Differential and Integral Equations Volume 19, Number 5 (2006), 481–516

ON A CLASS OF FREE BOUNDARY PROBLEMS OF TYPE $div(a(X)\nabla u) = -div(\chi(u)H(X))$

S. Challal and A. Lyaghfouri
King Fahd University of Petroleum and Minerals
P.O. Box 728, Dhahran 31261, Saudi Arabia

(Submitted by: Jesus Ildefonso Diaz)

Abstract. We consider a class of two-dimensional free boundary problems of type $div(a(X)\nabla u) = -div(\chi(u)H(X))$, where H is a Lipschitz vector function satisfying $div(H(X)) \geq 0$. We prove that the free boundary $\partial[u > 0] \cap \Omega$ is represented locally by a family of continuous functions.

Introduction

In [4], we studied the following problem

$$(P_0)\begin{cases} \text{Find } (u,\chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that} \\ (i) \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(\chi - 1) = 0 \ \text{ a.e. in } \Omega \\ (ii) \quad \displaystyle\int_\Omega \big(a(X)\nabla u + \chi \mathtt{h}(X)\big).\nabla \xi dX \ \leq \ 0 \\ \qquad\qquad \forall \xi \in H^1(\Omega), \quad \xi = 0 \text{ on } \Gamma_1 \cup \Gamma_3, \quad \xi \geq 0 \text{ on } \Gamma_2, \end{cases}$$

where Ω is the open set $\{X = (x,y) \in \mathbb{R}^2 : y \in (a_0,b_0),\ \gamma_1(y) < x < \gamma_2(y)\}$ with $\gamma_1, \gamma_2 \in C^0(a_0,b_0)$, $\Gamma_1 = \{(\gamma_1(y),y) : y \in (a_0,b_0)\}$, $\Gamma_2 = \{(\gamma_2(y),y) : y \in (a_0,b_0)\}$ and $\Gamma_3 = \partial\Omega \setminus \big(\Gamma_1 \cup \Gamma_2\big)$. The two-by-two matrix $a = (a_{ij})_{i,j=1,2}$ satisfies the assumptions (1.1), (1.2), (4.1) and (4.2). The function $\mathtt{h} : \Omega \longrightarrow \mathbb{R}$ satisfies

$$0 < \underline{h} \leq \mathtt{h}(X) \leq \bar{h} \quad \text{ for a.e. } X \in \Omega$$
$$\mathtt{h}_x(X) \in L^p_{loc}, \quad p > 2, \qquad \mathtt{h}_x(X) \geq 0 \quad \text{ for a.e. } X \in \Omega.$$

Under these assumptions we proved that the free boundary $\partial[u > 0] \cap \Omega$ is a continuous curve $x = \Phi(y)$.

In this paper, we would like to consider the more general class of free boundary problems of type $div(a(X)\nabla u) = -div(\chi(u)H(X))$, where H is a Lipschitz continuous vector function with $(divH)(X) \geq 0$. Our objective

Accepted for publication: October 2005.
AMS Subject Classifications: 35R35.

is to prove that the free boundary can be parameterized by a family of continuous functions.

In the study of the problem (P_0), the monotonicity of χ with respect to x, i.e., $\chi_x \leq 0$ in $\mathcal{D}'(\Omega)$, was essential to define the free boundary as a function $x = \Phi(y)$. In the problem we are considering, we shall prove a more general monotonicity result for χ. For this purpose we introduce, for each $h \in \pi_y(\Omega)$ and $\omega \in \pi_x(\Omega \cap [y = h])$, the differential equation $(E(\omega, h))$: $X'(t) = H(X(t))$ with the initial condition $X(0) = (\omega, h)$. We show that the mappings $T_h : (t, \omega) \longmapsto X(t, \omega)$ are $C^{0,1}$ homeomorphisms from domains D_h into $T_h(D_h)$ and the family $(T_h(D_h))_h$ is a covering of Ω. Using the change of variables T_h, we prove that χ is non-increasing along the orbits of $(E(\omega, h))$. This allows us to define a local parameterization of the free boundary by a family of functions $(\phi_h)_h$.

In the first section, we state the problem. In the second section, we show the monotonicity of χ. In Section 3, we define the free boundary and establish some properties. In Section 4, we construct a barrier function that will be used to establish a key lemma for the proof of the continuity of the functions ϕ_h, which is done in Section 5.

We end the paper with some remarks. First, when H is $C^{1,1}$, T_h is a C^1 diffeomorphism and the use of this change of variables leads to a problem of type (P_0), i.e.,

$$div(\mathbb{A}(t, \omega)\nabla(uoT_h)) = -(\chi o T_h.\mathtt{h})_t$$

with $\mathbb{A}$ and $\mathtt{h}$ satisfying the assumptions of [4]. However when H is only $C^{0,1}$, the matrix $\mathbb{A}$ is not necessarily $C^{0,\alpha}$, and we are not in the framework of [4].

Finally, when $H(X) = a(X)e$, i.e., for the dam problem, we propose a proof for Lemma 4.4 and thus for Theorem 5.1 which does not require the assumptions (4.1)-(4.2).

1. Statement of the problem

Let Ω be a Lipschitz bounded domain of $\mathbb{R}^2$. Set $\partial\Omega = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \cup \overline{\Gamma}_3$, with Γ_1, Γ_2 and Γ_3 relatively open connected subsets of $\partial\Omega$. We are concerned by the study of the following problem

$$(P)\begin{cases} \text{Find } (u, \chi) \in H^1(\Omega) \times L^\infty(\Omega) \text{ such that} \\ (i) \quad u \geq 0, \quad 0 \leq \chi \leq 1, \quad u(\chi - 1) = 0 \ \text{ a.e. in } \Omega \\ (ii) \quad u = \varphi \quad \text{ on } \Gamma_2 \cup \Gamma_3 \\ (iii) \quad \displaystyle\int_\Omega \big(a(X)\nabla u + \chi H(X)\big).\nabla\xi dX \ \leq \ 0 \\ \qquad\qquad \forall \xi \in H^1(\Omega), \quad \xi \geq 0 \text{ on } \Gamma_2, \quad \xi = 0 \text{ on } \Gamma_3 \end{cases}$$

where $a = (a_{ij})_{i,j=1,2}$ is a two-by-two matrix satisfying

$$a_{ij} \in L^\infty(\Omega), \quad |a|_\infty \leq M \tag{1.1}$$

$$a(X)\xi.\xi \geq \lambda|\xi|^2 \quad \forall \xi \in \mathbb{R}^2, \quad \text{for a.e. } X \in \Omega, \tag{1.2}$$

with λ and M positive constants, φ a nonnegative Lipschitz function such that $\varphi = 0$ on Γ_2 and $\varphi > 0$ on Γ_3. The function $H = (H_1, H_2) : \Omega \longrightarrow \mathbb{R}^2$ satisfies for positive constants $\underline{h}$ and $\bar{h}$:

$$|H_1(X)| \leq \bar{h}, \qquad 0 < \underline{h} \leq H_2(X) \leq \bar{h} \quad \text{for a.e. } X \in \Omega \tag{1.3}$$

$$div(H(X)) \in L^\infty(\Omega), \qquad div(H(X)) \geq 0 \quad \text{for a.e. } X \in \Omega. \tag{1.4}$$

The existence of a solution of (P) is classical. We start by giving the following properties:

Proposition 1.1. *Let (u, χ) be a solution of (P). We have*

i)

$$div\big(a(X)\nabla u\big) = -div(\chi H(X)) \qquad in \quad \mathcal{D}'(\Omega). \tag{1.5}$$

ii)

$$div(\chi H(X)) - \chi([u > 0])div(H(X)) \leq 0 \qquad in \quad \mathcal{D}'(\Omega). \tag{1.6}$$

iii) $u \in C^{0,\alpha}_{loc}(\Omega \cup \Gamma_2 \cup \Gamma_3)$ *for all* $\alpha \in (0,1)$.

iv) $[u > 0]$ *is an open set.*

v) If $a \in C^{0,\alpha}_{loc}(\Omega)$, *then* $u \in C^{1,\alpha}_{loc}([u > 0])$.

Proof. *i)* This is an immediate consequence of taking $\pm\xi$, with $\xi \in \mathcal{D}(\Omega)$, as test functions for (P).

ii) Let $\xi \in \mathcal{D}(\Omega)$, $\xi \geq 0$ and let $F_\epsilon(s) = \min(\frac{s^+}{\epsilon}, 1)$, $\epsilon > 0$. Taking $\pm F_\epsilon(u)\xi$ as test functions for (P), we obtain

$$\int_\Omega \big(a(X)\nabla u + \chi H(X)\big).\nabla(F_\epsilon(u)\xi)dX = 0$$

which can be written, by taking into account $(P)(i)$, (1.2) and the fact that F_ϵ is nondecreasing,

$$\int_\Omega \big[F_\epsilon(u)a(X)\nabla u.\nabla\xi - (F_\epsilon(u)\xi)divH(X)\big]dX \leq 0.$$

Letting $\epsilon \to 0$, we obtain

$$\int_\Omega \big[a(X)\nabla u.\nabla\xi - \chi([u > 0])divH(X)\xi\big]dX \leq 0.$$

Combining the last inequality and (1.5), we get (1.6).

iii) This is a consequence of $(P)(ii)$, (1.5) and the regularity theory of elliptic problems (see [5], Theorem 8.29 for example).

iv) This is a consequence of *iii*).

v) Using $(P)(i)$ and (1.5) we obtain $div\big(a(X)\nabla u\big) = -div(H(X))$ in $\mathcal{D}'([u>0])$. Hence, the result becomes a consequence of the regularity theory of elliptic problems (see [5], Corollary 8.36). □

2. A monotonicity property of χ

In all follows, we shall assume that

$$H \in C^{0,1}(\overline{\Omega}). \tag{2.1}$$

We consider the following differential system

$$(E(\omega,h)) \begin{cases} X'(t,\omega,h) &= H(X(t,\omega,h)) \\ X(0,\omega,h) &= (\omega,h), \end{cases}$$

where $h \in \pi_y(\Omega)$ and $\omega \in \pi_x(\Omega \cap [y=h])$. π_x and π_y are respectively the orthogonal projections on the x and y axes.

By the classical theory of ordinary differential equations, there exists a unique maximal solution $X(.,\omega,h)$ of $E(\omega,h)$ defined on $(\alpha_-(\omega,h),\alpha_+(\omega,h))$ and continuous on the open set

$$\{(t,\omega,h) : \alpha_-(\omega,h) < t < \alpha_+(\omega,h),\ h \in \pi_y(\Omega),\ \omega \in \pi_x(\Omega\cap[y=h])\}.$$

Since H is bounded and continuous on $\overline{\Omega}$, $X(.,\omega,h)$ is defined on $[\alpha_-(\omega,h), \alpha_+(\omega,h)]$ (see Lemma 2.1 page 16 of [6]). Moreover, by Corollary 7.7, page 103 of [1], we know that $X(\alpha_-(\omega,h),\omega,h) \in \partial\Omega\cap[y<h]$, $X(\alpha_+(\omega,h),\omega,h) \in \partial\Omega\cap[y>h]$ (see Figure 1).

For simplicity, we will denote in the sequel $X(t,\omega,h)$, $\alpha_-(\omega,h)$, and $\alpha_+(\omega,h)$ respectively by $X(t,\omega)$, $\alpha_-(\omega)$ and $\alpha_+(\omega)$. We shall also denote by $\gamma(\omega)$ the orbit of $X(.,\omega)$.

Remark 2.1. Note that α_+ and α_- are bounded. Indeed, we have by (1.3)

$$\underline{h}\alpha_+(\omega) \le \int_0^{\alpha_+(\omega)} H_2(X(s,\omega))ds = X_2(\alpha_+(\omega),\omega) - h$$
$$X_2(\alpha_-(\omega),\omega) - h = -\int_{\alpha_-(\omega)}^0 H_2(X(s,\omega))ds \le \underline{h}\alpha_-(\omega).$$

Hence,

$$\frac{1}{\underline{h}}\Big(\inf_{y\in\overline{\pi_y(\Omega)}} y - h\Big) \le \alpha_-(\omega) < 0 < \alpha_+(\omega) \le \frac{1}{\underline{h}}\Big(\sup_{y\in\overline{\pi_y(\Omega)}} y - h\Big).$$

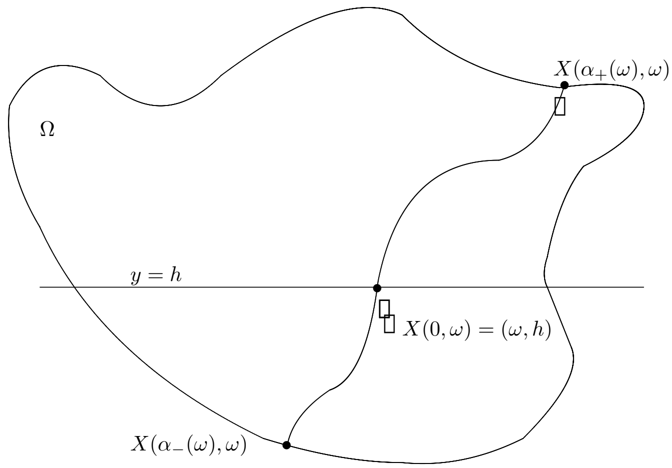

Figure 1

Definition 2.1. *For each $h \in \pi_y(\Omega)$ we define the set*

$$D_h = \{(t,\omega) : \omega \in \pi_x(\Omega \cap [y = h]),\, t \in (\alpha_-(\omega), \alpha_+(\omega))\}$$

and consider the mapping

$$\begin{aligned} T_h \,:\, D_h &\longrightarrow T_h(D_h) \\ (t,\omega) &\longmapsto T_h(t,\omega) = (T_h^1, T_h^2)(t,\omega) = X(t,\omega). \end{aligned}$$

Clearly, each $(x,y) \in \Omega$ can be written as $(x,y) = X(0,\omega) = T_h(0,\omega)$ with $\omega = x$ and $h = y$. So

$$\Omega = \bigsqcup_{h \in \pi_y(\Omega)} T_h(D_h). \tag{2.2}$$

Moreover, we have

Proposition 2.1. *T_h is continuous and one to one.*

Proof. By the previous remarks on the regularity of X, we have $T_h \in C^0(D_h)$. Now let (t_1,ω_1), $(t_2,\omega_2) \in D_h$ such that $T_h(t_1,\omega_1) = T_h(t_2,\omega_2)$ i.e. $X(t_1,\omega_1) = X(t_2,\omega_2)$. Then we consider the following ordinary differential equation

$$Z'(t) = H(Z(t)), \quad Z(0) = X(t_1,\omega_1).$$

Clearly, we have

$$Z(t) = X(t + t_1,\omega_1) = X(t + t_2,\omega_2)$$

$$\forall t \in [\alpha_-(\omega_1) - t_1, \alpha_+(\omega_1) - t_1] = [\alpha_-(\omega_2) - t_2, \alpha_+(\omega_2) - t_2].$$

In particular $t_2 - t_1 \in (\alpha_-(\omega_2), \alpha_+(\omega_2))$. Indeed we have

$$\alpha_-(\omega_1) - t_1 = \alpha_-(\omega_2) - t_2 \quad \text{and} \quad \alpha_+(\omega_1) - t_1 = \alpha_+(\omega_2) - t_2.$$

Then since $\alpha_-(\omega_1) < 0$ and $\alpha_+(\omega_1) > 0$, we get

$$\alpha_-(\omega_2) < \alpha_-(\omega_2) - \alpha_-(\omega_1) = t_2 - t_1 = \alpha_+(\omega_2) - \alpha_+(\omega_1) < \alpha_+(\omega_2).$$

So we can write $Z(-t_1) = X(0, \omega_1) = X(t_2 - t_1, \omega_2)$, i.e.,

$$(\omega_1, h) = (\omega_2, h) + \int_0^{t_2 - t_1} H(X(s, \omega_2))ds.$$

Therefore,

$$\int_0^{t_2 - t_1} H_2(X(s, \omega_2))ds = 0,$$

which leads by (1.3) to $t_2 = t_1$. We then deduce that $\omega_1 = \omega_2$. □

Proposition 2.2. *T_h and T_h^{-1} are $C^{0,1}$.*

Proof. The proof is done in several steps. For the Lipschitz continuity of T_h, we refer to [1], Theorem 8.3 page 110.

Step 1. *Extension.* Since $H \in C^{0,1}(\overline{\Omega})$, there exists by Kirszbraun's theorem (see [2], Theorem 2.10.43 page 210) an extension $\widetilde{H} \in C^{0,1}(\mathbb{R}^2)$ of H with the same Lipschitz constant L. Then

$$\overline{H} = \big(\min(\bar{h}, \max(\widetilde{H}_1, -\bar{h})), \min(\bar{h}, \max(\widetilde{H}_2, \underline{h}))\big) \in C^{0,1}(\mathbb{R}^2)$$
$$\text{with} \quad |\overline{H}_1| \le \bar{h} \quad \text{and} \quad \underline{h} \le \overline{H}_2 \le \bar{h}.$$

Step 2. *Regularization.* Let $H_\epsilon = \rho_\epsilon * \overline{H}$, where ρ_ϵ is the usual mollifier function. Then, it is well known that $H_\epsilon \in C^\infty(\mathbb{R}^2)$ and satisfies

$$\begin{cases} |H_\epsilon^1(X)| \le \bar{h}, \qquad \underline{h} \le H_\epsilon^2(X) \le \bar{h} \qquad \forall X \in \mathbb{R}^2 \\ H_\epsilon \longrightarrow \overline{H} \quad \text{uniformly on each compact set of } \mathbb{R}^2 \quad \text{as } \epsilon \to 0 \\ \|\nabla H_\epsilon\|_{L^\infty(\mathbb{R}^2)} \le \|\nabla \overline{H}\|_{L^\infty(\mathbb{R}^2)} \le L. \end{cases}$$

Now, for $(\omega, h) \in \mathbb{R}^2$, let X_ϵ and $\overline{X}$ be respectively the unique solutions of the differential equations

$$\begin{cases} X_\epsilon'(t, \omega) = H_\epsilon(X_\epsilon(t, \omega)) \\ X_\epsilon(0, \omega) = (\omega, h) \end{cases} \quad \text{and} \quad \begin{cases} \overline{X}'(t, \omega) = \overline{H}(\overline{X}(t, \omega)) \\ \overline{X}(0, \omega) = (\omega, h). \end{cases}$$

X_ϵ and $\overline{X}$ are defined on the maximal interval $(-\infty, +\infty)$. Moreover, X_ϵ is C^∞ with respect to $t \in \mathbb{R}$ and the initial value $(w, h) \in \mathbb{R}^2$.

Step 3. *Local uniform convergence.* Let K be a compact set of $\mathbb{R}^2$. There exists $T>0$, $\omega_1,\omega_2\in\mathbb{R}$ such that $K\subset\subset[-T,T]\times[\omega_1,\omega_2]=K'$. For each $(t,\omega)\in K$, we have

$$\begin{aligned}|X_\epsilon(t,\omega)-\overline{X}(t,\omega)| &= \Big|\int_0^t(H_\epsilon(X_\epsilon(s,\omega))-\overline{H}(\overline{X}(s,\omega)))ds\Big|\\ &\le \Big|\int_0^t(H_\epsilon(X_\epsilon(s,\omega))-H_\epsilon(\overline{X}(s,\omega)))ds\Big|\\ &\quad+\Big|\int_0^t(H_\epsilon(\overline{X}(s,\omega))-\overline{H}(\overline{X}(s,\omega)))ds\Big|\\ &\le \Big|\int_0^t L|X_\epsilon(s,\omega)-\overline{X}(s,\omega)|ds\Big|+|t||H_\epsilon-\overline{H}|_{\infty,\overline{X}(K')}.\end{aligned}$$

By Gronwall's Lemma (see [1], page 90), we obtain

$$|X_\epsilon(t,\omega)-\overline{X}(t,\omega)|\le|t||H_\epsilon-\overline{H}|_{\infty,\overline{X}(K')}\exp(L|t|).$$

So we have with $C=T\exp(LT)$

$$|X_\epsilon-\overline{X}|_{\infty,K}\le C|H_\epsilon-\overline{H}|_{\infty,\overline{X}(K')}\to 0 \qquad \text{when } \epsilon\to 0.$$

Step 4. *$X_\epsilon:\mathbb{R}^2\longrightarrow\mathbb{R}^2$ is a C^1 diffeomorphism.* • $X_\epsilon(\mathbb{R}^2)=\mathbb{R}^2$: Indeed let $(x_0,y_0)\in\mathbb{R}^2$ and let $Z=(Z_1,Z_2)$ be the unique maximal solution of the following differential equation

$$Z'(t)=H_\epsilon(Z(t)),\qquad Z(0)=(x_0,y_0).$$

It is not difficult to see that Z is defined on $(-\infty,+\infty)$ and that

$$\lim_{t\to\pm\infty}Z_2(t)=\pm\infty.$$

Moreover, since $Z_2'(t)=H_\epsilon^2(Z(t))\ge\underline{h}>0$, we deduce that Z_2 is bijective from $\mathbb{R}$ into $\mathbb{R}$. Therefore, there exists $t_0\in\mathbb{R}$ such that $Z_2(t_0)=h$.

Let $\omega_0=Z_1(t_0)$. Then $Z(t_0)=(\omega_0,h)$ and it is easy to verify that $X_\epsilon(t,\omega_0)=Z(t+t_0)$. In particular, $X_\epsilon(-t_0,\omega_0)=Z(0)=(x_0,y_0)$.

• Since X_ϵ is onto, it suffices then to verify that $det(\mathcal{J}X_\epsilon)$ does not vanish. Here we denote by $\mathcal{J}F$ the Jacobian matrix of the mapping F and by $det(\mathcal{J}F)$ the determinant of $\mathcal{J}F$.

One can easily check that

$$\begin{aligned}&Y_h^\epsilon(t,\omega)=det(\mathcal{J}X_\epsilon)=H_\epsilon^1(X_\epsilon(t,\omega))\frac{\partial X_{2\epsilon}}{\partial\omega}-H_\epsilon^2(X_\epsilon(t,\omega))\frac{\partial X_{1\epsilon}}{\partial\omega},\\ &\frac{\partial Y_h^\epsilon}{\partial t}(t,\omega)=Y_h^\epsilon(t,\omega).(div(H_\epsilon))(X_\epsilon(t,\omega)).\end{aligned}$$

Therefore,

$$Y_h^\epsilon(t,\omega) = Y_h^\epsilon(0,\omega).\exp(\int_0^t \{div(H_\epsilon)\}(X_\epsilon(s,\omega))ds). \tag{2.3}$$

Since $Y_h^\epsilon(0,\omega) = -H_\epsilon^2(X_\epsilon(0,\omega)) = -H_\epsilon^2(\omega,h) < 0$, we get $Y_h^\epsilon(t,\omega) < 0$ for all $(t,\omega) \in \mathbb{R}^2$.

Step 5. We have

$$||\mathcal{J}X_\epsilon^{-1}(x,y)||_\infty \le \frac{1}{\underline{h}}\big(\exp(\frac{L|y-h|}{\underline{h}}) + \bar{h}\big) \qquad \forall (x,y) \in \mathbb{R}^2.$$

Indeed, we have for $(t,\omega) = X_\epsilon^{-1}(x,y)$

$$\mathcal{J}X_\epsilon^{-1}(x,y) = \frac{1}{Y_h^\epsilon(t,\omega)} \begin{pmatrix} \dfrac{\partial X_{2\epsilon}}{\partial\omega}(t,\omega) & -\dfrac{\partial X_{1\epsilon}}{\partial\omega}(t,\omega) \\ -H_\epsilon^2(X_\epsilon(t,\omega)) & H_\epsilon^1(X_\epsilon(t,\omega)) \end{pmatrix}.$$

$$|H_\epsilon^i(X_\epsilon(t,\omega))| \le \bar{h}, \quad H_\epsilon^2(X_\epsilon(t,\omega)) \ge \underline{h}, \quad \text{and} \quad div(H_\epsilon) \ge 0.$$

It follows that

$$\frac{1}{|Y_h^\epsilon(t,\omega)|} = \frac{1}{H_\epsilon^2(\omega,h)}\exp(-\int_0^t \{div(H_\epsilon)\}(X_\epsilon(s,\omega))ds) \le \frac{1}{\underline{h}}.$$

We claim that $|\frac{\partial X_\epsilon}{\partial\omega}| \le \exp(L|t|)$. Indeed for ω_1, $\omega_2 \in \mathbb{R}$, we have

$$\begin{aligned} &|X_\epsilon(t,\omega_1) - X_\epsilon(t,\omega_2)| \\ &\qquad = \Big|(\omega_1-\omega_2,0) + \int_0^t (H_\epsilon(X_\epsilon(s,\omega_1)) - H_\epsilon(X_\epsilon(s,\omega_2)))ds\Big| \\ &\qquad \le |\omega_1-\omega_2| + L\int_0^t |X_\epsilon(s,\omega_1) - X_\epsilon(s,\omega_2)|ds. \end{aligned}$$

By Gronwall's lemma, we obtain

$$|X_\epsilon(t,\omega_1) - X_\epsilon(t,\omega_2)| \le |\omega_1-\omega_2|\exp(L|t|).$$

Now we conclude that

$$\begin{aligned} &\|\mathcal{J}X_\epsilon^{-1}(x,y)\|_\infty \\ &= \max\Big\{\frac{1}{|Y_h^\epsilon|}\Big(\Big|\frac{\partial X_{2\epsilon}}{\partial\omega}\Big| + |H_\epsilon^2 o X_\epsilon|\Big), \frac{1}{|Y_h^\epsilon|}\Big(\Big|\frac{\partial X_{1\epsilon}}{\partial\omega}\Big| + |H_\epsilon^1 o X_\epsilon|\Big)\Big\}(t,\omega) \\ &\le \frac{1}{\underline{h}}\big(\exp(L|t|) + \bar{h}\big) \le \frac{1}{\underline{h}}\big(\exp(\frac{L|y-h|}{\underline{h}}) + \bar{h}\big) \end{aligned}$$

since

$$|y-h| = \Big|\int_0^t H_\epsilon^2(X_\epsilon(s,\omega))ds\Big| \ge \underline{h}|t|.$$

Step 6. X_ϵ^{-1} *is uniformly Lipschitz continuous on each compact set with a Lipschitz constant independent of* ϵ. Let K be a compact subset of $\mathbb{R}^2$ and (x_1, y_1), (x_2, y_2) be two points in K. We denote by $|(x, y)|_\infty = \max(|x|, |y|)$. Then we have

$$|X_\epsilon^{-1}(x_1, y_1) - X_\epsilon^{-1}(x_2, y_2)|_\infty = \Big|\int_0^1 \frac{d}{d\tau} X_\epsilon^{-1}(\tau(x_1, y_1) + (1-\tau)(x_2, y_2))d\tau\Big|_\infty$$
$$= \Big|\int_0^1 \mathcal{J}X_\epsilon^{-1}(\tau(x_1, y_1) + (1-\tau)(x_2, y_2)).(x_1 - x_2, y_1 - y_2)d\tau\Big|_\infty$$
$$\leq \int_0^1 |\mathcal{J}X_\epsilon^{-1}(\tau(x_1, y_1) + (1-\tau)(x_2, y_2))|_\infty.|(x_1, y_1) - (x_2, y_2)|_\infty d\tau$$
$$\leq \Big(\frac{1}{\underline{h}}\int_0^1 \big(\exp(\frac{L}{\underline{h}}|(1-\tau)y_2 + \tau y_1 - h|) + \bar{h}\big)d\tau\Big)|(x_1, y_1) - (x_2, y_2)|_\infty$$
$$\leq c(K)|(x_1, y_1) - (x_2, y_2)|_\infty,$$

with $c(K) = \frac{1}{\underline{h}}\big(\exp(\frac{L}{\underline{h}}[m + h]) + \bar{h}\big)$ and $m = \max\{|y| : y \in \pi_y(K)\}$.

Step 7. *Conclusion.* There exists a subsequence $(X_{\epsilon_n})_{n\geq 0}$ such that $(X_{\epsilon_n}^{-1})_n$ converges uniformly to an element $X^* \in C_{loc}^{0,1}(\mathbb{R}^2)$ on each compact subset of $\mathbb{R}^2$.

We claim that $X^* = \overline{X}^{-1}$. Indeed we have

$$X_\epsilon o X_\epsilon^{-1}(x, y) = (x, y) \quad \text{and} \quad X_\epsilon^{-1} o X_\epsilon(t, \omega) = (t, \omega) \quad \forall (x, y), (t, \omega) \in \mathbb{R}^2.$$

Passing to the limit, we obtain

$$\overline{X} o X^*(x, y) = (x, y) \quad \text{and} \quad X^* o \overline{X}(t, \omega) = (t, \omega) \qquad \forall (x, y), (t, \omega) \in \mathbb{R}^2.$$

Since $\overline{X}_{|D_h} = X = T_h$, we have $T_h^{-1} = \overline{X}^{-1}{}_{|T_h(D_h)} \in C^{0,1}(T_h(D_h))$. □

Now we have the following proposition:

Proposition 2.3. *Let* $X(., \omega)$ *be the maximal solution of* $E(\omega, h)$*. We have*

i)

$$\mathcal{J}T_h = \begin{pmatrix} H_1(X(t,\omega)) & \dfrac{\partial X_1}{\partial \omega}(t,\omega) \\ H_2(X(t,\omega)) & \dfrac{\partial X_2}{\partial \omega}(t,\omega) \end{pmatrix} \in L^\infty(D_h)$$

$$Y_h(t,\omega) = det\mathcal{J}T_h = H_1(X(t,\omega))\frac{\partial X_2}{\partial \omega}(t,\omega) - H_2(X(t,\omega))\frac{\partial X_1}{\partial \omega}(t,\omega)$$

in $L^\infty(D_h)$.

ii) $\dfrac{\partial Y_h}{\partial t}(t,\omega) = Y_h(t,\omega)(divH)(X(t,\omega)) \qquad a.e.\ in\ D_h.$

$iii)$ $Y_h(t,\omega) = -H_2(\omega,h)\exp\Big(\int_0^t (divH)(X(s,\omega))ds\Big)$ $\qquad a.e.\ in\ D_h.$

$iv)$ $\underline{h} \le -Y_h(t,\omega) \le C\bar{h}, \qquad C > 0.$

Proof. i) Note that since $T_h \in C^{0,1}(D_h)$, we have $T_h \in W^{1,\infty}(D_h)$ and therefore we can talk about $\mathcal{J}T_h$. The formula is trivial.

ii) Given that H, T_h, T_h^{-1} are $C^{0,1}$, we can use the chain rule for $H_i oX$ (see [7]) to get

$$\frac{\partial(H_i oX)}{\partial t} = H_1\frac{\partial H_i}{\partial x} + H_2\frac{\partial H_i}{\partial y}. \tag{2.4}$$

Moreover, $H_\epsilon, X_\epsilon \in C^\infty(\mathbb{R}^2)$ and

$$X_\epsilon(t,\omega) = (\omega,h) + \int_0^t H_\epsilon(X_\epsilon(s,\omega))ds.$$

So we have

$$\frac{\partial X_\epsilon}{\partial \omega}(t,\omega)$$
$$= (1,0) + \int_0^t \Big(\frac{\partial X_{1\epsilon}}{\partial \omega}(s,\omega)\frac{\partial H_\epsilon}{\partial x}(X_\epsilon(s,\omega)) + \frac{\partial X_{2\epsilon}}{\partial \omega}(s,\omega)\frac{\partial H_\epsilon}{\partial y}(X_\epsilon(s,\omega))\Big)ds.$$

Since H_ϵ and X_ϵ converge uniformly to H and X respectively in Ω and D_h, $\dfrac{\partial X_\epsilon}{\partial \omega}$ and ∇H_ϵ converge to $\frac{\partial X}{\partial \omega}$ and ∇H respectively in $L^p(D_h)$ and $L^p(\Omega)$ for each $p \ge 1$, we obtain for almost every $(t,\omega) \in D_h$, by letting $\epsilon \to 0$

$$\frac{\partial X}{\partial \omega}(t,\omega)=(1,0)+\int_0^t \Big(\frac{\partial X_1}{\partial \omega}(s,\omega)\frac{\partial H}{\partial x}(X(s,\omega))+\frac{\partial X_2}{\partial \omega}(s,\omega)\frac{\partial H}{\partial y}(X(s,\omega))\Big)ds. \tag{2.5}$$

It follows from (2.5) that

$$\frac{\partial^2 X}{\partial t\partial \omega}(t,\omega) = \frac{\partial X_1}{\partial \omega}(t,\omega).\frac{\partial H}{\partial x}(X(t,\omega))+\frac{\partial X_2}{\partial \omega}(t,\omega).\frac{\partial H}{\partial y}(X(t,\omega)) \text{ in } L^\infty(D_h). \tag{2.6}$$

Now, since $H_i oX \in W^{1,\infty}(D_h)$ and $\frac{\partial^2 X_i}{\partial t\partial\omega} \in L^\infty(D_h)$, we obtain

$$\frac{\partial}{\partial t}\Big(H_i oX.\frac{\partial X_i}{\partial \omega}\Big) = \frac{\partial}{\partial t}(H_i oX).\frac{\partial X_i}{\partial \omega} + (H_i oX).\frac{\partial^2 X_i}{\partial t\partial \omega}. \tag{2.7}$$

Using (2.4)-(2.7), we obtain

$$\frac{\partial Y_h}{\partial t}(t,\omega)$$
$$= \Big(H_1(X(t,\omega))\frac{\partial H_1}{\partial x}(X(t,\omega)) + H_2(X(t,\omega))\frac{\partial H_1}{\partial y}(X(t,\omega))\Big)\frac{\partial X_2}{\partial \omega}(t,\omega)$$

$$- \Big(H_1(X(t,\omega))\frac{\partial H_2}{\partial x}(X(t,\omega)) + H_2(X(t,\omega))\frac{\partial H_2}{\partial y}(X(t,\omega))\Big)\frac{\partial X_1}{\partial \omega}(t,\omega)$$

$$+ H_1(X(t,\omega))\frac{\partial^2 X_2}{\partial t \partial \omega}(t,\omega) - H_2(X(t,\omega))\frac{\partial^2 X_1}{\partial t \partial \omega}(t,\omega)$$

$$= \Big(H_1(X(t,\omega))\frac{\partial X_2}{\partial \omega}(t,\omega) - H_2(X(t,\omega))\ \frac{\partial X_1}{\partial \omega}(t,\omega)\Big)\Big(\frac{\partial H_1}{\partial x} + \frac{\partial H_2}{\partial y}\Big)(X(t,\omega))$$

$$= Y_h(t,\omega)(divH)(X(t,\omega)).$$

iii) By using the product formula and chain rule, we obtain

$$\frac{\partial}{\partial t}\Big(Y_h(t,\omega).\exp\big(-\int_0^t (divH)(X(s,\omega))ds\big)\Big)$$

$$= \frac{\partial Y_h}{\partial t}\exp\big(-\int_0^t (divH)(X(s,\omega))ds\big)$$

$$+ Y_h(t,\omega)(-(divH)(X(t,\omega)))\exp\big(-\int_0^t (divH)(X(s,\omega))ds\big) = 0.$$

Then

$$Y_h(t,\omega)\exp\big(-\int_0^t (divH)(X(s,\omega))ds\big) = Cst = Y_h(0,\omega)$$

which exists because $Y_h \in C^0(\alpha_-(\omega),\alpha_+(\omega))$. Since we have $\frac{\partial X}{\partial \omega}(0,\omega) = (1,0)$, then $Y_h(0,\omega) = -H_2(X(0,\omega)) = -H_2(\omega,h)$.

iv) Since $0 \le divH \le L$, it follows that

$$\Big|\int_0^t (divH)(X(s,\omega))ds\Big| \le 2L|t| \le 2L\max(\alpha_+(\omega), -\alpha_-(\omega)) \le L'.$$

We deduce, since $\underline{h} \le H_2(\omega,h) \le \bar{h}$, that $\underline{h} \le -Y_h(t,\omega) \le \bar{h}\exp(L')$. □

Now we can prove the main result of this section.

Theorem 2.1. *Let (u,χ) be a solution of (P). We have for each $h \in \pi_y(\Omega)$*

$$\frac{\partial}{\partial t}\big(\chi o T_h\big) \le 0 \qquad in\ \mathcal{D}'(D_h).$$

Proof. Let $\varphi \in \mathcal{D}(D_h), \varphi \ge 0$. By (1.6), we have

$$\int_{T_h(D_h)}\big(-\chi H(X).\nabla(\varphi o T_h^{-1}) - \chi([u>0])divH(X).\varphi o T_h^{-1}\big)dX \le 0.$$

Since $T_h, T_h^{-1} \in C^{0,1}$, we can use T_h as a change of variables (see [7]) to obtain

$$\int_{D_h}\Big(-\chi o T_h\frac{\partial \varphi}{\partial t} - \chi([uoT_h > 0])(divH)oT_h.\varphi\Big)(-Y_h(t,\omega))dtd\omega \le 0.$$

Given that $\frac{\partial Y_h}{\partial t} = Y_h.(divH)oT_h$, we obtain

$$\begin{aligned}
&\int_{D_h} \chi oT_h \frac{\partial(-Y_h.\varphi)}{\partial t} dtd\omega \\
&= \int_{D_h} \chi oT_h \frac{\partial\varphi}{\partial t}(-Y_h) + \chi oT_h.(divH)oT_h.\varphi.(-Y_h)dtd\omega \\
&\geq \int_{D_h} (\chi oT_h - \chi([uoT_h > 0])).(divH)oT_h.\varphi.(-Y_h)dtd\omega \geq 0.
\end{aligned}$$

By approximation the last inequality remains valid for all nonnegative functions φ with compact support and such that $\varphi_t \in L^1(D_h)$. Since $Y_h \in L^\infty(D_h)$ and does not vanish, one can choose $\varphi = -\frac{\psi}{Y_h}$, with $\psi \in \mathcal{D}(D_h)$ and $\psi \geq 0$. Thus we get the result. □

3. DEFINITION OF THE FREE BOUNDARY AND SOME TECHNICAL RESULTS

In this section, we use the monotonicity result of the previous section and the continuity of u to define the free boundary. We also give some other results. First, we have the following key proposition.

Proposition 3.1. *Let* (u, χ) *be a solution of* (P) *and* $X_0 = (x_0, y_0) = T_h(t_0, \omega_0) \in T_h(D_h)$.

i) If $u(X_0) = uoT_h(t_0, \omega_0) > 0$, *then there exists* $\epsilon > 0$ *such that*

$$uoT_h(t,\omega) > 0 \qquad \forall(t,\omega) \in C_\epsilon = \{(t,\omega) \in D_h : |\omega - \omega_0| < \epsilon,\ t < t_0 + \epsilon\}.$$

ii) If $u(X_0) = uoT_h(t_0, \omega_0) = 0$, *then* $uoT_h(t, \omega_0) = 0 \quad \forall t \geq t_0$.

Proof. It suffices to verify $i)$. By continuity, there exists $\epsilon > 0$ such that

$$uoT_h(t,\omega) > 0 \qquad \forall(t,\omega) \in (t_0 - \epsilon, t_0 + \epsilon) \times (\omega_0 - \epsilon, \omega_0 + \epsilon) = Q_\epsilon.$$

Then $\chi oT_h(t, \omega) = 1$ for almost every $(t, \omega) \in Q_\epsilon$. By Theorem 2.1 and since $\chi oT_h \leq 1$, we get $\chi oT_h = 1$ almost everywhere in C_ϵ, i.e., $\chi = 1$ almost everywhere in $T_h(C_\epsilon)$.

From (1.4) and (1.5), we have $div(a(X)\nabla u) = -div(H(X)) \leq 0$ in $\mathcal{D}'(T_h(C_\epsilon))$. Then by the strong maximum principle we deduce, since $u \geq 0$ in Ω and $u > 0$ in $T_h(Q_\epsilon) \subset T_h(C_\epsilon)$, that $u > 0$ in $T_h(C_\epsilon)$ (see Figure 2). □

Thanks to Proposition 3.1, we can define, for each $h \in \pi_y(\Omega)$, the following function ϕ_h on $\pi_x(\Omega \cap [y = h])$ by

$$\phi_h(\omega) = \begin{cases} \sup\{t : (t,\omega) \in D_h, \quad uoT_h(t,\omega) > 0\} \\ \qquad\qquad \text{if this set is not empty} \\ \alpha_-(\omega) \qquad\qquad \text{otherwise.} \end{cases} \tag{3.1}$$

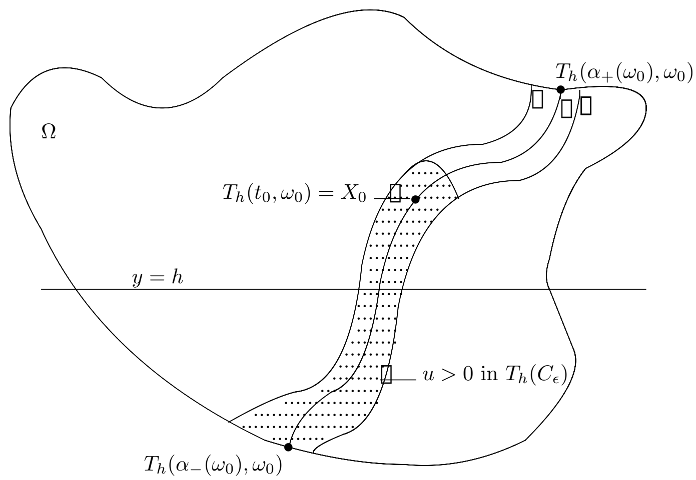

Figure 2

Remark 3.1. Since $u = \varphi > 0$ on Γ_3 and $u \in C^0(\Omega \cup \Gamma_3)$, we have $u > 0$ below Γ_3 in the following sense :

$$u(X(t,\omega)) > 0 \qquad \forall t \in [\alpha_-(\omega), \alpha_+(\omega)] \quad \text{such that} \quad X(\alpha_+(\omega),\omega) \in \Gamma_3.$$

Consequently, if $X(t_0,\omega_0) \in \Omega$ and $u(X(t_0,\omega_0)) = 0$, we have necessarily $X(\alpha_+(\omega_0),\omega_0) \in \overline{\Gamma}_1 \cup \overline{\Gamma}_2$.

Arguing as in [3], we have the following results

Proposition 3.2. *ϕ_h is lower semi-continuous on each $\omega \in \pi_x(\Omega \cap [y = h])$ such that $T_h(\phi_h(\omega),\omega) \in \Omega$. Moreover, $[uoT_h(t,\omega) > 0] \cap D_h = [t < \phi_h(\omega)]$.*

The following important lemmas will be useful in Sections 4 and 5. Some of them are extensions of lemmas in [3].

Lemma 3.1. *Let $h \in \pi_y(\Omega)$, $\omega_1, \omega_2 \in \pi_x(\Omega \cap [y = h])$ with $\omega_1 < \omega_2$, and $\underline{y} \in \pi_y(\Omega)$. We denote by $t_{\underline{y}}(\omega)$ the unique t (if it exists) at which the orbit $\gamma(\omega)$ meets the line $[y = \underline{y}]$.*

Assume that for $i = 1, 2$, $\gamma(\omega_i) \cap [y = \underline{y}] \neq \emptyset$ and that

$$[X(t_{\underline{y}}(\omega_1),\omega_1), X(t_{\underline{y}}(\omega_2),\omega_2)] \subset\subset \Omega.$$

Then we have

i) $\quad \gamma(\omega) \cap [y = \underline{y}] \neq \emptyset \quad \forall \omega \in [\omega_1, \omega_2]$

$$ii) \quad [X(t_{\underline{y}}(\omega_1),\omega_1), X(t_{\underline{y}}(\omega_2),\omega_2)] = \{X(t_{\underline{y}}(\omega),\omega) : \omega \in [\omega_1,\omega_2]\}.$$

Proof. $i)$ First note that it is enough to prove the assertion for $\omega \in (\omega_1, \omega_2)$. Moreover, if $\underline{y} = h$, then the assertion is trivial since in this case for all $\omega \in [\omega_1, \omega_2]$, $\gamma(\omega) \cap [y = \underline{y}] = \{(\omega, h)\}$.

So we assume that $\underline{y} \neq h$ and discuss the two cases:

$\underline{y} > h$: For each $\omega \in (\omega_1, \omega_2)$, the half orbit $\gamma^+(\omega) = \gamma(\omega) \cap [t \geq 0]$ is enclosed between $\gamma^+(\omega_1)$ and $\gamma^+(\omega_2)$. So if $\gamma^+(\omega) \cap [y = \underline{y}] = \emptyset$, then $\gamma^+(\omega)$ will never reach $\partial\Omega$, which contradicts $X(\alpha_+(\omega), \omega) \in \partial\Omega \cap [y > h]$.

$\underline{y} < h$: For each $\omega \in (\omega_1, \omega_2)$, the half orbit $\gamma^-(\omega) = \gamma(\omega) \cap [t \leq 0]$ is enclosed between $\gamma^-(\omega_1)$ and $\gamma^-(\omega_2)$. So if $\gamma^-(\omega) \cap [y = \underline{y}] = \emptyset$, then $\gamma^-(\omega)$ will never reach $\partial\Omega$, which contradicts $X(\alpha_-(\omega), \omega) \in \partial\Omega \cap [y < h]$.

$ii)$ First note that it is enough to show that

$$(X(t_{\underline{y}}(\omega_1),\omega_1), X(t_{\underline{y}}(\omega_2),\omega_2)) = \{X(t_{\underline{y}}(\omega),\omega) : \omega \in (\omega_1,\omega_2)\},$$

where $(X(t_{\underline{y}}(\omega_1),\omega_1), X(t_{\underline{y}}(\omega_2),\omega_2))$ denotes the line segment without the extreme points.

• Let $\omega \in (\omega_1, \omega_2)$. Since the orbit $\gamma(\omega)$ is strictly enclosed between the orbits $\gamma(\omega_1)$ and $\gamma(\omega_2)$, and meets the line $[y = \underline{y}]$ at the point $X(t_{\underline{y}}(\omega), \omega)$, we have

$$X_1(t_{\underline{y}}(\omega_1),\omega_1) < X_1(t_{\underline{y}}(\omega),\omega) < X_1(t_{\underline{y}}(\omega_2),\omega_2)$$

and therefore, $X(t_{\underline{y}}(\omega),\omega) \in (X(t_{\underline{y}}(\omega_1),\omega_1), X(t_{\underline{y}}(\omega_2),\omega_2))$.

• Let $(x_*, \underline{y}) \in (X(t_{\underline{y}}(\omega_1),\omega_1), X(t_{\underline{y}}(\omega_2),\omega_2))$. We consider $X(., x_*, \underline{y})$ the maximal solution of the differential equation $X'(t) = H(X(t))$, $X(0) = (x_*, \underline{y})$.

Note that the orbit $\gamma(x_*, \underline{y})$ of $X(., x_*, \underline{y})$ has no intersection with $\gamma(\omega_i)$, $i = 1, 2$, because otherwise we will have $\gamma(x_*, \underline{y}) = \gamma(\omega_1)$ or $\gamma(x_*, \underline{y}) = \gamma(\omega_2)$, which is impossible since $x_* \in (X_1(t_{\underline{y}}(\omega_1),\omega_1), X_1(t_{\underline{y}}(\omega_2),\omega_2))$. Hence $\gamma(x_*, \underline{y})$ is strictly enclosed between $\gamma(\omega_1)$ and $\gamma(\omega_2)$. Therefore, it meets the line $[y = h]$ at the point (ω_*, h), with $\omega_* \in (\omega_1, \omega_2)$. It follows that $X(t, x_*, \underline{y}) = X(t + t_{\underline{y}}(\omega_*), \omega_*, h)$ and in particular $(x_*, \underline{y}) = X(0, x_*, \underline{y}) = X(t_{\underline{y}}(\omega_*), \omega_*, h)$. □

Lemma 3.2. *Let* (u, χ) *be a solution of* (P). *Let* $h \in \pi_y(\Omega)$, $\omega_1, \omega_2 \in \pi_x(\Omega \cap [y = h])$ *with* $\omega_1 < \omega_2$. *Let* $\underline{y} \in \pi_y(\Omega)$ *such that* $[y = \underline{y}] \cap \gamma(\omega_i) \neq \emptyset$ $i = 1, 2$.

Set $D_{\underline{y}} = T_h\big(\{(t,\omega) \in D_h : \omega \in (\omega_1,\omega_2),\ t > t_{\underline{y}}(\omega)\}\big) = T_h([\omega_1 < \omega < \omega_2]) \cap [y > \underline{y}]$, *and assume that* $\overline{D}_{\underline{y}} \cap \overline{\Gamma}_3 = \emptyset$ *(see Figure* 3*). Then if*

$uoT_h(t_{\underline{y}}(\omega_i), \omega_i) = 0$ *for* $i = 1, 2$, *we have*

$$\int_{D_{\underline{y}}} \big(a(X)\nabla u + \chi H(X)\big).\nabla\zeta dX \le 0$$
$$\forall \zeta \in H^1(D_{\underline{y}}), \quad \zeta \ge 0, \quad \zeta(x, \underline{y}) = 0 \quad \textit{for a.e. } (x, \underline{y}) \in \overline{D}_{\underline{y}}.$$

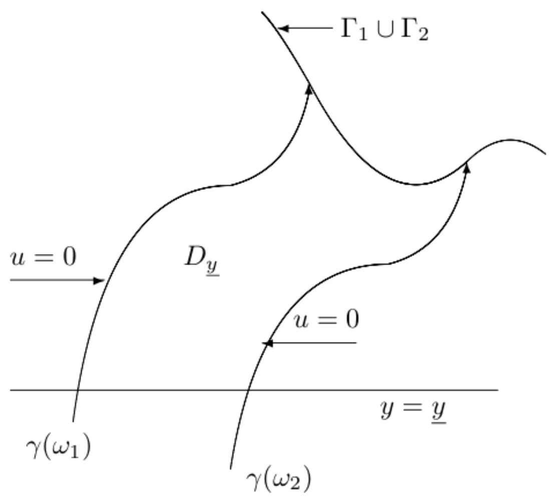

Figure 3

Proof. First note that $D_{\underline{y}}$ is well defined since by Lemma 3.1 *i*), $t_{\underline{y}}(\omega)$ exists for each $\omega \in (\omega_1, \omega_2)$. Next we claim that

$$\int_{D_{\underline{y}}} \big(a(X)\nabla u + \chi([u>0])H(X)\big).\nabla\zeta dX \le \int_{\omega_1}^{\omega_2} -(Y_h.\zeta oT_h)(\phi_h(\omega), \omega) d\omega$$
$$\forall \zeta \in H^1(D_{\underline{y}}) \cap C^0(\overline{D}_{\underline{y}}), \quad \zeta \ge 0, \quad \zeta(x, \underline{y}) = 0 \quad \text{for all } (x, \underline{y}) \in \overline{D}_{\underline{y}}. \quad (3.2)$$

Indeed, we deduce from $uoT_h(t_{\underline{y}}(\omega_i), \omega_i) = 0$, $i = 1, 2$ and Proposition 3.1 *ii*) that $uoT_h(t, \omega_i) = 0$, for all $t \ge t_{\underline{y}}(\omega_i)$, $i = 1, 2$. Therefore for $\epsilon > 0$, $\chi(D_{\underline{y}}).\min\big(\frac{u}{\epsilon}, \zeta\big)$ is a test function for (P) and we have

$$\int_{D_{\underline{y}}\cap[u\ge\epsilon\zeta]} a(X)\nabla u.\nabla\zeta dX + \int_{D_{\underline{y}}} \chi([u>0])H(X).\nabla\zeta dX$$
$$\le \int_{D_{\underline{y}}} \chi([u>0])H(X).\nabla\big(\zeta - \frac{u}{\epsilon}\big)^+ dX = I_\epsilon.$$

Using the change of variables T_h and the second mean value theorem, we obtain

$$\begin{aligned} I_\epsilon &= \int_{J=\{\omega\in(\omega_1,\omega_2):\phi_h(\omega)>t_{\underline{y}}(\omega)\}} \int_{t_{\underline{y}}(\omega)}^{\phi_h(\omega)} \frac{\partial}{\partial t}\Big(\big(\zeta-\frac{u}{\epsilon}\big)^+ oT_h\Big).(-Y_h(t,\omega))dtd\omega \\ &= \int_J (-Y_h(\phi_h(\omega),\omega))\Big\{\int_{t^*(\omega)}^{\phi_h(\omega)} \frac{\partial}{\partial t}\Big(\big(\zeta-\frac{u}{\epsilon}\big)^+ oT_h\Big)(t,\omega)dt\Big\}d\omega \\ &\le \int_{\omega_1}^{\omega_2} -Y_h(\phi_h(\omega),\omega).\zeta oT_h(\phi_h(\omega),\omega)d\omega, \qquad t^*(\omega)\in[t_{\underline{y}}(\omega),\phi_h(\omega)]. \end{aligned}$$

Then by letting ϵ go to 0, the inequality (3.2) holds.

Now to prove the lemma, it suffices to do it for $\zeta\in H^1(D_{\underline{y}})\cap C^0(\overline{D}_{\underline{y}})$, $\zeta\ge 0$, $\zeta(x,\underline{y})=0$ for all $(x,\underline{y})\in\overline{D}_{\underline{y}}$ and conclude by density. So let $\epsilon>0$ and $h_\epsilon=\theta_\epsilon oT_h^{-1}$, with $\theta_\epsilon(\omega)=\min\big(\frac{(\omega-\omega_1)^+}{\epsilon},1\big).\min\big(\frac{(\omega_2-\omega)^+}{\epsilon},1\big)$. Since $\chi(D_{\underline{y}}).\zeta.h_\epsilon$ is a test function for (P), we have

$$\begin{aligned} &\int_{D_{\underline{y}}}(a(X)\nabla u+\chi H(X)).\nabla\zeta dX \le \int_{D_{\underline{y}}}(a(X)\nabla u+\chi H(X)).\nabla((1-h_\epsilon)\zeta)dX \\ &= \int_{D_{\underline{y}}}(a(X)\nabla u+\chi([u>0])H(X)).\nabla((1-h_\epsilon)\zeta)dX \\ &\qquad + \int_{D_{\underline{y}}}(\chi-\chi([u>0]))H(X).\nabla((1-h_\epsilon)\zeta)dX = I_\epsilon^1+I_\epsilon^2. \end{aligned}$$

Using (3.2) and the fact that $\theta_\epsilon \underset{\epsilon\to 0}{\longrightarrow} 1$, we obtain the lemma since we have

$$\begin{aligned} I_\epsilon^1 &\le \int_{\omega_1}^{\omega_2} -Y_h(\phi_h(\omega),\omega).\zeta oT_h(\phi_h(\omega),\omega).(1-\theta_\epsilon(\omega))d\omega, \\ I_\epsilon^2 &= \int_{T_h^{-1}(D_{\underline{y}})}(\chi oT_h-\chi([uoT_h>0]))(-Y_h(t,\omega)).\frac{\partial}{\partial t}(\zeta oT_h).(1-\theta_\epsilon(\omega))dtd\omega. \end{aligned}$$

Lemma 3.3. *Let (u,χ) be a solution of (P) and $X_0=(x_0,y_0)=T_h(t_0,\omega_0)$ be a point of $T_h(D_h)$. We denote by $B_r(t_0,\omega_0)$ a ball with center (t_0,ω_0) and radius r contained in D_h. If $uoT_h=0$ in $B_r(t_0,\omega_0)$, then*

$$uoT_h=0 \quad \text{in } C_r \quad \text{and} \quad \chi oT_h=0 \quad \text{a.e. in } C_r$$

where $C_r=\{(t,\omega)\in D_h : |\omega-\omega_0|<r, \quad t>t_0\}\cup B_r(t_0,\omega_0)$. In other words if $u=0$ in $T_h(B_r(t_0,\omega_0))$, then $u=0$ and $\chi=0$ almost everywhere in $T_h(C_r)$ (see Figure 4).

Proof. By Proposition 3.1, we have $uoT_h=0$ in C_r. Applying Lemma 3.2 with domains $D_{\underline{y}}=T_h([\omega_1<\omega<\omega_2])\cap[y>\underline{y}]\subset T_h(C_r)$, $(\underline{y}\in\pi_y(\Omega))$

satisfying $[y = \underline{y}] \cap \gamma(\omega) \neq \emptyset$ for all $\omega \in [\omega_1, \omega_2]$ and taking $\zeta = (y - \underline{y})\chi(D_{\underline{y}})$, we obtain

$$\int_{D_{\underline{y}}} \chi H_2(X) dX \leq 0.$$

From (1.3), we deduce that $\chi = 0$ almost everywhere in $D_{\underline{y}}$. This holds for all domains $D_{\underline{y}}$ in $T_h(C_r)$. Hence $\chi = 0$ almost everywhere in $T_h(C_r)$. □

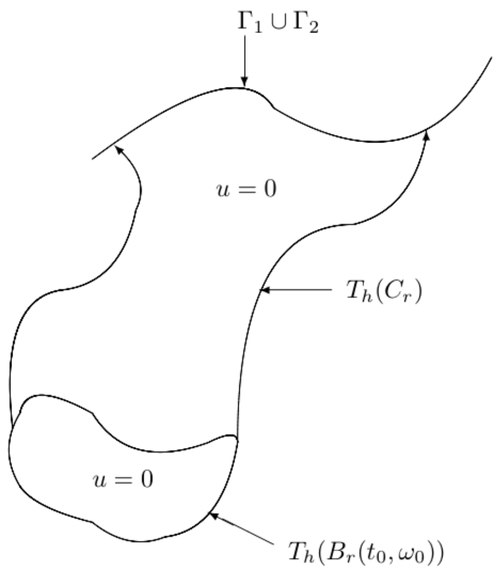

Figure 4

Lemma 3.4. *Let* (u, χ) *be a solution of* (P), $X_0 = (x_0, y_0) = T_h(t_0, \omega_0)$ *be a point of* Ω *and* B_r *the open ball in* D_h *with center* (t_0, ω_0) *and radius* r. *Then we cannot have the following situations* (*see Figure* 5)

$$(i) \begin{cases} uoT_h(t, \omega_0) = 0 & \forall t \in (t_0 - r, t_0 + r) \\ uoT_h(t, \omega) > 0 & \forall (t, \omega) \in B_r \setminus S, \qquad S = (t_0 - r, t_0 + r) \times \{\omega_0\}, \end{cases}$$

$$(ii) \begin{cases} uoT_h(t, \omega) = 0 & \forall (t, \omega) \in B_r \cap [\omega \leq \omega_0] \\ uoT_h(t, \omega) > 0 & \forall (t, \omega) \in B_r \cap [\omega > \omega_0], \end{cases}$$

$$(iii) \begin{cases} uoT_h(t, \omega) = 0 & \forall (t, \omega) \in B_r \cap [\omega \geq \omega_0] \\ uoT_h(t, \omega) > 0 & \forall (t, \omega) \in B_r \cap [\omega < \omega_0]. \end{cases}$$

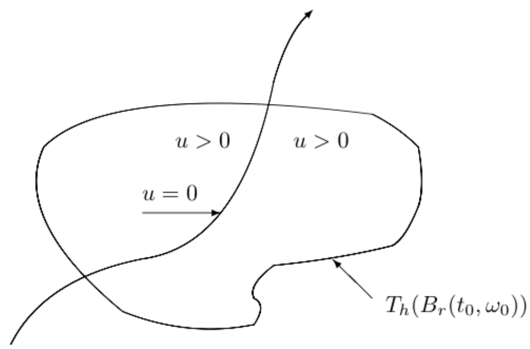

Figure 5 (i)

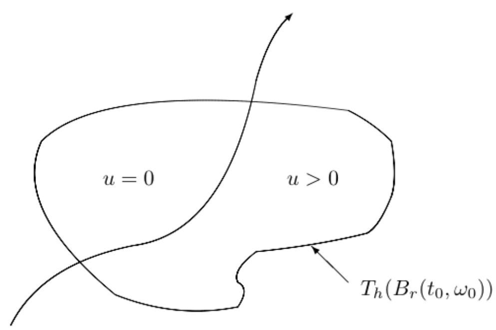

Figure 5 (ii)

Proof. Assume ii) holds. The proof of i) and iii) is based on the same arguments. Let $\zeta \in \mathcal{D}(T_h(B_r))$, $\zeta \geq 0$. Using the fact that, by Lemma 3.3, $\chi o T_h = 0$ almost everywhere on $B_r \cap [\omega < \omega_0]$ and $\pm\zeta$ are test functions for (P), we obtain after using the change of variables T_h

$$\int_{T_h(B_r)} a(X)\nabla u.\nabla\zeta dX = \int_{B_r \cap [\omega > \omega_0]} \frac{\partial}{\partial t}(-Y_h(t,\omega))\zeta o T_h dt d\omega \geq 0.$$

We deduce that $div(a(X)\nabla u) \leq 0$ in $\mathcal{D}'(T_h(B_r))$. By the strong maximum principle, we have either $u > 0$ or $u = 0$ in $T_h(B_r)$, which contradicts the assumption. □

Lemma 3.5. *Let (u, χ) be a solution of (P), $X_0 = (x_0, y_0) = T_h(t_0, \omega_0)$ be a point of Ω such that $uoT_h(t_0, \omega_0) = 0$. Then there exists $\rho > 0$ such that*

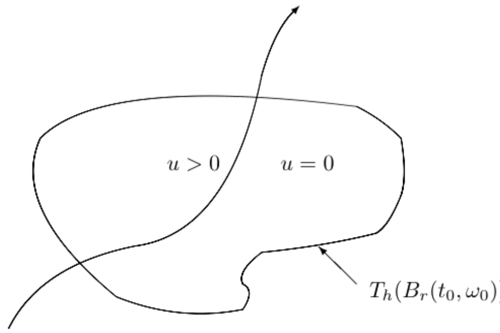

Figure 5 (iii)

one of the following situations holds

$$(i)\begin{cases} uoT_h(t,\omega) > 0 \quad \forall (t,\omega) \in B_\rho(t_0,\omega_0) \cap [\omega < \omega_0], \\ \text{there exists a sequence } (t_n,\omega_n)_{n\geq 1} \subset B_\rho(t_0,\omega_0) \cap [\omega > \omega_0] \\ \text{such that} \quad \forall n \geq 1 \quad uoT_h(t_n,\omega_n) = 0 \quad \text{and} \quad X(t_n,\omega_n) \underset{n\to\infty}{\longrightarrow} X_0 \end{cases}$$

$$(ii)\begin{cases} uoT_h(t,\omega) > 0 \quad \forall (t,\omega) \in B_\rho(t_0,\omega_0) \cap [\omega > \omega_0], \\ \text{there exists a sequence } (t_n,\omega_n)_{n\geq 1} \subset B_\rho(t_0,\omega_0) \cap [\omega < \omega_0] \\ \text{such that} \quad \forall n \geq 1 \quad uoT_h(t_n,\omega_n) = 0 \quad \text{and} \quad X(t_n,\omega_n) \underset{n\to\infty}{\longrightarrow} X_0 \end{cases}$$

$$(iii)\begin{cases} \text{There exist two sequences } (t_n^\pm,\omega_n^\pm)_{n\geq 1} \subset B_\rho(t_0,\omega_0), \\ \text{such that} \quad \forall n \geq 1 \quad \omega_n^- < \omega_0 < \omega_n^+ \\ uoT_h(t_n^-,\omega_n^-) = uoT_h(t_n^+,\omega_n^+) = 0 \\ \text{and} \quad X(t_n^-,\omega_n^-) \underset{n\to\infty}{\longrightarrow} X_0, \quad X(t_n^+,\omega_n^+) \underset{n\to\infty}{\longrightarrow} X_0. \end{cases}$$

Proof. Let $\eta > 0$ such that $B_\eta(t_0,\omega_0) \subset D_h$. By Proposition 3.1, we have $uoT_h(t,\omega_0) = 0$ for all $t \geq t_0$. Then for any $\rho \in (0,\eta)$, by Lemma 3.4, one of the following situations holds necessarily

$$\alpha) \quad \exists (t_1^-,\omega_1^-) \in B_\rho(t_0,\omega_0) \cap [\omega < \omega_0] \quad \text{such that} \quad uoT_h(t_1^-,\omega_1^-) = 0$$
$$\beta) \quad \exists (t_1^+,\omega_1^+) \in B_\rho(t_0,\omega_0) \cap [\omega > \omega_0] \quad \text{such that} \quad uoT_h(t_1^+,\omega_1^+) = 0.$$

We discuss the following cases:

• If α) and β) hold simultaneously for any $\rho \in (0,\eta)$, then we are in the situation *iii*).

• If for example α does not hold for some $\rho \in (0,\eta)$, then $uoT_h > 0$ in $B_\rho(t_0,\omega_0) \cap [\omega < \omega_0]$. Moreover by Lemma 3.4, β) holds for any $\rho' \in (0,\rho)$. In this case we are in the situation *i*).

• If for example β does not hold, then we show as in the previous case that we obtain the situation *ii*). □

4. A comparison result

In all what follows, we assume that

$$a \in C^{0,\alpha}_{loc}(\Omega) \qquad (0 < \alpha < 1) \tag{4.1}$$

$$\exists c_0 \in \mathbb{R} \quad / \quad \forall Y \in \Omega \quad : \quad div(a(X)(X - Y)) \le c_0 \quad \text{in } \mathcal{D}'(\Omega). \tag{4.2}$$

Note that (4.2) is satisfied in particular if $a \in C^{0,1}$ or simply if $div(a(X)e_1)$, $div(a(X)e_2) \in L^\infty(\Omega)$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Moreover, one can adapt the proof in [4] (see Remark 2.2 of this reference) to verify that $u \in C^{0,1}_{loc}(\Omega)$. The main result of this section is the comparison Lemma 4.4. First, we construct a barrier function and establish some of its properties.

Lemma 4.1. *Let $k > 0$, $(x_1, \underline{y})$, $(x_2, \underline{y}) \in \Omega$ with $x_1 < x_2$ and $x_2 - x_1 = 2k\epsilon$, where ϵ is small enough so that $(x_1 - \epsilon, x_2 + \epsilon) \times (\underline{y}, \underline{y} + 2\epsilon) \subset\subset \Omega$. Let $Z = (x_1 - \epsilon, x_2 + \epsilon) \times (\underline{y}, \underline{y} + \epsilon)$ and denote by v the unique solution in $H^1(Z)$ of*

$$\begin{cases} div(a(X)\nabla v) = -div(H(X)) & \text{in } Z \\ v = \epsilon(\underline{y} + \epsilon - y)^+ & \text{on } \partial Z. \end{cases} \tag{4.3}$$

Then, there exists a positive constant C independent of ϵ such that

$$\begin{array}{lll} i) & 0 < v \le C\epsilon^2 & \text{in} \quad Z \\ ii) & |\nabla v(X)| \le C\epsilon & \forall X \in T = [x_1, x_2] \times \{\underline{y} + \epsilon\}. \end{array}$$

Proof. *i*) Since $div(a(X)\nabla v) = -div(H(X)) \le 0$ in Z and due to the boundary condition, we deduce by the weak and strong maximum principles (see [5]) that $v > 0$ in Z.

To prove the second inequality, we introduce the function

$$\begin{aligned} \omega \quad : \quad & \widehat{Z} = (0, 2k + 2) \times (0, 1) \longrightarrow \mathbb{R}^+ \\ & X' = (x', y') \longmapsto \omega(X') = v(x_1 - \epsilon + \epsilon x', \underline{y} + \epsilon y'). \end{aligned}$$

It is not difficult to check that

$$\begin{cases} div(\widehat{a}(X')\nabla \omega) = -\epsilon^2 \widehat{divH} & \text{in } \widehat{Z} \\ \omega = \epsilon^2(1 - y')^+ & \text{on } \partial\widehat{Z}, \end{cases} \tag{4.4}$$

where

$$\widehat{a}(X') = a(x_1 - \epsilon + \epsilon x', \underline{y} + \epsilon y'), \quad \widehat{divH}(X') = (divH)(x_1 - \epsilon + \epsilon x', \underline{y} + \epsilon y').$$

Moreover, we have

$$\widehat{a}(X')\xi.\xi \geq \lambda|\xi|^2, \qquad \forall \xi \in \mathbb{R}^2, \quad \forall X' \in \widehat{Z}$$
$$|\widehat{a}|_{\infty,\widehat{Z}} \leq M, \qquad 0 \leq \widehat{divH}(X') \leq C = |divH|_\infty, \qquad \text{a.e. } X' \in \widehat{Z}.$$

Applying Theorem 8.16 page 191 of [5], we get

$$\sup_{\widehat{Z}} \omega \leq \sup_{\partial \widehat{Z}} \omega + C_1 \frac{|\epsilon^2 \widehat{divH}|_{L^{q/2}}}{\lambda},$$

where $q > 2$ and C_1 is a positive constant depending only on Y. So

$$\sup_Z v = \sup_{\widehat{Z}} \omega \leq \epsilon^2 + C_2\epsilon^2 = C\epsilon^2.$$

ii) Let $S = (\frac{1}{2}, 2k + \frac{3}{2}) \times \{1\}$ and $\widehat{Z}' = (\frac{1}{2}, 2k + \frac{3}{2}) \times (\frac{1}{2}, 1)$. Since S is a $C^{1,\alpha}$ boundary portion of $\partial\widehat{Z}$ and $\omega = 0$ on S, we deduce from (4.4) by applying Corollary 8.36 page 212 [5] that $\omega \in C^{1,\alpha}(\widehat{Z} \cup S)$ with the following estimate

$$|\omega|_{1,\alpha,\widehat{Z}'} \leq C\Big(|\omega|_{0,\widehat{Z}} + |\epsilon^2 \widehat{divH}|_{0,\widehat{Z}}\Big),$$

where $C = C(\lambda, M, K, d', S)$ is a constant independent of ϵ, $d' = d(\widehat{Z}', \partial\widehat{Z} \backslash S)$ and $K = \max_{i,j}(|a_{ij}|_{0,\alpha})$.

Taking into account the estimate in *i*), we obtain

$$|\nabla\omega|_{0,\widehat{Z}'} \leq |\omega|_{1,\alpha,\widehat{Z}'} \leq C\epsilon^2$$

which, in particular, leads to

$$|\nabla\omega(x', 1)| \leq C\epsilon^2 \qquad \forall x' \in [1, 1 + 2k].$$

Therefore,

$$|\nabla v(x, \underline{y} + \epsilon)| = \frac{1}{\epsilon}\Big|\nabla\omega\Big(\frac{x - x_1 + \epsilon}{\epsilon}, 1\Big)\Big| \leq C\,\epsilon \qquad \forall x \in [x_1, x_2].$$

Lemma 4.2. *Let $h \in \pi_y(\Omega)$, $\omega_1, \omega_2 \in \pi_x(\Omega \cap [y = h])$ with $\omega_1 < \omega_2$.*

Let $\underline{y} \in \pi_y(\Omega)$ such that $\gamma(\omega_i) \cap [y = \underline{y}] \neq \emptyset$ $i = 1, 2$.

Set $D_{\underline{y}} = T_h([\omega_1 < \omega < \omega_2]) \cap [y > \underline{y}]$. Assume that $D_{\underline{y}} \cap [y < \underline{y} + \epsilon] \subset (x_1, x_2) \times (\underline{y}, \underline{y} + \epsilon) \subset Z$ with Z defined in Lemma 4.1. Then after extending v by 0 to $D_{\underline{y}}$, we obtain

$$\int_{D_{\underline{y}}} (a(X)\nabla v + \chi([v > 0])H(X))\nabla\zeta dX \geq 0$$

for all $\zeta \in H^1(D_{\underline{y}})$, $\zeta \geq 0$, $\zeta = 0$ on $\partial D_{\underline{y}} \cap \Omega$.

Proof. Set $T' = [y = \underline{y} + \epsilon] \cap \overline{D}_{\underline{y}} \subset T$ and let ν be the outward unit normal vector to T. We have by Lemma 4.1 *ii)* $a(X)\nabla v.\nu + H(X).\nu = a(X)\nabla v.e_y + H_2(X) \geq -C\epsilon + \underline{h} \geq 0$ on T' for ϵ small enough. Now, for $\zeta \in H^1(D_{\underline{y}})$, $\zeta \geq 0$, $\zeta = 0$ on $\partial D_{\underline{y}} \cap \Omega$, we have

$$\int_{D_{\underline{y}}} (a(X)\nabla v + \chi([v>0])H(X))\nabla\zeta dX = \int_{T'} (a(X)\nabla v.\nu + H(X).\nu)\zeta dX \geq 0.$$

The following lemma extends a lemma proved in [4] for $H(X) = h(X)e_2$.

Lemma 4.3. *Let (u,χ) be a solution of (P). Assume that the hypothesis of Lemma 4.2 holds, $\overline{D}_{\underline{y}} \cap \Gamma_3 = \emptyset$ and (see Figure 6)*

$$\begin{aligned} &uoT_h(t_{\underline{y}}(\omega_1),\omega_1) = uoT_h(t_{\underline{y}}(\omega_2),\omega_2) = 0\\ &uoT_h(t_{\underline{y}}(\omega),\omega) \leq \epsilon^2 = voT_h(t_{\underline{y}}(\omega),\omega) \qquad \forall\omega \in (\omega_1,\omega_2). \end{aligned}$$

Then we have

$$\lim_{\delta\to 0}\frac{1}{\delta}\int_{D_{\underline{y}}\cap[v>0]\cap[0<u-v<\delta]} a(X)\nabla(u-v)^+.\nabla(u-v)^+ dX = 0.$$

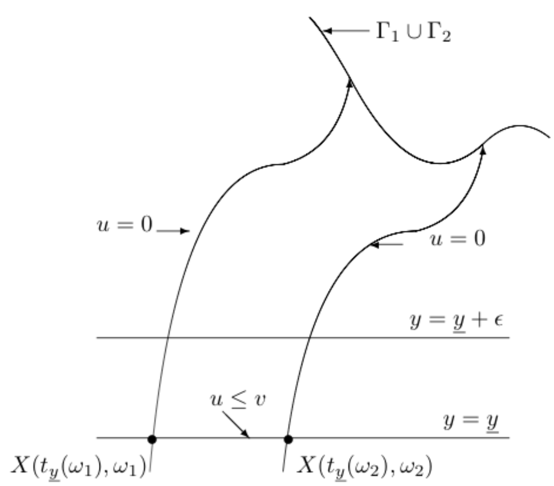

Figure 6

Proof. For $\delta,\eta > 0$, let $F_\delta(s)$ be the function introduced in the proof of Proposition 1.1, $d_\eta(y) = F_\eta(y - \bar{y})$ and $\bar{y} = \underline{y} + \epsilon$. By applying Lemma 3.2

and Lemma 4.2 for $\zeta = F_\delta(u-v) + d_\eta(1-F_\delta(u))$ and for $\zeta = F_\delta(u-v)$ respectively, we get

$$\int_{D_{\underline{y}}} \big(a(X)\nabla u + \chi H(X)\big).\nabla(F_\delta(u-v))dX \\ \leq -\int_{D_{\underline{y}}} \big(a(X)\nabla u + \chi H(X)\big).\nabla(d_\eta(1-F_\delta(u)))dX. \quad (4.5)$$

$$\int_{D_{\underline{y}}} \big(a(X)\nabla v + \chi([v>0])H(X)\big).\nabla(F_\delta(u-v))dX \geq 0. \quad (4.6)$$

Using (4.5) and (4.6), we get since $d_\eta = 0$ on $[v>0]$

$$\int_{D_{\underline{y}}\cap[v>0]} F'_\delta(u-v)a(X)\nabla(u-v).\nabla(u-v)dX \\ \leq -\int_{D_{\underline{y}}\cap[v=0]} (1-d_\eta)\big(a(X)\nabla u + \chi H(X)\big).\nabla(F_\delta(u))dX \\ -\int_{D_{\underline{y}}\cap[v=0]} (1-F_\delta(u))\big(a(X)\nabla u + \chi H(X)\big).\nabla d_\eta dX = I_1^{\delta\eta} + I_2^{\delta\eta}.$$

Since

$$|I_1^{\delta\eta}| \leq \int_{D_{\underline{y}}\cap[\bar{y}<y<\bar{y}+\eta]} |\big(a(X)\nabla u + \chi H(X)\big).\nabla(F_\delta(u))|dX,$$

we obtain $\lim_{\eta\to 0} I_1^{\delta\eta} = 0$.

As for $I_2^{\delta\eta}$, we have

$$I_2^{\delta\eta} = -\int_{D_{\underline{y}}\cap[u=v=0]} \chi H(X).\nabla d_\eta dX \\ -\int_{D_{\underline{y}}\cap[u>0=v]} (1-F_\delta(u))\big(a(X)\nabla u + H(X)\big).\nabla d_\eta dX = I_3^{\delta\eta} + I_4^{\delta\eta} \leq I_4^{\delta\eta},$$

since

$$I_3^\delta = -\int_{D_{\underline{y}}\cap[u=v=0]} H_2(X).\chi.\partial_y d_\eta dX = \frac{-1}{\eta}\int_{D_{\underline{y}}\cap[u=v=0]\cap[\bar{y}<y<\bar{y}+\eta]} H_2(X)\chi dX \leq 0.$$

Moreover, since $u \in C^{0,1}_{loc}(\Omega)$, one has for some constant C

$$|I_4^{\delta\eta}| \leq \frac{C}{\eta}\int_{D_{\underline{y}}\cap[u>v=0]\cap[\bar{y}<y<\bar{y}+\eta]} (1-F_\delta(u))dX$$

$$= \frac{C}{\eta}\int_J\int_{t_{\bar{y}}(\omega)}^{\min(\phi_h(\omega),t_{\bar{y}+\eta}(\omega))}(1-F_\delta(uoT_h))(t,\omega).(-Y_h(t,\omega))dtd\omega$$

$$\leq C\int_J\Big(\frac{1}{\eta}\int_{t_{\bar{y}}(\omega)}^{t_{\bar{y}}(\omega)+\frac{\eta}{\underline{h}}}(1-F_\delta(uoT_h))dt\Big)d\omega,$$

where $J=\{\omega\in(\omega_1,\omega_2):\phi_h(\omega)>t_{\bar{y}}(\omega)\}$.

Since the function $t\mapsto 1-F_\delta(uoT_h(t,\omega))$ is continuous, we obtain

$$\limsup_{\eta\to 0}|I_4^{\delta\eta}|\leq C\int_J(1-F_\delta(uoT_h(t_{\bar{y}}(\omega),\omega)))d\omega.$$

Hence,

$$\int_{D_{\underline{y}}\cap[v>0]\cap[0<u-v<\delta]}\frac{1}{\delta}a(X)\nabla(u-v)^+.\nabla(u-v)^+dX$$
$$\leq C\int_J(1-F_\delta(uoT_h(t_{\bar{y}}(\omega),\omega)))d\omega.$$

But given that $\omega\in J$, we have $uoT_h(t_{\bar{y}}(\omega),\omega)>0$. Thus,

$$\lim_{\delta\to 0}(1-F_\delta(uoT_h(t_{\bar{y}}(\omega),\omega)))=0$$

and the result follows. □

Lemma 4.4. *Let (u,χ) be a solution of (P). Assume that the hypothesis of Lemma 4.3 holds. Then we have*

$$u\equiv 0 \qquad \text{in } D_{\underline{y}}\cap[y>\underline{y}+\epsilon].$$

Proof. Let

$$D^+=D_{\underline{y}}\cap[v>0]=D_{\underline{y}}\cap[\underline{y}<y<\underline{y}+\epsilon]$$
$$\triangle=T_h\big(\{(t,\omega)\in D_h:\omega\in(\omega_1,\omega_2)\,,\ \alpha_-(\omega)<t<t_{\underline{y}+\epsilon}(\omega)\}\big)$$
$$w=\begin{cases}(u-v)^+ & \text{in} \quad D^+\\ 0 & \text{in} \quad \triangle\setminus\overline{D}^+.\end{cases}$$

We have $w\in H^1(\triangle)$ since by assumption $u\leq v$ on $\triangle\cap[y=\underline{y}]$.

Let $\zeta\in\mathcal{D}(\triangle)$. We have

$$\int_\triangle a(X)\nabla w.\nabla\zeta dX=\int_{D^+}a(X)\nabla(u-v)^+.\nabla\zeta dX$$
$$=\lim_{\delta\to 0}\int_{D^+}F_\delta(u-v)a(X)\nabla(u-v)^+.\nabla\zeta dX\ =\ \lim_{\delta\to 0}I_\delta.$$

Note that

$$I_\delta = \int_{D^+} a(X)\nabla(u-v)^+.\nabla(F_\delta(u-v)\zeta)dX$$
$$-\frac{1}{\delta}\int_{D^+\cap[0<u-v<\delta]} \zeta a(X)\nabla(u-v).\nabla(u-v)dX = I_\delta^1 - I_\delta^2.$$

By Lemma 4.3, $\lim_{\delta\to 0} I_\delta^2 = 0$, since we have

$$|I_\delta^2| \leq \sup_{\triangle}|\zeta|\,.\,\frac{1}{\delta}\int_{D^+\cap[0<u-v<\delta]} a(X)\nabla(u-v).\nabla(u-v)dX.$$

Moreover, we have since $F_\delta(u-v)\zeta \in H_0^1(D^+)$,

$$I_\delta^1 = \int_{D^+} a(X)\nabla u.\nabla(F_\delta(u-v)\zeta)dX - \int_{D^+} a(X)\nabla v.\nabla(F_\delta(u-v)\zeta)dX$$
$$= -\int_{D^+}\chi H(X).\nabla(F_\delta(u-v)\zeta)dX + \int_{D^+} H(X).\nabla(F_\delta(u-v)\zeta)dX$$
$$= 0 \qquad \text{since} \quad \chi = 1 \quad \text{a.e. in} \quad [u>0].$$

It follows that

$$\int_{\triangle} a(X)\nabla w.\nabla\zeta dX \;=\; 0 \qquad \forall\zeta\in\mathcal{D}(\triangle).$$

Since $\omega = 0$ in $\Delta\setminus\overline{D^+}$, we obtain by the strong maximum principle : $w=0$ in $\triangle$. Consequently, $u\leq v$ in D^+ and then $uoT_h(t_{\underline{y}+\epsilon}(\omega),\omega) = 0$ for all $\omega\in[\omega_1,\omega_2]$. Therefore,

$$uoT_h(t,\omega) = 0 \qquad \forall t\geq t_{\underline{y}+\epsilon}(\omega) \qquad \forall\omega\in[\omega_1,\omega_2].$$

Combining Lemma 3.5 and Lemma 4.4, we obtain the following useful lemma:

Lemma 4.5. *Let $X_0 = T_h(t_0,\omega_0) = (x_0,y_0)\in\Omega$, $\omega_{01},\omega_{02}\in\pi_y(\Omega\cap[y=h])$ such that $u(X_0)=0$, $\omega_{01}<\omega_0<\omega_{02}$ and $\gamma(\omega_{0i})\cap[y=y_0]\neq\emptyset$, $i=1,2$.*

Let $\epsilon>0$ and $D_{y_0} = T_h([\omega_{01}<\omega<\omega_{02}])\cap[y>y_0]$. We assume that for some $k>0$, $D_{y_0}\cap[y<y_0+\epsilon]\subset(x_0-2k\epsilon,x_0+2k\epsilon)\times(y_0,y_0+2\epsilon)\subset\subset\Omega$, and for all $\omega\in(\omega_{01},\omega_{02})$ $uoT_h(t_{y_0}(\omega),\omega)\leq\epsilon^2$. Then the following situations cannot hold

$$(i)\begin{cases}\textit{There exists a sequence } (t_n,\omega_n)_{n\geq1}\subset B_{\rho_0}(t_0,\omega_0)\cap[\omega<\omega_0]\\ \textit{satisfying } uoT_h(t_n,\omega_n)=0\quad\forall n\geq1,\quad X(t_n,\omega_n)\underset{n\to\infty}{\longrightarrow}X_0,\\ \forall n\geq1,\ X(\alpha_+(\omega_n),\omega_n)\textit{ does not belong to the connected component}\\ \textit{of } \Gamma_1\cup\Gamma_2 \textit{ which contains } X(\alpha_+(\omega_0),\omega_0).\end{cases}$$

$$(ii)\begin{cases} \textit{There exists a sequence } (t_n,\omega_n)_{n\geq 1}\subset B_{\rho_0}(t_0,\omega_0)\cap[\omega>\omega_0] \\ \ \textit{satisfying } uoT_h(t_n,\omega_n)=0 \quad \forall n\geq 1, \qquad X(t_n,\omega_n)\underset{n\to\infty}{\longrightarrow} X_0 \\ \forall n\geq 1, \quad X(\alpha_+(\omega_n),\omega_n) \textit{ does not belong to the connected} \\ \textit{component of } \Gamma_1\cup\Gamma_2 \textit{ which contains } X(\alpha_+(\omega_0),\omega_0). \end{cases}$$

Proof. We will consider only the first situation. The second one can be treated similarly.

Let $(x^*,y^*)\in M$ such that $y^*>y_0+\epsilon$, where M is the domain enclosed between $\gamma(\omega_{01})$, $\gamma(\omega_0)$, $[y=y_0]$ and $\partial\Omega$. Consider the maximal solution $X(.,x^*,y^*)$ of $X'(t)=H(X(t))$, $X(0)=(x^*,y^*)$. The orbit $\gamma(x^*,y^*)$ of $X(.,x^*,y^*)$ leaves M from the top at a point of $\partial\Omega$ and from the bottom at a point (x_*,y_0) of $[y=y_0]$.

From Lemma 3.1 *ii*), we know that $(x_*,y_0)=X(t_{y_0}(\omega_*),\omega_*,h)$ for some $\omega_*\in(\omega_{01},\omega_0)$. It follows that the two orbits $\gamma(x^*,y^*)$ and $\gamma(\omega_*,h)$ coincide. Therefore, we have $X(t,x^*,y^*)=X(t+t^*,\omega_*,h)$, where $t^*=t_{y^*}(\omega_*)$ is defined by $(x^*,y^*)=X(t^*,\omega_*,h)$. We have

$$X_1(t_{y^*}(\omega_{01}),\omega_{01},h)<x^*<X_1(t_{y^*}(\omega_0),\omega_0,h)$$

and $X_{1n}(t_{y^*}(\omega_0),\omega_n,h)$ converges to $X_1(t_{y^*}(\omega_0),\omega_0,h)$ when $n\to\infty$. So there exists $n_1>1$ such that $x^*<X_{1n_1}(t_{y^*}(\omega_0),\omega_{n_1},h)$.

We deduce that $(x^*,y^*)\in M_{n_1}$: the domain enclosed between $\gamma(\omega_{01})$, $\gamma(\omega_{n_1})$, $[y=y_0]$ and $\partial\Omega$.

It follows, by Lemma 4.4, that $u\equiv 0$ in $M_{n_1}\cap[y\geq y_0+\epsilon]$. In particular, we obtain $u(x^*,y^*)=0$. This holds for any point of M. Then $u\equiv 0$ in $M\cap[y\geq y_0+\epsilon]$. But (see Remark 3.1), this contradicts $\overline{M}\cap\Gamma_3\neq\emptyset$ and $u>0$ on Γ_3. □

Remark 4.1. Lemma 4.5 becomes trivial if α_+ is continuous. However we know only that α_+ is lower semi-continuous (see Lemma 10.5 page 125, [1]). Of course one can have more regularity for α_+ if one assumes more regularity on H and the boundary of Ω. Actually one can verify that α_+ is C^1 if $H\in C^1(\overline{\Omega})$, $\partial\Omega$ is C^1 and $H(X).\nu$ does not vanish on $\partial\Omega$ (see Proposition 2.1, [3]).

5. Continuity of the free boundary

The main result of this section is the continuity of the functions ϕ_h representing the free boundary. Note that by Remark 3.1, if $X(\phi_h(\omega),\omega)\in\Omega$, then $X(\alpha_+(\omega),\omega)\in\partial\Omega\setminus\Gamma_3$. Here we will consider the case where $X(\alpha_+(\omega),\omega)\in\partial\Omega\setminus\overline{\Gamma}_3$.

Theorem 5.1. *For each $h \in \pi_y(\Omega)$, the function ϕ_h is continuous at each $\omega \in \pi_x(\Omega \cap [y = h])$ such that $X(\phi_h(\omega), \omega) \in \Omega$ and $X(\alpha_+(\omega), \omega) \in \partial\Omega \setminus \overline{\Gamma}_3$.*

Proof. Let $\omega_0 \in \pi_x(\Omega \cap [y = h])$ such that $X(\phi_h(\omega_0), \omega_0) = T_h(\phi_h(\omega_0), \omega_0) = T_h(t_0, \omega_0) = (x_0, y_0) = X_0 \in \Omega$ and $X(\alpha_+(\omega_0), \omega_0) \in \partial\Omega \setminus \overline{\Gamma}_3$.

Let $0 < \epsilon < \min\left(\frac{h}{3}(\alpha_+(\omega_0) - t_0), \frac{h}{2}(t_0 - \alpha_-(\omega_0))\right)$. Since $u(X_0) = 0$ and u is continuous, there exists $\rho^* \in (0, \epsilon)$ such that

$$u(X) \le \epsilon^2 \qquad \forall X \in B_{\rho^*}(X_0) \subset T_h(D_h). \tag{5.1}$$

Since (t_0, ω_0) belongs to the open set $T_h^{-1}\big(B_{\rho^*}(X_0)\big)$, there exists $\eta_1 \in (0, \rho^*)$ such that

$$B_{\eta_1}(t_0, \omega_0) \subset\subset T_h^{-1}\big(B_{q\rho^*}(X_0)\big) \qquad \text{with } q = \underline{h}/4\bar{h}. \tag{5.2}$$

By Theorem 3.4 page 24 in [6], there exists $\eta_2 \in (0, \eta_1)$ such that

$$X(t, \omega) \text{ exists for all } (t, \omega) \in [\alpha_-(\omega_0), \alpha_+(\omega_0)] \times (\omega_0 - \eta_2, \omega_0 + \eta_2) \tag{5.3}$$
$$\text{and } \ (t, \omega) \longmapsto X(t, \omega) \quad \text{ is continuous.}$$

So there exists $\eta_3 \in (0, \eta_2)$ such that

$$|X(t, \omega) - X(t_0, \omega_0)| < \epsilon \qquad \forall (t, \omega) \in B_{\eta_3}(t_0, \omega_0). \tag{5.4}$$

Set $\rho = \eta_3 < \epsilon$. By Lemma 3.4, one of the following situations is true:

i) $\exists (t_1, w_1) \in B_\rho(t_0, \omega_0)$ such that $\omega_1 < \omega_0$ and $uoT_h(t_1, \omega_1) = 0$;

ii) $\exists (t_2, \omega_2) \in B_\rho(t_0, \omega_0)$ such that $\omega_2 > \omega_0$ and $uoT_h(t_2, \omega_2) = 0$.

We will consider only the case where *i)* holds (see Figure 7). The other case can obviously be treated in a similar way. Note that $X(t_1, \omega_1)$ is at the left-hand side of the orbit $\gamma(\omega_0)$ since $X(0, \omega_1) = (\omega_1, h)$, $X(0, \omega_0) = (\omega_0, h)$ and $\omega_1 < \omega_0$.

Set $\underline{y} = \max(X_2(t_0, \omega_0), X_2(t_1, \omega_1))$. Then

$$uoT_h(t_{\underline{y}}(\omega_i), \omega_i) = 0 \qquad i = 0, 1. \tag{5.5}$$

Consider the set

$$\mathcal{O} = \{(x, y) = X(t, \omega) \in T_h(D_h) : |\omega - \omega_0| < \rho\} \cap [\underline{y} < y < \underline{y} + \epsilon].$$

Then we have:

Lemma 5.1. *For all ω in $(\omega_0 - \rho, \omega_0 + \rho)$, we have*

$$\gamma(\omega) \cap [y = \underline{y}] \neq \emptyset \quad \text{ and } \quad X_2(\alpha_-(\omega_0), \omega) < \underline{y} < \underline{y} + \epsilon < X_2(\alpha_+(\omega_0), \omega).$$

Moreover, the open set $\mathcal{O}$ can be written

$$\mathcal{O} = T_h\Big(\{(t, \omega) \in D_h : |\omega - \omega_0| < \rho,\ t_{\underline{y}}(\omega) < t < t_{\underline{y}+\epsilon}(\omega)\}\Big).$$

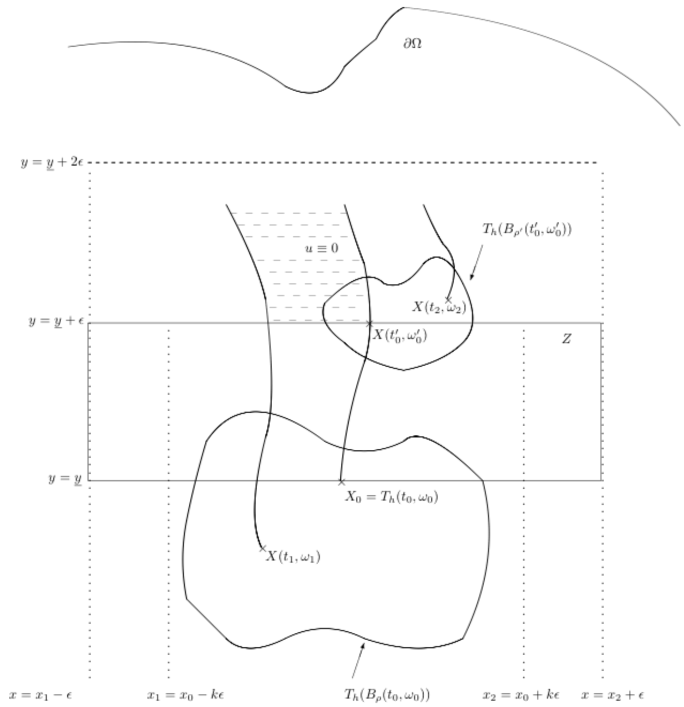

Figure 7

Proof. *i*) First we show that

$$X_2(\alpha_-(\omega_0),\omega) < \underline{y} < \underline{y} + \epsilon < X_2(\alpha_+(\omega_0),\omega) \quad \forall \omega \in (\omega_0 - \rho, \omega_0 + \rho). \quad (5.6)$$

Indeed we have for $\omega \in (\omega_0 - \rho, \omega_0 + \rho)$

$$X_2(\alpha_+(\omega_0),\omega) - X_2(t_0,\omega) = \int_{t_0}^{\alpha_+(\omega_0)} H_2(X(s,\omega))ds \geq \underline{h}(\alpha_+(\omega_0) - t_0)$$
$$X_2(\alpha_-(\omega_0),\omega) - X_2(t_0,\omega) = -\int_{\alpha_-(\omega_0)}^{t_0} H_2(X(s,\omega))ds \leq -\underline{h}(t_0 - \alpha_-(\omega_0)).$$

Using (5.4), we get

$$X_2(\alpha_+(\omega_0),\omega) \geq X_2(t_0,\omega_0) - \epsilon + \underline{h}(\alpha_+(\omega_0) - t_0)$$
$$X_2(\alpha_-(\omega_0),\omega) \leq X_2(t_0,\omega_0) + \epsilon - \underline{h}(t_0 - \alpha_-(\omega_0)).$$

Since $|X_2(t_0,\omega_0)-\underline{y}| \le |X_2(t_0,\omega_0)-X_2(t_1,\omega_1)| < \rho < \epsilon$ (by (5.4)), we get

$$X_2(\alpha_+(\omega_0),\omega) \ge \underline{y} - 2\epsilon + \underline{h}(\alpha_+(\omega_0)-t_0)$$
$$X_2(\alpha_-(\omega_0),\omega) \le \underline{y} + 2\epsilon - \underline{h}(t_0-\alpha_-(\omega_0)).$$

To conclude it is enough to verify that

$$-2\epsilon + \underline{h}(\alpha_+(\omega_0)-t_0) > \epsilon \qquad \text{and} \qquad 2\epsilon - \underline{h}(t_0-\alpha_-(\omega_0)) < 0,$$

which is assured by the choice of ϵ.

As a consequence of (5.6), we obtain by the intermediate value theorem that $\gamma(\omega)\cap[y=\underline{y}] \ne \emptyset$ for all $\omega \in (\omega_0-\rho,\omega_0+\rho)$.

ii) Clearly $\mathcal{O}' = T_h\Big(\{(t,\omega)\in D_h : |\omega-\omega_0|<\rho,\ t_{\underline{y}}(\omega) < t < t_{\underline{y}+\epsilon}(\omega)\}\Big) \subset \mathcal{O}$. To prove that $\mathcal{O}\subset\mathcal{O}'$, it is enough to show that for all $(\omega,y)\in(\omega_0-\rho,\omega_0+\rho)\times[\underline{y},\underline{y}+\epsilon]$ there exists $t_y(\omega)\in(\alpha_-(\omega_0),\alpha_+(\omega_0)):\quad X_2(t_y(\omega),\omega) = y$ which is a consequence of (5.6) and the continuity of the function $t\longmapsto X_2(t,\omega)$. □

Lemma 5.2. *We have*

$$|t_{\underline{y}}(\omega)-t_0| < \frac{2\epsilon}{\underline{h}} \qquad \forall\omega\in(\omega_0-\rho,\omega_0+\rho). \tag{5.7}$$

Proof. Let $\omega\in(\omega_0-\rho,\omega_0+\rho)$. We have

$$\underline{y} - X_2(t_0,\omega) = X_2(t_{\underline{y}}(\omega),\omega) - X_2(t_0,\omega) = \int_{t_0}^{t_{\underline{y}}(\omega)} H_2(X(s,\omega))ds.$$

If $\underline{y} = X_2(t_0,\omega_0)$, then by (5.4), $\quad |\underline{y} - X_2(t_0,\omega)| < \epsilon$.

If $\underline{y} = X_2(t_1,\omega_1)$, then we have

$$|\underline{y} - X_2(t_0,\omega)| \le |X_2(t_1,\omega_1)-X_2(t_0,\omega_0)| + |X_2(t_0,\omega_0)-X_2(t_0,\omega)|.$$

Using (5.4) and the fact that $(t_1,\omega_1)\in B_\rho(t_0,\omega_0)$, we deduce that $|\underline{y} - X_2(t_0,\omega)| < 2\epsilon$.

We conclude by distinguishing the cases $t_{\underline{y}}(\omega) > t_0$, $t_{\underline{y}}(\omega) < t_0$ and use (1.3) to conclude. □

We claim that

Lemma 5.3.

$$\mathcal{O}\subset(x_0-k\epsilon,x_0+k\epsilon)\times(\underline{y},\underline{y}+\epsilon), \qquad k = c_0\big(1+\frac{2}{\underline{h}}\big)+\frac{\bar{h}}{\underline{h}}.$$

Proof. Indeed, let $X(t,\omega)\in\mathcal{O}$. By definition of $\mathcal{O}$, we have $\underline{y} < X_2(t,\omega) < \underline{y}+\epsilon$. So we only need to verify that $|X_1(t,\omega)-x_0| < k\epsilon$.

Note that, since $T_h \in C^{0,1}(D_h)$, we have

$$|X_1(t_{\underline{y}}(\omega),\omega) - X_1(t_0,\omega_0)| \le c_0(|t_{\underline{y}}(\omega) - t_0| + |\omega - \omega_0|).$$

Using (5.7), we get for all ω in $(\omega_0 - \rho, \omega_0 + \rho)$

$$|X_1(t_{\underline{y}}(\omega),\omega) - X_1(t_0,\omega_0)| < c_0(\frac{2\epsilon}{\underline{h}} + \rho) < c_0\big(1 + \frac{2}{\underline{h}}\big)\epsilon. \tag{5.8}$$

We also have for $\omega \in (\omega_0 - \rho, \omega_0 + \rho)$ and $t_{\underline{y}}(\omega) < t = t_y(\omega) < t_{\underline{y}+\epsilon}(\omega)$

$$\epsilon \ge y - \underline{y} = X_2(t,\omega) - X_2(t_{\underline{y}}(\omega),\omega) = \int_{t_{\underline{y}}(\omega)}^t H_2(X(s,\omega))ds \ge \underline{h}(t - t_{\underline{y}}(\omega))$$

$$|X_1(t,\omega) - X_1(t_{\underline{y}}(\omega),\omega)| = \Big|\int_{t_{\underline{y}}(\omega)}^t H_1(X(s,\omega))ds\Big| \le \bar{h}(t - t_{\underline{y}}(\omega)) \le \frac{\bar{h}}{\underline{h}}\epsilon. \tag{5.9}$$

Combining (5.8) and (5.9), we obtain

$$\begin{aligned}
&|X_1(t,\omega) - X_1(t_0,\omega_0)| \\
&\le |X_1(t,\omega) - X_1(t_{\underline{y}}(\omega),\omega)| + |X_1(t_{\underline{y}}(\omega),\omega) - X_1(t_0,\omega_0)| \\
&< \big(c_0\big(1 + \frac{2}{\underline{h}}\big) + \frac{\bar{h}}{\underline{h}}\big)\epsilon = k\epsilon \qquad \forall X(t,\omega) \in \mathcal{O}.
\end{aligned}$$

□

From now on, we assume that ϵ is small enough to ensure that

$$(x_0 - (k+1)\epsilon, x_0 + (k+1)\epsilon) \times (\underline{y}, \underline{y} + 2\epsilon) \subset\subset \Omega.$$

We set

$$\begin{aligned}
&x_1 = x_0 - k\epsilon, \qquad x_2 = x_0 + k\epsilon \\
&Z = (x_1 - \epsilon, x_2 + \epsilon) \times (\underline{y}, \underline{y} + \epsilon) \\
&D_{\underline{y}} = T_h\Big(\big\{(t,\omega) \in D_h : \omega \in (\omega_1,\omega_0),\ t > t_{\underline{y}}(\omega)\big\}\Big).
\end{aligned}$$

We have:

Lemma 5.4. *The line segment*

$$S = [X(t_{\underline{y}}(\omega_1),\omega_1), X(t_{\underline{y}}(\omega_0),\omega_0)] \subset B_{\rho^*}(X_0).$$

Proof. Since $B_{\rho^*}(X_0)$ is convex, it suffices to prove that $X(t_{\underline{y}}(\omega_1),\omega_1)$, $X(t_{\underline{y}}(\omega_0),\omega_0) \in B_{\rho^*}(X_0)$.

First, we have $(t_1,\omega_1) \in B_\rho(t_0,\omega_0)$, and by (5.2), we are led to $X(t_1,\omega_1) \in B_{q\rho^*}(X_0)$. Using the definition of $\underline{y}$, we get

$$|\underline{y} - X_2(t_0,\omega_0)| < q\rho^* < \rho^*/4.$$

In the same way we have

$$|\underline{y} - X_2(t_1, \omega_1)| \leq |X_2(t_0, \omega_0) - X_2(t_1, \omega_1)| < q\rho^* < \rho^*/4.$$

Now, for $i = 0, 1$, we have

$$X_1(t_{\underline{y}}(\omega_i), \omega_i) - X_1(t_i, \omega_i) = \int_{t_i}^{t_{\underline{y}}(\omega_i)} H_1(X(s, \omega_i))ds$$

$$\underline{y} - X_2(t_i, \omega_i) = X_2(t_{\underline{y}}(\omega_i), \omega_i) - X_2(t_i, \omega_i) = \int_{t_i}^{t_{\underline{y}}(\omega_i)} H_2(X(s, \omega_i))ds \geq 0,$$

from which we deduce that

$$|X_1(t_{\underline{y}}(\omega_i), \omega_i) - X_1(t_i, \omega_i)| \leq \frac{\bar{h}}{\underline{h}}|\underline{y} - X_2(t_i, \omega_i)| < \rho^*/4.$$

Hence

$$\begin{aligned}
&|X_1(t_{\underline{y}}(\omega_0), \omega_0) - X_1(t_0, \omega_0)| < \rho^*/4 \\
&|X_1(t_{\underline{y}}(\omega_1), \omega_1) - X_1(t_0, \omega_0)| \\
&\quad \leq |X_1(t_{\underline{y}}(\omega_1), \omega_1) - X_1(t_1, \omega_1)| + |X_1(t_1, \omega_1) - X_1(t_0, \omega_0)| \\
&\quad < \rho^*/4 + q\rho^* < \rho^*/2.
\end{aligned}$$

We conclude that $|X(t_{\underline{y}}(\omega_i), \omega_i) - X(t_0, \omega_0)| \leq \sqrt{(\rho^*/4)^2 + (\rho^*/2)^2} < \rho^*$. □

End of the Proof of Theorem 5.1. As a consequence of Lemma 3.1 *ii*) and Lemma 5.4, we have

$$uoT_h(t_{\underline{y}}(\omega), \omega) \leq \epsilon^2 \qquad \forall \omega \in (\omega_1, \omega_0). \tag{5.10}$$

Moreover, by Lemma 5.3, we have $D_{\underline{y}} \cap [\underline{y} < y < \underline{y} + \epsilon] \subset (x_1, x_2) \times (\underline{y}, \underline{y} + \epsilon)$.

We discuss the following cases:

1^{st} *case*: $\overline{D}_{\underline{y}} \cap \Gamma_3 = \emptyset$.

Applying Lemma 4.4, we deduce that $u \equiv 0$ in $D_{\underline{y}} \cap [y \geq \underline{y} + \epsilon]$.

Set $X_0' = X(t_0', \omega_0') = X(t_{\underline{y}+\epsilon}(\omega_0), \omega_0)$. Arguing as before, one can find $(t_2, \omega_2) \in B_{\rho'}(t_0', \omega_0') \cap [\omega > \omega_0]$ such that $uoT_h(t_2, \omega_2) = 0$. We define $\underline{y}' = \max(X_2(t_0', \omega_0'), X_2(t_2, \omega_2))$ and $D_{\underline{y}'} = T_h([\omega_0 < \omega < \omega_2]) \cap [y > \underline{y}']$.

- If $D_{\underline{y}'} \cap \Gamma_3 = \emptyset$, then $u \equiv 0$ in $T_h([\omega_0 < \omega < \omega_2]) \cap [y > \underline{y}' + \epsilon]$. So for all $\omega \in (\omega_1, \omega_2)$, we have

$$\begin{aligned}
\phi_h(\omega) &\leq t_{\underline{y}'+\epsilon}(\omega) \leq t_{\underline{y}'}(\omega) + \frac{\epsilon}{\underline{h}} < t_0' + \frac{3\epsilon}{\underline{h}} \\
&< t_{\underline{y}}(\omega_0) + \frac{\epsilon}{\underline{h}} + \frac{3\epsilon}{\underline{h}} < t_0 + 2\frac{\epsilon}{\underline{h}} + \frac{4\epsilon}{\underline{h}} = \phi_h(\omega_0) + \frac{6\epsilon}{\underline{h}}
\end{aligned}$$

which is the upper semi-continuity (u.s.c) of ϕ_h at ω_0.

• If $D_{\underline{y'}} \cap \Gamma_3 \neq \emptyset$, then $X(\alpha_+(\omega_2), \omega_2)$ does not belong to the same connected component of $\Gamma_1 \cup \Gamma_2$ containing $X(\alpha_+(\omega_0), \omega_0)$. Moreover, we are now in the situation iii) of Lemma 3.5. So there exists $(t_n^+, \omega_n^+) \in B_{\rho'}(t_0', \omega_0) \cap [\omega > \omega_0]$ satisfying $uoT_h(t_n^+, \omega_n^+) = 0$ and $X(t_n^+, \omega_n^+) \underset{n\to\infty}{\longrightarrow} X_0'$. By Lemma 4.5, there exists $n_0 \geq 1$ such that $X(\alpha_+(\omega_{n_0}), \omega_{n_0})$ belongs to the same connected component of $\Gamma_1 \cup \Gamma_2$ which is containing $X(\alpha_+(\omega_0), \omega_0)$. Necessarily, the set $\{X(\alpha_+(\omega), \omega),\ \omega \in [\omega_0, \omega_{n0}^+]\}$ is contained in this connected component. Then, by considering $D_{\underline{y'}} \cap T_h([\omega_0 < \omega < \omega_{n0}^+])$, we can argue as in the previous case, since $\overline{D_{\underline{y'}} \cap T_h([\omega_0 < \omega < \omega_{n0}^+])} \cap \Gamma_3 = \emptyset$, to show that ϕ_h is u.s.c at ω_0.

2^{nd} *case*: $\overline{D}_{\underline{y}} \cap \Gamma_3 \neq \emptyset$.

From Lemma 3.5, we can have a sequence $(t_n^-, \omega_n^-)_{n\geq 1}$ in $B_\rho(t_0, \omega_0) \cap [\omega < \omega_0]$ or a sequence $(t_n^+, \omega_n^+)_{n\geq 1}$ in $B_\rho(t_0, \omega_0) \cap [\omega > \omega_0]$ or both of them, converging to X_0 and such that uoT_h vanishes on each point of the sequences. By Lemma 4.5, we can find $\omega_{n_1}^- < \omega_0$ or $\omega_{n_2}^+ > \omega_0$ such that $X(\alpha_+(\omega_{n_1}^-), \omega_{n_1}^-)$ or $X(\alpha_+(\omega_{n_2}^+), \omega_{n_2}^+)$ or both of them belong to the same connected component of $\Gamma_1 \cup \Gamma_2$ which is containing $X(\alpha_+(\omega_0), \omega_0)$. We conclude for the last case by considering $D_{\underline{y'}} \cap T_h([\omega_{n_1}^- < \omega < \omega_{n_2}^+])$. For the other cases, we are back to the 1^{st} one. □

6. Some remarks

In this section we first propose a different proof for Theorem 5.1 when H is more regular. Then we show that conditions (4.1)-(4.2) are not sharp. Finally we show that in condition (1.3), one can replace the direction $e = (0, 1)$ by any other direction.

Remark 6.1. When $H \in C^{1,1}(\overline{\Omega})$, it is possible to give another proof for Theorem 5.1 much simpler than the above one. It consists on using the change of variables T_h, which is now a $C^{1,1}$ diffeomorphism, to reduce the problem to a problem of type (P_0).

Proof of Theorem 5.1 when $H \in C^{1,1}(\overline{\Omega})$. Indeed let $h \in \pi_y(\Omega)$, $\xi \in H^1(D_h)$, $\xi = 0$ on $(\partial D_h \cap T_h^{-1}(\Gamma_3)) \cup (\partial D_h \cap \Omega)$ and $\xi \geq 0$ on $\partial D_h \cap T_h^{-1}(\Gamma_2)$. Then $\xi o T_h^{-1} \chi(T_h(D_h))$ is a test function for (P) and we have

$$\int_{T_h(D_h)} (a(X)\nabla u + \chi H(X)).\nabla(\xi o T_h^{-1}) dX \leq 0$$

which can be written using the change of variables T_h

$$\int_{D_h} (\mathbb{A}(t, \omega)\nabla(uoT_h) + \chi o T_h.\mathtt{h}(t, \omega) e_t).\nabla \xi dt d\omega \leq 0,$$

where the matrix $\mathbb{A}$ and the function $\mathtt{h}$ are given by

$$\mathtt{h}(t,\omega) = |Y_h(t,\omega)|, \qquad e_t = (1,0)$$
$$\mathbb{A}(t,\omega) = |Y_h(t,\omega)|^t P(t,\omega).a(X(t,\omega)).P(t,\omega)$$
$$\text{with} \quad P = ({}^t\mathcal{J}T_h)^{-1} = \frac{1}{Y_h(t,\omega)} \begin{pmatrix} \dfrac{\partial X_2}{\partial \omega}(t,\omega) & -H_2(X(t,\omega)) \\ -\dfrac{\partial X_1}{\partial \omega}(t,\omega) & H_1(X(t,\omega)) \end{pmatrix}.$$

Note that from Proposition 2.3, the function $\mathtt{h}$ satisfies

$$\begin{cases} 0 < \underline{h} \le \mathtt{h}(t,\omega) \le C\bar{h} & \text{for a.e } (t,\omega) \in D_h \\ 0 \le \mathtt{h}_t(t,\omega) \le C\bar{h} & \text{for a.e } (t,\omega) \in D_h. \end{cases}$$

From the proof of Proposition 2.3, $\dfrac{\partial X}{\partial \omega} = U(t,\omega)$ satisfies the following differential equation

$$U'(t,\omega) = DH(X(t,\omega)).U(t,\omega) \qquad U(0,\omega) = (1,0).$$

Arguing as in the proof of Proposition 2.3, we deduce, since $DH \in C^{0,1}(\overline{\Omega})$, that $\frac{\partial X}{\partial \omega} \in C^{0,1}(D_h)$. Moreover,

$$\frac{1}{Y_h(t,\omega)} = -\frac{1}{H_2(\omega,h)} \exp\big(- \int_0^t (divH)(X(s,\omega))ds\big)$$

clearly, belongs to $C^{0,1}(D_h)$. Hence, the matrix $\mathbb{A}$ satisfies

$$\mathbb{A} \in C^{0,1}(D_h) \qquad \text{and} \qquad |\mathbb{A}(t,\omega)| \le C,$$

where C is a positive constant. To conclude, it remains to verify the following ellipticity condition

$$\mathbb{A}(t,\omega)\xi.\xi \ge \mu|\xi|^2 \quad \forall \xi \in \mathbb{R}^2, \quad \text{for a.e. } (t,\omega) \in D_h,$$

for some positive constant μ. So, let $\xi \in \mathbb{R}^2$. We have

$$\mathbb{A}(t,\omega)\xi.\xi = |Y_h|. < aoT_h.P\xi, P\xi > \ge \ \lambda|Y_h||P\xi|^2 = \lambda|Y_h| <^t PP\xi, \xi > .$$

Denote by Q the matrix ${}^tPP$. Since Q is symmetric, its eigenvalues κ_1 and κ_2 are real numbers. Moreover, we have

$$\kappa_1.\kappa_2 = detQ = (detP)^2 = \frac{1}{Y_h^2} \tag{6.1}$$

$$\kappa_1 + \kappa_2 = trQ = \frac{1}{Y_h^2}\Big(H_1^2 + H_2^2 + \big(\frac{\partial X_1}{\partial \omega}\big)^2 + \big(\frac{\partial X_2}{\partial \omega}\big)^2\Big). \tag{6.2}$$

Then $\kappa_1 > 0$ and $\kappa_2 > 0$. Assume for example that $\kappa_1 \le \kappa_2$ and set $m = \inf_{(t,\omega)\in D_h} \kappa_1(t,\omega)$. Suppose $m = 0$. There exists a sequence $(t_n, \omega_n) \in$

D_h such that $m = \lim_{n\to\infty} \kappa_1(t_n, \omega_n) = 0$. Since H and $\frac{\partial X}{\partial \omega}$ are bounded, we deduce from (6.2) that the sequence $\kappa_2(t_n, \omega_n)$ is bounded in $\mathbb{R}^+$. So there exists a subsequence $(n_k)_{k\geq 1}$ such that $\lim_{k\to\infty}\kappa_2(t_{n_k}, \omega_{n_k}) = \kappa^*$ with $0 \leq \kappa^* < \infty$. Now, letting $k \to \infty$ in (6.1), we get $\lim_{k\to\infty} \frac{1}{Y_h^2(t_{n_k},\omega_{n_k})} = 0$ which is a contradiction with Proposition 2.3 iv). So $m > 0$.

Now since Q is symmetric, there exists an orthogonal matrix O (i.e., $O^tO =^t OO = I_2$) such that $Q = ODO^{-1}$, D is a diagonal matrix with diagonal coefficients equal to the eigenvalues of Q. Then we have

$$< Q\xi, \xi >=< DO^{-1}\xi,^t O\xi >=< D^tO\xi,^t O\xi >\geq m|^tO\xi|^2 = m|\xi|^2.$$

Hence,

$$< \mathbb{A}\xi, \xi >\geq \lambda m|Y_h||\xi|^2 \geq \lambda m\underline{h}|\xi|^2 \qquad \forall\xi \in \mathbb{R}^2.$$

We conclude (see [4]) that the free boundary $\partial[uoT_h > 0]\cap D_h$ is a continuous curve $[t = \phi_h(\omega)]$. □

Remark 6.2. The conditions under which Theorem 5.1 is proved are not sharp. Indeed we present below a proof when $H(X) = a(X)e$, that is to say when (P) is the weak formulation of the dam problem with Dirichlet boundary conditions, with $a(X)$ satisfying (1.1)-(1.2), $a(X)e \in C^{0,1}(\overline{\Omega})$, but not the assumptions (4.1)-(4.2). Note that only the proof of Lemma 4.4 requires the last assumptions. Actually the proof given in Section 4 is based on the comparison of u with respect to the barrier function defined by (4.3). It uses the local Lipschitz continuity of u which requires the assumptions (4.1)-(4.2). For this special case, we propose another proof using an explicit barrier function. Moreover the assumption "$uoT_h(t_{\underline{y}}(\omega), \omega) \leq \epsilon^2$ $\forall\omega \in (\omega_1, \omega_2)$" in Lemma 4.3, will be modified by changing ϵ^2 to ϵ.

Proof of Lemma 4.4 when $H(X) = a(X)e$. Let $v(y) = (\epsilon + \underline{y} - y)^+$ and $\xi(x, y) = \chi(D_{\underline{y}})(u - v)^+$. Since $v \geq 0 = u$ on $(\partial D_{\underline{y}} \setminus ([y = \underline{y}])) \cap \Omega$, we have $\xi = 0$ on $(\partial D_{\underline{y}} \setminus ([y = \underline{y}])) \cap \Omega$. Moreover $v(\underline{y}) = \epsilon \geq u(x, \underline{y})$ and then $\xi(x, \underline{y}) = 0$. It follows that $\xi = 0$ on $(\partial D_{\underline{y}} \cap \Omega) \cup (\partial D_{\underline{y}} \cap \Gamma_2)$, and $\pm\xi$ are test functions for (P). So we have

$$\int_{D_{\underline{y}}} \big(a(X)\nabla u + \chi a(X)e\big).\nabla(u - v)^+ dX \leq 0. \tag{6.3}$$

We also have

$$\int_{D_{\underline{y}}} \big(a(X)\nabla v + \chi([v > 0])a(X)e\big).\nabla(u - v)^+ dX = 0. \tag{6.4}$$

Subtracting (6.4) from (6.3), we obtain

$$\int_{D_{\underline{y}}\cap[v>0]} a(X)\nabla(u-v).\nabla(u-v)^{+}dX$$
$$+\int_{D_{\underline{y}}\cap[v=0]} a(X)(\nabla u+\chi e).\nabla u dX \le 0. \tag{6.5}$$

By Lemma 3.2, we have for $D_{\underline{y}+\epsilon}=[y>\underline{y}+\epsilon]\cap D_{\underline{y}}=D_{\underline{y}}\cap[v=0]$ and $\zeta=y-(\underline{y}+\epsilon)$

$$\int_{D_{\underline{y}}\cap[v=0]} a(X)(\nabla u+\chi e).e dX \le 0. \tag{6.6}$$

Adding (6.5) and (6.6), we get by taking into account $(P)i)$

$$\int_{D_{\underline{y}}\cap[v>0]} a(X)\nabla(u-v).\nabla(u-v)^{+}dX$$
$$+\int_{D_{\underline{y}}\cap[u>v=0]} a(X)(\nabla u+e).(\nabla u+e)dX+\int_{D_{\underline{y}}\cap[u=v=0]}\chi a(X)e.e dX\le 0,$$

or by (1.2)

$$\int_{D_{\underline{y}}\cap[v>0]}|\nabla(u-v)^{+}|^2dX+\int_{D_{\underline{y}}\cap[u>v=0]}|\nabla u+e|^2dX+\int_{D_{\underline{y}}\cap[u=v=0]}\chi dX\le 0.$$

Since the three integrals in the left-hand side of the above inequality are all nonnegative, we obtain $\nabla(u-v)^{+}=0$ almost everywhere in $D_{\underline{y}}\cap[v>0]$ and then, since $(u-v)^{+}=0$ on $\partial D_{\underline{y}}\cap[y=\underline{y}]$, we get $u\le v$ in $D_{\underline{y}}\cap[v>0]$. This leads to $u(x,\underline{y}+\epsilon)=0$ for all $x\in\pi_x(D_{\underline{y}}\cap[y=\underline{y}+\epsilon])$. Hence $u=0$ in $D_{\underline{y}}\cap[y\ge\underline{y}+\epsilon]$. □

Remark 6.3. The assumption (1.3) can be replaced by the more general one

$$|H_1(X)|\le\overline{h},\qquad 0<\underline{h}\le H(X).\nu\le\overline{h}\qquad \text{a.e. } X\in\Omega \tag{6.7}$$

where $\nu\neq 0$ is a constant vector.

Proof of Remark 6.3. Indeed, set $\nu=(\nu_1,\nu_2)$, $n=(-\nu_2,\nu_1)$. We can assume that $|\nu|=\nu_1^2+\nu_2^2=1$. Clearly (n,ν) is an orthonormal basis of $\mathbb{R}^2$.

For a point $M\in\Omega$, we denote by X (respectively Y) its coordinates in the canonical (respectively new) basis (e_1,e_2) (respectively (n,ν)). We have

$$X=RY\quad\text{with}\quad R=R^{-1}=\begin{pmatrix}-\nu_2 & \nu_1\\ \nu_1 & \nu_2\end{pmatrix}.$$

Consider the change of variables $\theta : Y \longmapsto X = RY$ from $\theta^{-1}(\Omega) = \widetilde{\Omega}$ into Ω. Let $\xi \in H^1(\widetilde{\Omega})$, $\xi = 0$ on $\widetilde{\Gamma_3}$, $\xi \geq 0$ on $\widetilde{\Gamma_2}$, where $\widetilde{\Gamma}_i = \theta^{-1}(\Gamma_i)$ for $i = 1, 2, 3$. Using $\xi o\theta^{-1}$ as a test function for (P), we obtain

$$\begin{aligned}&\int_{\Omega}(a(X)\nabla u + \chi H(X)).\nabla(\xi o\theta^{-1})dX\\ &= \int_{\theta^{-1}(\Omega)}(R.ao\theta.R\nabla_Y(uo\theta) + \chi o\theta R.Ho\theta).\nabla_Y \xi dY\\ &= \int_{\widetilde{\Omega}}(\widetilde{a}(Y)\nabla\widetilde{u} + \widetilde{\chi}\widetilde{H}(Y))\nabla\xi dY,\end{aligned}$$

where $\widetilde{a}(Y) = R.ao\theta(Y).R$, $\widetilde{u} = uo\theta$, $\widetilde{\chi} = \chi o\theta$, and $\widetilde{H}(Y) = R.Ho\theta(Y)$. Note that $Ho\theta = H_1 o\theta e_1 + H_2 o\theta e_2 = \widetilde{H}_1(Y)n + \widetilde{H}_2(Y)\nu = R.\widetilde{H}(Y)$. Then $\widetilde{H}_1(Y) = -\nu_2 H_1 o\theta(Y) + \nu_1 H_2 o\theta(Y)$ and $\widetilde{H}_2(Y) = \nu_1 H_1 o\theta(Y) + \nu_2 H_2 o\theta(Y) = Ho\theta(Y).\nu$. We deduce that

$$|\widetilde{H}_1(Y)| \leq 2\overline{h}, \qquad 0 < \underline{h} \leq \widetilde{H}_2(Y) \leq \bar{h} \qquad \text{a.e. } Y \in \theta^{-1}(\Omega).$$

Finally, one can check easily that $(div_Y \widetilde{H})(Y) = (div_X H)(X)$ from which we deduce that

$$div_Y \widetilde{H} \in L^\infty(\theta^{-1}(\Omega)) \quad \text{and} \quad (div_Y \widetilde{H}(Y)) \geq 0 \qquad \text{a.e. } Y \in \theta^{-1}(\Omega).$$

Similarly, one can check that $\widetilde{a}(Y)$ satisfies the assumptions (1.1)-(1.2) and (4.1)-(4.2). □

Acknowledgments. We would like to thank KFUPM for its continuing support and facilities provided.

References

[1] H. Amann, "Ordinary Differential Equations, An Introduction to Nonlinear Analysis" Walter de Gruyter, Berlin, New York 1990.

[2] H. Federer, "Geometric Measure Theory," Springer-Verlag 1969.

[3] S. Challal and A. Lyaghfouri, *A Filtration Problem through a Heterogeneous Porous Medium,* Interfaces and Free Boundaries, 6 (2004), 55–79.

[4] S. Challal and A. Lyaghfouri, *Continuity of the Free Boundary in Problems of type* $div(a(x)\nabla u) = -(\chi(u)h(x))_{x_1}$, Nonlinear Analysis, Theory, Methods & Applications, 62 (2005), 283–300.

[5] D. Gilbarg and N.S. Trudinger, "Elliptic Partial Differential Equations of Second Order," Springer-Verlag, 1983.

[6] J. Hale, "Ordinary Differential Equations," Second Edition. Robert E. Krieger Publishing Co., Inc., Huntington, N.Y., 1980.

[7] W.P. Ziemer, "Weakly Differential Functions," Graduate Texts in Mathematics 120, Springer-Verlag 1989.