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EXISTENCE RESULT FOR A MODEL OF PROTEUS MIRABILIS SWARM

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Abstract. In this paper we present a modification of the usual *Proteus mirabilis* Swarm model. For the obtained model (which is a two phase model with a non-linear diffusion term containing memory) we set up a collection of a priori estimates. Those estimates allow us to get an existence and uniqueness result.

1. Introduction and results

Proteus mirabilis is a bacterium that can be either a short cell we call "swimmer" or an elongated cell capable of translocation we call "swarmer". A model of behaviour of a *Proteus mirabilis* colony has been proposed by Esipov and Shapiro [8], based on ideas of Gurtin [10].

In this paper, we prove an existence result for a model which is, in a way, a generalization but also a regularization of the Esipov and Shapiro [8] model.

The model under consideration here is a two phase model with a non-linear diffusion term containing memory for one of the two phases. It involves two functions ρ and Q. The function $\rho = \rho(t, a, x)$ refers, at time $t \in [0, T)$, $0 < T < +\infty$, to the density of swarmers of age $a \in [0, A)$, $0 < A \leq +\infty$ at position $x \in \Omega$, where Ω is a regular sub-domain of $\mathbb{R}^2$, with boundary $\partial\Omega$. In each point x of $\partial\Omega$, $\overrightarrow{\nu} = \overrightarrow{\nu}(x)$ stands for the unit normal vector pointing outside Ω. The function $Q = Q(t, x)$ stands, at time t, for the biomass density of swimmers on Ω.

For a constant τ, those two functions are supposed to satisfy the following system:

$$\frac{\partial Q}{\partial t} = \frac{1-\xi}{\tau} Q + \int_0^A \rho(\cdot, a, \cdot) e^{a/\tau} \mu(\cdot, a, \cdot)\, da + \chi(A) \rho(\cdot, A, \cdot) e^{A/\tau},$$

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$$\text{on } [0,T) \times \Omega, \quad (1.1)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} = -\mu\rho + \nabla \cdot \left[\big(D(\mathcal{M}, Q, P) + d\big)\nabla\rho\right], \quad \text{on } [0,T) \times [0,A) \times \Omega, \tag{1.2}$$

$$\rho(\cdot, 0, \cdot) = \frac{\xi}{\tau} Q, \quad \text{on } [0,T) \times \Omega, \tag{1.3}$$

$$\rho(0, \cdot, \cdot) = \rho_0, \quad \text{on } [0,A) \times \Omega, \tag{1.4}$$

$$Q(0, \cdot) = Q_0, \quad \text{on } \Omega, \tag{1.5}$$

$$\frac{\partial \rho}{\partial \overrightarrow{\nu}} = 0, \quad \text{on } [0,T) \times [0,A) \times \partial\Omega. \tag{1.6}$$

Above, $\chi(A)$ is an artifice allowing us to take into account a possible maximum age A beyond which swarmers cannot exist. It has the following definition:

$$\chi(A) = 1, \ \forall A \in \mathbb{R}, \ \chi(+\infty) = 0. \tag{1.7}$$

Denoting by C_b^k the space of functions having continuous and bounded derivatives up to order k, $\mu = \mu(t, a, x)$ is a function such that

$$\mu \in C_b^2([0,T) \times [0,A) \times \Omega), \ \mu \geq 0, \ \lim_{a \to A} \mu(t, a, x) = \overline{\mu} \text{ uniformly in } x \text{ and } t, \tag{1.8}$$

with $\overline{\mu} \geq c(1 - \chi(A))$, for a constant $c > 0$. The function $\xi = \xi(t, Q)$ satisfies

$$\xi \in C_b^2([0,T) \times \mathbb{R}), \ 0 \leq \xi \leq 1. \tag{1.9}$$

In the second equation, ∇ stands for the gradient and $\nabla \cdot$ for the divergence with respect to the $x-$variable. The diffusion coefficient is the sum of a constant, a priori small,

$$\begin{aligned} &d > 0, \ \text{and of} \\ &D = D(\mathcal{M}, P, Q), \ \text{a nonnegative } C_b^1 \text{ function of its arguments.} \end{aligned} \tag{1.10}$$

In (1.10), P is defined for $0 \leq a_{min} < A$ by

$$P(t,x) = \int_{a_{min}}^{A} \rho(t, a, x) e^{a/\tau} da, \tag{1.11}$$

and Q is given by the first equation of the system.

The memory (or hysteresis) term $\mathcal{M} = \mathcal{M}_{[P]}(t,x)$ keeps information on the value of P in the past. For four thresholds, $P_{min} < p_{min} < p_{max} < P_{max}$, with P_{min} close to p_{min} and P_{max} close to p_{max}, $\mathcal{M}$ is defined as the solution to:

$$\frac{\partial \mathcal{M}}{\partial t} = \frac{1}{P_{max} - p_{max}} \; H_r\Big(\frac{P - p_{max}}{P_{max} - p_{max}}\Big) \; H_r(1-\mathcal{M}) - \frac{1}{p_{min} - P_{min}} \; H_r\Big(\frac{p_{min} - P}{p_{min} - P_{min}}\Big) \; H_r(\mathcal{M}),$$
$$\mathcal{M}(0,\cdot) = \mathcal{M}_0, \tag{1.12}$$

with, denoting $P_0 = P(0,\cdot)$,

$$\mathcal{M}_0 \in C^1_b(\Omega),\; 0 \le \mathcal{M}_0 \le 1,\; \mathcal{M}_0 = 0 \text{ where } P_0 < P_{min} \text{ and } \mathcal{M}_0 = 1 \text{ where } P_0 > P_{max}, \tag{1.13}$$

and with

$$H_r(p) = 0 \text{ if } p \le 0,\; H_r(p) = p \text{ if } 0 \le p \le 1 \text{ and } H_r(p) = 1 \text{ if } p \ge 1. \tag{1.14}$$

We now turn to the statement of the main result of this paper.

Theorem 1.1. *Under assumptions* (1.8), (1.9), (1.10) *and* (1.13), *if* $\rho_0 \ge 0 \in L^1 \cap W^{2,2} \cap W^{1,4}([0,A) \times \Omega,\; e^{a/\tau} da dx)$ *satisfies*

$$\begin{aligned} &\int_\Omega (\rho_0)^2 e^{2a/\tau} dx \le b \int_\Omega (\rho_0)^2 e^{a/\tau} dx,\; \forall a \in [0,A), \\ &\int_0^A \int_\Omega |\nabla \rho_0|^4 e^{4a/\tau} dadx \le b \int_0^A \int_\Omega |\nabla \rho_0|^4 e^{a/\tau} dadx, \\ &\int_0^A \int_\Omega |\nabla \rho_0|^2 e^{2a/\tau} dx \le b \int_0^A \int_\Omega |\nabla \rho_0|^2 e^{a/\tau} dx, \end{aligned} \tag{1.15}$$

for a constant b, *and if* $Q_0 \ge 0 \in L^1 \cap W^{2,2} \cap W^{1,4}(\Omega)$, *then, there exists a unique solution* $(Q,\rho) \in L^\infty\big(0,T;((L^1 \cap W^{1,2}(\Omega)) \times (L^1 \cap W^{1,2}([0,A) \times \Omega,\; e^{a/\tau} dadx))\big)$ *to system* (1.1)–(1.6) *coupled with* (1.11) *and* (1.12). *Moreover,* $Q \ge 0$ *and* $\rho \ge 0$.

The precise definitions of the spaces at work in the theorem are given in the beginning of Section 4.

We now give references where modelling and mathematical methods are developed on the age-structured population problem: Gurtin and Mac Camy [11], Marcati [22], Andreasen [1, 2, 3]; possibly with diffusion: Gurtin [10], Di Blasio and Lamberti [7], Di Blasio [6], Mac Camy [21], Gurtin and Mac Camy [12], Busenberg and Iannelli [5], Langlais [17, 18], Kubo and Langlais [15], Huang [13] and Esipov and Shapiro [8]. For simulation methods we refer to Lopez and Trigiante [20], Milner [24], Kim [14], Esipov and Shapiro [8], Medvedev, Kapper and Koppel [23], Ayati and Dupont [4]. Concerning the biological description of *Proteus mirabilis* colony behaviour, we refer for instance to Rauprich *et al* [25], Gué, Dupont, Dufour and Sire [9] and their references.

The paper is organized as follows: in Section 2 we present the way to go from the Esipov and Shapiro model to system (1.1)-(1.6). Then Section 3 is devoted to a priori estimates for the solution to (1.1)-(1.6). By a usual procedure consisting in linearizing and passing to the limit, we prove the theorem in Section 4.

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2. Model

Proteus mirabilis is a pathogenic bacterium of the urinary tract that, when standing in a liquid medium, consists of a usual short "swimmer cell" or "swimmer". When placed on an agar medium, if the bacterium density is large enough, it may undergo a differentiation process producing an elongated cell with several nuclei called "swarmer cells" or "swarmers". Those swarmers are capable of translocation allowing the bacterial colony to colonise the medium.

The macroscopic model built by Esipov and Shapiro [8] describes this swarm phenomenon at the colony scale. We shall explain this model now. The swarmer behaviour depends on their own age. This dependence is taken into account by introducing the age dependent density of swarmers $\rho(t,a,x)$. The link between this density ρ and the biomass density is a consequence of the fact that the mass of each cell is in direct proportion to its length and that the length growth of a swarmer is also in direct proportion to its length. Then the biomass density, at time t, of swarmers of age a at position x is $\rho(t,a,x)e^{a/\tau}$, where τ is the growth rate of the biomass. The first age-depending behaviour of the swarmers is that they actively participate in group migration only after an age a_{min}. Then the definition of the biomass density P of swarmers capable of active translocation is given by (1.11). The second age-depending behaviour is that the swarmers dedifferentiate themselves and produce swimmers. On this topic, Esipov and Shapiro [8] consider two situations. In the first one (Model A), the swarmers dedifferentiate themselves at a given age a_{max}. The second situation considers that swarmers may dedifferentiate at each time with a probability $1/\overline{a}$ (Model B).

Now, we are able to write the swimmer evolution equation for the biomass density Q. Its evolution results from the classical cellular division, with a characteristic time which is the biomass growth rate τ, subtracting the proportion of bacteria undergoing differentiation and adding the dedifferentiation product. In the case of Model A, the evolution equation for Q is then

$$\frac{\partial Q}{\partial t} = \frac{1-\xi}{\tau} Q + \rho(\cdot, a_{max}, \cdot) e^{a_{max}/\tau}, \; Q(0,\cdot) = Q_0. \tag{2.1}$$

Here, Q_0 stands for the initial swimmer density and ξ/τ is the fraction of swimmer population to produce swarmers. In the case of Model B, the evolution equation is

$$\frac{\partial Q}{\partial t} = \frac{1-\xi}{\tau} Q + \int_0^t \rho(\cdot, a, \cdot) \, \frac{e^{a/\tau}}{\overline{a}} da, \; Q(0,\cdot) = Q_0. \tag{2.2}$$

We turn to the evolution of the swarmer density ρ. Its evolution is linked with ageing and the dedifferentiation process, but also to swarm. This last phenomenon is modelled by a nonlinear diffusion term with memory. The evolution equation for ρ is then

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} = \nabla \cdot \big[D(\mathcal{M}, Q, P) \nabla \rho \big], \tag{2.3}$$

in the case of Model A; and, in the case of Model B, it is

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial a} = -\frac{1}{\overline{a}} \rho + \nabla \cdot \big[D(\mathcal{M}, Q, P) \nabla \rho \big]. \tag{2.4}$$

Both of those equations are equipped with the following initial and boundary conditions:

$$\text{(a): } \rho(\cdot, 0, \cdot) = \frac{\xi}{\tau} Q, \; \text{(b): } \rho(0, \cdot, \cdot) = 0, \quad \text{and (c): } \frac{\partial \rho}{\partial \overrightarrow{\nu}}\Big|_{\partial\Omega} = 0. \tag{2.5}$$

The first of those three conditions means the fraction ξ/τ of swimmers undergoing the differentiation process produces swarmers of age 0. The initial condition on ρ means that, at the beginning of the process, there is no swarmer. The boundary condition means that a swarmer cannot leave the domain Ω. In the diffusion term $\nabla \cdot \big[D(\mathcal{M}, Q, P) \nabla \rho \big]$ appearing in (2.3) and (2.4), and modelling the swarm, the diffusion factor $D(\mathcal{M}, Q, P)$ depends on the present value of Q and P but also on the history of P.

The term $\mathcal{M}$ then keeps in memory information concerning the history of the swarmer density. Esipov and Shapiro [8] defines $\mathcal{M}$ as being set to 1 in a given point x if the value of P in x reaches a threshold P_{max}. Then it remains at the value 1 until the value of P in x reaches another value $P_{min} < P_{max}$. Then they suggest taking

$$D(\mathcal{M}, Q, P) = \overline{D_0}\,\mathcal{M}\,\gamma\Big(\frac{P}{P_{max}}\Big)\ \exp\Big(\frac{-Q}{Q_{sat}}\Big), \tag{2.6}$$

for given values of $\overline{D_0}$ and Q_{sat} and with, H being such that $H(p) = 0$ if $p < 0$ and $H(p) = 1$ if $p > 0$,

$$\gamma(p) = (p - \frac{P_{min}}{P_{max}})H(p - \frac{P_{min}}{P_{max}}) \text{ or } \gamma(p) = p - \frac{P_{min}}{P_{max}} \text{ or } \gamma(p) = p^2 \text{ or } \gamma(p) = 1. \tag{2.7}$$

We also mention that Medvedev, Kapper and Koppel [23] studied this model taking, for a given value of k,

$$D(\mathcal{M}, Q, P) = D(Q, P) = \frac{\overline{D_0}P}{P + kQ}. \tag{2.8}$$

Now, we explain in what sense the model (1.1)-(1.6) is a generalization and a regularization of the Esipov and Shapiro [8] model.

First, because of the initial condition (2.5.b), it is an easy game to see that, at least formally, the solution ρ to (2.3) or (2.4) satisfies

$$\rho(t, a, x) = 0 \text{ when } a > t. \tag{2.9}$$

Hence we can replace (2.2) by

$$\frac{\partial Q}{\partial t} = \frac{1 - \xi}{\tau}Q + \int_0^{+\infty} \rho(\cdot, a, \cdot)\,\frac{e^{a/\tau}}{\overline{a}}da,\ Q(0, \cdot) = Q_0. \tag{2.10}$$

Making this observation allows us to take under consideration, with no loss of consistency, initial data $\rho(0, \cdot, \cdot)$ that are not 0 coming to the more general initial and boundary conditions (1.3)-(1.6).

Secondly, equations (2.1) and (2.10) are particular cases of the general equation (1.1) with assumption (1.8). The case of equation (2.1) is recovered by setting $\mu = 0$ and $A = a_{max}$ and the case of (2.10) is recovered by setting $A = +\infty$ and $\mu = 1/\overline{a}$. We also see that (1.2) with $d = 0$ is a general framework inside which (2.3) and (2.4) may enter directly.

Concerning the fraction ξ of swimmers to produce swarmers, it seems to depend on experimental conditions and to be 0 when the swimmer density is high. It seems then to be reasonable to say that ξ satisfies (1.9).

The first regularization effect we consider in our model (1.1)-(1.6) consists in adding $d > 0$ in (1.2). This may be justified by experimental arguments saying that swarmers always experience a small random motion even before and after swarming.

The second regularization effect which constitutes the most visible modification of the model concerns the memory term $\mathcal{M}$. In order to explain the way to go from the definition of $\mathcal{M}$ by Esipov and Shapiro [8] to the definition (1.12), we first notice that the term $\mathcal{M}$ of Esipov and Shapiro could be defined formally, in any x, as the solution to the following equation:

$$\begin{aligned}&\frac{\partial \mathcal{M}}{\partial t}(\cdot,x) = (1-\mathcal{M}(\cdot,x))\delta_{\{t/P(t,x)=P_{max}\}} - \mathcal{M}(\cdot,x)\delta_{\{t/P(t,x)=P_{min}\}},\\ &\mathcal{M}(0,x)=0,\end{aligned} \tag{2.11}$$

where $\delta_{\{t/P(t,x)=P_{max}\}}$ stands for the Dirac measure in the instant t where $P(t,x) = P_{max}$. Of course this equation has no real mathematical meaning. But formally, if $\mathcal{M} = 0$ and if the value P_{max} is reached at a given time $\tilde{t}$, the solution of (2.11) experiences a jump $+1$. When $P = P_{max}$, and $\mathcal{M} = 1$, nothing happens and this is what is needed. In the same way, if at a given time $\tilde{t}$, $P = P_{min}$, nothing happens if $\mathcal{M} = 0$ and $\mathcal{M}$ experiences a jump -1 if $\mathcal{M} = 1$. Now, it is easy to see that the right-hand side of (1.12) is nothing but a regularization of the right-hand value of (2.11). This is the reason why we make this choice to define $\mathcal{M}$.

3. A priori estimates

The key point to get the existence result is a collection of a priori estimates satisfied by (Q,ρ). They are mathematical translations of biological properties. In order to set those estimates comfortably, we assume that the assumptions and the conclusions of Theorem 1.1 are satisfied.

3.1. L^1 and L^2 estimates.

For $p > 0$, we denote

$$\|\rho(t)\|_p = \Big(\int_0^A \int_\Omega |\rho(t,a,x)|^p\, e^{a/\tau} dx da\Big)^{\frac{1}{p}}, \quad \|Q(t)\|_p = \Big(\int_\Omega |Q(t,x)|^p\, dx\Big)^{\frac{1}{p}}, \tag{3.1}$$

and we have the following estimates saying that the total biomass grows exponentially with a growth rate τ.

Lemma 3.1. *If the assumptions of Theorem* 1.1 *are valid, and if the solution* (Q,ρ) *given by this same theorem exists, then it satisfies*

$$\|\rho(t)\|_1 + \|Q(t)\|_1 = \big(\|\rho_0\|_1 + \|Q_0\|_1\big)e^{t/\tau}. \tag{3.2}$$

Proof. Multiplying equation (1.2) by $e^{a/\tau}$, integrating then with respect to x and a yields:

$$\frac{d\|\rho\|_1}{dt} + \lim_{a\to A}\Big(\int_\Omega \rho\, e^{a/\tau} dx\Big) - \int_\Omega \rho(\cdot,0,\cdot)\, dx = \frac{1}{\tau}\|\rho\|_1 - \int_0^A \int_\Omega \mu\rho\, e^{a/\tau} dxda, \tag{3.3}$$

which also reads

$$\frac{d\|\rho\|_1}{dt} + \chi(A)e^{A/\tau} \int_\Omega \rho(\cdot,A,\cdot)\, dx - \int_\Omega \frac{\xi}{\tau} Q\, dx = \frac{1}{\tau}\|\rho\|_1 - \int_0^A \int_\Omega \mu\rho\, e^{a/\tau} dxda. \tag{3.4}$$

Integrating now (1.1) with respect to x, we obtain

$$\frac{d\|Q\|_1}{dt} = \int_\Omega \frac{1-\xi}{\tau} Q\, dx + \int_0^A \int_\Omega \mu\rho\, e^{a/\tau} dxda + \chi(A)e^{A/\tau} \int_\Omega \rho(\cdot,A,\cdot)\, dx. \tag{3.5}$$

Summing up (3.4) and (3.5) finally gives

$$\frac{d\big(\|\rho\|_1 + \|Q\|_1\big)}{dt} = \frac{1}{\tau}\big(\|\rho\|_1 + \|Q\|_1\big), \tag{3.6}$$

proving the lemma. □

The second lemma concerns the L^2−norms of ρ and Q. It mathematically translates that the biomass cannot be so gathered that a null area set contains a positive biomass quantity.

Lemma 3.2. *If the assumptions of Theorem* 1.1 *are realized and if the solution* (Q,ρ) *exists, then for any* $t \in [0,T)$*, it satisfies*

$$\|\rho(t)\|_2 + \|Q(t)\|_2 \le c\big(\|\rho_0\|_2 + \|Q_0\|_2\big), \tag{3.7}$$

for a constant c *(*c *only depends on* A*,* τ*,* T *and* $\tilde\mu = \sup\{\mu(t,a,x) : t \in [0,T), a \in [0,A), x \in \Omega\}$*).*

Proof. First, integrating (1.2) with respect to x, we get:

$$\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\Big)\Big(\int_\Omega \rho\, dx\Big) = -\int_\Omega \mu\rho\, dx \le 0. \tag{3.8}$$

Doing the same, after multiplying (1.2) by ρ, gives:

$$\frac{1}{2}\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\Big)\Big(\int_\Omega \rho^2\, dx\Big) = -\int_\Omega \mu\rho^2\, dx - \int_\Omega (D+d)|\nabla\rho|^2\, dx \le 0. \tag{3.9}$$

Defining, for a fixed α and for $p = 1$ or 2,

$$r_p : s \mapsto \int_\Omega \rho^p(s, \alpha+s, x)\, dx, \tag{3.10}$$

we have

$$r_p'(s) = \Big(\Big(\frac{\partial}{\partial t} + \frac{\partial}{\partial a}\Big)\Big(\int_\Omega \rho^p \, dx\Big)\Big)(s, \alpha + s) \leq 0. \tag{3.11}$$

Now, since for fixed t and a and with $\alpha = a - t$ we have

$$\int_\Omega \rho^p(t, a, x) \, dx = r_p(t),$$

if $t < a$ the relation $r_p(t) \leq r_p(0) = \int_\Omega \rho^p(0, a-t, x) \, dx$ reads

$$\int_\Omega \rho(t, a, x) \, dx \leq \int_\Omega \rho_0(a - t, x) \, dx, \ \int_\Omega \rho^2(t, a, x) \, dx \leq \int_\Omega \rho_0^2(a - t, x) \, dx. \tag{3.12}$$

In the case when $t > a$, the relation $r_p(t) \leq r_p(t - a) = \int_\Omega \rho^p(t - a, 0, x) \, dx$ gives

$$\begin{aligned} \int_\Omega \rho(t, a, x) \, dx &\leq \int_\Omega \frac{\xi}{\tau} Q(t - a, x) \, dxda, \\ \int_\Omega \rho^2(t, a, x) \, dx &\leq \int_\Omega \frac{\xi^2}{\tau^2} Q^2(t - a, x) \, dxda. \end{aligned} \tag{3.13}$$

Secondly, we multiply equation (1.2) by $e^{a/\tau}\rho$ and we integrate in x and a. Since

$$\int_0^A \frac{\partial \rho}{\partial a} \rho \, e^{a/\tau} da = -\int_0^A \frac{\partial \rho}{\partial a} \rho \, e^{a/\tau} da - \frac{1}{\tau} \int_0^A \rho^2 \, e^{a/\tau} da + \Big[\rho^2 \, e^{a/\tau}\Big]_{a=0}^{a=A}, \tag{3.14}$$

and since

$$\int_\Omega \rho^2(t, 0, x) \, dx = \int_\Omega \frac{\xi^2}{\tau^2} Q^2 \, dx, \tag{3.15}$$

we obtain

$$\begin{aligned} \frac{d\|\rho\|_2^2}{dt} + \chi(A) e^{A/\tau} \int_\Omega \rho^2(\cdot, A, \cdot) \, dx + 2 \int_0^A \int_\Omega (D + d) |\nabla \rho|^2 \, e^{a/\tau} dxda \\ + 2 \int_0^A \int_\Omega \mu \rho^2 \, e^{a/\tau} dxda = \int_\Omega \frac{\xi^2}{\tau^2} Q^2 \, dx + \frac{1}{\tau} \|\rho\|_2^2. \end{aligned} \tag{3.16}$$

As the second, third and fourth terms of the left-hand side of equation (3.16) are nonnegative, we may deduce

$$\frac{d\|\rho\|_2^2}{dt} \leq \frac{1}{\tau^2} \|Q\|_2^2 + \frac{1}{\tau} \|\rho\|_2^2. \tag{3.17}$$

In the same way, multiplying now (1.1) by Q and integrating yields

$$\frac{d\|Q\|_2^2}{dt} = 2 \int_\Omega \frac{1 - \xi}{\tau} Q^2 \, dx + 2 \int_0^A \int_\Omega \mu \rho Q \, e^{a/\tau} dxda$$

$$+ 2\chi(A)e^{A/\tau}\int_\Omega \rho(\cdot,A,\cdot)Q\,dx. \quad (3.18)$$

Concerning the second term of the right-hand side of the last equality, using Young's inequality and formula (3.12) and (3.13), we get

$$\begin{aligned}\int_0^A\int_\Omega \mu\rho Q\,e^{a/\tau}dxda &\le \int_0^A \Big(\int_\Omega \mu^2\rho^2\,dx\Big)^{1/2}\Big(\int_\Omega Q^2\,dx\Big)^{1/2}e^{a/\tau}da\\ &\le \tilde\mu\Big[\int_0^t\Big(\int_\Omega \frac{\xi^2}{\tau^2}Q^2(t-a,\cdot)\,dx\Big)^{1/2}e^{a/\tau}da\\ &\qquad + H(A-t)e^{t/\tau}\int_t^A\Big(\int_\Omega \rho_0^2(a-t,\cdot)e^{2(a-t)/\tau}\,dx\Big)^{1/2}da\Big]\|Q\|_2\\ &\le \tilde\mu\max(e^{T/\tau},\frac{e^{T/\tau}}{\tau^2})\Big[\int_0^t\Big(\int_\Omega Q^2(t-a,\cdot)\,dx\Big)^{1/2}da\\ &\qquad + H(A-t)\int_t^A\Big(\int_\Omega \rho_0^2(a-t,\cdot)e^{2(a-t)/\tau}\,dx\Big)^{1/2}da\Big]\|Q\|_2,\end{aligned} \quad (3.19)$$

where $H(a)=0$ if $a<0$ and $H(a)=1$ if $a\in[0,+\infty]$ and where $\tilde\mu=\sup\{\mu(t,a,x): t\in[0,T), a\in[0,A), x\in\Omega\}$.

The third term of the right-hand side of (3.18) may also be estimated:

$$\chi(A)e^{A/\tau}\int_\Omega \rho(\cdot,A,\cdot)Q\,dx \le \chi(A)e^{A/\tau}\Big(\int_\Omega \rho^2(\cdot,A,\cdot)\,dx\Big)^{1/2}\Big(\int_\Omega Q^2\,dx\Big)^{1/2}. \quad (3.20)$$

Hence, applying again (3.12) and (3.13),

$$\begin{aligned}&\chi(A)e^{A/\tau}\int_\Omega \rho(\cdot,A,\cdot)Q\,dx\\ &\qquad\le \chi(A)e^{A/\tau}\Big(\int_\Omega Q^2(t-A,\cdot)\,dx\Big)^{1/2}\|Q\|_2,\ \text{if } t>A, \quad (3.21)\\ &\qquad\le \chi(A)e^{2t/\tau}\Big(\int_\Omega \rho_0^2(A-t,\cdot)e^{2(A-t)/\tau}\,dx\Big)^{1/2}\|Q\|_2,\ \text{if } t<A.\end{aligned}$$

Using (3.19), (3.21) and (1.15) in (3.18), for four nonnegative constant c_1, c_2, c_3 and c_4 we have

$$\begin{aligned}\frac{d\|Q\|_2^2}{dt} \le \Big(c_1\|Q\|_2 &+ \int_0^t\Big(\int_\Omega Q^2(t-a,\cdot)\,dx\Big)^{1/2}da + c_3\|\rho_0\|_2\\ &+ c_4\Big(H(t-A)\int_\Omega Q^2(t-A,\cdot)\,dx\Big)^{1/2}\Big)\|Q\|_2.\end{aligned} \quad (3.22)$$

Setting

$$F(t) = \sup_{s\in[0,T)} \|Q(s)\|_2^2 + \sup_{s\in[0,T)} \|\rho(s)\|_2^2, \tag{3.23}$$

we have

$$F'(t) \le \max(0, \frac{d\|Q(t)\|_2^2}{dt}) + \max(0, \frac{d\|\rho(t)\|_2^2}{dt}), \tag{3.24}$$

and from (3.17) and (3.22) we deduce

$$F'(t) \le \big((c_1+c_2T+c_3+c_4)\sqrt{F(t)}\big)\sqrt{F(t)} + \big(\frac{1}{\tau^2}+\frac{1}{\tau}\big)F(t) \le c_5F(t), \tag{3.25}$$

for a constant c_5, from which we deduce that

$$F(t) \le e^{c_5T}F(0), \tag{3.26}$$

and, as a consequence, that for a constant c_6,

$$\|Q(t)\|_2^2 + \|\rho(t)\|_2^2 \le c_6(\|Q_0\|_2^2 + \|\rho_0\|_2^2). \tag{3.27}$$

Finally we get (3.7) as a consequence of (3.27); this ends the proof. □

As a consequence of Lemma 3.2 we have the following corollary whose biological meaning is: Biomass cannot be created from nothing.

Corollary 3.3. *If the assumptions of Theorem* 1.1 *are realized and if* $\rho_0 = 0$ *and* $Q_0 = 0$ *then the solution* (Q, ρ) *given by the theorem satisfies*

$$\rho = 0 \text{ and } Q = 0. \tag{3.28}$$

In order to establish the previous estimates, we have assumed that $Q \ge 0$ and $\rho \ge 0$. We can show that this is a consequence of the nonnegativity of Q_0 and ρ_0.

Lemma 3.4. *If the assumptions of Theorem* 1.1 *are true and if there exists a solution* (Q, ρ) *to system* (1.1)–(1.6), *then* $Q \ge 0$ *and* $\rho \ge 0$.

The proof of this lemma is close to the one of Lemma 3.2.

Proof. We define

$$\rho^- = \min(\rho, 0),\ Q^- = \min(Q, 0). \tag{3.29}$$

Of course,

$$\rho^-(0,\cdot,\cdot) = 0,\ Q^-(0,\cdot) = 0,\ \rho^-(\cdot,0,\cdot) = \frac{\xi}{\tau}Q^-. \tag{3.30}$$

Now, multiplying (3.30) by $e^{a/\tau}\rho^-$ and integrating, since

$$\frac{\partial\rho}{\partial t}\rho^- = \frac{\partial\rho^-}{\partial t}\rho^-, \tag{3.31}$$

proceeding as while establishing (3.17), we get

$$\frac{d\|\rho^-\|_2^2}{dt} \leq \frac{1}{\tau^2}\|Q^-\|_2^2 + \frac{1}{\tau}\|\rho^-\|_2^2. \tag{3.32}$$

Multiplying now (1.1) by Q^- and integrating gives

$$\begin{aligned}\frac{d\|Q^-\|_2^2}{dt} &= 2\int_\Omega \frac{1-\xi}{\tau} Q^{-2}\,dx \\ &\quad + 2\int_0^A \int_\Omega \mu\rho Q^- e^{a/\tau}\,dxda + 2\chi(A)e^{A/\tau}\int_\Omega \rho(\cdot,A,\cdot)Q^-\,dx \\ &\leq 2\int_\Omega \frac{1-\xi}{\tau} Q^{-2}\,dx \\ &\quad + 2\int_0^A \int_\Omega \mu\rho^- Q^- e^{a/\tau}\,dxda + 2\chi(A)e^{A/\tau}\int_\Omega \rho^-(\cdot,A,\cdot)Q^-\,dx.\end{aligned} \tag{3.33}$$

Since, as we had (3.12) and (3.13), we have here:

$$\begin{aligned}&\int_\Omega \rho^{-2}\,dx \leq \int_\Omega \rho_0^{-2}\,dx \text{ if } t < a \text{ and} \\ &\int_\Omega \rho^{-2}\,dx \leq \int_\Omega \frac{\xi^2}{\tau^2} Q^{-2}(t-a,\cdot)\,dx \text{ if } t > a,\end{aligned} \tag{3.34}$$

we can finish the proof as in the proof of Lemma 3.2 and get

$$\|Q^-(t)\| + \|\rho^-(t)\| \leq c'\big(\|Q_0^-\| + \|\rho_0^-\|\big) = 0, \tag{3.35}$$

giving the lemma. □

3.2. **Estimates on the derivatives.** As a byproduct of the proof of Lemma 3.2, we can deduce from (3.16) the following result insuring a first control on the regularity of ρ.

Corollary 3.5. *If the assumptions and the conclusion of Theorem* 1.1 *are valid then, for any* $0 \leq s \leq T$,

$$\int_0^s \|\nabla\rho\|_2^2\,dt = \int_0^s \int_0^A \int_\Omega |\nabla\rho|^2 e^{a/\tau}\,dxdadt \leq \frac{c}{d}, \tag{3.36}$$

for a constant c (depending only on A, τ, T, $\tilde{\mu}$, $\|\rho_0\|_2$ *and* $\|Q_0\|_2$*).*

Proof. Integrating (3.16) from 0 to s yields:

$$\chi(A)e^{A/\tau}\int_0^s \int_\Omega \rho^2(\cdot,A,\cdot)\,dxdt + 2\int_0^s \int_0^A \int_\Omega (D+d)|\nabla\rho|^2\, e^{a/\tau} dxdadt$$

$$+\,2\int_0^s\int_0^A\int_\Omega \mu\rho^2\, e^{a/\tau} dxdadt$$
$$=\int_0^s\int_\Omega \frac{\xi^2}{\tau^2}Q^2\,dxdt+\frac{1}{\tau}\int_0^s \|\rho\|_2^2\,dt+\|\rho(0)\|_2^2-\|\rho(s)\|_2^2. \quad (3.37)$$

Using the previous estimate concerning ρ and Q, we may deduce

$$d\int_0^s\int_0^A\int_\Omega |\nabla\rho|^2\, e^{a/\tau} dxdadt \le c, \quad (3.38)$$

and (3.36) follows. □

Because of the form of the nonlinearity in (1.1)-(1.6), we need a supplementary estimate concerning

$$\int_0^T \|\Delta\rho\|_2^2\,dt=\int_0^T\int_0^A\int_\Omega |\Delta\rho|_2^2\,dxdadt. \quad (3.39)$$

This estimate is a consequence of an estimate on $\|\nabla\rho\|_4$ and on

$$\|\rho\|_\infty=\sup\big\{|\rho(t,a,x)| : t\in[0,T), a\in[0,A), x\in\Omega\big\}, \quad (3.40)$$
$$\|Q\|_\infty=\sup\big\{|Q(t,x)| : t\in[0,T), x\in\Omega\big\}, \quad (3.41)$$

that we now set.

Lemma 3.6. *The solution (Q,ρ) given by Theorem 1.1 satisfies*

$$\|\rho\|_\infty+\|Q\|_\infty\le k, \quad (3.42)$$

where k is a constant depending only on T, A, $\sup_{t\in[0,T)}\|\rho(t)\|_2$, and $\sup_{t\in[0,T)}\|Q(t)\|_2$ (which are estimated by Lemma 3.2)

Proof. As we already see that $\|\rho(t)\|_2$ is bounded, using a method similar to Ladyzenskaja, Solonnikov and Ural'ceva [16] (paragraph III-8), we deduce that

$$\big|\chi(A)\rho(\cdot,A,\cdot)e^{A/\tau}\big|\le k_1, \quad (3.43)$$

where the constant k_1 only depends only on A and $\sup_{t\in[0,T)}\|\rho(t)\|_2$. Then defining

$$\tilde P=\int_0^A \rho e^{a/\tau}\,da, \quad (3.44)$$

we deduce from (1.2) that $\tilde P$ is a solution to the following parabolic equation

$$\frac{\partial\tilde P}{\partial t}+\chi(A)\rho(\cdot,A,\cdot)e^{A/\tau}=-\int_0^A \mu\rho e^{a/\tau}\,da+\nabla\cdot\big((D+d)\nabla\tilde P\big)+\frac{\xi}{\tau}Q, \quad (3.45)$$

from which we get that

$$\|\tilde{P}\|_\infty = \sup\big\{|\tilde{P}(t,x)| : t \in [0,T), x \in \Omega\big\} \le k_2, \tag{3.46}$$

where k_2 only depends on $\sup_{t\in[0,T)} \|\rho(t)\|_2$, on $\sup_{t\in[0,T)} \|Q(t)\|_2$ and on $\sup_{x\in\Omega} |\tilde{P}_0| = \sup_{x\in\Omega} |\int_0^A \rho_0 e^{a/\tau}\, da|$ which is finite by assumption.

Then, (3.46) and (3.43) give that $\|Q\|_\infty$ is finite, and as a consequence, (1.2) and (1.3) finally give the bound on $\|\rho\|_\infty$, ending the proof. □

Lemma 3.7. *The solution (Q, ρ) given by Theorem 1.1 satisfies*

$$\|\nabla Q\|_4 + \|\nabla\rho\|_4 = \Big(\int_\Omega |\nabla Q|^4\, dx\Big)^{1/4} + \Big(\int_0^A \int_\Omega |\nabla\rho|^4\, e^{a/\tau} dx da\Big)^{1/4} \le \frac{C}{d}, \tag{3.47}$$

for a constant C (C does not depend on d).

Proof. Multiplying equation (1.2) by $-\nabla\cdot\big(|\nabla\rho|^2\nabla\rho\big)e^{a/\tau}$, and integrating in a and x gives:

$$\begin{aligned}
-\int_0^A\int_\Omega \frac{\partial\rho}{\partial t}\,\nabla\cdot\big(|\nabla\rho|^2\nabla\rho\big)\, e^{a/\tau}dxda - \int_0^A\int_\Omega \frac{\partial\rho}{\partial a}\,\nabla\cdot\big(|\nabla\rho|^2\nabla\rho\big)\, e^{a/\tau}dxda \\
= \int_0^A\int_\Omega \mu\rho\,\nabla\cdot\big(|\nabla\rho|^2\nabla\rho\big)\, e^{a/\tau}dxda \\
-\int_0^A\int_\Omega \nabla\cdot\big[\big(D+d\big)\nabla\rho\big]\,\nabla\cdot\big(|\nabla\rho|^2\nabla\rho\big)\, e^{a/\tau}dxda.
\end{aligned} \tag{3.48}$$

Making a double integration by parts, and following a straightforward computation procedure, we get

$$\begin{aligned}
&\int_0^A\int_\Omega \nabla\cdot\big[\big(D+d\big)\nabla\rho\big]\,\nabla\cdot\big(|\nabla\rho|^2\nabla\rho\big)\, e^{a/\tau}dxda \\
&= \int_0^A\int_\Omega \big(D+d\big)\big(|\nabla\rho|^2|\nabla^2\rho|^2 + 2\mathbf{H}(\nabla\rho,\nabla^2\rho)\big)\, e^{a/\tau}dxda + \mathbf{E},
\end{aligned} \tag{3.49}$$

where

$$|\nabla^2\rho|^2 = \big(\frac{\partial^2\rho}{\partial x_1^2}\big)^2 + 2\big(\frac{\partial^2\rho}{\partial x_1\partial x_2}\big)^2 + \big(\frac{\partial^2\rho}{\partial x_2^2}\big)^2, \tag{3.50}$$

$$\mathbf{H}(\nabla\rho,\nabla^2\rho) = \Big(\frac{\partial\rho}{\partial x_1}\frac{\partial^2\rho}{\partial x_1^2} + \frac{\partial\rho}{\partial x_2}\frac{\partial^2\rho}{\partial x_1\partial x_2}\Big)^2 + \Big(\frac{\partial\rho}{\partial x_1}\frac{\partial^2\rho}{\partial x_1\partial x_2} + \frac{\partial\rho}{\partial x_2}\frac{\partial^2\rho}{\partial x_2^2}\Big)^2, \tag{3.51}$$

and

$$\mathbf{E} = \int_0^A \int_\Omega \sum_{i,j=1}^{2} \Big(\frac{\partial D(\mathcal{M},Q,P)}{\partial x_i}\frac{\partial \rho}{\partial x_j}\Big) \times \Big(|\nabla\rho|^2 \frac{\partial^2\rho}{\partial x_i \partial x_j} + 2\big(\frac{\partial\rho}{\partial x_1}\frac{\partial^2\rho}{\partial x_1\partial x_j} + \frac{\partial\rho}{\partial x_2}\frac{\partial^2\rho}{\partial x_j \partial x_2}\big)\frac{\partial\rho}{\partial x_i}\Big)\, e^{a/\tau} dxda. \quad (3.52)$$

Since

$$\frac{\partial D(\mathcal{M},Q,P)}{\partial x_i} = \frac{\partial D}{\partial \mathcal{M}}\frac{\partial \mathcal{M}}{\partial x_i} + \frac{\partial D}{\partial Q}\frac{\partial Q}{\partial x_i} + \frac{\partial D}{\partial P}\frac{\partial P}{\partial x_i}, \quad (3.53)$$

in view of the regularity of D, of equation (1.12) that gives a control on $\partial\mathcal{M}/\partial x_i$ in terms of $\partial P/\partial x_i$, we get for a constant C_1

$$\begin{aligned} |\mathbf{E}| &\le C_1 \int_0^A \int_\Omega \big(|\nabla P| + |\nabla Q|\big)|\nabla\rho|^3\, |\nabla^2\rho|\, e^{a/\tau} dxda \\ &\le \frac{d}{4}\int_0^A \int_\Omega |\nabla\rho|^2\, |\nabla^2\rho|^2\, e^{a/\tau} dxda \\ &\qquad + \frac{C_1^2}{d}\int_0^A \int_\Omega \big(|\nabla P| + |\nabla Q|\big)^2 |\nabla\rho|^2\, e^{a/\tau} dxda; \end{aligned} \quad (3.54)$$

in order to get the last expression in (3.54), we used $UV \le \frac{d}{4}U^2 + \frac{1}{d}V^2$ with $V = C_1\big(|\nabla P| + |\nabla Q|\big)|\nabla\rho|$.

Concerning the other terms of (3.48), since

$$\frac{\partial|\nabla\rho|^4}{\partial t} = 2\frac{\partial|\nabla\rho|^2}{\partial t}|\nabla\rho|^2 = 4\frac{\partial\nabla\rho}{\partial t}\cdot\nabla\rho\,|\nabla\rho|^2,$$

making an integration by parts, we get

$$-\int_0^A \int_\Omega \frac{\partial\rho}{\partial t}\, \nabla\cdot\big(|\nabla\rho|^2\nabla\rho\big)\, e^{a/\tau} dxda = \frac{1}{4}\frac{d\|\nabla\rho\|_4^4}{dt}. \quad (3.55)$$

In a similar way,

$$\begin{aligned} -\int_0^A \int_\Omega \frac{\partial\rho}{\partial a}\, \nabla\cdot\big(|\nabla\rho|^2\nabla\rho\big)\, e^{a/\tau} dxda &= \frac{1}{4}\int_0^A\int_\Omega \frac{\partial(|\nabla\rho|^4\, e^{a/\tau})}{\partial a} dxda - \frac{1}{4\tau}\|\nabla\rho\|_4^4 \\ &= \frac{1}{4}\chi(A)\int_\Omega |\nabla\rho(\cdot,A,\cdot)|^4\, e^{A/\tau} dx \\ &\quad - \frac{1}{4\tau^4}\int_\Omega \big(\xi^4 + (\frac{\partial\xi}{\partial Q}Q)^4\big)|\nabla Q|^4\, dx - \frac{1}{4\tau}\|\nabla\rho\|_4^4, \end{aligned} \quad (3.56)$$

and

$$\Big| \int_0^A \int_\Omega \mu\rho\, \nabla\cdot\big(|\nabla\rho|^2\nabla\rho\big)\, e^{a/\tau} dx da\Big| \tag{3.57}$$
$$= \Big| -\int_0^A \int_\Omega \mu|\nabla\rho|^4\, e^{a/\tau} dx da - \int_0^A \int_\Omega \nabla\mu\,\rho\,|\nabla\rho|^2\, e^{a/\tau} dx da\Big|$$
$$\leq C_2\big(\|\nabla\rho\|_4^4 + 1\big),$$

using the regularity of μ and the estimate on $\sup(\rho)$ given by Lemma 3.6.

The regularity of ξ and the estimate on $\sup(Q)$ give $|\xi^4 + (\frac{\partial\xi}{\partial Q}Q)^4| \leq C_3$ for a constant C_3. Hence (3.48) yields

$$\frac{1}{4}\frac{d\|\nabla\rho\|_4^4}{dt} + \frac{1}{4}\chi(A)\int_\Omega |\nabla\rho(\cdot,A,\cdot)|^4\, e^{A/\tau} dx$$
$$+ \int_0^A \int_\Omega \big(D+d\big)\big(|\nabla\rho|^2|\nabla^2\rho|^2\big)\, e^{a/\tau} dx da$$
$$+ 2\int_0^A \int_\Omega \big(D+d\big)\, \mathbf{H}(\nabla\rho,\nabla^2\rho)\, e^{a/\tau} dx$$
$$\leq \frac{C_3}{4\tau^4}\int_\Omega |\nabla Q|^4\, dx + \frac{1}{4\tau}\|\nabla\rho\|_4^4 + C_2\big(\|\nabla\rho\|_4^4 + 1\big)$$
$$+ \frac{d}{4}\int_0^A \int_\Omega |\nabla\rho|^2\, |\nabla^2\rho|^2\, e^{a/\tau} dx da$$
$$+ \frac{C_1^2}{d}\Big(\int_0^A \int_\Omega |\nabla\rho|^4\, e^{a/\tau} dx da + C_4\big(\|\nabla Q\|_4^4 + \|\nabla\rho\|_4^4\big)\Big), \tag{3.58}$$

and passing the fourth term of the right-hand side to the left-hand side we can deduce

$$\frac{d\|\nabla\rho\|_4^4}{dt} \leq \frac{C_5}{d}\big(\|\nabla\rho\|_4^4 + \|\nabla Q\|_4^4 + 1\big), \tag{3.59}$$

for a constant C_5.

Multiplying equation (1.2) by $-\nabla\cdot\big(|\nabla\rho|^2\nabla\rho\big)e^{4a/\tau}$, and making the same operations as previously, we obtain an inequality which is (3.58) with $e^{a/\tau}$ replaced by $e^{4a/\tau}$ and $\|\nabla Q\|_4^4$ replaced by $\int_0^A \int_\Omega |\nabla\rho|^4\, e^{4a/\tau} dx da$. From this, we can deduce

$$\frac{d\big(\int_0^A \int_\Omega |\nabla\rho|^4\, e^{4a/\tau} dx da\big)}{dt} + \chi(A)\int_\Omega |\nabla\rho(\cdot,A,\cdot)|^4\, e^{4A/\tau} dx$$
$$\leq \frac{C_6}{d}\Big(\int_0^A \int_\Omega |\nabla\rho|^4\, e^{4a/\tau} dx da + \|\nabla Q\|_4^4 + 1\Big). \tag{3.60}$$

On the other hand, computing the gradient of (1.1), and multiplying by $\nabla Q|\nabla Q|^2$ yields

$$\begin{aligned}\frac{1}{4}\frac{d\|\nabla Q\|_4^4}{dt} = \frac{1}{\tau}\int_\Omega \frac{\partial \xi}{\partial Q} Q|\nabla Q|^4\,dx + \int_\Omega \frac{1-\xi}{\tau}|\nabla Q|^4\,dx \\ + \int_\Omega \Big(\int_0^A \nabla\mu\,\rho\,e^{a/\tau}da\Big)\cdot\nabla Q|\nabla Q|^2\,dx \\ + \int_\Omega \Big(\int_0^A \mu\nabla\rho\,e^{a/\tau}da\Big)\cdot\nabla Q|\nabla Q|^2\,dx \\ + \chi(A)\int_\Omega \Big(\nabla\rho(\cdot,A,\cdot)e^{A/\tau}\Big)\cdot\nabla Q|\nabla Q|^2\,dx. \quad (3.61)\end{aligned}$$

Because of the regularity of ξ and μ and Lemma 3.6, applying Young's inequality,

$$\begin{aligned}&\int_\Omega \Big(\int_0^A \mu\nabla\rho\,e^{a/\tau}da\Big)\cdot\nabla Q|\nabla Q|^2\,dx \\ &\quad\le \Big(\int_\Omega\Big(\int_0^A \mu|\nabla\rho|\,e^{a/\tau}da\Big)^4 dx\Big)^{1/4}\Big(\int_\Omega\Big(|\nabla Q|^3\Big)^{4/3}dx\Big)^{3/4} \\ &\quad\le \tilde\mu\Big(\int_0^A\int_\Omega|\nabla\rho|^4\,e^{4a/\tau}dxda\Big)^{1/4}\|\nabla Q\|_4^3, \quad (3.62)\end{aligned}$$

and

$$\begin{aligned}&\chi(A)\int_\Omega\big(\nabla\rho(\cdot,A,\cdot)\,e^{A/\tau}\big)\cdot\nabla Q|\nabla Q|^2\,dx \\ &\quad\le \chi(A)\Big(\int_\Omega|\nabla\rho(\cdot,A,\cdot)|^4\,e^{4A/\tau}\,dx\Big)^{1/4}\|\nabla Q\|_4^3 \\ &\quad\le \chi(A)\frac{C_6'}{d}\Big(\Big(\int_0^A\int_\Omega|\nabla\rho|^4\,e^{4a/\tau}dxda\Big)^{1/4}+\|\nabla Q\|_4+1\Big)\|\nabla Q\|_4^3, \quad (3.63)\end{aligned}$$

we deduce from (3.61)

$$\begin{aligned}\frac{d\|\nabla Q\|_4^4}{dt} &\le C_4(\|\nabla Q\|_4^4+1) \\ &\quad+\frac{C_5}{d}\Big(\Big(\int_0^A\int_\Omega|\nabla\rho|^4\,e^{4a/\tau}dxda\Big)^{1/4}+\|\nabla Q\|_4+1\Big)\|\nabla Q\|_4^3. \quad (3.64)\end{aligned}$$

Inequalities (3.59), (3.60), (3.64) and the assumptions on Q_0 and ρ_0 give

$$\|\nabla Q\|_4+\|\nabla\rho\|_4+\Big(\int_0^A\int_\Omega|\nabla\rho|^4\,e^{4a/\tau}dxda\Big)^{1/4}\le\frac{C}{d}, \quad (3.65)$$

and finally the lemma. □

Lemma 3.8. *The solution* (Q,ρ) *given by Theorem* 1.1 *satisfies*

$$\|\nabla Q(t)\|_2^2 \le \frac{c}{d},\ \|\nabla \rho(t)\|_2^2 \le \frac{c}{d} \text{ for any } 0 \le t \le T, \tag{3.66}$$

$$\int_0^s \|\Delta\rho(t)\|_2^2\,dt = \int_0^s \int_0^A \int_\Omega |\Delta\rho(t)|^2\, e^{a/\tau} dxdadt \le \frac{c}{d^2},\ \text{for any } 0 \le s \le T, \tag{3.67}$$

$$\int_0^s \Big\|\frac{\partial Q}{\partial t}(t)\Big\|_2^2 dt \le c,\ \int_0^s \Big\|\frac{\partial \rho}{\partial t}(t)\Big\|_2^2 dt \le \frac{c}{d^2} \text{ and } \int_0^s \Big\|\frac{\partial \rho}{\partial a}(t)\Big\|_2^2 dt \le \frac{c}{d^2},\ \text{for any } 0 \le s \le T, \tag{3.68}$$

for a constant c (*which does not depend on* d).

Proof. Multiplying (1.2) by $(-\Delta\rho)e^{a/\tau}$ and integrating in a and x gives

$$\begin{aligned}
&\frac{d\|\nabla\rho\|_2^2}{dt} + \chi(A)e^{A/\tau}\int_\Omega \big|\nabla\rho(\cdot,A,\cdot)\big|^2\,dx + 2\int_0^A\int_\Omega \mu\big|\nabla\rho\big|^2\,e^{a/\tau}dxda \\
&\quad + 2\int_0^A\int_\Omega (D+d)\big|\Delta\rho\big|^2\,e^{a/\tau}dxda = \int_0^A\int_\Omega \nabla\mu\cdot\nabla\rho\,\rho\,e^{a/\tau}dxda \\
&\quad - \int_0^A\int_\Omega \nabla\big[D(\mathcal{M},Q,P)\big]\cdot\nabla\rho\Delta\rho\,e^{a/\tau}dxda + \frac{1}{\tau}\|\nabla\rho\|_2^2 \\
&\quad + \int_\Omega \big(\xi^2 + (\frac{\partial\xi}{\partial Q}Q)^2\big)|\nabla Q|^2\,dx \\
&\le c_1\big(\|\nabla\rho\|_2 + \|\nabla\rho\|_2^2 + \|\nabla Q\|_2^2\big) \\
&\quad + \int_0^A\int_\Omega \Big(\frac{\partial D}{\partial\mathcal{M}}\nabla\mathcal{M} + \frac{\partial D}{\partial P}\nabla P + \frac{\partial D}{\partial Q}\nabla Q\Big)\nabla\rho\Delta\rho\,e^{a/\tau}dxda \\
&\le c_1\big(\|\nabla\rho\|_2 + \|\nabla\rho\|_2^2 + \|\nabla Q\|_2^2\big) \\
&\quad + \frac{1}{d}\int_0^A\int_\Omega \Big(\frac{\partial D}{\partial\mathcal{M}}\nabla\mathcal{M} + \frac{\partial D}{\partial P}\nabla P + \frac{\partial D}{\partial Q}\nabla Q\Big)^2\big|\nabla\rho\big|^2\,e^{a/\tau}dxda \\
&\quad + \frac{d}{4}\int_0^A\int_\Omega \big|\Delta\rho\big|^2\,e^{a/\tau}dxda,
\end{aligned} \tag{3.69}$$

for a constant c_1. Now transferring the last term of the right-hand side to the left-hand side and using the estimate on $\|\nabla\rho\|_4$ and $\|\nabla Q\|_4$ which also gives an estimate on $\|\nabla\mathcal{M}\|_4$ and $\|\nabla P\|_4$, we get, for a constant c_2

$$\frac{d\|\nabla\rho\|_2^2}{dt} + \frac{5d}{4}\|\Delta\rho\|_2^2 \le c_2\big(\|\nabla\rho\|_2^2 + \|\nabla Q\|_2^2 + \frac{1}{d}\big). \tag{3.70}$$

In a similar way, we can also get

$$\frac{d(\int_0^A \int_\Omega |\nabla \rho|^2 \, e^{2a/\tau} dx da)}{dt} \leq c_3 \Big(\int_0^A \int_\Omega |\nabla \rho|^2 \, e^{2a/\tau} dx da + \|\nabla Q\|_2^2 + \frac{1}{d} \Big), \tag{3.71}$$

and

$$\frac{d\|\nabla Q\|_2^2}{dt} \leq c_4 \Big(\int_0^A \int_\Omega |\nabla \rho|^2 \, e^{2a/\tau} dx da + \|\nabla Q\|_2^2 \Big). \tag{3.72}$$

From the three last inequalities we get (3.66). Integrating (3.70) from 0 to s gives (3.67). The estimate on $\|\nabla Q(t)\|_2$ is then obtained as the estimate on $\|\nabla Q(t)\|_4$. Estimate (3.68) is finally a direct consequence of equations (1.1) and (1.2). □

Remark 3.1. We could also prove that $\|\nabla Q\|_\infty$ and $\|\nabla \rho\|_\infty$ are bounded.

4. Existence and uniqueness of the solution

Once the a priori estimates are set, the proof of existence is classical and in the spirit of Ladyzenskaja, Solonnikov and Ural'ceva [16]. It essentially consists in linearization and passing to the limit using the estimates.

In the following $L^p(\Omega)$ and $L^p([0,A) \times \Omega, \; e^{a/\tau} dadx)$ are the functional spaces associated with the norms (3.1), and, $W^{k,p}(\Omega)$ and $W^{k,p}([0,A) \times \Omega, \; e^{a/\tau} dadx)$ are the Sobolev spaces composed of functions whose derivatives up to order k are in $L^p(\Omega)$ or $L^p([0,A) \times \Omega, \; e^{a/\tau} dadx)$. $L^p([0,T) \times \Omega)$ and $L^p([0,T) \times [0,A) \times \Omega, \; e^{a/\tau} dadxdt)$ are the spaces of functions having finite norm

$$\Big(\int_0^T \int_0^A \int_\Omega |\rho(t,a,x)|^p \, e^{a/\tau} dx da dt \Big)^{1/p} \text{ or } \Big(\int_0^T \int_\Omega |Q(t,x)|^p \, dx dt \Big)^{1/p}, \tag{4.1}$$

and $W^{k,p}([0,T) \times \Omega)$ and $W^{k,p}([0,T) \times [0,A) \times \Omega, \; e^{a/\tau} dadxdt)$ are their associated Sobolev spaces. Finally, for a functional space W, $L^\infty(0,T;W)$ stands for the functions whose norm in W is finite for any $t \in [0,T]$.

4.1. **Linearization.**

We linearize the system (1.1)-(1.6). Then using classical results on partial and ordinary differential equations, we give an existence and uniqueness result for the solution to this linearized system.

We set $Q^0 = Q_0$ and $\rho^0 = \rho_0$ and for $n \in \mathbb{N}^*$, we consider (Q^n, ρ^n), a solution to:

$$\frac{\partial Q^n}{\partial t} = \frac{1-\xi}{\tau} Q^n + \int_0^A \rho^n(\cdot,a,\cdot) e^{a/\tau} \mu(\cdot,a,\cdot) \, da + \chi(A) \rho^n(\cdot,A,\cdot) e^{A/\tau},$$

$$\text{on } [0,T)\times\Omega, \quad (4.2)$$

$$\frac{\partial\rho^n}{\partial t}+\frac{\partial\rho^n}{\partial a}=-\mu\rho^n+\nabla\cdot\big[\big(D(\mathcal{M}^{n-1},Q^{n-1},P^{n-1})+d\big)\nabla\rho^n\big],$$
$$\text{on } [0,T)\times[0,A)\times\Omega, \quad (4.3)$$

$$\rho^n(\cdot,0,\cdot)=\frac{\xi}{\tau}Q^n, \quad \text{on } [0,T)\times\Omega, \quad (4.4)$$

$$\rho^n(0,\cdot,\cdot)=\rho_0, \quad \text{on } [0,A)\times\Omega, \quad (4.5)$$

$$Q^n(0,\cdot)=Q_0, \quad \text{on } \Omega, \quad (4.6)$$

$$\frac{\partial\rho^n}{\partial\overrightarrow{\nu}}=0, \quad \text{on } [0,T)\times[0,A)\times\partial\Omega, \quad (4.7)$$

where for $n\in\mathbb{N}$,

$$P^n(t,x)=\int_{a_{min}}^{A}\rho^n(t,a,x)e^{a/\tau}da, \quad (4.8)$$

and $\mathcal{M}^n$ is a solution to

$$\frac{\partial\mathcal{M}^n}{\partial t}=\frac{1}{P_{max}-p_{max}}\;H_r\Big(\frac{P^n-p_{max}}{P_{max}-p_{max}}\Big)\;H_r(1-\mathcal{M}^n)$$
$$-\frac{1}{p_{min}-P_{min}}\;H_r\Big(\frac{p_{min}-P^n}{p_{min}-P_{min}}\Big)\;H_r(\mathcal{M}^n),$$
$$\mathcal{M}^n(0,\cdot)=\mathcal{M}_0. \quad (4.9)$$

Theorem 4.1. *Under assumptions* (1.8), (1.9), (1.10) *and* (1.13), *if* $\rho_0\geq 0\in L^1\cap W^{2,2}\cap W^{1,4}([0,A)\times\Omega,\ e^{a/\tau}dadx)$ *satisfies* (1.15) *and if* $Q_0\geq 0\in L^1\cap W^{2,2}\cap W^{1,4}(\Omega)$ *then for any* $n\in\mathbb{N}$, *there exists a unique solution* $(Q^n,\rho^n)\in C_b^0\big(0,T;(L^1\cap W^{1,2}(\Omega))\times(L^1\cap W^{1,2}([0,A)\times\Omega,\ e^{a/\tau}dadx))\big)$ *to system* (4.2)–(4.7) *coupled with* (4.8) *and* (4.9). *Moreover,* $Q^n\geq 0$, $\rho^n\geq 0$ *and* (Q^n,ρ^n) *satisfy estimates* (3.2), (3.7), (3.42), (3.47), (3.66), (3.67), *and* (3.68) *with constants independent of* n.

The proof of this theorem uses only classical partial and ordinary differential equation arguments. Hence we only sketch it.

Proof. The proof consists of an induction procedure. Because of the assumptions on (Q_0,ρ_0) and the definition of (Q^0,ρ^0), the theorem is true for $n=0$.

Then, if the theorem is true for $n-1$, by regularization arguments, we can get that $\mathcal{M}^{n-1}$ exists and is unique on $[0,T)\times\Omega$ and that $\mathcal{M}^{n-1}\in C_b^0(0,T;W^{1,2}(\Omega))\cap C_b^1(0,T;L^\infty(\Omega))$. Hence we deduce that, for each $l\in\mathbb{N}^*$, there exists a unique solution $(Q^{n,l},\rho^{n,l})\in C_b^0\big(0,T;(L^1\cap W^{1,2}(\Omega))\times(L^1\cap$

$W^{1,2}([0,A)\times\Omega,\ e^{a/\tau}dadx))\big)$ to

$$\frac{\partial Q^{n,l}}{\partial t} = \frac{1}{\tau}Q^{n,l} - \frac{\xi}{\tau}Q^{n,l-1} + \int_0^A \rho^{n,l}(\cdot,a,\cdot)e^{a/\tau}\mu(\cdot,a,\cdot)\,da + \chi(A)\rho^{n,l}(\cdot,A,\cdot)e^{A/\tau}, \quad (4.10)$$

$$\frac{\partial \rho^{n,l}}{\partial t} + \frac{\partial \rho^{n}}{\partial a} = -\mu\rho^{n,l} + \nabla\cdot\big[\big(D(\mathcal{M}^{n-1},Q^{n-1},P^{n-1})+d\big)\nabla\rho^{n,l}\big], \quad (4.11)$$

$$\rho^{n,l}(\cdot,0,\cdot) = \frac{\xi}{\tau}Q^{n,l-1},\ \rho^{n,l}(0,\cdot,\cdot)=\rho_0,\ \frac{\partial\rho^{n,l}}{\partial\overrightarrow{\nu}}\Big|_{\partial\Omega}=0, \quad (4.12)$$

$$Q^{n,l}(0,\cdot) = Q_0, \quad (4.13)$$

where $Q^{n,0}$ is defined as $Q^{n,0}=Q^{n-1}$. This deduction involves first a classical semi-group or Galerkin routine in order to deduce that there exists a unique solution $\rho^{n,l}$ to (4.11) - (4.12) as soon as the existence of $(Q^{n,l-1},\rho^{n,l-1})$ is achieved. These routines are explained in Lions and Magenes [19], Ladyzenskaja, Solonnikov and Ural'ceva [16], or –in a context close to ours– in Langlais [17]. Once the existence of $\rho^{n,l}$ is established, the existence and uniqueness of $Q^{n,l}$ follows. Now, following the way leading to (3.17) and (3.22) we deduce that $(Q^{n,l},\rho^{n,l})$ satisfies

$$\frac{d\|\rho^{n,l}\|_2^2}{dt} \le \frac{1}{\tau^2}\|Q^{n,l-1}\|_2^2 + \frac{1}{\tau}\|\rho^{n,l}\|_2^2, \quad (4.14)$$

$$\frac{d\|Q^{n,l}\|_2^2}{dt} \le \Big(c_1\|Q^{n,l}\|_2 + c_1'\|Q^{n,l-1}\|_2\int_0^t\Big(\int_\Omega (Q^{n,l})^2(t-a,\cdot)\,dx\Big)^{1/2}da$$
$$+c_3\|\rho_0\|_2\ + c_4\Big(H(t-A)\int_\Omega (Q^{n,l})^2(t-A,\cdot)\,dx\Big)^{1/2}\Big)\|Q^{n,l}\|_2, \quad (4.15)$$

which is enough to deduce that $(\|\rho^{n,l}\|_2+\|Q^{n,l}\|_2)$ is bounded. As a consequence of this bound, we get that, for a subsequence still denoted l, $(Q^{n,l},\rho^{n,l}) \longrightarrow (Q^n,\rho^n)$ in $L^\infty\big(0,T;(L^2(\Omega))\times(L^2([0,A)\times\Omega,\ e^{a/\tau}dadx))$ weakly$-*$, where (Q^n,ρ^n) is the solution to (4.2) - (4.7). Finally, the estimates are deduced in the same way as in Section 3. The uniqueness follows directly from the linear character of (4.2) - (4.7). Hence, the theorem is true for n.

The induction procedure is then straightforward to end the proof of the theorem. □

4.2. **Existence.** From estimates (3.66), (3.67) and (3.68) we deduce that the sequence (Q^n,ρ^n) is bounded in $W^{1,2}([0,T)\times\Omega)\times W^{1,2}([0,T)\times[0,A)\times\Omega)$. Hence, up to a subsequence still denoted n, we have $(Q^n,\rho^n)\longrightarrow(Q,\rho)$ in

$W^{1,2}([0,T)\times\Omega)\times W^{1,2}([0,T)\times[0,A)\times\Omega,\ e^{a/\tau}dadxdt)$ weakly, and then, in $L^2([0,T)\times\Omega)\times L^2([0,T)\times[0,A)\times\Omega,\ e^{a/\tau}dadxdt)$ strongly.

From this we can also deduce that $P^n\longrightarrow P$ in $L^2([0,T)\times\Omega)$ strongly, with P defined from ρ by (1.11). In view of (4.9), we can deduce that $(\mathcal{M}^n)$, $(\partial\mathcal{M}^n/\partial t)$ and, taking the gradient of (4.9), $(\nabla\mathcal{M}^n)$ are bounded in $L^2([0,T)\times\Omega)$. Then extracting again a subsequence still denoted n, we deduce $\mathcal{M}^n\longrightarrow\mathcal{M}$ strongly, where $\mathcal{M}$ is the solution to (1.12).

Using now the regularity of D, we obtain $D(\mathcal{M}^{n-1},Q^{n-1},P^{n-1})\longrightarrow D(\mathcal{M},Q,P)$ in $L^2([0,T)\times\Omega)$ strongly.

The regularity of trace operators gives $\chi(A)\rho^n(\cdot,A,\cdot)\longrightarrow\chi(A)\rho(\cdot,A,\cdot)$, $\rho^n(\cdot,0,\cdot)\longrightarrow\rho(\cdot,0,\cdot)$, $\rho^n(0,\cdot,\cdot)\longrightarrow\rho(0,\cdot,\cdot)$, $Q^n(0,\cdot)\longrightarrow Q^n(0,\cdot)$ weakly, and using (3.67), $\partial\rho^n/\partial\overrightarrow{\nu}_{|\partial\Omega}\longrightarrow\partial\rho/\partial\overrightarrow{\nu}_{|\partial\Omega}$ weakly.

Then passing to the limit in (4.2) - (4.9), we obtain that (Q,ρ) is the solution to (1.1) - (1.6) coupled with (1.11) and (1.12).

Once this existence result is established, using regularizations and truncations, we can start the computations of Section 3 giving the additional regularity and the nonnegativity of Q and ρ.

It now remains to prove the uniqueness of the solution.

4.3. **Uniqueness.** Consider (Q,ρ) with associated P and $\mathcal{M}$ and $(\hat{Q},\hat{\rho})$ with associated $\hat{P}$ and $\hat{\mathcal{M}}$ two solutions of (1.1) - (1.6). They both satisfy the estimates and the difference $(\tilde{Q},\tilde{\rho})=(Q-\hat{Q},\rho-\hat{\rho})$ satisfies

$$\frac{\partial\tilde{Q}}{\partial t}=\frac{1-\xi}{\tau}\tilde{Q}+\int_0^A\tilde{\rho}(\cdot,a,\cdot)e^{a/\tau}\mu(\cdot,a,\cdot)\,da+\chi(A)\tilde{\rho}(\cdot,A,\cdot)e^{A/\tau}, \tag{4.16}$$

$$\begin{aligned}\frac{\partial\tilde{\rho}}{\partial t}+\frac{\partial\tilde{\rho}}{\partial a}&=-\mu\tilde{\rho}+\nabla\cdot\big[\big(D(\mathcal{M},Q,P)+d\big)\nabla\tilde{\rho}\big]\\&\qquad-\nabla\cdot\big[\big(D(\hat{\mathcal{M}},\hat{Q},\hat{P})-D(\mathcal{M},Q,P)\big)\nabla\hat{\rho}\big],\end{aligned} \tag{4.17}$$

$$\tilde{\rho}(\cdot,0,\cdot)=\frac{\xi}{\tau}\tilde{Q},\ \tilde{\rho}(0,\cdot,\cdot)=0,\ \tilde{Q}(0,\cdot)=0,\ \frac{\partial\tilde{\rho}}{\partial\overrightarrow{\nu}}_{|\partial\Omega}=0. \tag{4.18}$$

Multiplying (4.17) by $\tilde{\rho}e^{a/\tau}$ and integrating gives

$$\begin{aligned}\frac{d\|\tilde{\rho}\|_2^2}{dt}&+\chi(A)e^{A/\tau}\int_\Omega\tilde{\rho}^2(\cdot,A,\cdot)\,dx\\&+2\int_0^A\int_\Omega(D(\tilde{\mathcal{M}},\tilde{Q},\tilde{P})+d)|\nabla\tilde{\rho}|^2\,e^{a/\tau}dxda\\&=-2\int_0^A\int_\Omega\mu\tilde{\rho}^2\,e^{a/\tau}dxda+\int_\Omega\frac{\xi^2}{\tau^2}\tilde{Q}^2\,dx+\frac{1}{\tau}\|\tilde{\rho}\|_2^2\end{aligned}$$

$$+\int_0^A\int_\Omega \big(D(\hat{\mathcal{M}},\hat{Q},\hat{P})-D(\mathcal{M},Q,P)\big)\nabla\hat\rho\cdot\nabla\tilde\rho\, e^{a/\tau}dxda$$
$$\le k_1\big(\|\tilde\rho\|_2^2+\|\tilde Q\|_2^2\big)+k_2\|\nabla\hat\rho\|_\infty\big(\|\tilde\rho\|_2+\|\tilde Q\|_2\big)\|\nabla\tilde\rho\|_2$$
$$\le k_1\big(\|\tilde\rho\|_2^2+\|\tilde Q\|_2^2\big)+\frac{k_2^2}{d}\|\nabla\hat\rho\|_\infty^2\big(\|\tilde\rho\|_2+\|\tilde Q\|_2\big)^2+\frac{d}{4}\|\nabla\tilde\rho\|_2^2, \quad (4.19)$$

for constants k_1 and k_2. Passing the term $\frac{d}{4}\|\nabla\tilde\rho\|_2^2$ to the left-hand side gives, for a constant k_3

$$\frac{d\|\tilde\rho\|_2^2}{dt}\le\frac{k_3}{d}\big(\|\tilde\rho\|_2^2+\|\tilde Q\|_2^2\big). \quad (4.20)$$

Doing the same, but multiplying (4.17) by $\tilde\rho e^{2a/\tau}$ yields

$$\frac{d\big(\int_0^A\int_\Omega|\tilde\rho|^2\,e^{2a/\tau}dxda\big)}{dt}+\chi(A)e^{2A/\tau}\int_\Omega\tilde\rho(\cdot,A,\cdot)\,dx$$
$$\le\frac{k_3}{d}\Big(\int_0^A\int_\Omega|\tilde\rho|^2\,e^{2a/\tau}dxda+\|\tilde Q\|_2^2\Big). \quad (4.21)$$

Multiplying (4.16) by $\tilde Q$ gives

$$\frac{d\|\tilde Q\|_2^2}{dt}\le\frac{k_4}{d}\Big(\int_0^A\int_\Omega|\tilde\rho|^2\,e^{2a/\tau}dxda+\|\tilde Q\|_2^2\Big), \quad (4.22)$$

for a constant k_4. From the last three inequalities we deduce, for a constant K,

$$\|\tilde Q\|_2^2+\|\tilde\rho\|_2^2+\int_0^A\int_\Omega|\tilde\rho|^2\,e^{2a/\tau}dxda$$
$$\le K\Big(\|\tilde Q_{|t=0}\|_2^2+\|\tilde\rho_{|t=0}\|_2^2+\int_0^A\int_\Omega|\tilde\rho_{|t=0}|^2\,e^{2a/\tau}dxda\Big)=0, \quad (4.23)$$

giving $\tilde Q=\tilde\rho=0$ and then the uniqueness of the solution to (1.1) - (1.6). □

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