DERIVATION OF HYDRODYNAMIC LIMIT FROM KNUDSEN GAS MODEL

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Abstract. We prove that the solutions of the kinetic equation modelling the reflection of particles according to a reversible law and considered on the n-torus converge to the diffusion equation when the main free path goes to zero. This extends the work of Bardos, Golse and Colonna [2] to the case of any n-dimensional torus ergodic automorphism.

1. Set-up and statement of result

This work is concerned with the following scaled kinetic equation

$$\epsilon\frac{\partial f_\epsilon}{\partial t}+a(\omega)\cdot\nabla_x f_\epsilon = \frac{1}{\epsilon}Q(f_\epsilon) \quad \text{in} \quad \mathbb{R}_t^+\times\mathbb{R}_x^d\times\mathbb{T}^n, \tag{1.1}$$

$$f_\epsilon(0,x,\omega) = \phi(x) \quad \text{in} \quad \mathbb{R}_x^d\times\mathbb{T}^n, \tag{1.2}$$

modelling the reflection of particles on the n torus $\mathbb{T}^n\equiv\mathbb{R}^n/(2\pi\mathbb{Z})^n$. We are interested in the behavior of the solutions $f_\epsilon(t,x,\omega)$ of (1.1)-(1.2) as the collision mean free path ϵ goes to 0. The unknown $f_\epsilon(t,x,\omega)$ is interpreted as a distribution function of particles (which may be, depending on the context, electrons, holes, gas molecules). Each particle has velocity $a(\omega)$, where $\omega\in\Omega=\mathbb{T}^n$. The variable ω might seem a little obscure, but one could think of ω as a wave vector in a Brillouin zone, $a:\mathbb{T}^n\to\mathbb{R}^d$ denoting a smooth enough vector field of zero mean. One could think of $a(\omega)$ as the gradient of the energy with respect to this wave vector. Here we deal with the situation where the operator Q reads as

$$Q(f) = f\circ T-f \tag{1.3}$$

and $f\circ T$ is shorthand notation for $(t,x,\omega)\longmapsto f(t,x,T\omega)$. The transformation $T:\Omega\to\Omega$ is a one-to-one C^∞ map, an ergodic automorphism of

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the torus. In two space dimensions, the above limit problem has been investigated by [2, 4], where $T : \mathbb{T}^2 \to \mathbb{T}^2$ is chosen to be the 'Arnold cat map' i.e., a hyperbolic map, defined by

$$T \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \quad (\text{mod. } 2\pi), \tag{1.4}$$

and

$$T^{-1} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \quad (\text{mod. } 2\pi), \tag{1.5}$$

where T^{-1} stands for the inverse of T, a one-to-one C^∞ map, leaving invariant the Lebesgue measure $d\omega_1 d\omega_2/4\pi^2$. Using the ergodicity property of T the authors derived a diffusion equation. The goal of this paper is to revise the approach in [2, 4] to study multidimensional limits and to remove the hyperbolic character of T.

Integrating (1.1) by the characteristic method, that is to say, by solving the ordinary differential equation associated to $a(\omega)$, leads to

$$f_\epsilon(t, x, \omega) = \phi\Big(x + \epsilon \sum_{k=0}^{\lfloor t/\epsilon^2 \rfloor} \tau_k a(T^n \omega)\Big) + O(\epsilon). \tag{1.6}$$

Therefore most of the analysis of the behavior of the solution of (1.1) is reduced to the study of the limit, for $\epsilon \to 0$, of the expression

$$\psi_\epsilon(t, x, \omega) = \phi\Big(x + \epsilon \sum_{k=0}^{\lfloor t/\epsilon^2 \rfloor} \tau_k a(T^k \omega)\Big) \quad \text{or} \tag{1.7}$$

$$u_\epsilon(t, x, \omega) = E_\mu \phi\Big(x + \epsilon \sum_{k=0}^{\lfloor t/\epsilon^2 \rfloor} \tau_k a(T^k \omega)\Big),$$

where $\tau_k,\ k = 0, \ldots$ is a sequence of independent, exponentially distributed random jump times, i.e., $\text{Prob}\{\tau_k > t\} = e^{-t}$ for all positive t. In [2], it is proved that a suitably scaled process converges weakly to a diffusion. The proof of the asymptotic limit is inspired by the proof of the Itô formula. The crucial question of the proof relies on decorrelation of the size of intervals. It will be important to, uniformly with respect to their size, decorrelate two intervals of time under the only hypothesis that their distance is large enough. This decorrelation is based on trigonometric polynomials. It claims that the transformation T which is hyperbolic, has the following property (see Proposition 6, formula (37) in [2]):

(P1) There exist two constants $\beta_0 > 0$ and β_1 such that for all $l, m \in \mathbb{N}$, $U \subset \{n, ..., n+l\}$, $V \subset \{n, ..., n+m\}$ and for all pairs of trigonometric polynomial P, Q, of degree less than R, such that , for $n \geq \beta_0 \log R + \beta_1$,

$$\langle \prod_{k\in U} P\circ T^{-k} \prod_{k\in V} Q\circ T^{k}\rangle - \langle \prod_{k\in U} P\circ T^{-k}\rangle\langle \prod_{k\in V} Q\circ T^{k}\rangle = 0. \tag{1.8}$$

Regarding the above property, we make some remarks. First, we need to make precise some rather basic facts on ergodic automorphisms of the n-torus. Unfamiliar concepts and notation are explained in Section 2. We recall that a total automorphism T is ergodic if and only if the matrix $M \in SL(n, \mathbb{R})$ associated to T does not have any eigenvalue root of unity [9]. Ergodic total automorphisms are automatically mixing as well [9]. In addition, their eigenvalues can not all lie on the unit circle: at least one of them has to have a modulus strictly bigger than 1. This is an immediate consequence of the Kronecker theorem (see [10], Theorem 2.1) applied to the characteristic polynomial of M. As a result, in the decomposition of $\mathbb{R}^n$ into M-invariant subspaces [8] given by $\mathbb{R}^n = E_- \oplus E_0 \oplus E_+$, where E_+ (respectively E_0, E_-) is the root space of M corresponding to eigenvalues of modulus strictly bigger than (respectively equal to, strictly smaller than) 1, we are sure that E_-, E_+ are non-trivial. A matrix M is said to be hyperbolic if and only if M has no modulus one eigenvalue or if and only if $E_0 = \{0\}$. If $E_0 \neq \{0\}$, M is called quasi-hyperbolic (see [8]).

Now, in [2], what makes possible the decorrelation property (1.8)? With the notation

$$X_U^- = \sum_{k\in U} M^{-k}\xi_k\,, \qquad X_V^+ = \sum_{k\in V} M^{k}\eta_k\,,$$

we should point out that (1.8) is equivalent to the following claim (see relation (39) in [2]):

(P2) There exist $\beta_0 > 0$ and β_1 such that: $X_U^- + X_V^+ = 0\,,\ \Rightarrow\ X_U^- = 0\,,$ and $X_V^+ = 0\,,$ $\forall n > \beta_0 \log R + \beta_1\,,\ \forall\, l, m \in \mathbb{N}$. Indeed, denoting by $I = \mathbb{R}e_+$ (respectively $S = \mathbb{R}e_-$) the unstable (respectively stable) manifold of M acting on $\mathbb{R}^2$, it is easy to check that for ξ and η in $K_R = \{k \in \mathbb{Z}^2 : \sup(|k_1|, |k_2|) \leq R\}$, we have

$$|X_U^- \cdot e_+| \leq \sqrt{2}R\frac{\lambda_-^n}{1-\lambda_-}\,, \qquad |X_V^+ \cdot e_-| \leq \sqrt{2}R\frac{\lambda_-^n}{1-\lambda_-}.$$

This implies that X_U^- belongs to a neighborhood S_R^n of S while X_V^+ belongs to a neighborhood I_R^n of I given by the formulas:

$$I_R^n = \{X \in \mathbb{R}^2 : |X \cdot e_-| \leq \sqrt{2} R \frac{\lambda_-^n}{1-\lambda_-}\},$$

$$S_R^n = \{X \in \mathbb{R}^2 : |X \cdot e_+| \leq \sqrt{2} R \frac{\lambda_-^n}{1-\lambda_-}\}.$$

The hyperbolicity of T implies that since $X_U^- + X_V^+ = 0$, both X_U^- and X_V^+ belong to $K'_{R,n} = S_R^n \cap I_R^n$ which for an n greater than a given value N_0 is contained in $K_{R,n}$. The proof breaks down when T is not hyperbolic, since in that case $K'_{R,n} \neq S_R^n \cap I_R^n$. The following example gives an example of an ergodic but non hyperbolic automorphism of $\mathbb{T}^4$.

$$N = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{pmatrix}.$$

It is then natural to inquire about the validity of the convergence in that case. The question we will be interested in is: do the functions f_ϵ converge to f and in which sense, if T is replaced by any total quasi-hyperbolic transformation? Are there conditions on T ensuring this or that type of convergence? This is the main problem discussed in this article. We extend the aforementioned result of [2]. Using a kinetic approach, we show that the diffusion equation is preserved for any automorphism transformation (even if it is not hyperbolic). The basic tools are a generalization of a lemma on diophantine approximation [7, 5], the ergodic theorem and in certain places a property of trigonometric polynomials.

In the sequel, we denote by brackets as in $\langle \cdot \rangle$ the average in ω. From now, T will denote quasi-hyperbolic (i.e. ergodic automorphism non hyperbolic) transformations of the torus $\mathbb{T}^n (n \geq 2)$.

Here now is the precise statement of our result.

Theorem 1.1. *Let $a \in \mathcal{C}^{d+2}(\mathbb{T}^n, \mathbb{R}^d)$ be such that $\langle a \rangle = 0$. Assume that $\phi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$. Then as $\epsilon \to 0$, the family f_ϵ converges in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^d \times \mathbb{T}^n)$ weak-* to the solution u of the diffusion equation*

$$\partial_t u = \frac{1}{2} D(a) : \nabla_x^2 u, \qquad x \in \mathbb{R}^d, \quad t > 0, \tag{1.9}$$

$$u(0,x) = \phi(x), \qquad x \in \mathbb{R}^d, \tag{1.10}$$

where the diffusion coefficient is given by

$$D(a) = 2 \sum_{n \geq 0} \langle a \circ T^n a \rangle. \tag{1.11}$$

The matrix $D(a)$ is strictly positive for $a \in L^2(\mathbb{T}^2)$.

The positivity of the diffusion tensor $D(a)$ guarantees that the problem (1.9)-(1.10) is well posed. For detailed discussions on the model we refer to [1, 2]. A rather complete presentation of the problem as well as a deep discussion at the relevant scaling can also be found in [4]. We also refer to the work of V. P. Leonov [6] where related problems are considered. Finally, let us mention that this work can be regarded as a contribution to the theory of diffusive limits for Knudsen gas developed in [2, 3, 4].

The outline of the paper is as follows. Section 2 contains the proof of Theorem 1.1 which is split into several steps. In section 2.2 we discuss some notation, introduce function spaces and recall some needed properties of these function spaces. We review also the definition of a co-boundary in Hilbert space. In section 2.3 we state the needed estimate for the solutions of (1.1)-(1.2). In section 2.4 we introduce and analyze auxiliary problems. In section 2.5 we carry out the property of diffusivity. In section 2.6 we prove the convergence of diffusion. A conclusion is then drawn in section 3.

2. Proof of Theorem 1.1

2.1. **Clarification of notation and preliminaries.** We recall the main notation that we will use through the article and we refer, if necessary, for more details to [2].

We state one notation for matrices. Let r and s be two integers. For A (respectively B) the vectors in $\mathbb{R}^r$ (respectively $\mathbb{R}^s$), we let the matrix $A \otimes B := AB^T$ be of type $r \times s$ and $A^{\otimes 2}$ be the symmetric matrix $A \otimes A$. Letting M and N be two matrices of type $r \times s$, we define the real $M; N = N; M = \sum_{i,j} M_{ij} N_{ij}$. $|\cdot|_\infty$ is the supremum norm relatively to the canonical basis of $\mathbb{R}^r$. Partial space derivatives of order n of a function $f : \mathbb{R}^n \to \mathbb{R}^p$ are denoted by $\nabla_x^n f$. $L^2 = L^2(\mathbb{T}^n)$ is the space of square integrable (vector- or matrix -valued) functions on the n-dimensional unit torus. For nonnegative integers m and p, $H^m = H^m(\mathbb{R}^p, \mathbb{R}^r)$ or $H^m = H(\mathbb{T}^p, \mathbb{R}^r)$, $\|f\|_{H^m} := (\sum_{|k| \leq m} \|D^k f\|_{L^2}^2)^{\frac{1}{2}}$, where $D^k f = \frac{\partial^{|k|}}{\partial x_1^{k_1} \ldots \partial x_d^{k_d}}$ with $|k| = k_1 + \ldots + k_d$. $C_b^3(\mathbb{R}^n)$ is a vector space with the norm $\|f\|_{C_b^3(\mathbb{R}^n)} = \sum_{|k| \leq 3} \|D^k f\|_\infty$. Partial time derivatives will sometimes be denoted by u_t

instead $\partial_t u$. We let $(v_1, \ldots, v_n)$ be a basis of $\mathbb{R}^n$ in which M is represented by a real Jordan matrix. In $\mathbb{R}^n$, we fix the following norm: when $x = \sum_{i=1}^n x_i v_i$, $\|x\| = (\sum_{i=1}^n |x|^2)^{1/2}$. We let p (respectively q, respectively s) the dimension of E_+ (respectively E_-, respectively E_0). For $\alpha \in \mathbb{R}^n$, we denote by $(\alpha_+, \alpha_-, \alpha_0)$ the unique element of $E_+ \times E_- \times E_0$ such that $\alpha = \alpha_+ + \alpha_- + \alpha_0$. In addition, we let

$$\rho := \frac{\min\{\|k\| \ / \ k \in \mathbb{Z}^n \setminus \{0\}\}}{2\|M\|}$$

be the spectral radius of M and ρ_s the spectral radius of $M|_{E_-}$. It is of considerable interest to introduce the connection of a unitary operator associated to a transformation preserving the measure, since we need to study a possible limit (in the sense that will be precise later) of the mean $\frac{1}{n}S_n(f)$. For a measurable function f on Ω, we define for any integer $n \geq 1$ the partial sum $S_n(f) \stackrel{\text{def}}{=} \sum_{k=0}^{n-1} f(T^k x)$. We define on $L^2(\Omega)$ the operator U_T by $U_T f \stackrel{\text{def}}{=} f \circ T$, for all $f \in L^2(\Omega)$. It is easy to verify that U_T is linear, and an isometry of $L^2(\Omega)$. In addition if T is invertible and bi-measurable (in particular, if T is an automorphism of Lebesgue space), U_T is itself invertible, so it is a unitary operator on $L^2(\Omega)$, therefore its adjoint is given by $U_T^* f = U_T^{-1} f = f \circ T^{-1}$. We introduce the following definitions:

Definition 2.1. *A co-boundary in $L^2(\Omega)$ is any function $f \in L^2(\Omega)$ such that there exists a function $g \in L^2(\Omega)$ such that $f = g - g \circ T$.*
Two functions f and g belonging to $L^2(\Omega)$ are homologous if and only if $f - g$ is a co-boundary and this equivalence relation will be denoted $f \sim g$.

The next proposition describes the elementary properties of what will be, in the limit $\epsilon \to 0$, the diffusion coefficient. It is essentially based on the analysis of the ergodic and mixing properties of the mapping T. We have

Proposition 2.2. *Let $T : \Omega \to \Omega$ be an ergodic transformation of $L^2(\Omega)$. Then, the set*
$C = \{f - f \circ T;\ f \in L^2(\Omega)\}$ of co-boundary in $L^2(\Omega)$ is dense in $\{f \in L^2(\Omega) : \langle f \rangle = 0\}$.

Proof. It is sufficient to prove that if f is a vector, orthogonal to the set $I = \{U \in L^2 : Uf = f\}$ and C, then f is a null vector. But, if f is such a vector, as $Uf - f$ is a co-boundary , we have $\langle f, Uf - f \rangle = 0$, therefore $\langle f, Uf \rangle = \|f\|^2$, from which we deduce that $f = Uf$; in other words, $f \in I$. Since $f \perp I$, we deduce that $f = 0$. $\square$

2.2. **Decorrelation property.** To prove Theorem 1.1, we will need the following result from [5], which is the generalization to higher dimensions of the obvious diophantine inequality satisfied in the case $n = 2$ by the slopes of the stable and unstable directions of M.

Proposition 2.3. *Let M be a $n \times n$ matrix with integral entries. Let $\mathbb{R}^n = V_1 \oplus V_2$ with V_i invariant spaces for M such that*

(i) *$M|_{V_1}$ and $M|_{V_2}$ don't have common eigenvalues;*
(ii) *$V_1 \cap \mathbb{Z}^n = \{0\}$.*

There exists a constant C such that for every $k \in \mathbb{Z}^n \setminus \{0\}$, we have

$$d(k, V) \geq C\|k\|^{-\dim(V)}.$$

From Proposition 2.3, we therefore immediately obtain the following corollary which we shall need later in our proof.

Corollary 2.4. *Let T be ergodic. Then we have*

$$\mathbb{Z}^n \cap (E_0 \oplus E_+) = \{0\} \qquad \textit{and} \qquad \mathbb{Z}^n \cap (E_0 \oplus E_-) = \{0\}. \tag{2.1}$$

In addition, there exist two constants $\overline{K}_1 \equiv K_{0,-} > 0$, $\overline{K}_2 \equiv K_{0,-} > 0$ such that, for $\alpha \in \mathbb{Z}^n \setminus \{0\}$

$$\|\alpha_+\| \geq \frac{\overline{K}_1}{\|\alpha\|^{s+q}} \geq \frac{\overline{K}_2}{\|\alpha\|^n}.$$

Proof of Corollary 2.4. It is well known that if M is a matrix with integral entries with all eigenvalues having modulus 1, these eigenvalues are the roots of unity. In particular, if T is ergodic, then $E_0 \neq \mathbb{R}^n$.

We argue by contradiction. Assume that there exists a non-null integer vector $\alpha \in E_0 \oplus E_+$. Let W be the subspace of $\mathbb{R}^n$, with dimension equal to r and generated by $\{M^k\alpha;\ \ k \in \mathbb{N}\}$. Thanks to the Cayley-Hamilton theorem, W is the subspace of $\{M^k\alpha;\ \ k = 0, ..., n-1\}$, stable under M and M^{-1}. By the theory of linear algebra, it is well known that there exists a matrix P with integral entries and determinant equal to 1 such that the first r-vectors columns of P is a basis of W. Denote by $L = P^{-1}Q^TP$ the matrix of M in this basis. L has the form

$$L = \begin{pmatrix} A & B \\ \mathbf{0} & C \end{pmatrix}$$

where A is a matrix of type $r \times r$ and C of type $(n-r) \times (n-r)$ with $\det(L) = \pm 1$. Since A is the matrix of $M_{|W}$ in the basis of P, $\det(A) = \pm 1$. Since $W \subset E_0 \oplus E_+$, we have necessarily $W \cap E_+ = \{0\}$ (because $W \cap E_+ = \{0\}$ and $\det(A) = \pm 1$). We deduce that W is included in E_0.

This contradicts the property of T, since T ergodic implies $E_0 \neq \mathbb{R}^n$. We conclude that $\mathbb{Z}^n \cap (E_0 \oplus E_+) = \{0\}$. The second claim of (2.1) is proved in the same manner. The last part of the corollary is an immediate consequence of Proposition 2.3. This completes the proof. $\square$

Proposition 2.5. *Let T be an ergodic automorphism, f a positive polynomial function on $\mathbb{N}$, and $R \geq 1$. For all sequences of trigonometric polynomials $(P^{(k)})_k$, $(Q^{(l)})_l$, of degree less than R, there exists an integer $K(R,f) > 0$ such that for all $K \geq K(R,f)$ and $N \in \mathbb{N}$, we have*

$$\langle \prod_{k=K}^{K+N} P \circ T^{-k} \prod_{l=K}^{K+f(K)} Q \circ T^{k} \rangle - \langle \prod_{k=K}^{K+N} P \circ T^{-k} \rangle \langle \prod_{l=K}^{K+f(K)} Q \circ T^{k} \rangle = 0. \quad (2.2)$$

Proof. Consider a vector of integers $\xi_1, \ldots, \xi_{K+N}, \eta_1, \ldots, \eta_{K+f(K)} \in \mathbb{Z}^n$ whose coordinates in the basis $(v_1, \ldots, v_n)$ are bounded by R and such that X_ξ^- and X_η^+ are non-nulls. It is easy to check that the frequencies appearing in (2.2) have the forms

$$X_\xi^- := \sum_{k=K}^{K+N} M^{-k} \xi_k \,, \qquad X_\eta^+ := \sum_{k=K}^{K+f(K)} M^k \eta_k.$$

We denote by $r_+ := \rho\left(M^{-1}|_{E_+}\right)$ the spectral radius of $M^{-1}|_{E_+}$ and we fix $r_0 \in]r_+, 1[$. There exists an integer $K \geq 1$ such that for $k \geq K$ we have $\|M^{-k}|_{E_+}\| \leq r_0^k$. To prove (2.2) it is clearly sufficient to prove that

$$X_\xi^- + X_\eta^+ = 0 \Longrightarrow X_\xi^- = X_\eta^+ = 0.$$

Actually, we shall prove that

$$\left\|(X_\xi^-)_+\right\| > \left\|(X_\eta^+)_+\right\|. \quad (2.3)$$

On the one hand, for K greater than a given value K_0, we have

$$\|(X_\xi^-)_+\| \leq \sum_{k=K}^{k+N} \|M^{-k}(\xi_k)_+\| \leq \sum_{k=K}^{k+N} \|M^{-k}_{|_{E_+}}\| \cdot \|(\xi_k)_+\| \leq R \sum_{k \geq K} r_0^k = R \frac{r_0^K}{1 - r_0}. \quad (2.4)$$

Denote $x_{0,-} = x_0 + x_-$ and observe that

$$\left(M^{-K} X_\eta^+\right)_{0,-} = \Big(\sum_{l=0}^{f(K)} M^l \eta_{K+l} \Big)_{0,-} = \Big(\sum_{l=0}^{f(K)} M^l (\eta_{K+l}) \Big)_{0,-}.$$

There exists a constant C such that for $l \in \mathbb{N}$, we have $\|M^l|_{E_-+E_0}\| \leq Cl^n$. Since $(E_- \otimes E_+) \cap \mathbb{Z}^n = \{0\}$, if $X_\eta^+ \neq 0$, we get

$$0 < \left\|\left(M^{-K}X_\eta^+\right)_{0,-}\right\| \leq R + CR\sum_{l=1}^{f(K)} l^n \leq (C+1)R \cdot (f(K)+K)^{n+1}. \quad (2.5)$$

We claim that, there exists a constant $C_1 > 0$ such that for K greater than a given value $K_1(f)$, we have

$$\left\|(X_\eta^+)_+\right\| > C_1 r_0^{-K} R^{-n}(f(K)+K)^{-n(n+1)}. \quad (2.6)$$

Indeed let $C' > 0$. If K is an integer such that $K \geq K_1(f)$ satisfying

$$\left\|\left(M^{-K}X_\eta^+\right)_+\right\| < C'(f(K)+K)^{-n(n+1)},$$

then we have

$$\begin{aligned}\left\|\left(M^{-K}X_\eta^+\right)_+\right\| &\leq (C+1)(f(K)+K)^{n+1} + C'(f(K)+K)^{-n(n+1)}\\ &\leq (C'+C+1)(f(K)+K)^{n+1}.\end{aligned}$$

Hence thanks to corollary 2.4, we have

$$\left\|\left(M^{-K}X_\eta^+\right)_+\right\| \geq \frac{\overline{K}_1}{(C'+C+1)}(f(K)+K)^{n+1}.$$

There exists $K_2(f)$ such that $(f(K)+K)^{n(n+1)} < r_0^{-K}$, for all integers $K \geq K_2(f)$. Since $K > 0$ is sufficiently greater, bringing together (2.4) and (2.6), we have

$$R\frac{r_0^K}{1-r_0} > C_1 r_0^{-k} R^{-n} r_0^K,$$

from which we deduce that $k > \frac{\ln(C_1(1-r_0))}{\ln(r_0)} - \frac{n+1}{\ln(r_0)}\ln R$. Therefore, writing

$$\beta_0(f) = -\frac{n+1}{\ln(r_0)} \quad \text{and} \quad \beta_1(f) = \max\left(\frac{\ln(C_1(1-r_0))}{\ln(r_0)}; K_0; K_1(f); K_2(f)\right)$$

we get, for $K \geq \beta_0(f)\ln(R) + \beta_1(f)$,

$$r_0^{2K}(f(K)+K)^{n(n+1)} < r_0^K \leq C_1(1-r_0)R^{-(n+1)},$$

so that

$$R\frac{r_0^K}{1-r_0} < C_1 r_0^{-K} R^{-n}(f(K)+K)^{-n(n+1)}.$$

Consequently, for K sufficiently greater and $\xi, \eta \in (\mathbb{Z}^n)^N$, we obtain the inequality (2.3). This completes the proof of the proposition. □

We now turn our attention to the following proposition expressing the properties of the self correlation coefficient.

Lemma 2.6. *Let $0 < \chi(R)$ be a decreasing positive function going to 0 for R going to infinity. Let T be an ergodic automorphism of $\mathbb{T}^n$. Define the class of functions H_χ by*

$$H_\chi = \Big\{ f \in L^2(\mathbb{T}^n) : \sum_{\|k\| \geq R} |c_k(f)|^2_\infty \leq \chi(R)^2 \|f\|_2^2, \quad \forall R > 0 \Big\}.$$

(i) If ρ_0 satisfies $\rho_s < \rho_0 < 1$, there exist constants $C_1 > 0$, $C_0 > 0$ such that, for any functions f and g in H_χ with $\langle f \rangle = \langle g \rangle = 0$, and for $n \in \mathbb{N}$,

$$|\langle f \circ T^n \cdot g \rangle|_\infty \leq C_1 \|f\|_{L^2} \|g\|_{L^2} \chi\big(C_0 \rho_0^{-\frac{n}{2}}\big). \tag{2.7}$$

(ii) For $s \geq 1$, there exists a constant C_2 such that, for f, $g \in H^s(T^n, \mathbb{R}^r)$ with mean value $\langle f \rangle = \langle g \rangle = 0$ and $n \in \mathbb{N}$

$$|\langle f \circ T^n \cdot g \rangle|_\infty \leq C_2 \|f\|_{H^s} \|g\|_{H^s} e^{-sn\gamma} \qquad \textit{with} \quad \gamma = -\frac{\ln \rho_0}{2}. \tag{2.8}$$

In particular, the self correlation coefficient $C_f(n) := \langle f \circ T^n \cdot f \rangle$ of any function $f \in H^s(\mathbb{T}^n)$ satisfies

$$|\langle C_f(n) \rangle|_\infty \leq C^{te} \|f\|^2_{H^s} e^{-sn\gamma}. \tag{2.9}$$

Proof. The proof is quite flexible since it is a minor modification of the proof in [2], hence is left to the reader. □

2.3. **A priori estimates.** In this section, we derive formally the uniform estimates which we need for the asymptotic results. Obviously, equation (1.1) gives

$$\int_{\mathbb{T}^n} Q(f)(\omega) d\omega = 0, \tag{2.10}$$

since T leaves invariant the Lebesgue measure. For the same reasons, omitting x and t in f_ϵ, we can write, since it is a parameter,

$$\int_{T^n} Q(f_\epsilon)(\omega) f_\epsilon(\omega) d\omega = \int_{T^n} f_\epsilon^2(\omega) d\omega - \int_{\mathbb{T}^n} f_\epsilon(T\omega) f_\epsilon(\omega) d\omega \tag{2.11}$$

$$\int_{\mathbb{T}^n} \frac{1}{2} \left[f_\epsilon^2(\omega) + f^2(T\omega) \right] d\omega - \int_{\mathbb{T}^n} f_\epsilon(T\omega) f_\epsilon(\omega) d\omega$$

$$= \frac{1}{2} \int_{\mathbb{T}^n} \left(f_\epsilon(\omega) - f_\epsilon(T\omega) \right)^2 d\omega.$$

Integrating (1.1) with respect to x, ω yields the following 'mass conservation' property

$$\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{T}^n} f_\epsilon dx d\omega = 0. \tag{2.12}$$

Let $\varphi(r)$, $r \geq 0$, be a convex regular function ($\varphi'' \geq 0$). Multiplying both side of (1.1) by $\varphi'(g_\epsilon)$ after integrating with respect to x and ω we obtain

$$\begin{aligned}\int_{\mathbb{R}^d\times\mathbb{T}^n} \frac{\partial f_\epsilon}{\partial t}\varphi'(f_\epsilon)dxd\omega &= \frac{1}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{T}^n} Q(f_\epsilon)\varphi'(f_\epsilon)dxd\omega \qquad (2.13)\\ &= \frac{1}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{T}^n} Q(f_\epsilon)(\varphi'(f_\epsilon)-\varphi'(f_\epsilon(T\omega))dxd\omega.\end{aligned}$$

Since φ' is non-decreasing, we obtain at once that

$$\begin{aligned}&\frac{d}{dt}\int_{\mathbb{R}^d\times\mathbb{T}^n} \varphi(f_\epsilon)dxd\omega \qquad (2.14)\\ &= -\frac{1}{\epsilon^2}\int_{\mathbb{R}^d\times\mathbb{T}^n} Q(f_\epsilon)(\varphi'(f_\epsilon)-\varphi'(f_\epsilon(T\omega))dxd\omega \leq 0.\end{aligned}$$

In particular, if the initial data $\phi \in L^\infty(\mathbb{R}^d)$, taking $\varphi(r) = r^p$, for any $p \geq 1$ and letting p go to $+\infty$, we deduce the following bound

$$\|f_\epsilon\|_{L^\infty(\mathbb{R}^d\times\mathbb{T}^n\times\mathbb{R})} \leq \|\phi\|_{L^\infty(\mathbb{R}^d)}. \tag{2.15}$$

Hence, multiplying both sides of (1.1) by f_ϵ, integrating in x and ω, and taking into account the fact that each function f_ϵ vanishes at infinity in x as does ϕ, we get

$$\frac{1}{2}\|f_\epsilon(t)\|_{L^2(\mathbb{R}^d\times\mathbb{T}^n)} + \frac{1}{2\epsilon^2}\int_0^t\int_{\mathbb{R}^d}\langle(f_\epsilon - f_\epsilon\circ T)^2\rangle dxds = \frac{1}{2}\|\phi\|_{L^2(\mathbb{R}^d)}. \tag{2.16}$$

Therefore, we can average the transport equation (1.1) with respect to ω to obtain the following equation of continuity

$$\partial_t\langle f_\epsilon\rangle + \frac{1}{\epsilon}\nabla_x\cdot\int_{\mathbb{T}^n} a(\omega)f_\epsilon d\omega = 0, \tag{2.17}$$

which holds in the distribution sense on $\mathbb{R}_+^*\times\mathbb{T}^n$. We summarize what we showed as the following

Proposition 2.7. *Suppose the initial data ϕ is bounded in $L^2(\mathbb{R}^d)$. Then*
(i) the sequence $(f_\epsilon)_\epsilon$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}^d\times\mathbb{T}^n))$;
(ii) the sequence $\langle f_\epsilon\rangle$ is bounded in $L^\infty(0,T;L^2(\mathbb{R}^d))$.

2.4. **Auxiliary problem.** The peculiarity of the asymptotic limit of (1.7) is the presence of two kinds of auxiliary problems. The first one is used to compute the diffusion coefficient and is connected with the corresponding equation: Find the unknown function $b_\epsilon \equiv b_\epsilon(\omega)$ such that

$$\epsilon^2 b_\epsilon + b_\epsilon - b_\epsilon \circ T^{-1} = a. \tag{2.18}$$

We prove:

Proposition 2.8. *For any $\epsilon > 0$, the solution of the problem* (2.18) *satisfies*

$$\langle b_\epsilon \rangle = 0, \qquad \epsilon \langle b_\epsilon^2 \rangle = O(1). \tag{2.19}$$

Proof. Since the transformation T leaves invariant Lebesgue measure on $\mathbb{T}^n$, the operator $a \mapsto a \circ T$ is a unitary transformation of L^2 and b_ϵ can be defined by the "inverse formula"

$$b_\epsilon = \sum_{n \geq 0} \frac{1}{(1+\epsilon^2)^{n+1}} a \circ T^{-n},$$

which is normally a converging series. The first statement of (2.19) is a trivial consequence of (2.18). For the second statement, we write

$$\epsilon^2 \langle b_\epsilon^2 \rangle + \langle b_\epsilon (b_\epsilon - b_\epsilon \circ T) \rangle = \langle b_\epsilon a \rangle, \tag{2.20}$$

and we observe that $\langle b_\epsilon (b_\epsilon - b_\epsilon \circ T) \rangle = \frac{1}{2} \langle (b_\epsilon - b_\epsilon \circ T)^2 \rangle \geq 0.$ □

Once Proposition 2.8 is proved, it remains to check that

$$\langle b_\epsilon a \rangle \to \frac{1}{2} D(a) \qquad \text{as} \quad \epsilon \to 0. \tag{2.21}$$

To this end, we introduce the second kind of auxiliary problem which allows us to ensure the independence of asymptotic increases. It turns out to be connected to the decorrelation property and is described by the following proposition.

Proposition 2.9. *The family f_ϵ defined by* (1.7) *satisfies*

$$\langle b_\epsilon a f_\epsilon \rangle - \langle b_\epsilon a \rangle \langle f_\epsilon \rangle \to 0 \tag{2.22}$$

$$\langle \epsilon b_\epsilon f_\epsilon \rangle - \langle \epsilon b_\epsilon \rangle \langle f_\epsilon \rangle \to 0. \tag{2.23}$$

Sketch of Proof. The proof involves two main ingredients. First, it employs the mixing properties that are enjoyed by the transformation T, namely Proposition 2.6. Secondly, using the representation of solutions of f_ϵ, i.e., the relation (1.7), we must decorrelate two intervals of time under only the hypothesis that their distance is large enough. This is an immediate consequence of Proposition 2.5. □

2.5. **Properties of the diffusivity tensor.** The purpose of this subsection is to establish some properties of the diffusion coefficient $D(a)$. To this end we prove the following proposition.

Proposition 2.10. *Let T be an ergodic automorphism of the torus and $a \in H^s(\mathbb{T}^n)$ with $s \geq 1$. Then, the following matrix-valued series converges to a non-negative matrix. More precisely we have*

$$D(a) = \frac{1}{2}\langle a^2\rangle + \sum_{k\geq 1}\langle a \circ T^k \otimes a\rangle = \frac{1}{2}\Big\langle\Big(\frac{1}{\sqrt{N}}\sum_{k=0}^{\infty} a \circ T^k\Big)^{\otimes 2}\Big\rangle \geq 0\,.$$

Proof. It is a consequence of Proposition 2.8 and a minor modification of the proof of [2]. □

Proposition 2.11. *Let T be an ergodic automorphism of the torus and $a \in H^s(\mathbb{T}^n, \mathbb{R}^d)$ with $\langle a\rangle = 0$. Let $\xi \in \mathbb{R}^d$ and $(\xi^T \cdot f_N)$ be a sequence of functions in L^2 with $\xi^T \cdot f_N = \sum_{n=0}^{N-1} \xi^T a \circ T^n$. The following properties are equivalent:*

(i) $D(a)\xi = 0$.

(ii) $\xi^T D(a)\xi = 0$.

(iii) $\xi^T f_N$ is bounded (uniformly with respect to N) in $L^2(\mathbb{T}^n)$.

(iv) $\xi^T a$ is a co-boundary of $L^2(\mathbb{T}^n)$.

Remark 2.12. Under the hypothesis of the previous proposition:

(i) If $d = 1$, the $D(a) = 0$ if and only if a is a co-boundary.

(ii) If $d \in \mathbb{N}^*$, $D(a)$ is non degenerate if and only if for $\xi \neq 0$, $\xi^T D(a)\xi \neq 0$, i.e., from Proposition 2.11, for $\xi \neq 0$, $\xi^T a$ is not a coboundry in $L^2(\mathbb{T}^n)$.

Proof of Proposition 2.11. Clearly $(i) \Longrightarrow (ii)$. We prove $(ii) \Longrightarrow (iii)$. To this end, we introduce

$$r(z) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\langle \xi^T a \circ T^k \cdot \xi\rangle\, e^{ikz}\,, \qquad z \in \mathbb{T}^1, \quad \xi \in \mathbb{R}^d$$

and we observe that

$$r(0) = \frac{1}{2\pi}\sum_{k\in\mathbb{Z}}\langle \xi^T a \circ T^k \cdot \xi\rangle = \frac{1}{\pi}\xi^T \cdot D(a) \cdot \xi = \pi r(0).$$

Denote by $\mathcal{F}_n$ the Féjer kernel, i.e., $\mathcal{F}_n(t) := \frac{1}{n}\left(\frac{\sin \pi n t}{\sin \pi t}\right)^2$. From Lemma 2.6, we deduce that $r(t)$ is an even holomorphic function in the strip $\{z \in \mathbb{C} : \|\mathrm{Im}(z)\| < \alpha s\}$. Since T leaves invariant the Lebesgue measure on $\mathbb{T}^n$, its

Fourier transform is given $\widehat{r}(k) = \langle \xi^T a \circ T^k \cdot \xi \rangle$. Therefore,

$$\|\xi^T f_N\|^2_{L^2(\mathbb{T}^2)} = \Big\langle \Big(\sum_{k=0}^{N-1} \xi^T a \circ T^k \Big)^2 \Big\rangle = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \langle \xi^T a \circ T^k \cdot \xi^T a \circ T^l \rangle$$
$$= \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \widehat{r}(k-l) = \int_{-\pi}^{+\pi} r(t) \Big(\sum_{k=0}^{N-1} \sum_{l=0}^{N-1} e^{-i(k-l)t} \Big) dt$$
$$= \int_{-\pi}^{+\pi} \Big| \frac{1-e^{iNt}}{1-e^{it}} \Big|^2 r(t) dt = N \int_{-\pi}^{+\pi} \mathcal{F}_N(t) r(t) dt$$
$$= N \int_{-\pi}^{+\pi} \mathcal{F}_N(t) \frac{r(t)+r(-t)-2r(0)}{2} dt + N \int_{-\pi}^{+\pi} \mathcal{F}_N(t) r(0) dt.$$

Consequently,

$$\|\xi^T f_N\|^2_{L^2(\mathbb{T}^2)} \quad = \quad N r(0) + \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{r(t)+r(-t)-2r(0)}{2\sin^2 \pi t} \left(\sin(\pi N t)\right)^2 dt\,.$$

Computing the Taylor expansion up to the second order of r, we are led to

$$\int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{r(t)+r(-t)-2r(0)}{2\sin^2 \pi t} dt \le \|r''\|_\infty \int_{-\frac{1}{2}}^{+\frac{1}{2}} \frac{t^2}{2\sin^2 \pi t} dt < +\infty \qquad (2.24)$$

since the integral in the right-hand side of (2.24) is finite. Therefore the relation

$$0 = \xi^T \cdot D(a) \cdot \xi = \pi r(0)$$

implies that the series $\xi^T f_N$ is (uniformly with respect to N) bounded in $L^2(\mathbb{T}^n)$. Next, we prove $(iii) \Longrightarrow (iv)$ Assume that $\big(\xi^T f_N\big)_N$ is bounded in L^2. Thanks to the Banach-Alaoglu theorem $\big(\xi^T f_N\big)_N$ is relatively compact in L^2 for the weak-* topology. Let f^ξ be a limit point of this family such that $\xi^T f_N$ converges weakly to an element $f^\xi \in L^2$. We can assume, possibly at the extracting subsequences, that for any function $g \in L^2$

$$\langle (f^\xi - f^\xi \circ T) g \rangle = \lim_{N \to +\infty} \langle (\xi^T f_N - \xi^T f_N \circ T) g \rangle = \langle \xi^T a g \rangle - \lim_{N \to +\infty} \langle \xi^T a \circ T^N \cdot g \rangle.$$

Since by Cauchy-Schwartz's inequality, $\sum_{n=0}^N \xi^T a \circ T^n g$ is bounded in L^1, we get

$$\lim_{N \to +\infty} \langle \xi^T\, a \circ T^N g \rangle = 0.$$

Thus, we have

$$\langle (f^\xi - f^\xi \circ T) g \rangle \;=\; \langle \xi^T\, a g \rangle,$$

from which we deduce that $\xi^T a$ is a co-boundary in L^2. Now, we prove $(iv) \Longrightarrow (i)$. Assume that $\xi^T ag$ has the form $\xi^T a = g - g \circ T$ with $g \in L^2$; since T leaves invariant Lebesgue measure, we have

$$\begin{aligned} D(a) \cdot \xi &= \lim_{N \to +\infty} \langle \sum_{n \in \mathbb{Z}} (a \circ T^n \otimes a) \xi \rangle = \lim_{N \to +\infty} \langle \sum_{n \in \mathbb{Z}} a \circ T^n \ a \xi^T \rangle \\ &= \lim_{N \to +\infty} \langle \sum_{n \in \mathbb{Z}} a \circ T^n (g - g \circ T) \rangle \\ &= \lim_{N \to +\infty} \langle a \circ T^N - a \circ T^{-N-1}) g \rangle = 0. \end{aligned}$$

Thanks to (2.8) and the fact that the Sobolev space H^s is dense in L^2, we led to $\lim_{N \to \pm\infty} \langle a \circ T^N \cdot g \rangle = 0$. This completes the proof of the proposition.

2.6. **Convergence proof.** Set $u_\epsilon(t, x) = \langle \psi_\epsilon(t, x, \cdot) \rangle$ and note that, since $\varphi \in C_b^3$, we can write Taylor's formula with Lagrange remainder of order 2. This allows us to identify the term $\langle \nabla_x^2 \psi_\epsilon(t, x, \cdot) \rangle$. By the dominated convergence theorem, we can deduce that u_ϵ is $C^2(\mathbb{R}^d)$ and since $\nabla^2 \varphi$ is bounded, $\nabla_x^2 u_\epsilon(t, x) = \langle \nabla_x^2 \psi_\epsilon(t, x, \cdot) \rangle$. Next, we define

$$F_{\epsilon,\tau}(t) := \Big\langle \frac{u_\epsilon(\cdot + \tau, \cdot) u_\epsilon(\cdot, \cdot)}{\tau}, f \Big\rangle$$

and we check that, for any compact $K \subset \mathbb{R}_x^d$, since $f \in C_K^\infty$, the relation

$$F_\tau(t) = \int_{\mathbb{R}^+ \times \mathbb{R}^d} \left(\nabla_x^2 u(t, x) \, : \, D(a) \right) f(t, x) dt dx + O(\tau^{\frac{1}{2}})$$

holds. Bringing together Proposition 2.8 and Proposition 2.9, plus the weak compactness of the family $\langle f_\epsilon \rangle$ guaranteed by the maximum principle (2.15), we can assume – possibly at the cost of extracting subsequences f_ϵ still abusively denoted f_ϵ – that the family of averages $\langle f_\epsilon \rangle$ converges to the solution f of (1.9)-(1.10). Finally, by convexity and properties of the weak limit, (2.16) shows that any weak-* limit point of f_ϵ as $\epsilon \to 0$ is invariant under the transformation T acting on the ω variable. Since T is ergodic, this shows that such weak-* limit points as $\epsilon \to 0$ must be independent of ω and therefore equal to the solution f of (1.9)-(1.10). Letting $n \to +\infty$ and $\tau \to 0$ to make the term $O(\tau^{\frac{1}{2}})$ disappear, we obtain

$$\partial_t u = \frac{1}{2} D(a) \, : \, \nabla_x^2 u.$$

To obtain the Cauchy data, we observe that $\partial_t \langle f_\epsilon \rangle = -\nabla_x \langle a f_\epsilon \rangle$ is bounded in $L^2(0, T; H^{-1}(\mathbb{R}^d))$ while $\langle f_\epsilon \rangle$ is bounded in $L^\infty(0, T, L^2(\mathbb{R}^d))$. These bounds ensure the compactness of $\langle f_\epsilon \rangle$ in $\mathcal{C}^0([0,T]); H^{-1}_{loc}(\mathbb{R}^d))$ (see [11]). This achieves the proof of Theorem 1.1.

3. Conclusion

We have derived the hydrodynamic limit from the kinetic model. In order to gain some insight, we have analyzed a model where technical difficulties are reduced: the so-called Knudsen gas. A Knudsen gas consists of interaction point particles moving freely between two infinite plates. By analogy with the Sinaï billiard, the reflection law at the plates is chosen to have convenient mixing properties. More specifically, when the particles impact the top or the bottom plate, their velocities are reversed and modified by the right action of the toral automorphism T. We have proved that, if T is quasi-hyperbolic, under some condition based on trigonometric polynomials, we derived a diffusion equation. This can be compared to the pioneer paper [2] where T is only hyperbolic.

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