2,...,x_n)$

$$w\Big(x,-\frac{|x|^2}{2n}\Big)\to 0\qquad\text{as}\quad x\to 0.$$

In this case we would obtain

$$\lim_{x_i\to 0}\frac{w}{\frac{x_i^2}{2}}=\lim_{x_i\to 0}\Big(\frac{w_{x_i}}{x_i}-\frac{w_t}{n}\Big).$$

Hence if we assume that $w_{x_i}(0,0)=0$ then

$$\lim_{x_i\to 0}\frac{w}{\frac{x_i^2}{2}}=\lim_{x_i\to 0}\Big(w_{x_ix_i}-\frac{x_i}{n}\,w_{x_i,t}-\frac{w_t}{n}\Big).$$

After collecting the limits over the remaining directions

$$\Delta w-\frac{1}{n}x.\nabla w_t-w_t-2w\Big(\sum_{i=1}^{n}\frac{1}{x_i^2}\Big)=o(1).$$

By using equation (4.1) we end up with

$$\frac{1}{n}x.\nabla w_t+2\Big(\sum_{i=1}^{n}\frac{1}{x_i^2}\Big)w+\frac{|\nabla w|^2}{w}\cdot\frac{\varphi''(w)}{\varphi'(w)}w+\frac{f(\varphi(w)}{\varphi'(w)}=o(1).$$

On the other hand

$$\lim_{x_i\to 0} w_{x_i} = \lim_{x_i\to 0} \frac{w}{x_i} + \frac{1}{n} x_i\, w_t.$$

That is, $\frac{w_{x_i}^2}{w} = \frac{w}{x_i^2}\left(1+\frac{1}{n}x_i^2\frac{w_t}{w}\right)^2 + o(1)$, out of which we deduce that

$$\frac{|\nabla w|^2}{w} = \Big(\sum_{i=1}^{n}\frac{1}{x_i^2}\Big[1+\frac{1}{n}x_i^2\frac{w_t}{w}\Big]^2\Big)w + o(1).$$

Assuming that $\left|\frac{w_t x_i^2}{w}\right| \to 0$ as $x_i \to 0$ we obtain

$$\frac{|\nabla w|^2}{w} \approx \Big(\sum_{i=1}^{n}\frac{1}{x_i^2}\Big)w \quad \text{as} \quad w\to 0.$$

Let us also assume that

$$\lim_{w\to 0}\frac{f(\varphi(w))}{\varphi'(w)} = -b \neq 0,\ \lim_{w\to 0}\frac{w\varphi''(w)}{\varphi'(w)} = d \quad \text{and} \quad \frac{b}{2+d} > 0. \tag{4.2}$$

Hence after collecting all terms and assuming that

$$\frac{x^2 w_t}{w} \to 0 \text{ and } x.\nabla w_t \to 0 \text{ as } x\to 0 \text{ on the path } \ t = -\frac{|x|^2}{2n}, \tag{4.3}$$

it follows that

$$\lim_{x\to 0} 2\Big(\sum_{i=1}^{n}\frac{1}{x_i^2}\Big)w - \frac{1}{n}x.\nabla w_t + d\Big(\sum_{i=1}^{n}\frac{1}{x_i^2}\Big)w = b$$

and if $\frac{1}{n}x.\nabla w_t \to 0$ as $x\to 0$ then

$$w(2+d)\Big(\sum\frac{1}{x_i^2}\Big) \approx b;$$

i.e.,

$$w\Big(x, -\frac{|x|^2}{2n}\Big) \approx \frac{b}{(2+d)}\Big(\sum_{i=1}^{n}\frac{1}{x_i^2}\Big)^{-1}.$$

Going back to the solution u yields

$$u\Big(x, -\frac{|x|^2}{2n}\Big) \approx \varphi\Big(\frac{b}{(2+d)}\Big(\sum_{i=1}^{n}\frac{1}{x_i^2}\Big)^{-1}\Big) \text{ as } \ |x|\to 0.$$

Proposition 2. *Assume that the limits in* (4.2) *and* (4.3) *hold, then*

$$u\Big(x, -\frac{|x|^2}{2n}\Big) \approx \varphi\Big(\frac{b}{(2+d)}\Big(\sum_{i=1}^{n}\frac{1}{x_i^2}\Big)^{-1}\Big) \quad \textit{as} \quad |x|\to 0. \tag{4.4}$$

Example. We now work out an example treated in [8]. Assume that $u(x,t)$ is a solution of

$$u_t = \Delta u + |u|^{p-1}u$$

which blows up at the point $(0,0)$ say, where $x \in \mathbb{R}^n$. By choosing $u(x,t) = \varphi(w) = w^\alpha$ where $\alpha = \frac{-1}{p-1}$, it follows that

$$d = \frac{w\varphi''(w)}{\varphi'(w)} = \frac{-p}{p-1} \quad \text{and} \qquad -b = \frac{f(\varphi(w))}{\varphi'(w)} = 1-p.$$

Thus we have

$$\frac{b}{2+d} = \frac{p-1}{2-\frac{p}{p-1}} = \frac{(p-1)^2}{p-2}.$$

On the path $t = \frac{-|x|^2}{2n}$, by (4.4) we have the following asymptotic

$$u\Big(x, \frac{-|x|^2}{2n}\Big) \approx \Big(\frac{(p-1)^2}{p-2}\Big(\sum_{i=1}^n \frac{1}{x_i^2}\Big)^{-1}\Big)^{\frac{-1}{p-1}} \quad \text{as} \;\; |x| \to 0.$$

Observe that if we choose $x_i = a$, we approach the blow-up point through a straight line in x then

$$\begin{aligned} u(a,...,a,-\frac{a^2}{2}) \;&\approx\; \Big(\frac{(p-1)^2}{p-2}\frac{a^2}{n}\Big)^{\frac{-1}{p-1}} = \Big(\frac{(p-2)n}{(p-1)^2}a^{-2}\Big)^{\frac{1}{p-1}} \qquad (4.5)\\ &\approx\; c\, a^{\frac{-2}{p-1}} \quad \text{as } a \to 0, \end{aligned}$$

which was obtained by Giga and Kohn in [8]. Observe that since we want the solution $u \geq 0$ in (4.5), we need at least that $p > 2$. It was shown by Giga-Matsui and Sasayama, [7], that the optimal range for p is $1 < p < \frac{n+2}{n-2}$ for $n \geq 3$, or $1 < p < \infty$ if $n \leq 2$. In the light of this result, (4.5) should hold for

$$2 < p \quad \text{if } n \leq 2 \text{ or } 2 < p < \frac{n+2}{n-2} \quad \text{for } n = 3,4,5.$$

The purpose of the above analysis is to extend the method of regular variation to partial differential equations, [5], and compute the rate of blow-up in a simple manner, assuming of course, it happens. For the latest results, on how and when a solution blows up, we refer to the paper [7], for more advanced methods that use local energy estimates and bootstrap arguments. We may also conjecture that when $p > \frac{n+2}{n-2}$ for $n \geq 3$ the limits in (4.3) may not hold.

REFERENCES

[1] U.G. Abdulla, *Evolution of interfaces and explicit asymptotics at infinity for the fast diffusion equation with absorption,* Nonlinear Anal., 50, Ser. A (2002), 541–560.

[2] A. Friedman and J.B. Mcleod, *Blow-up of positive solution of semilinear heat equations,* Indiana Univ Math J., 34 (1985), 425–447.

[3] V.A. Galaktionov and S.A. Poshakov, *Single point blow-up for N- dimensional quasilinear equations with gradient diffusion and source,* Indiana. Univ. Math. J., 40 (1991), 1041–1060.

[4] V.A. Galaktionov and J.L. Vazquez, "A Stability Technique for Evolution Partial Differential Equations," Progress in Nonlinear Differential equations and their applications, Vol 56, Birkhauser, (2004).

[5] J.L. Geluk, *Note on a theorem of Avakumović.* Proc. Amer. Math. Soc., 112 (1991), 429–431.

[6] J.L. Geluk and L. Haan, "Regular Variation, Extensions and Tauberian Theorems," CWI Tract, 40. Amsterdam, (1987)

[7] Y. Giga, S. Matsui, and S. Sasayama, *Blow up rate for semilinear heat equations with subcritical nonlinearity,* Indiana Univ. Math. J., 53 (2004), 483–514.

[8] Y. Giga and R.V. Kohn, *Asymptotic self-similar blow-up of semilinear heat equations,* Comm. Pure. Appl. Math., 38 (1985), 297–319.

[9] V. Maric, "Regular Variation and Differential Equations," Lecture Notes in Mathematics, 1726. Springer, Berlin, (2000).