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ROUGH SOLUTIONS OF A SCHRÖDINGER - BENJAMIN - ONO SYSTEM

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Abstract. The Cauchy problem for a coupled Schrödinger and Benjamin-Ono system is shown to be globally well posed for a class of data without finite energy. The proof uses the I-method introduced by Colliander, Keel, Staffilani, Takaoka, and Tao.

0. Introduction

Consider the following weakly coupled dispersive system

$$i\partial_t u + \partial_x^2 u = \alpha u v \tag{0.1}$$

$$\partial_t v + \nu \partial_x |\partial_x| v = \beta \partial_x (|u|^2) \tag{0.2}$$

with Cauchy data

$$u(x,0) = u_0(x)\,,\ v(x,0) = v_0(x), \tag{0.3}$$

where $x,t \in \mathbf{R}$, $\alpha, \beta, \nu \in \mathbf{R}$.

This system was introduced by Funakoshi and Oikawa [10] to model the interaction of two fluids described by a short wave term $u : \mathbf{R} \times \mathbf{R} \to \mathbf{C}$, which fulfills a Schrödinger type equation and a long wave term $v : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, which fulfills a Benjamin - Ono type equation. Bekiranov, Ogawa and Ponce [1] showed local well posedness for data $u_0 \in H^s(\mathbf{R})$, $v_0 \in H^{s-\frac{1}{2}}(\mathbf{R})$ and $|\nu| \neq 1$, $s \geq 0$. Because the system satisfies three conservation laws (cf. (2.1) - (2.3) below) it is not difficult to see that this solution exists globally if $\nu > 0$ and $\frac{\alpha}{\beta} < 0$ in the case $s \geq 1$ (finite energy solutions).

In this paper we first show local well posedness also in the case $|\nu| = 1$, if $s > 0$. Then we use the Fourier restriction norm method and especially the so-called I-method to show global well posedness for data with infinite energy

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(and $\nu > 0$, $\frac{\alpha}{\beta} < 0$); we assume only $s > \frac{1}{3}$. This method was introduced by Colliander, Keel, Staffilani, Takaoka, and Tao and successfully applied in various situations ([2], [3], [4], [5], [6], [7], [8], [9], [12], [13], [15]), in most of the cases using a scaling invariance of the problem. Such an invariance is also very helpful in our case. One introduces for given $0 < s < 1$ and $N >> 1$ the Fourier multiplier $\widehat{I_N f}(\xi) := m_N(\xi)\widehat{f}(\xi)$, where m_N is a smooth, radially symmetric and nonincreasing function of $|\xi|$ and

$$m(\xi) := m_N(\xi) := \begin{cases} 1 & |\xi| \le N \\ (\frac{N}{|\xi|})^{1-s} & |\xi| \ge 2N. \end{cases}$$

Then $I = I_N$ is a smoothing operator which maps $H^s(\mathbf{R})$ to $H^1(\mathbf{R})$ in the sense that

$$\|u\|_{H^s} \le c\|Iu\|_{H^1} \le cN^{1-s}\|u\|_{H^s}$$

and similarly

$$\|v\|_{H^{s-\frac{1}{2}}} \le c\|Iv\|_{H^{\frac{1}{2}}} \le cN^{1-s}\|v\|_{H^{s-\frac{1}{2}}} .$$

One then considers the conserved functionals L and E (cf. (2.2) and (2.3) below) replacing u and v by $I_N u$ and $I_N v$, so that they make sense for $u \in H^s$, $v \in H^{s-\frac{1}{2}}$, whereas they originally are only defined for $u \in H^1$, $v \in H^{\frac{1}{2}}$. These modified functionals are then shown to be almost conserved in the sense that their increment on a local existence interval is bounded by cN^{-1}. One can show that this is enough to make the continuation process by reapplying the local existence theorem uniformly, provided s is close enough to 1, namely $s > \frac{1}{3}$.

We use the following norms (for $s \in \mathbf{R}$, $-1 < b < 1$):

$$\|u\|_{X^{s,b}} := \|\langle\tau + \xi^2\rangle^b \langle\xi\rangle^s \widehat{u}(\xi,\tau)\|_{L^2(\mathbf{R}^2)}$$

$$\|v\|_{Y^{s,b}} := \|\langle\tau + \nu\xi|\xi|\rangle^b \langle\xi\rangle^s \widehat{v}(\xi,\tau)\|_{L^2(\mathbf{R}^2)}$$

belonging to the Schrödinger and Benjamin - Ono equation, respectively. We also need the local in time norm $\|u\|_{X^{s,b}_\delta} := \inf_{\psi_{|[0,\delta]}=f} \|\psi\|_{X^{s,b}}$ and similarly $\|v\|_{Y^{s,b}_\delta}$.

The standard facts about the Fourier restriction norm method which we use without further comments can be found in [1], Chapter 2. The Strichartz estimates for the homogeneous Schrödinger and Benjamin - Ono equation read

$$\|e^{it\partial_x^2}u_0\|_{L^6_{xt}} \le c\|u_0\|_{L^2_x} \quad \text{and} \quad \|e^{it\nu\partial_x|\partial_x|}u_0\|_{L^6_{xt}} \le c\|u_0\|_{L^2_x}$$

(cf. [14] and [11]), which immediately imply $\|u\|_{L^p_{xt}} \le c\|u\|_{X^{0,b}}$ and $\|v\|_{L^p_{xt}} \le c\|v\|_{Y^{0,b}}$ for $2 \le p \le 6$ and $b > \frac{1}{2}$. We also use the following bilinear Strichartz type estimate for the Schrödinger equation

$$\|D_x^{1/2}(u_1\overline{u_2})\|_{L^2_{xt}} \le c\|u_1\|_{X^{0,b}}\|u_2\|_{X^{0,b}}, \qquad b > \tfrac{1}{2} \tag{0.4}$$

(for a proof cf. e.g. [1], Lemma 3.2).

We denote by $a+$ and $a-$ a number slightly larger and smaller than a, respectively.

1. Local existence

Proposition 1.1. *For* $|\nu| = 1$ *we have*

$$\|uv\|_{X^{s,a}} \le c\|u\|_{X^{s,b}}\|v\|_{Y^{s-\frac{1}{2},b}}$$

if $-\frac{1}{2} < a < 0 < \frac{1}{2} < b$ *and* $s > 1 - 2|a|$ ($\Leftrightarrow |a| > \frac{1-s}{2}$) (*especially* $s > 0$).

Proof (along the lines of [1], Lemma 3.4). Assume first $\nu = 1$. We have to prove the following estimate

$$\Big|\int\int\int\int \frac{\langle\xi\rangle^s g(\sigma,\eta) f(\tau-\sigma,\xi-\eta)\overline{\phi}(\tau,\xi)\, d\sigma d\eta d\tau d\xi}{\langle\eta\rangle^{s-\frac{1}{2}}\langle\xi-\eta\rangle^s\langle\tau+\xi^2\rangle^{|a|}\langle\sigma+\nu\eta|\eta|\rangle^b\langle\tau-\sigma+(\xi-\eta)^2\rangle^b}\Big| \le c\|g\|_{L^2}\|f\|_{L^2}\|\phi\|_{L^2}. \tag{1.1}$$

We split $(\tau,\sigma,\xi,\eta) \in \mathbf{R}^4$ into several regions:

$$\begin{aligned} A &= \{|\eta| < 1\} \\ B &= \{\eta < 0\,,\ |\xi| \ge \tfrac{1}{2}|\eta|\,,\ |\eta| \ge 1\} \\ C &= \{\eta < 0\,,\ |\xi| < \tfrac{1}{2}|\eta|\,,\ |\eta| \ge 1\} \\ D &= \{\eta > 0\,,\ |\xi-\eta| \le \tfrac{1}{2}|\eta|\,,\ |\eta| \ge 1\} \\ E &= \{\eta > 0\,,\ |\xi-\eta| > \tfrac{1}{2}|\eta|\,,\ |\eta| \ge 1\}. \end{aligned}$$

Now in E we have

$$|\nu\eta|\eta| + \eta^2 - 2\xi\eta| = 2|\eta^2 - \xi\eta| = 2|\eta||\eta-\xi| > |\eta|^2$$

and thus

$$|\tau+\xi^2| + |\sigma+\nu\eta|\eta|| + |\tau-\sigma+(\xi-\eta)^2| \ge |\nu\eta|\eta| + \eta^2 - 2\xi\eta| > |\eta|^2\,.$$

According to which of the terms on the left-hand side is dominant we split E into 3 parts:

$$E_1 = E \cap \{|\tau+\xi^2| \ge |\sigma+\nu\eta|\eta||, |\tau-\sigma+(\xi-\eta)^2|\ ,\ |\tau+\xi^2| \ge \tfrac{1}{3}|\eta|^2\}$$

$E_2 = E \cap \{|\sigma + \nu\eta|\eta|| \geq |\tau + \xi^2|, |\tau - \sigma + (\xi - \eta)^2| \, , \, |\sigma + \nu\eta|\eta|| \geq \frac{1}{3}|\eta|^2\}$

$E_3 = E \cap \{|\tau - \sigma + (\xi - \eta)^2| \geq |\tau + \xi^2|, |\sigma + \nu\eta|\eta||, |\tau - \sigma + (\xi - \eta)^2| \geq \frac{1}{3}|\eta|^2\}.$

Define $R_1 = A \cup B \cup D \cup E_1$, $R_2 = C \cup E_2$, $R_3 = E_3$. In order to prove (1.1) in the region R_1 it is sufficient to show

$$\Big\| \frac{\langle \xi \rangle^s}{\langle \tau + \xi^2 \rangle^{|a|}} \Big(\int\int \frac{\langle \eta \rangle \chi_{R_1} \, d\sigma d\eta}{\langle \eta \rangle^{2s} \langle \xi - \eta \rangle^{2s} \langle \sigma + \nu\eta|\eta| \rangle^{2b} \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \Big)^{\frac{1}{2}} \Big\|_{L^\infty_\tau (L^\infty_\xi)} < \infty. \tag{1.2}$$

Similarly, in order to prove (1.1) in R_2 we have to show

$$\Big\| \frac{\langle \eta \rangle^{\frac{1}{2}}}{\langle \eta \rangle^s \langle \sigma + \nu\eta|\eta| \rangle^b} \Big(\int\int \frac{\langle \xi \rangle^{2s} \chi_{R_2} \, d\tau d\xi}{\langle \xi - \eta \rangle^{2s} \langle \tau + \xi^2 \rangle^{2|a|} \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \Big)^{\frac{1}{2}} \Big\|_{L^\infty_\sigma (L^\infty_\eta)} < \infty. \tag{1.3}$$

Finally, in order to prove (1.1) in R_3 we use the transformed region

$$\widetilde{R_3} = \{(\rho, \sigma, \zeta, \eta) \in \mathbf{R}^4 : |\eta| \geq 1 \, , \, |\rho - \zeta^2| \geq \frac{1}{3}|\eta|^2\} \, ,$$

where $\rho := \sigma - \tau$, $\zeta := \eta - \xi$. We have to show

$$\Big\| \frac{1}{\langle \zeta \rangle^s \langle \rho - \zeta^2 \rangle^b} \Big(\int\int \frac{\langle \eta - \zeta \rangle^{2s} \chi_{\widetilde{R_3}} \, d\sigma d\eta}{\langle \eta \rangle^{2s-1} \langle \sigma + \nu\eta|\eta| \rangle^{2b} \langle \sigma - \rho + (\eta - \zeta)^2 \rangle^{2|a|}} \Big)^{\frac{1}{2}} \Big\|_{L^\infty_\rho (L^\infty_\zeta)} < \infty. \tag{1.4}$$

We start to prove (1.2). In the regions A, B and D we use the estimate $\langle \xi \rangle \leq \langle \eta \rangle \langle \xi - \eta \rangle$ so that it suffices to show

$$\Big\| \int\int \frac{\langle \eta \rangle \, d\sigma d\eta}{\langle \sigma + \nu\eta|\eta| \rangle^{2b} \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \Big\|_{L^\infty_\tau (L^\infty_\xi)} < \infty \, .$$

Performing the σ-integration we get by [1], Lemma 2.5 (2.11):

$$\int\int \frac{\langle \eta \rangle \, d\sigma d\eta}{\langle \sigma + \nu\eta|\eta| \rangle^{2b} \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \leq c \int \frac{\langle \eta \rangle \, d\eta}{\langle \tau + \xi^2 + \nu\eta|\eta| + \eta^2 - 2\xi\eta \rangle^{2b}} \, .$$

This is trivially bounded in the region A, whereas in region B we substitute $\tau + \xi^2 + \nu\eta|\eta| + \eta^2 - 2\xi\eta = \tau + \xi^2 - 2\xi\eta =: \eta'$, so that $\frac{d\eta'}{d\eta} = -2\xi$, and we get the bound using $|\xi| \geq \frac{1}{2}|\eta|$ and $|\eta| \geq 1$:

$$c \int \frac{\langle \eta \rangle \, d\eta'}{\langle \eta' \rangle^{2b} |\xi|} \leq c \int \frac{d\eta'}{\langle \eta' \rangle^{2b}} < \infty$$

for $b > \frac{1}{2}$. In the region D we have $\tau + \xi^2 + \nu\eta|\eta| + \eta^2 - 2\xi\eta = \tau + \xi^2 + 2\eta^2 - 2\xi\eta =: \eta'$, so that $|\frac{d\eta'}{d\eta}| = |4\eta - 2\xi| = |2(2\eta - \xi)| = 2|\eta + (\eta - \xi)| \geq$

$2(|\eta| - |\eta - \xi|) \geq |\eta|$, because $|\xi - \eta| \leq \frac{1}{2}|\eta|$. Thus we get the bound

$$c \int \frac{\langle \eta \rangle \, d\eta'}{\langle \eta' \rangle^{2b} |\eta|} < \infty$$

for $b > \frac{1}{2}$ and $|\eta| \geq 1$.

It remains to prove (1.2) in the region E_1. First we consider the case $0 < s < \frac{1}{2}$. This implies $|a| > \frac{1-s}{2} > \frac{1}{4}$. We use again the estimate $\langle \xi \rangle \leq \langle \eta \rangle \langle \xi - \eta \rangle$, so that it suffices to show

$$\Big\| \frac{1}{\langle \tau + \xi^2 \rangle^{|a|}} \left(\int \int \frac{\langle \eta \rangle \, d\sigma d\eta}{\langle \sigma + \nu\eta|\eta| \rangle^{2b} \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \right)^{\frac{1}{2}} \Big\|_{L^\infty_\tau(L^\infty_\xi)} < \infty \, .$$

Performing the σ-integration as above it remains to bound

$$\begin{aligned} &\frac{1}{\langle \tau + \xi^2 \rangle^{|a|}} \Big(\int \frac{\langle \eta \rangle \, d\eta}{\langle \tau + \xi^2 + \nu\eta|\eta| + \eta^2 - 2\xi\eta \rangle^{2b}} \Big)^{\frac{1}{2}} \\ \leq \quad & c \Big(\int \frac{d\eta}{\langle \tau + \xi^2 + 2\eta^2 - 2\xi\eta \rangle^{2b}} \Big)^{\frac{1}{2}} \leq c \end{aligned}$$

using $\langle \tau + \xi^2 \rangle^{|a|} \geq c \langle \eta \rangle^{2|a|} \geq c \langle \eta \rangle^{\frac{1}{2}}$.

Next we consider the case $s \geq \frac{1}{2}$ in the region E_1. First of all, consider the subregion $|\xi| \geq \frac{3}{2}|\eta|$. In this case we get the following bound for (1.2) performing the σ-integration:

$$\begin{aligned} & c \Big(\int \int \frac{\langle \eta \rangle \, d\sigma d\eta}{\langle \eta \rangle^{2s} \langle \sigma + \nu\eta|\eta| \rangle^{2b} \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \Big)^{\frac{1}{2}} \\ \leq & c \Big(\int \frac{\langle \eta \rangle \, d\eta}{\langle \eta \rangle^{2s} \langle \tau + \xi^2 + \nu\eta|\eta| + \eta^2 - 2\xi\eta \rangle^{2b}} \Big)^{\frac{1}{2}} \\ \leq & c \Big(\int \frac{d\eta}{\langle \tau + \xi^2 + 2\eta^2 - 2\xi\eta \rangle^{2b}} \Big)^{\frac{1}{2}} \leq c \, . \end{aligned}$$

In the subregion $|\xi| \leq \frac{3}{2}|\eta|$ we get the following bound for (1.2) performing the σ-integration:

$$\begin{aligned} & c \Big(\int \int \frac{\langle \eta \rangle \, d\sigma d\eta}{\langle \sigma + \nu\eta|\eta| \rangle^{2b} \langle \tau - \sigma + (\xi - \eta)^2 \rangle^{2b}} \Big)^{\frac{1}{2}} \\ \leq \quad & c \Big(\int \frac{\langle \eta \rangle \, d\eta}{\langle \tau + \xi^2 + \nu\eta|\eta| + \eta^2 - 2\xi\eta \rangle^{2b}} \Big)^{\frac{1}{2}} . \end{aligned}$$

Substituting $\tau + \xi^2 + \nu\eta|\eta| + \eta^2 - 2\xi\eta = \tau + \xi^2 + 2\eta^2 - 2\xi\eta =: \eta'$ we have $|\frac{d\eta'}{d\eta}| = |4\eta - 2\xi| = 2(|2\eta| - |\xi|) \geq 2(|2\eta| - \frac{3}{2}|\eta|) = |\eta| \sim \langle\eta\rangle$, and we get the bound $c\left(\int \langle\eta'\rangle^{-2b} d\eta'\right)^{\frac{1}{2}} < \infty$.

Next we have to prove (1.3). In the region C it suffices to show

$$\Big\| \frac{\langle\eta\rangle^{\frac{1}{2}}}{\langle\eta\rangle^{2s}} \Big(\int\int \frac{\langle\xi\rangle^{2s}\chi_C \, d\tau d\xi}{\langle\tau+\xi^2\rangle^{2|a|}\langle\tau-\sigma+(\xi-\eta)^2\rangle^{2b}} \Big)^{\frac{1}{2}} \Big\|_{L^\infty_\sigma L^\infty_\eta} < \infty \,. \tag{1.5}$$

The integration with respect to τ gives by [1], Lemma 2.5 (2.11) and Hölder:

$$\begin{aligned} \int\int \frac{\langle\xi\rangle^{2s}\chi_C \, d\tau d\xi}{\langle\tau+\xi^2\rangle^{2|a|}\langle\tau-\sigma+(\xi-\eta)^2\rangle^{2b}} &\leq c \int \frac{\langle\xi\rangle^{2s}\chi_C \, d\xi}{\langle\sigma-\eta^2+2\xi\eta\rangle^{2|a|}} \\ &\leq c \Big(\int_{|\xi|\leq\frac{1}{2}|\eta|} \langle\xi\rangle^{2s\widehat{p}} \, d\xi \Big)^{\frac{1}{\widehat{p}}} \Big(\int \frac{d\xi}{\langle\sigma-\eta^2+2\xi\eta\rangle^{2|a|\widehat{q}}} d\xi \Big)^{\frac{1}{\widehat{q}}} . \end{aligned}$$

Choosing $\frac{1}{\widehat{q}} = 2|a|-$, $\frac{1}{\widehat{p}} = 1+2|a|+$ and substituting $\eta' = \sigma - \eta^2 + 2\xi\eta$ we get the bound

$$c\langle\eta\rangle^{\frac{2s\widehat{p}+1}{\widehat{p}}} \Big(\int \frac{d\eta'}{\langle\eta'\rangle^{1+}|\eta|} \Big)^{\frac{1}{\widehat{q}}} \leq c\langle\eta\rangle^{2s+\frac{1}{\widehat{p}}-\frac{1}{\widehat{q}}} = c\langle\eta\rangle^{2s+1-\frac{2}{\widehat{q}}} = c\langle\eta\rangle^{2s+1-4|a|+} .$$

Thus, we get the following bound for (1.5):

$$\langle\eta\rangle^{\frac{1}{2}-2s+s+\frac{1}{2}-2|a|+} = \langle\eta\rangle^{1-s-2|a|+} \leq c$$

because $|a| > \frac{1-s}{2}$.

Next we prove (1.3) in the region E_2. Using the estimate $\langle\xi\rangle \leq \langle\eta\rangle\langle\xi-\eta\rangle$ and performing the τ-integration it suffices to get a bound on E_2 for

$$\frac{\langle\eta\rangle^{\frac{1}{2}}}{\langle\sigma+\nu\eta|\eta|\rangle^b} \Big(\int \frac{d\xi}{\langle\sigma-\eta^2+2\eta\xi\rangle^{2|a|}} \Big)^{\frac{1}{2}} .$$

By substitution of $\eta' := \sigma - \eta^2 + 2\eta\xi$ and using the definition of E_2 we get

$$\begin{aligned} |\eta'| = |\sigma-\eta^2+2\eta\xi| &= |(\tau+\xi^2) - (\tau-\sigma+(\xi-\eta)^2)| \\ &\leq |\tau+\xi^2| + |\tau-\sigma+(\xi-\eta)^2| \leq 2|\sigma+\nu\eta|\eta|| \end{aligned}$$

and thus

$$\begin{aligned} &\frac{\langle\eta\rangle^{\frac{1}{2}}}{\langle\sigma+\nu\eta|\eta|\rangle^b} \Big(\int \frac{d\xi}{\langle\sigma-\eta^2+2\eta\xi\rangle^{2|a|}} \Big)^{\frac{1}{2}} \\ &\leq c \frac{\langle\eta\rangle^{\frac{1}{2}}}{\langle\sigma+\nu\eta|\eta|\rangle^b} \Big(\int_{|\eta'|\leq 2|\sigma+\nu\eta|\eta||} \frac{d\eta'}{\langle\eta'\rangle^{2|a|}|\eta|} \Big)^{\frac{1}{2}} \leq c \frac{\langle\sigma+\nu\eta|\eta|\rangle^{\frac{1}{2}-|a|}}{\langle\sigma+\nu\eta|\eta|\rangle^b} \leq c \,. \end{aligned}$$

It remains to prove (1.4) in the region $\widetilde{R_3}$. Using the estimate $\langle \eta-\zeta\rangle \le \langle\zeta\rangle\langle\eta\rangle$ and performing the σ-integration it is enough to give the following bound in $\widetilde{R_3}$:

$$\frac{1}{\langle\rho-\zeta^2\rangle^b}\Big(\int\frac{\langle\eta\rangle\,d\eta}{\langle\rho-\zeta^2+\nu\eta|\eta|-\eta^2+2\zeta\eta\rangle^{2|a|}}\Big)^{\frac12}\le c\Big(\int\frac{d\eta}{\langle\eta\rangle^{4b-1}}\Big)^{\frac12}\le c\,.$$

The case $\nu=-1$ can be treated in the same way by replacing $\eta<0$ by $\eta>0$ in the regions B and C and $\eta>0$ by $\eta<0$ in D and E.

Moreover, the following estimates for the nonlinearities are true (cf. [1], Corollary 3.3 and Lemma 3.4).

Proposition 1.2.

(1) *For arbitrary* ν *and* $s\ge 0$, $b>\frac12$:

$$\|(|u|^2)_x\|_{Y^{s-\frac12,0}}\le c\|u\|^2_{X^{s,b}}\,.$$

(2) *For* $|\nu|\ne 1$ *and* $s\ge 0$, $b>\frac12$:

$$\|uv\|_{X^{s,-\frac14}}\le c\|u\|_{X^{s,b}}\|v\|_{Y^{s-\frac12,b}}\,.$$

These propositions imply by standard arguments the following local existence result.

Theorem 1.1. *Let* $s>0$ *in the case* $|\nu|=1$, *and* $s\ge 0$ *in the case* $|\nu|\ne 1$. *For any* $(u_0,v_0)\in H^s(\mathbf{R})\times H^{s-\frac12}(\mathbf{R})$ *there exists* $b>\frac12$ *and* $\delta=\delta(\|u_0\|_{H^s},\|v_0\|_{H^{s-\frac12}})>0$ *such that the Cauchy problem* (0.1), (0.2), (0.3) *has a unique solution* $(u,v)\in X^{s,b}_\delta\times Y^{s-\frac12,b}_\delta$ *and* $(u,v)\in C^0([0,\delta],H^s\times H^{s-\frac12})$.

Applying the operator I to the system (0.1),(0.2),(0.3) we get the problem

$$iI\partial_t u+I\partial_x^2 u = \alpha I(uv) \tag{1.6}$$

$$I\partial_t v+\nu I(\partial_x|\partial_x|u) = \beta I\partial_x(|u|^2) \tag{1.7}$$

$$Iu(0)=Iu_0\quad,\quad Iv(0)=Iv_0\,. \tag{1.8}$$

For this system the following modified local existence result holds:

Proposition 1.3. *Assume* $1\ge s>0$, *if* $|\nu|=1$, *and* $s\ge 0$ *otherwise. For any* $(u_0,v_0)\in H^s\times H^{s-\frac12}$ *there exists* $\delta\le 1$ *and* $\delta\sim(\|Iu_0\|_{H^1}+\|Iv_0\|_{H^{\frac12}})^{-\frac2s-}$, *if* $|\nu|=1$, *and* $\delta\sim(\|Iu_0\|_{H^1}+\|Iv_0\|_{H^{\frac12}})^{-4-}$, *if* $|\nu|\ne 1$, *such that the system* (1.6), (1.7), (1.8) *has a unique local solution in the time interval* $[0,\delta]$ *with the property*

$$\|Iu\|_{X^{1,b}_\delta}+\|Iv\|_{Y^{\frac12,b}_\delta}\le\widehat{c}(\|Iu_0\|_{H^1}+\|Iv_0\|_{H^{\frac12}})\,,\quad \textit{where } b=\tfrac12+.$$

Proof. We construct a fixed point of the mapping $S = (\tilde{S}_0, \tilde{S}_1)$ induced by the integral equations belonging to the system (1.6), (1.7), (1.8):

$$\begin{aligned}\tilde{S}_0(Iu, Iv)(t) &:= e^{it\partial_x^2} Iu_0 - i \int_0^t e^{i(t-s)\partial_x^2} \alpha I(u(s)v(s))\, ds \\ \tilde{S}_1(Iu, Iv)(t) &:= e^{-\nu t \partial_x |\partial_x|} Iv_0 + \int_0^t e^{-\nu(t-s)\partial_x|\partial_x|} \beta I(|u(s)|^2)_x\, ds\,.\end{aligned}$$

The estimates for the nonlinearities in (1.1) and (1.2) carry over to corresponding estimates including the I-operators by the interpolation lemma of [7], namely

$$\begin{aligned}\|I(|u|^2)_x\|_{Y^{\frac{1}{2},0}} &\leq c\|Iu\|^2_{X^{1,b}} \\ \|I(uv)\|_{X^{1,-|a|}} &\leq c\|Iu\|_{X^{1,b}} \|Iv\|_{Y^{\frac{1}{2},b}}\end{aligned}$$

(with $|a| = \frac{1-s}{2}+$, if $|\nu| = 1$, and $|a| = \frac{1}{4}$ otherwise). This implies

$$\begin{aligned}\|\tilde{S}_0(Iu, Iv)\|_{X^{1,b}_\delta} &\leq c\|Iu_0\|_{H^1} + c|\alpha| \|Iu\|_{X^{1,b}_\delta} \|Iv\|_{Y^{\frac{1}{2},b}_\delta} \delta^{\frac{1}{2}-|a|-} \\ \|\tilde{S}_1(Iu, Iv)\|_{Y^{\frac{1}{2},b}_\delta} &\leq c\|Iv_0\|_{H^{\frac{1}{2}}} + c|\beta| \|Iu\|^2_{X^{1,b}_\delta} \delta^{\frac{1}{2}-}\,.\end{aligned}$$

This gives the desired bounds, provided

$$c\delta^{\frac{1}{2}-|a|-}(\|Iu_0\|_{H^1} + \|Iv_0\|_{H^{\frac{1}{2}}}) < 1\,.$$

2. Conservation laws

Our system has the following conserved quantities:

$$M := \|u\| \tag{2.1}$$

$$L(u,v) := -\frac{\alpha}{2\beta}\|v\|^2 - \Im \int u\overline{u_x}\, dx \tag{2.2}$$

$$E(u,v) := \|u_x\|^2 - \frac{\alpha\nu}{2\beta}\|D_x^{\frac{1}{2}} v\|^2 + \alpha \int v|u|^2\, dx\,. \tag{2.3}$$

From now on, we assume $\nu > 0$ and $\frac{\alpha}{\beta} < 0$.

Then L and E are controlled by $\|u\|_{H^1}$ and $\|v\|_{H^{\frac{1}{2}}}$, and vice versa, as one concludes as follows:

$$|L(u,v)| \leq c\|v\|^2 + M\|u_x\| \tag{2.4}$$

and

$$\|v\|^2 \leq c(|L| + M\|u_x\|)\,. \tag{2.5}$$

Concerning E we have by Gagliardo - Nirenberg

$$\int |vu^2|\,dx \le \|v\|\|u\|^{\frac{3}{2}}\|u_x\|^{\frac{1}{2}} \le c(|L|^{\frac{1}{2}}M^{\frac{3}{2}}\|u_x\|^{\frac{1}{2}} + M^2\|u_x\|)$$
$$\le c(|L|^{\frac{2}{3}}M^2 + M^4) + \epsilon\|u_x\|^2 \le c(|L|^{\frac{4}{3}} + M^4) + \epsilon\|u_x\|^2$$

and thus

$$\|u_x\|^2 + |\frac{\alpha\nu}{\beta}|\|D_x^{\frac{1}{2}}v\|^2 \le |E| + c(|L|^{\frac{4}{3}} + M^4) + \epsilon\|u_x\|^2 \,,$$

consequently,

$$\|u_x\|^2 + \|D_x^{\frac{1}{2}}v\|^2 \le c(|E| + |L|^{\frac{4}{3}} + M^4)\,. \tag{2.6}$$

Similarly,

$$|E| \le c(\|u_x\|^2 + \|D_x^{\frac{1}{2}}v\|^2 + |L|^{\frac{4}{3}} + M^4)\,. \tag{2.7}$$

From (2.4) and (2.7) we get

$$\begin{aligned} |E| &\le c(\|u_x\|^2 + \|D_x^{\frac{1}{2}}v\|^2 + \|v\|^{\frac{8}{3}} + M^{\frac{4}{3}}\|u_x\|^{\frac{4}{3}} + M^4) \\ &\le c(\|u_x\|^2 + \|D_x^{\frac{1}{2}}v\|^2 + \|v\|^{\frac{8}{3}} + M^4)\,. \end{aligned} \tag{2.8}$$

From (2.5) and (2.6) we have

$$\|v\|^2 \le c(|L| + M(|E|^{\frac{1}{2}} + |L|^{\frac{2}{3}} + M^2)) \le c(|L| + M|E|^{\frac{1}{2}} + M^3 + 1)\,. \tag{2.9}$$

Finally, from (2.6) and (2.9) we arrive at

$$\|u\|^2_{H^1} + \|v\|^2_{H^{\frac{1}{2}}} \le c(|E| + |L|^{\frac{4}{3}} + M^4 + 1)\,. \tag{2.10}$$

These estimates imply a-priori-bounds for the H^1 norm of u and the $H^{\frac{1}{2}}$ norm of v for data with finite energy E, finite L and finite $\|u_0\|$. This is the case for H^1 data u_0 and $H^{\frac{1}{2}}$ data v_0. Thus our local existence result implies

Theorem 2.1. *For data* $(u_0, v_0) \in H^1 \times H^{\frac{1}{2}}$ *and* $\nu > 0$, $\frac{\alpha}{\beta} < 0$ *there exists* $b > \frac{1}{2}$ *such that* (0.1), (0.2), (0.3) *has a unique global solution* $(u, v) \in X^{1,b} \times Y^{\frac{1}{2},b}$ *with* $(u, v) \in C^0(\mathbf{R}^+, H^1 \times H^{\frac{1}{2}})$.

In order to get a corresponding result for less regular data we consider the modified functionals $E(Iu, Iv)$ and $L(Iu, Iv)$.

Using the modified system (1.6), (1.7) an elementary calculation shows

$$\frac{d}{dt}E(Iu, Iv) = \alpha\nu\langle I(|u|^2) - |Iu|^2, D_x I v_x\rangle + \alpha\beta\langle I(|u|^2)_x - (|Iu|^2)_x, |Iu|^2\rangle$$

$$- 2\alpha^2 \Im\langle IvIu, I(vu) - IvIu\rangle - 2\alpha\Im\langle Iu_x, I(vu)_x - (IvIu)_x\rangle =: \sum_{j=1}^{4} I_j \tag{2.11}$$

and

$$\frac{d}{dt}L(Iu, Iv) = -\alpha(\langle Iv, (I(|u|^2) - (|Iu|^2)_x\rangle + 2\Re\langle I(vu) - IvIu, Iu_x\rangle) . \tag{2.12}$$

3. ALMOST CONSERVATION

Proposition 3.1. *If* (u, v) *is a solution of* (0.1), (0.2), (0.3) *in* $[0, \delta]$ *in the sense of Proposition* 1.3, *then the following estimate holds for* $N \geq 1$ *and* $s \geq \frac{1}{4}$:

$$\begin{aligned} &|E(Iu(\delta), Iv(\delta)) - E(Iu(0), Iv(0))| + |L(Iu(\delta), Iv(\delta)) - L(Iu(0), Iv(0))| \\ &\leq \; cN^{-1}\Big(\|Iu\|^2_{X_\delta^{1,\frac{1}{2}+}} \|Iv\|_{Y_\delta^{\frac{1}{2},\frac{1}{2}+}} + \|Iu\|^4_{X_\delta^{1,\frac{1}{2}+}} + \|Iu\|^2_{X_\delta^{1,\frac{1}{2}+}} \|Iv\|^2_{Y_\delta^{\frac{1}{2},\frac{1}{2}+}}\Big). \end{aligned}$$

Proof. Integrating (2.11) over $t \in [0, \delta]$ we have to estimate the various terms on the right-hand side. We assume without loss of generality the Fourier transforms of all the functions to be nonnegative. We drop δ from the notation $X_\delta^{s,b}$ and $Y_\delta^{s,b}$.

Estimate of I_1: We have to show

$$\begin{aligned} &\int_0^\delta \int_* \left|\frac{m(\xi_1 + \xi_2) - m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)}\right| |\xi_1 + \xi_2| \widehat{u_1}(\xi_1, t)\widehat{\overline{u_2}}(\xi_2, t)|\xi_3|\widehat{v}(\xi_3, t)\, d\xi dt \\ &\leq cN^{-1}\|u_1\|_{X^{1,\frac{1}{2}+}} \|u_2\|_{X^{1,\frac{1}{2}+}} \|v\|_{Y^{\frac{1}{2},\frac{1}{2}+}} . \end{aligned}$$

Here and in the sequel * denotes integration over the set $\sum \xi_i = 0$. We may assume $|\xi_1| \geq N$ or $|\xi_2| \geq N$, because otherwise the multiplier term vanishes, and also the two largest frequencies are equivalent.

Case 1: $|\xi_1| << |\xi_2| \sim |\xi_3|$, $|\xi_1| \leq N$, $|\xi_2| \geq N$.

Using the mean value theorem the multiplier term is estimated by

$$\left|\frac{m(\xi_1 + \xi_2) - m(\xi_2)}{m(\xi_2)}\right| \leq c\left|\frac{(\nabla m)(\xi_2)\xi_1}{m(\xi_2)}\right| \leq c\frac{|\xi_1|}{|\xi_2|} \leq c\frac{|\xi_1|}{N} .$$

Thus, by use of (0.4) the integral is bounded by

$$\begin{aligned} &\frac{c}{N}\int_0^\delta \int_* |\xi_1 + \xi_2|^{\frac{1}{2}} |\xi_1|\widehat{u_1}(\xi_1, t)|\xi_2|\widehat{\overline{u_2}}(\xi_2, t)|\xi_3|^{\frac{1}{2}}\widehat{v}(\xi_3, t)\, d\xi dt \\ &\leq \frac{c}{N}\|D_x^{\frac{1}{2}}(D_x u_1 D_x \overline{u_2})\|_{L^2_{xt}} \|D_x^{\frac{1}{2}} v\|_{L^2_{xt}} \end{aligned}$$

$$\leq \frac{c}{N}\|u_1\|_{X^{1,\frac{1}{2}+}}\|u_2\|_{X^{1,\frac{1}{2}+}}\|v\|_{Y^{\frac{1}{2},\frac{1}{2}+}}.$$

Case 2: $|\xi_1| << |\xi_2| \sim |\xi_3|$, $|\xi_1|, |\xi_2| \geq N$.

The multiplier is bounded by $\frac{c}{m(\xi_1)} \leq c\frac{|\xi_1|}{N}$. Thus we can conclude as in Case 1.

Case 3: $|\xi_1| \sim |\xi_2| \geq cN$, $|\xi_1 + \xi_2| \leq 2N$ ($\Longrightarrow |\xi_3| \leq c|\xi_1|, c|\xi_2|$).

The multiplier is bounded by $\frac{c}{m(\xi_1)m(\xi_2)} \leq c\frac{|\xi_1||\xi_2|}{N^2}$. Thus, we get the following bound for the integral using (0.4):

$$\begin{aligned}
&\frac{c}{N^2}\int_0^\delta \int_* N|\xi_1|\widehat{u_1}(\xi_1,t)|\xi_2|\widehat{\overline{u_2}}(\xi_2,t)|\xi_1+\xi_2|^{\frac{1}{2}}|\xi_3|^{\frac{1}{2}}\widehat{v}(\xi_3,t)\,d\xi dt \\
&\leq \frac{c}{N}\|D_x^{\frac{1}{2}}(D_x u_1 D_x \overline{u_2})\|_{L^2_{xt}}\|D_x^{\frac{1}{2}}v\|_{L^2_{xt}} \\
&\leq \frac{c}{N}\|u_1\|_{X^{1,\frac{1}{2}+}}\|u_2\|_{X^{1,\frac{1}{2}+}}\|v\|_{Y^{\frac{1}{2},\frac{1}{2}+}}.
\end{aligned}$$

Case 4: $|\xi_1| \sim |\xi_2| \geq cN$, $|\xi_1 + \xi_2| \geq 2N$.

The multiplier is bounded by

$$\frac{m(\xi_1+\xi_2)}{m(\xi_1)m(\xi_2)} + 1 \leq c\frac{|\xi_1|^{1-s}|\xi_2|^{1-s}N^{1-s}}{N^{1-s}N^{1-s}|\xi_1+\xi_2|^{1-s}} = c\frac{|\xi_1|^{1-s}|\xi_2|^{1-s}}{N^{1-s}|\xi_1+\xi_2|^{1-s}},$$

which gives the integral bound

$$\begin{aligned}
&\frac{c}{N^{1-s}}\int_0^\delta \int_* |\xi_1+\xi_2|^s|\xi_1|^{1-s}\widehat{u_1}(\xi_1,t)|\xi_2|^{1-s}\widehat{\overline{u_2}}(\xi_2,t)|\xi_3|\widehat{v}(\xi_3,t)\,d\xi dt \\
&\leq \frac{c}{N^{1-s}}\int_0^\delta \int_* |\xi_1|\widehat{u_1}(\xi_1,t)\frac{|\xi_2|}{N^s}\widehat{\overline{u_2}}(\xi_2,t)|\xi_1+\xi_2|^{\frac{1}{2}}|\xi_3|^{\frac{1}{2}}\widehat{v}(\xi_3,t)\,d\xi dt \\
&\leq \frac{c}{N}\|D_x^{\frac{1}{2}}(D_x u_1 D_x \overline{u_2})\|_{L^2_{xt}}\|D_x^{\frac{1}{2}}v\|_{L^2_{xt}} \\
&\leq \frac{c}{N}\|u_1\|_{X^{1,\frac{1}{2}+}}\|u_2\|_{X^{1,\frac{1}{2}+}}\|v\|_{Y^{\frac{1}{2},\frac{1}{2}+}}.
\end{aligned}$$

Estimate of I_4: It is sufficient to show

$$\begin{aligned}
&\int_0^\delta \int_* \left|\frac{m(\xi_1+\xi_2)-m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)}\right| |\xi_1+\xi_2|\widehat{v}(\xi_1,t)\widehat{u_2}(\xi_2,t)|\xi_3|\widehat{\overline{u_3}}(\xi_3,t)\,d\xi dt \\
&\leq cN^{-1}\|u_2\|_{X^{1,\frac{1}{2}+}}\|u_3\|_{X^{1,\frac{1}{2}+}}\|v\|_{Y^{\frac{1}{2},\frac{1}{2}+}}.
\end{aligned}$$

Case 1: $|\xi_1| << |\xi_2| \sim |\xi_3|$, $|\xi_2| \geq N$ ($\Longrightarrow |\xi_1 + \xi_2| \sim |\xi_2|$).

If $|\xi_1| \le N$, the multiplier is bounded by

$$\left|\frac{m(\xi_1+\xi_2)-m(\xi_2)}{m(\xi_2)}\right| \le c\left|\frac{(\nabla m)(\xi_2)\xi_1}{m(\xi_2)}\right| \le c\frac{|\xi_1|}{|\xi_2|} \le c\frac{|\xi_1|}{N}$$

and, if $|\xi_1| \ge N$, we have the bound $\frac{c}{m(\xi_1)} \le c\frac{|\xi_1|}{N}$, so that the integral is bounded by

$$\begin{aligned}
&\frac{c}{N}\int_0^\delta \int_* |\xi_1||\xi_1+\xi_2|\widehat{v}(\xi_1,t)\widehat{u_2}(\xi_2,t)|\xi_3|\widehat{\overline{u_3}}(\xi_3,t)\,d\xi dt \\
&\quad\le \frac{c}{N}\int_0^\delta \int_* |\xi_1|^{\frac12}|\xi_2+\xi_3|^{\frac12}\widehat{v}(\xi_1,t)|\xi_2|\widehat{u_2}(\xi_2,t)|\xi_3|\widehat{\overline{u_3}}(\xi_3,t)\,d\xi dt \\
&\quad\le \frac{c}{N}\|D_x^{\frac12}v\|_{L^2_{xt}}\|D_x^{\frac12}(D_xu_2D_x\overline{u_3})\|_{L^2_{xt}} \\
&\quad\le \frac{c}{N}\|u_2\|_{X^{1,\frac12+}}\|u_3\|_{X^{1,\frac12+}}\|v\|_{Y^{\frac12,\frac12+}}\,.
\end{aligned}$$

Case 2: $|\xi_1| >> |\xi_2|$, $|\xi_3| \sim |\xi_1|$, $|\xi_1| \ge N$ ($\Longrightarrow |\xi_1+\xi_2| \sim |\xi_1|$).

Similarly, as in Case 1 the multiplier is bounded by $c\frac{|\xi_2|}{N}$ and the integral by

$$\begin{aligned}
&\frac{c}{N}\int_0^\delta \int_* |\xi_1+\xi_2|\widehat{v}(\xi_1,t)|\xi_2|\widehat{u_2}(\xi_2,t)|\xi_3|\widehat{\overline{u_3}}(\xi_3,t)\,d\xi dt \\
&\quad\le \frac{c}{N}\int_0^\delta |\xi_1|^{\frac12}\widehat{v}(\xi_1,t)|\xi_2|\widehat{u_2}(\xi_2,t)|\xi_3|\widehat{\overline{u_3}}(\xi_3,t)|\xi_2+\xi_3|^{\frac12}\,d\xi dt\,,
\end{aligned}$$

the same bound as in Case 1, using $|\xi_1+\xi_2| \le c|\xi_1| = c|\xi_1|^{\frac12}|\xi_2+\xi_3|^{\frac12}$.

Case 3: $|\xi_1| \sim |\xi_2| \ge N$, $|\xi_1+\xi_2| \le 2N$.

The multiplier bound $\frac{c}{m(\xi_1)m(\xi_2)} \le c\frac{|\xi_1|}{N}\frac{|\xi_2|}{N}$ implies the integral bound

$$\begin{aligned}
&cN^{-2}\int_0^\delta \int_* |\xi_1||\xi_2||\xi_1+\xi_2|\widehat{v}(\xi_1,t)\widehat{u_2}(\xi_2,t)|\xi_3|\widehat{\overline{u_3}}(\xi_3,t)\,d\xi dt \\
&\quad\le cN^{-2}\int_0^\delta \int_* |\xi_1|^{\frac12}|\xi_2+\xi_3|^{\frac12}|\xi_2|2N\widehat{v}(\xi_1,t)\widehat{u_2}(\xi_2,t)|\xi_3|\widehat{\overline{u_3}}(\xi_3,t)\,d\xi dt \\
&\quad\le cN^{-1}\|D_x^{\frac12}v\|_{L^2_{xt}}\|D_x^{\frac12}(D_xu_2D_x\overline{u_3})\|_{L^2_{xt}} \\
&\quad\le cN^{-1}\|v\|_{Y^{\frac12,\frac12+}}\|u_2\|_{X^{1,\frac12+}}\|u_3\|_{X^{1,\frac12+}}\,.
\end{aligned}$$

Case 4: $|\xi_1| \sim |\xi_2| \ge N$, $|\xi_1+\xi_2| \ge 2N$.

The multiplier is bounded by

$$\frac{m(\xi_1+\xi_2)}{m(\xi_1)m(\xi_2)}+1 \le c\frac{|\xi_1|^{1-s}|\xi_2|^{1-s}N^{1-s}}{N^{1-s}N^{1-s}|\xi_1+\xi_2|^{1-s}} = c\frac{|\xi_1|^{1-s}|\xi_2|^{1-s}}{N^{1-s}|\xi_1+\xi_2|^{1-s}}$$

and the integral by

$$\begin{aligned}
&\frac{c}{N^{1-s}}\int_0^\delta\int_* |\xi_1+\xi_2|^s|\xi_1|^{1-s}\widehat{v}(\xi_1,t)|\xi_2|^{1-s}\widehat{u_2}(\xi_2,t)|\xi_3|\widehat{\overline{u_3}}(\xi_3,t)\,d\xi dt\\
&\le \frac{c}{N}\int_0^\delta\int_* |\xi_1|^{\frac12}\widehat{v}(\xi_1,t)|\xi_2+\xi_3|^{\frac12}|\xi_2|\widehat{u_2}(\xi_2,t)|\xi_3|\widehat{\overline{u_3}}(\xi_3,t)\,d\xi dt\\
&\le cN^{-1}\|D_x^{\frac12}v\|_{L^2_{xt}}\|D_x^{\frac12}(D_xu_2D_x\overline{u_3})\|_{L^2_{xt}}\\
&\le cN^{-1}\|v\|_{Y^{\frac12,\frac12+}}\|u_2\|_{X^{1,\frac12+}}\|u_3\|_{X^{1,\frac12+}}
\end{aligned}$$

using $|\xi_1+\xi_2|^s|\xi_1|^{1-s}\le c|\xi_1| = c|\xi_1|^{\frac12}|\xi_2+\xi_3|^{\frac12}$ and $|\xi_2|^{1-s}\le|\xi_2|N^{-s}$.

Estimate of I_2: We have to show

$$\int_0^\delta\int_* \frac{m(\xi_1+\xi_2)-m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)}|\xi_1+\xi_2|\widehat{u_1}(\xi_1,t)\widehat{\overline{u_2}}(\xi_2,t)\widehat{u_3}(\xi_3,t)\widehat{\overline{u_4}}(\xi_4,t)d\xi dt$$
$$\le cN^{-1}\prod_{i=1}^4\|u_i\|_{X^{1,\frac12+}}.$$

The multiplier is bounded by $c|\xi_1||\xi_2|N^{-2}$, if $|\xi_1|,|\xi_2|\ge N$, and the integral by

$$cN^{-2}\int_0^\delta\int_* |\xi_1|\widehat{u_1}(\xi_1,t)|\xi_2|\widehat{\overline{u_2}}(\xi_2,t)(|\xi_3|\widehat{u_3}(\xi_3,t)\widehat{\overline{u_4}}(\xi_4,t)+\widehat{u_3}(\xi_3,t)|\xi_4|\widehat{\overline{u_4}}(\xi_4,t))d\xi dt$$

using $|\xi_1+\xi_2|=|\xi_3+\xi_4|\le|\xi_3|+|\xi_4|$. Strichartz' estimate gives the bound

$$\begin{aligned}
&cN^{-2}\|D_xu_1\|_{L^4_{xt}}\|D_xu_2\|_{L^4_{xt}}(\|D_xu_3\|_{L^4_{xt}}\|u_4\|_{L^4_{xt}}+\|u_3\|_{L^4_{xt}}\|D_xu_4\|_{L^4_{xt}})\\
&\le cN^{-2}\prod_{i=1}^4\|u_i\|_{X^{1,\frac12+}}.
\end{aligned}$$

If however $|\xi_1|\ge N$, $|\xi_2|\le N$, the multiplier bound $c|\xi_1|N^{-1}$ similarly gives the bound $cN^{-1}\prod_{i=1}^4\|u_i\|_{X^{1,\frac12+}}$.

Estimate of I_3: It suffices to show

$$\begin{aligned}
&\int_0^\delta\int_* \frac{m(\xi_1+\xi_2)-m(\xi_1)m(\xi_2)}{m(\xi_1)m(\xi_2)}\widehat{v_1}(\xi_1,t)\widehat{u_2}(\xi_2,t)\widehat{v_3}(\xi_3,t)\widehat{\overline{u_4}}(\xi_4,t)\,d\xi dt\\
&\le cN^{-1}\|v_1\|_{Y^{\frac12,\frac12+}}\|u_2\|_{X^{1,\frac12+}}\|v_3\|_{Y^{\frac12,\frac12+}}\|u_4\|_{X^{1,\frac12+}}.
\end{aligned}$$

Case 1: $|\xi_1| \geq N$, $|\xi_2| \geq N$.

The multiplier bound

$$c\left(\frac{|\xi_1|}{N}\right)^{\frac{3}{4}}\left(\frac{|\xi_2|}{N}\right)^{\frac{3}{4}} = c\frac{|\xi_1|^{\frac{1}{2}}|\xi_2+\xi_3+\xi_4|^{\frac{1}{4}}|\xi_2|^{\frac{3}{4}}}{N^{\frac{3}{4}}N^{\frac{3}{4}}}$$

allows us to estimate the integral by

$$cN^{-\frac{3}{2}}\int_0^\delta\int_* |\xi_1|^{\frac{1}{2}}\widehat{v_1}(\xi_1,t)\langle\xi_2\rangle\widehat{u_2}(\xi_2,t)\langle\xi_3\rangle^{\frac{1}{4}}\widehat{v_3}(\xi_3,t)\langle\xi_4\rangle^{\frac{1}{4}}\widehat{\overline{u_4}}(\xi_4,t)\,d\xi dt\,.$$

Using Hölder's inequality with exponent 4 and Strichartz' estimate easily gives the desired bound.

Case 2: $|\xi_1| \geq N$, $|\xi_2| \leq N$ (or similarly $|\xi_1| \leq N$, $|\xi_2| \geq N$).

The multiplier bound $c|\xi_1|N^{-1} \leq c|\xi_1|^{\frac{1}{2}}|\xi_2+\xi_3+\xi_4|^{\frac{1}{2}}N^{-1}$ allows us to estimate the integral by

$$cN^{-1}\int_0^\delta\int_* |\xi_1|^{\frac{1}{2}}\widehat{v_1}(\xi_1,t)\langle\xi_2\rangle^{\frac{1}{2}}\widehat{u_2}(\xi_2,t)\langle\xi_3\rangle^{\frac{1}{2}}\widehat{v_3}(\xi_3,t)\langle\xi_4\rangle^{\frac{1}{2}}\widehat{\overline{u_4}}(\xi_4,t)\,d\xi dt\,.$$

Similarly as in Case 1 this gives the desired estimate.

Concerning the estimate for L we remark that the first term on the right-hand side of (2.12) can be handled like I_1 and the second term like I_4 (with one derivative less). This completes the proof.

4. GLOBAL EXISTENCE

One easily checks

$$\|I_N u\|_{\dot{H}^1} \leq cN^{1-s}\|u\|_{\dot{H}^s}$$

and also for $0 < s \leq \frac{1}{2}$:

$$\|I_N v\|_{L^2} \leq cN^{\frac{1}{2}-s}\|v\|_{H^{s-\frac{1}{2}}}\,.$$

Trivially one has

$$\|I_N u\|_{L^2} \leq c\|u\|_{L^2}$$

and also

$$\|I_N u\|_{L^4} \leq c\|u\|_{L^4}$$

by Mikhlin's multiplier theorem. This implies immediately for $1 > s \geq \frac{1}{2}$:

$$\begin{aligned}|E(I_N u, I_N v)| &\leq c(\|I_N u_x\|^2 + \|D_x^{\frac{1}{2}} I_N v\|^2 + \|I_N v\|_{L^2}\|I_N u\|_{L^4}^2)\\ &\leq c\left[N^{2(1-s)}(\|u\|_{\dot{H}^s}^2 + \|v\|_{\dot{H}^{s-\frac{1}{2}}}^2) + \|v\|_{L^2}\|u\|_{L^4}^2\right]\end{aligned}$$

and

$$|L(I_N u, I_N v)| \le c(\|I_N v\|^2 + \|I_N u\| \|I_N u_x\|) \le c(\|v\|^2 + \|u\| N^{1-s} \|u\|_{\dot{H}^s}) .$$

Similarly, for $0 < s \le \frac{1}{2}$ we get

$$|E(I_N u, I_N v)| \le c \left[N^{2(1-s)} (\|u\|^2_{\dot{H}^s} + \|v\|^2_{H^{s-\frac{1}{2}}}) + N^{\frac{1}{2}-s} \|v\|_{H^{s-\frac{1}{2}}} \|u\|^2_{L^4} \right]$$

and

$$|L(I_N u, I_N v)| \le c(N^{2(\frac{1}{2}-s)} \|v\|^2_{H^{s-\frac{1}{2}}} + \|u\| N^{1-s} \|u\|_{\dot{H}^s}) .$$

We note that our system has a scaling invariance, i.e., if (u, v) is a solution, then also

$$u^{(\lambda)}(x,t) := \lambda^{-\frac{3}{2}} u(\frac{x}{\lambda}, \frac{t}{\lambda^2}) \quad , \quad v^{(\lambda)}(x,t) := \lambda^{-2} v(\frac{x}{\lambda}, \frac{t}{\lambda^2})$$

for $\lambda > 0$, as one easily checks. Then

$$\|u_0^{(\lambda)}\|_{\dot{H}^s} = \lambda^{-\frac{3}{2}} \|u_0(\frac{x}{\lambda})\|_{\dot{H}^s} = c\lambda^{-(s+1)} \|u_0\|_{\dot{H}^s}$$

and

$$\|v_0^{(\lambda)}\|_{\dot{H}^{s-\frac{1}{2}}} = \lambda^{-2} \|v_0(\frac{x}{\lambda})\|_{\dot{H}^{s-\frac{1}{2}}} = c\lambda^{-(s+1)} \|v_0\|_{\dot{H}^{s-\frac{1}{2}}} \quad (\text{for } s \ge \frac{1}{2})$$

as well as

$$\|u_0^{(\lambda)}\|_{L^4} = c\lambda^{-\frac{5}{4}} \|u_0\|_{L^4} \, , \ \|v_0^{(\lambda)}\|_{L^2} = c\lambda^{-\frac{3}{2}} \|v_0\|_{L^2} \, , \ \|u_0^{(\lambda)}\|_{L^2} = c\lambda^{-1} \|u_0\|_{L^2} .$$

We also need

Lemma 4.1. *For $s \le \frac{1}{2}$ and $\lambda \ge 1$ the following estimate holds*:

$$\|v_0^{(\lambda)}\|_{H^{s-\frac{1}{2}}} \le c\lambda^{-(s+1)} \|v_0\|_{H^{s-\frac{1}{2}}} .$$

Proof.

$$\begin{aligned}
\|v_0^{(\lambda)}\|_{H^{s-\frac{1}{2}}} &= \lambda^{-2} \|v_0(\frac{x}{\lambda})\|_{H^{s-\frac{1}{2}}} = \lambda^{-2} \|\langle\xi\rangle^{s-\frac{1}{2}} \widehat{v_0(\frac{x}{\lambda})}\|_{L^2} \\
&= \lambda^{-1} \|\langle\xi\rangle^{s-\frac{1}{2}} \widehat{v_0}(\lambda\xi)\|_{L^2} = \lambda^{-\frac{3}{2}} \Big(\int \Big| \langle\frac{\eta}{\lambda}\rangle^{s-\frac{1}{2}} \widehat{v_0}(\eta \Big|^2 d\eta \Big)^{\frac{1}{2}} \\
&\le c\lambda^{-\frac{3}{2}} \Big[\Big(\int_{|\eta|\le 1} |\widehat{v_0}(\eta)|^2 d\eta \Big)^{\frac{1}{2}} + \Big(\int_{|\eta|\ge 1} \Big| |\frac{\eta}{\lambda}|^{s-\frac{1}{2}} \widehat{v_0}(\eta) \Big|^2 d\eta \Big)^{\frac{1}{2}} \Big] \\
&\le c\lambda^{-\frac{3}{2}} (1 + \lambda^{-(s-\frac{1}{2})}) \|v_0\|_{H^{s-\frac{1}{2}}} \ \le \ c\lambda^{-(s+1)} \|v_0\|_{H^{s-\frac{1}{2}}} .
\end{aligned}$$

This implies the following bounds for the modified functionals E and L for $\lambda \ge 1$:

a) In the case $1 \geq s \geq \frac{1}{2}$, we get

$$\begin{aligned} &|E(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})| \\ &\quad \leq c\Big(N^{2(1-s)}(\|u_0^{(\lambda)}\|^2_{\dot H^s} + \|v_0^{(\lambda)}\|^2_{\dot H^{s-\frac{1}{2}}}) + \|v_0^{(\lambda)}\|_{L^2}\|u_0^{(\lambda)}\|^2_{L^4}\Big) \\ &\quad \leq c\left(N^{2(1-s)}\lambda^{-2(s+1)}(\|u_0\|_{\dot H^s} + \|v_0\|_{\dot H^{s-\frac{1}{2}}})^2 + \lambda^{-4}\|u_0\|^2_{L^4}\|v_0\|^2\right). \end{aligned}$$

Thus,

$$|E(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})| \leq c_0^2 N^{2(1-s)}\lambda^{-2(s+1)}(1 + \|u_0\|_{H^s} + \|v_0\|_{H^{s-\frac{1}{2}}})^4 .$$

Similarly,

$$\begin{aligned} |L(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})| &\leq c(\lambda^{-3}\|v_0\|^2 + \lambda^{-1}\|u_0\| N^{1-s}\lambda^{-(s+1)}\|u_0\|_{\dot H^s}) \\ &\leq cN^{1-s}\lambda^{-(s+1)}(\|v_0\|^2 + \|u_0\|\|u_0\|_{\dot H^s}) \\ &\leq c_0 N^{1-s}\lambda^{-(s+1)}(1 + \|u_0\|_{H^s} + \|v_0\|_{H^{s-\frac{1}{2}}})^2. \end{aligned}$$

b) In the case $\frac{1}{4} \leq s < \frac{1}{2}$ we get by Lemma 4.1 :

$$\begin{aligned} &|E(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})| \\ &\leq c[N^{2(1-s)}\lambda^{-2(s+1)}(\|u_0\|_{\dot H^s} + \|v_0\|_{H^{s-\frac{1}{2}}})^2 + N^{\frac{1}{2}-s}\lambda^{-(s+1)}\|v_0\|_{H^{s-\frac{1}{2}}}\lambda^{-\frac{5}{2}}\|u_0\|^2_{L^4}] \\ &\leq c_0^2 N^{2(1-s)}\lambda^{-2(s+1)}(1 + \|u_0\|_{H^s} + \|v_0\|_{H^{s-\frac{1}{2}}})^4. \end{aligned}$$

Moreover, we crudely estimate

$$\begin{aligned} &|L(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})| \\ &\quad \leq c(N^{2(\frac{1}{2}-s)}\lambda^{-2(s+1)}\|v_0\|^2_{H^{s-\frac{1}{2}}} + \lambda^{-1}\|u_0\| N^{1-s}\lambda^{-(s+1)}\|u_0\|_{\dot H^s}) \\ &\quad \leq c_0 N^{1-s}\lambda^{-(s+1)}(1 + \|v_0\|_{H^{s-\frac{1}{2}}} + \|u_0\|_{H^s})^2 . \end{aligned}$$

Now assume $N >> 1$ is given (to be chosen later), we choose $\lambda = \lambda(N, \|u_0\|_{H^s}, \|v_0\|_{H^{s-\frac{1}{2}}})$ as follows:

$$\lambda = N^{\frac{1-s}{1+s}}(4c_0)^{\frac{1}{1+s}}(1 + \|v_0\|_{H^{s-\frac{1}{2}}} + \|u_0\|_{H^s})^{\frac{2}{s+1}} ,$$

so that $|E(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})| \leq \frac{1}{4}$ and $|L(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})| \leq \frac{1}{4}$. Such a bound implies by (2.10) the following estimate for the scaled initial data:

$$\begin{aligned} &\|I_N u_0^{(\lambda)}\|^2_{H^1} + \|I_N v_0^{(\lambda)}\|^2_{H^{\frac{1}{2}}} \\ &\leq \overline{c}^2(|E(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})| + |L(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})|^{\frac{4}{3}} + \|I_N u_0^{(\lambda)}\|^4_{L^2} + 1) \qquad (4.1) \\ &\leq \overline{c}^2(2 + \|u_0\|^4_{L^2}) , \end{aligned}$$

because $\|I_N u_0^{(\lambda)}\|_{L^2} \le \|u_0^{(\lambda)}\|_{L^2} = \lambda^{-1}\|u_0\|_{L^2} \le \|u_0\|_{L^2}$ for $\lambda \ge 1$. Thus, the local existence theorem Proposition 1.3 gives a solution on a time interval of length $\delta = \delta(\|u_0\|_{L^2})$ and

$$\|I_N u^{(\lambda)}\|_{X_\delta^{1,b}} + \|I_N v^{(\lambda)}\|_{Y_\delta^{\frac{1}{2},b}} \le \widehat{c}\,\overline{c}(2 + \|u_0\|_{L^2}^4)^{\frac{1}{2}} . \tag{4.2}$$

Thus, Proposition 3.1 shows

$$\begin{aligned}&|E(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta))| + |L(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta))|\\ \le\ & CN^{-1} + |E(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})| + |L(I_N u_0^{(\lambda)}, I_N v_0^{(\lambda)})|,\end{aligned}$$

where $C = C(\|u_0\|_{L^2})$. We choose N large enough, so that

$$|E(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta))| + |L(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta))| < 1$$

and such that we can reapply the local existence theorem with time intervals of equal length (note that $\|u_0\|_{L^2}$ is conserved) N^{1-} times before the size of $|E(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta))| + |L(I_N u^{(\lambda)}(\delta), I_N v^{(\lambda)}(\delta))|$ reaches 1). During the whole iteration process the bounds for the iterated solutions on the right-hand side of (4.1) and (4.2) can be chosen uniformly. Now, given any finite time T we are able to get a solution in this way on $[0, T]$, provided $N^{1-}\delta\lambda^{-2} = T$, taking the scaling into account. Using the definition of λ above, this means that $N^{1-}\delta N^{-\frac{2(1-s)}{1+s}} = T$. This can be fulfilled for a sufficiently large N, provided $1 > \frac{2(1-s)}{1+s} \iff 1 + s > 2(1-s) \iff s > \frac{1}{3}$.

Thus, we have proven the following global existence result:

Theorem 4.1. *For $1 > s > \frac{1}{3}$ and $(u_0, v_0) \in H^s \times H^{s-\frac{1}{2}}$ there exists $b > \frac{1}{2}$ such that the Cauchy problem* (0.1), (0.2), (0.3) *has a unique global solution $(u, v) \in X^{s,b} \times Y^{s-\frac{1}{2},b}$ with $(u, v) \in C^0_{loc}(\mathbf{R}^+, H^s \times H^{s-\frac{1}{2}})$.*

It is also possible to show that this global solution grows at most polynomially in t. The procedure above namely shows

$$|E(I_N u^{(\lambda)}(N^{1-}\delta), I_N v^{(\lambda)}(N^{1-}\delta))| + |E(I_N u^{(\lambda)}(N^{1-}\delta), I_N v^{(\lambda)}(N^{1-}\delta))| \le 1 .$$

This implies by (2.10):

$$\begin{aligned}&\|I_N u^{(\lambda)}(N^{1-}\delta)\|_{H^1}^2 + \|I_N v^{(\lambda)}(N^{1-}\delta)\|_{H^{\frac{1}{2}}}^2 \le c(1 + \|I_N u^{(\lambda)}(N^{1-}\delta)\|_{L^2}^4)\\ &\le c(1 + \|u^{(\lambda)}(N^{1-}\delta)\|_{L^2}^4) \le c(1 + \|u_0^{(\lambda)}\|_{L^2}^4)\\ &\le c(1 + \lambda^{-4}\|u_0\|_{L^2}^4) \le c(1 + \|u_0\|_{L^2}^4)\end{aligned}$$

for $\lambda \geq 1$. Thus we get

$$\|u^{(\lambda)}(N^{1-}\delta)\|_{H^s} + \|v^{(\lambda)}(N^{1-}\delta)\|_{H^{s-\frac{1}{2}}} \leq c\,.$$

But now

$$\|u^{(\lambda)}(N^{1-}\delta)\|_{H^s} = \lambda^{-\frac{3}{2}}\|u(\frac{x}{\lambda},T)\|_{H^s} \geq c\lambda^{-(1+s)}\|u(T)\|_{H^s}$$

and similarly for $s \geq \frac{1}{2}$:

$$\|v^{(\lambda)}(N^{1-}\delta)\|_{H^{s-\frac{1}{2}}} = \lambda^{-2}\|v(\frac{x}{\lambda},T)\|_{H^{s-\frac{1}{2}}} \geq c\lambda^{-(1+s)}\|v(T)\|_{H^{s-\frac{1}{2}}}\,,$$

whereas for $s < \frac{1}{2}$:

$$\|v^{(\lambda)}(N^{1-}\delta)\|_{H^{s-\frac{1}{2}}} \geq c\lambda^{-\frac{3}{2}}\|v(T)\|_{H^{s-\frac{1}{2}}}\,.$$

Because $\lambda \sim N^{\frac{1-s}{1+s}}$ and $T \sim N^{1-}\lambda^{-2} \sim N^{1-}N^{-\frac{2(1-s)}{1+s}} = N^{\frac{3s-1}{s+1}-}$, so that $N \sim T^{\frac{s+1}{3s-1}+}$, we get $\lambda \sim T^{\frac{1-s}{3s-1}+}$, so that we have proven

Theorem 4.2. *The global solution of Theorem* 4.1 *fulfills for* $t \in \mathbf{R}$ *:*

$$\|u(t)\|_{H^s} \quad \leq \quad c(1+t^{\frac{(s+1)(1-s)}{3s-1}+}) \quad \text{for} \quad 1 > s > \frac{1}{3}$$

and

$$\begin{aligned} \|v(t)\|_{H^{s-\frac{1}{2}}} &\leq c(1+t^{\frac{(s+1)(1-s)}{3s-1}+}) \quad \text{for} \quad 1 > s \geq \frac{1}{2}\,, \\ \|v(t)\|_{H^{s-\frac{1}{2}}} &\leq c(1+t^{\frac{\frac{3}{2}(1-s)}{3s-1}+}) \qquad \text{for} \quad \frac{1}{2} > s > \frac{1}{3}\,. \end{aligned}$$

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