MULTIPLE POSITIVE SOLUTIONS FOR CLASSES OF ELLIPTIC SYSTEMS WITH COMBINED NONLINEAR EFFECTS

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(Submitted by: Jean Mawhin)

Abstract. We study the existence of multiple positive solutions to systems of the form

$$\begin{cases} -\Delta u = \lambda f(v), & \text{in } \Omega, \\ -\Delta v = \lambda g(u), & \text{in } \Omega, \\ u = 0 = v, & \text{on } \partial\Omega. \end{cases}$$

Here Δ is the Laplacian operator, λ is a positive parameter, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary and f, g belong to a class of positive functions that have a combined sublinear effect at ∞. Our results also easily extend to the corresponding p-Laplacian systems. We prove our results by the method of sub and super solutions.

1. Introduction

Consider the boundary-value problem

$$\begin{cases} -\Delta u = \lambda f(v), & \text{in } \Omega, \\ -\Delta v = \lambda g(u), & \text{in } \Omega, \\ u = 0 = v, & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and λ is a positive parameter. We assume that $f,\ g$ are nontrivial $C^1([0,\infty))$ functions satisfying the following assumptions:

(H1) $f(0) \geq 0,\ g(0) \geq 0$ and f and g are nondecreasing.

Accepted for publication: February 2006.
AMS Subject Classifications: 35J55, 35J70.

(H2) $\lim_{x\to\infty} \frac{f(Mg(x))}{x} = 0$ for all $M > 0$ (a combined sublinear effect at ∞).

For such a system with $f(0)$ or $g(0)$ strictly positive it is easy to deduce the existence of a positive solution for every $\lambda > 0$ (see [5]). In addition from [3] it follows that if $\frac{x}{f(x)}$ and $\frac{x}{g(x)}$ are nondecreasing for all $x \geq 0$, then (1.1) will have at most one positive solution. In this paper, we will focus on the case when either $\frac{x}{f(x)}$ or $\frac{x}{g(x)}$ is decreasing for a certain range of x. In particular, under certain combined effects of $\frac{x}{f(x)}$ and $\frac{x}{g(x)}$ we study the existence of multiple positive solutions to (1.1). Namely for $a_1 < a_2, b_1 < b_2$, we define

$$Q_1(a_1, b_1) := \min\left\{\frac{a_1}{f(b_1)},\ \frac{b_1}{g(a_1)}\right\}$$

$$Q_2(a_2, b_2) := \max\left\{\frac{a_2}{f(b_2)},\ \frac{b_2}{g(a_2)}\right\}$$

and establish the following result:

Theorem 1.1. *Let $f(0)$ or $g(0)$ be strictly positive. There exists $C(\Omega) > 0$ such that if $Q_1/Q_2 > C$ for some $a_i, b_i,\ i = 1, 2$, then (1.1) has at least three positive solutions for*

$$\frac{CQ_2}{\|e\|_\infty} \leq \lambda \leq \frac{Q_1}{\|e\|_\infty}, \tag{1.2}$$

where e is the unique solution of $-\Delta e = 1$ in Ω, $e = 0$ on $\partial\Omega$.

If $f(0) = g(0) = 0$ then $(0, 0)$ itself becomes a solution. However, using the arguments in the previous theorem we can still assert the existence of two positive solutions. Namely, we establish:

Theorem 1.2. *Let $f(0) = 0 = g(0)$. There exists $C(\Omega) > 0$ such that if $Q_1/Q_2 > C$ for some $a_i, b_i,\ i = 1, 2$, then (1.1) has at least two positive solutions for*

$$\frac{CQ_2}{\|e\|_\infty} \leq \lambda \leq \frac{Q_1}{\|e\|_\infty}. \tag{1.3}$$

In the above case, if we have more information on the derivatives of the nonlinearities at 0, namely, if $f'(0) = g'(0) = 0$ then we can get two positive solutions for λ large. In particular, we prove:

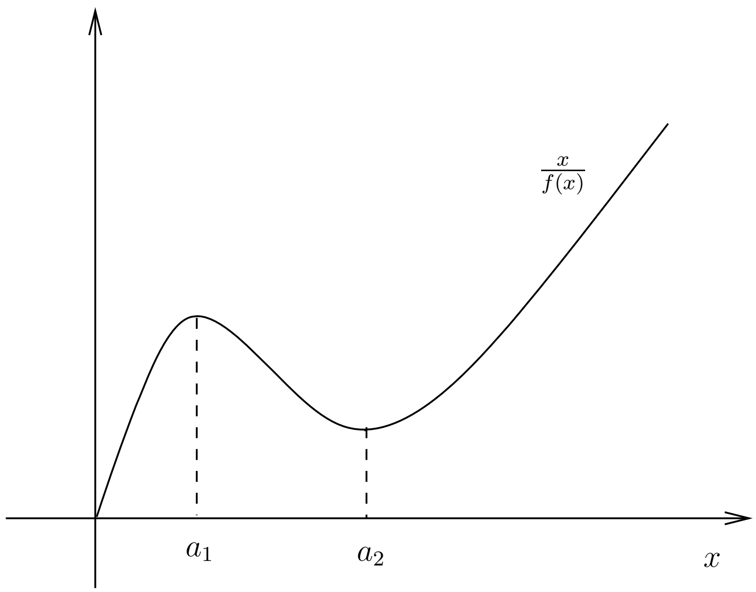
Theorem 1.3. *Let $f(0) = 0 = g(0) = f'(0) = g'(0)$. There exists $C_1(\Omega) > 0$ such that if $\lambda > C_1 Q$, (1.1) has at least two positive solutions, where $Q := \inf_{r>0,s>0} \max\{\frac{r}{f(s)}, \frac{s}{g(r)}\}$.*

Remark 1.1. Note that in the application of Theorem 1.1, one may take $a_i = b_i,\ i = 1, 2$, i.e., look for a_1, a_2 such that $a_1 < a_2$ and

$$\left(\min\left\{\frac{a_1}{f(a_1)},\ \frac{a_1}{g(a_1)}\right\}\Big/\max\left\{\frac{a_2}{f(a_2)},\ \frac{a_2}{g(a_2)}\right\}\right) > C(\Omega).$$

Remark 1.2. In the case of systems, our result shows that multiplicity occurs even when one of the function $\frac{x}{f(x)}$ or $\frac{x}{g(x)}$ is nondecreasing (see Example 2 in Section 6).

Remark 1.3. Note that in the case $f = g,\ u = v$, taking $a_i = b_i,\ i = 1, 2$, we get $Q_1/Q_2 = \frac{a_1}{f(a_1)}/\frac{a_2}{f(a_2)}$, and Theorem 1.1 coincides with the multiplicity result established for the single equation in [2].



Graph of $\frac{x}{f(x)}$

We establish our results via the method of sub and supersolutions. By a subsolution (supersolution) of (1.1) we mean a pair of functions $(z, \overline{z}) \in W^{1,2}(\Omega) \cap C(\overline{\Omega}) \times W^{1,2}(\Omega) \cap C(\overline{\Omega})$ such that $z = 0 = \overline{z}$ on $\partial\Omega$ and

$$\begin{cases} \displaystyle\int_\Omega \nabla z \cdot \nabla q \le (\ge) \int_\Omega \lambda f(\overline{z}) & \text{in } \Omega, \\ \displaystyle\int_\Omega \nabla \overline{z} \cdot \nabla q \le (\ge) \int_\Omega \lambda g(z) & \text{in } \Omega, \end{cases}$$

for every $q \in \{\eta \in C_0^\infty(\Omega) : \eta \geq 0 \text{ in } \Omega\}$. It is well known that, if there exist sub and supersolutions $(\psi, \overline{\psi})$ and $(\phi, \overline{\phi})$ respectively such that $(\psi, \overline{\psi}) \leq (\phi, \overline{\phi})$ then (1.1) has a solution (u, v) such that $(u, v) \in [(\psi, \overline{\psi}), (\phi, \overline{\phi})]$. (Here by $(\psi, \overline{\psi}) \leq (\phi, \overline{\phi})$ we mean $\psi \leq \phi$ and $\overline{\psi} \leq \overline{\phi}$.) We prove our multiplicity results by the following result discussed in [6].

Lemma 1.1. *Let f and g be nonnegative and nondecreasing, and suppose there exist a subsolution $(\psi_1, \overline{\psi}_1)$, a strict supersolution $(\phi_1, \overline{\phi}_1)$, a strict subsolution $(\psi_2, \overline{\psi}_2)$, and a supersolution $(\phi_2, \overline{\phi}_2)$ for* (1.1) *such that $(\psi_1, \overline{\psi}_1) \leq (\phi_1, \overline{\phi}_1) \leq (\phi_2, \overline{\phi}_2)$, $(\psi_1, \overline{\psi}_1) \leq (\psi_2, \overline{\psi}_2) \leq (\phi_2, \overline{\phi}_2)$, and $(\psi_2, \overline{\psi}_2) \not\leq (\phi_1, \overline{\phi}_1)$. Then* (1.1) *has at least three distinct solutions (u_i, v_i), $i = 1, 2, 3$, such that $(u_1, v_1) \in [(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)]$, $(u_2, v_2) \in [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$, and $(u_3, v_3) \in [(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big)$.*

See also [1] and [8] where such a result was established in the single equation case. Using this result, in [2], a multiplicity result similar to Theorem 1.1 was established for the single equation case, and recently in [7] it was extended to p-Laplacian single equations. In [2], the Green's function played a crucial role in the construction of a subsolution, while in [7] this was avoided in order to extend the result for the p-Laplacian. We will adopt and extend the method in [7] to systems, but for simplicity we restrict the proofs in this paper for Laplacian systems.

We will prove Theorem 1.1 in Section 2 when Ω is a ball and in Section 3 when Ω is a bounded domain. In Section 4, we prove Theorem 1.2 and 1.3, and in Section 5, we discuss p-Laplacian systems. In Section 6, we will discuss various examples.

2. Proof of Theorem 1.1 (when Ω is a ball)

Proof. We will establish a pair of subsolutions $(\psi_1, \overline{\psi}_1), (\psi_2, \overline{\psi}_2)$ and a pair of supersolutions $(\phi_1, \overline{\phi}_1), (\phi_2, \overline{\phi}_2)$ satisfying Lemma 1.1. Clearly $(\psi_1, \overline{\psi}_1) = (0, 0)$ is a subsolution of (1.1) since $f(0) \geq 0$ and $g(0) \geq 0$.

We next construct a positive supersolution $(\phi_1, \overline{\phi}_1)$ of (1.1) when $\lambda \leq \frac{1}{\|e\|_\infty} \min\{\frac{a_1}{f(b_1)}, \frac{b_1}{g(a_1)}\} = A$ (say). Let $e \in C^2(\overline{\Omega})$ be the solution of

$$\begin{cases} -\Delta e = 1, & \text{in } \Omega \\ e = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.1}$$

Let $(\phi_1, \overline{\phi}_1) = (a_1 \frac{e}{\|e\|_\infty}, b_1 \frac{e}{\|e\|_\infty})$. Since $\lambda \leq \frac{1}{\|e\|_\infty} \frac{a_1}{f(b_1)}$, we have $-\Delta\phi_1 = \frac{a_1}{\|e\|_\infty} \geq \lambda f(b_1) \geq \lambda f(\overline{\phi}_1)$. A similar proof shows that $\overline{\phi}_1$ satisfies $-\Delta\overline{\phi}_1 \geq$

$\lambda g(\phi_1)$. This proves that $(\phi_1, \overline{\phi}_1)$ is a positive supersolution of (1.1). Note that $\|\phi_1\|_\infty = a_1, \|\overline{\phi}_1\|_\infty = b_1$.

Now we construct a positive subsolution $(\psi_2, \overline{\psi}_2)$ of (1.1) when $\lambda \geq C_1 \max\{\frac{b_2}{g(a_2)}, \frac{a_2}{f(b_2)}\} = B$ (say) where $C_1 := \inf_\epsilon \frac{N}{\epsilon^N}\frac{R^{N-1}}{R-\epsilon}$. For $0 < \epsilon < R$, $\alpha, \beta > 1$ define $\rho(r) : [0, R] \to [0, 1]$ by

$$\rho(r) = \begin{cases} 1, & 0 \leq r \leq \epsilon \\ 1 - (1 - (\frac{R-r}{R-\epsilon})^\beta)^\alpha & \epsilon < r \leq R. \end{cases}$$

Thus,

$$\rho'(r) = \begin{cases} 0, & 0 \leq r \leq \epsilon \\ -\frac{\alpha\beta}{R-\epsilon}(1 - (\frac{R-r}{R-\epsilon})^\beta)^{\alpha-1}(\frac{R-r}{R-\epsilon})^{\beta-1} & \epsilon < r \leq R. \end{cases}$$

Note that $|\rho'(r)| \leq \frac{\alpha\beta}{R-\epsilon}$.

Let $w(r) = a_2\rho(r)$, $\overline{w}(r) = b_2\rho(r)$, define $\psi_2(r), \overline{\psi}_2(r)$ as the radially symmetric C^2 solutions of

$$\begin{cases} -\Delta\psi_2 = \lambda f(\overline{w}), & \text{in } B(0,R) \\ -\Delta\overline{\psi}_2 = \lambda g(w), & \text{in } B(0,R) \\ \psi_2 = 0 = \overline{\psi}_2, & \text{on } \partial B(0,R). \end{cases} \tag{2.2}$$

Then ψ_2, $\overline{\psi}_2$ satisfies

$$-\psi_2'(r) = \frac{\lambda}{r^{N-1}}\int_0^r s^{N-1} f(\overline{w}(s))ds$$
$$-\overline{\psi}_2'(r) = \frac{\lambda}{r^{N-1}}\int_0^r s^{N-1} g(w(s))ds.$$

Since $|\rho'(r)| \leq \frac{\alpha\beta}{R-\epsilon}$, we have

$$|w'(r)| \leq \frac{a_2\alpha\beta}{R-\epsilon},\ |\overline{w}'(r)| \leq \frac{b_2\alpha\beta}{R-\epsilon}. \tag{2.3}$$

Note that for $0 \leq r \leq \epsilon$, clearly $\psi_2'(r) \leq w'(r)$, $\overline{\psi}_2'(r) \leq \overline{w}'(r)$. Now for $r > \epsilon$,

$$\begin{aligned} -\psi_2'(r) &= \frac{\lambda}{r^{N-1}}\int_0^r s^{N-1} f(\overline{w}(s))ds \\ &\geq \frac{\lambda}{r^{N-1}}\int_0^\epsilon s^{N-1} f(\overline{w}(s))ds \geq \frac{\lambda}{R^{N-1}} f(b_2)\frac{\epsilon^N}{N}. \end{aligned} \tag{2.4}$$

Similar calculations show that

$$-\overline{\psi_2}'(r) \geq \frac{\lambda}{R^{N-1}} g(a_2) \frac{\epsilon^N}{N}. \tag{2.5}$$

Note that the $\inf_{\epsilon} \frac{N}{\epsilon^N} \frac{R^{N-1}}{R-\epsilon}$ is attained at $\epsilon_0 = \frac{N}{N+1} R$. We let $\epsilon = \epsilon_0$. Hence $C_1 = \frac{N}{\epsilon_0^N} \frac{R^{N-1}}{R-\epsilon_0}$. Suppose $\lambda > C_1 \max\{\frac{b_2}{g(a_2)}, \frac{a_2}{f(b_2)}\}$. Then choose $\alpha, \beta > 1$ so that $\lambda \geq \alpha\beta C_1 \max\{\frac{b_2}{g(a_2)}, \frac{a_2}{f(b_2)}\}$. Hence, $\lambda \geq \alpha\beta \frac{N}{\epsilon_0^N} \frac{R^{n-1}}{R-\epsilon_0} \frac{b_2}{g(a_2)}$ and by (2.5),

$$-\overline{\psi_2}'(r) \geq \frac{\lambda}{R^{N-1}} g(a_2) \frac{\epsilon_0^N}{N} \geq \frac{\alpha\beta b_2}{R-\epsilon_0} \geq -\overline{w}'(r).$$

That is, we have $\overline{\psi_2}'(r) \leq \overline{w}'(r); 0 \leq r \leq R$. Similarly we can establish $\psi_2'(r) \leq w'(r); 0 \leq r \leq R$ by using (2.4) and (2.3). Using the continuity argument on λ these inequalities hold for $\lambda \geq B$. Since $\psi_2(R) = \overline{\psi}_2(R) = 0 = w(R) = \overline{w}(R)$, it is easy to see that

$$\psi_2 \geq w, \overline{\psi}_2 \geq \overline{w}, \text{ for } 0 \leq r \leq R. \tag{2.6}$$

Now since f and g are nondecreasing, using (2.2) and (2.6), we have

$$\begin{cases} -\Delta\psi_2 = \lambda f(\overline{w}) \leq \lambda f(\overline{\psi}_2) & \text{in } B(0,R), \\ -\Delta\overline{\psi}_2 = \lambda g(w) \leq \lambda g(\psi_2) & \text{in } B(0,R), \\ \psi_2 = 0 = \overline{\psi}_2 & \text{on } \partial B(0,R), \end{cases} \tag{2.7}$$

and hence $(\psi_2, \overline{\psi}_2)$ is a positive subsolution of (1.1). We also note that $\|\psi_2\|_\infty \geq \|w\|_\infty = a_2 > a_1, \|\overline{\psi}_2\|_\infty \geq b_2$, hence $(\psi_2, \overline{\psi}_2) \not\leq (\phi_1, \overline{\phi}_1)$.

Finally, using (H2), we will construct a large positive supersolution $(\phi_2, \overline{\phi}_2)$. If both f and g are bounded, let $(\phi_2, \overline{\phi}_2) = (\lambda C_\lambda \frac{e}{\|e\|_\infty}, \lambda C_\lambda \frac{e}{\|e\|_\infty})$ and choose C_λ so large that $\frac{C_\lambda}{\|e\|_\infty} > \max\{\|f\|_\infty, \|g\|_\infty\}$. Then it is easy to see that $(\phi_2, \overline{\phi}_2)$ is a positive supersolution of (1.1). Suppose $g(x) \to \infty$ as $x \to \infty$ and let $(\phi_2, \overline{\phi}_2) = (C_\lambda e, \lambda g(C_\lambda \|e\|_\infty) e)$. Then by $(H2)$, choosing C_λ large we have

$$\frac{f\big(\lambda \|e\|_\infty g(C_\lambda \|e\|_\infty)\big)}{C_\lambda \|e\|_\infty} \leq \frac{1}{\lambda \|e\|_\infty}.$$

Thus we have $-\Delta\phi_2 = C_\lambda \geq \lambda f\big(\lambda \|e\|_\infty g(C_\lambda \|e\|_\infty)\big) \geq \lambda f\big(\lambda g(C_\lambda \|e\|_\infty) e\big) = \lambda f(\overline{\phi}_2)$. Also we have $-\Delta\overline{\phi}_2 = \lambda g(C_\lambda \|e\|_\infty) \geq \lambda g(\phi_2)$, showing that $(\phi_2, \overline{\phi}_2)$ is a supersolution of (1.1). (If $g(x)$ is bounded and $f(x) \to \infty$ as $x \to \infty$, then $\lim_{x\to\infty} \frac{g(Mf(x))}{x} = 0$ for all $M > 0$ and we can prove that $(\phi_2, \overline{\phi}_2) = (\lambda f(C_\lambda \|e\|_\infty) e, C_\lambda e)$ is a supersolution.) Also since $e > 0$ in Ω and $\frac{\partial e}{\partial \eta} <$

0 on $\partial\Omega$, for C_λ large enough, in all the above cases we have $(\phi_2, \overline{\phi}_2) \geq (\phi_1, \overline{\phi}_1)$ and $(\phi_2, \overline{\phi}_2) \geq (\psi_2, \overline{\psi}_2)$. Now by Lemma 1.1, if $C_1 Q_2 < \frac{Q_1}{\|e\|_\infty}$; that is, $Q_1/Q_2 > C(\Omega) := C_1\|e\|_\infty$; then for all λ satisfying (1.2), (1.1) has a solution on each of the following components $[(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)]$, $[(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$, and $[(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big)$. Clearly the solutions in $[(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$ and $[(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big)$ are positive. Since $f(0)$ or $g(0)$ is strictly positive, the solutions in $[(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)]$ are also positive. $\square$

3. Proof of Theorem 1.1 (when Ω is a general bounded domain)

In this section we will prove the main theorem. First we construct a positive subsolution $(z, \overline{z})$ of (1.1) in Ω with $\|z\|_\infty \geq a_2, \|\overline{z}\|_\infty \geq b_2$. Let B_R be the largest inscribed ball in Ω. Assume $\lambda \geq C_1 Q_2$ and let $(\psi_2(r), \overline{\psi}_2(r))$ be the subsolution of (1.1) constructed in B_R of the previous theorem.
Now define

$$z(x) = \begin{cases} \psi_2(|x|)\ ; & x \in B_R \\ 0\ ; & x \in \Omega - B_R, \end{cases} \quad \text{and} \quad \overline{z}(x) = \begin{cases} \overline{\psi}_2(|x|)\ ; & x \in B_R \\ 0\ ; & x \in \Omega - B_R. \end{cases}$$

Then $z, \overline{z} \in W^{1,2}(\Omega) \cap C(\overline{\Omega})$ and $z = 0 = \overline{z}$ on $\partial\Omega$. Further, on B_R we have

$$\begin{cases} -\Delta z = -\Delta \psi_2 \leq \lambda f(\overline{\psi}_2) = \lambda f(\overline{z}), \\ -\Delta \overline{z} = -\Delta \overline{\psi}_2 \leq \lambda g(\psi_2) = \lambda g(z), \end{cases}$$

while outside B_R we have

$$\begin{cases} -\Delta z = 0 < \lambda f(0) = \lambda f(\overline{z}), \\ -\Delta \overline{z} = 0 < \lambda g(0) = \lambda g(z). \end{cases}$$

Hence $(z, \overline{z})$ is a subsolution of (1.1) in Ω for $\lambda \geq C_1 Q_2$ with $\|z\|_\infty \geq a_2, \|\overline{z}\|_\infty \geq b_2$. The rest of the proof is identical to the previous case except that here for the second subsolution we will use $(z, \overline{z})$ described above.

4. Proof of Theorem 1.2 and 1.3

4.1. **Proof of Theorem 1.2.** Since $f(0) = g(0) = 0$, it is obvious that $(\psi_1, \overline{\psi}_1) = (0, 0)$ is a solution to (1.1), and constructing sub and supersolutions as in Theorem 1.1, it is easy to see that (1.1) has at least two positive solutions in $[(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$ and $[(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big)$.

4.2. **Proof of Theorem 1.3.** Since $f(0) = g(0) = 0$, we have the first subsolution (solution) $(\psi_1, \overline{\psi}_1) = (0,0)$ for (1.1) for any $\lambda > 0$. Now let $\lambda > C_1 \inf_{r>0,s>0} \max\{\frac{r}{f(s)}, \frac{s}{g(r)}\}$ be fixed. Then there exists $a_2 > 0, b_2 > 0$ such that $\lambda > C_1 Q_2(a_2, b_2)$. Hence we have the second positive subsolution $(\psi_2, \overline{\psi}_2)$ of (1.1) (as discussed in the proof of Theorem 1.1) with $\|\psi_2\|_\infty \geq a_2$, $\|\overline{\psi}_2\|_\infty \geq b_2$. Since $f(0) = g(0) = 0 = f'(0) = g'(0)$, we have $\frac{x}{f(x)}$ and $\frac{x}{g(x)} \to \infty$ as $x \to 0$. Thus there exists $a < a_2$ (or b_2) such that $\big(\min\{\frac{a}{f(a)}, \frac{a}{g(a)}\}\big)/\|e\|_\infty \geq \lambda$. Let $(\phi_1, \overline{\phi}_1) = (a\frac{e}{\|e\|_\infty}, a\frac{e}{\|e\|_\infty})$. Then it is easy to see that $(\phi_1, \overline{\phi}_1)$ is a supersolution of (1.1). Note that $(\psi_2, \overline{\psi}_2) \not\leq (\phi_1, \overline{\phi}_1)$ since $a < a_2$. Also using $(H2)$ as in the proof of Theorem 1.1 there exists a large positive supersolution $(\phi_2, \overline{\phi}_2)$ of (1.1) such that $(\phi_1, \overline{\phi}_1) \leq (\phi_2, \overline{\phi}_2)$ and $(\psi_2, \overline{\psi}_2) \leq (\phi_2, \overline{\phi}_2)$. Now by Lemma 1.1, (1.1) has at least two positive solutions in $[(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]$ and $[(\psi_1, \overline{\psi}_1), (\phi_2, \overline{\phi}_2)] \setminus \big([(\psi_1, \overline{\psi}_1), (\phi_1, \overline{\phi}_1)] \cup [(\psi_2, \overline{\psi}_2), (\phi_2, \overline{\phi}_2)]\big)$.

5. p-Laplacian systems

Here we consider the boundary-value problem,

$$\begin{cases} -\Delta_p u = \lambda f(v), & \text{in } \Omega, \\ -\Delta_p v = \lambda g(u), & \text{in } \Omega, \\ u = 0 = v, & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

where $\Delta_p z := \operatorname{div}(|\nabla z|^{p-2}\nabla z)$; $p > 1$, Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$ and λ is a positive parameter. We assume that f, g are nontrivial $C^1([0,\infty))$ functions satisfying the following assumptions:

$(\widetilde{H}1)$ $f(0) \geq 0$, $g(0) \geq 0$ and f and g are nondecreasing.

$(\widetilde{H}2)$ $\lim_{x\to\infty} \frac{f(M(g(x)^{\frac{1}{p-1}}))}{x^{p-1}} = 0$ $\forall M > 0$ (a combined p-sublinear effect at ∞).

Again using the method of sub and supersolution (extended appropriately for the p-Laplacian (see [7], [4])) the ideas in the proofs of this paper easily extend to give the following result for p-Laplacian systems:

Theorem 5.1. *Let $f(0)$ or $g(0)$ be strictly positive. There exists $\widetilde{C}(\Omega) > 0$ such that if $\widetilde{Q}_1/\widetilde{Q}_2 > \widetilde{C}$ for some a_i, b_i, $i = 1, 2$, then* (5.1) *has at least three positive solutions for*

$$\frac{\widetilde{C}\widetilde{Q}_2}{\|e\|_\infty^{p-1}} \leq \lambda \leq \frac{\widetilde{Q}_1}{\|e\|_\infty^{p-1}}, \tag{5.2}$$

with $\widetilde{C} := \widetilde{C}_1 \|e\|_\infty^{p-1}$, *where* $\widetilde{C}_1 := \inf_\epsilon \frac{N}{\epsilon^N} \frac{R^{N-1}}{(R-\epsilon)^{p-1}}$, *and*

$$\widetilde{Q}_1(a_1, b_1) := \min\Big\{ \frac{a_1^{p-1}}{f(b_1)},\ \frac{b_1^{p-1}}{g(a_1)} \Big\}$$

$$\widetilde{Q}_2(a_2, b_2) := \max\Big\{ \frac{a_2^{p-1}}{f(b_2)},\ \frac{b_2^{p-1}}{g(a_2)} \Big\}.$$

Here e *is the unique solution of* $-\Delta_p e = 1$ *in* Ω, $e = 0$ *on* $\partial\Omega$.

Theorem 5.2. *Let* $f(0) = 0 = g(0)$. *Then there exists* $\widetilde{C}(\Omega) > 0$ *such that if* $\widetilde{Q}_1/\widetilde{Q}_2 > \widetilde{C}$ *for some* $a_i,\ b_i,\ i = 1, 2$, *then* (1.1) *has at least two positive solutions for*

$$\frac{\widetilde{C}\widetilde{Q}_2}{\|e\|_\infty^{p-1}} \leq \lambda \leq \frac{\widetilde{Q}_1}{\|e\|_\infty^{p-1}}. \tag{5.3}$$

Here $\widetilde{Q}_1$ *and* $\widetilde{Q}_2$ *are as in Theorem* 5.1.

Theorem 5.3. *Let* $f(0) = g(0) = 0 = f^{(k)}(0) = g^{(k)}(0)$ *for* $k = 1, 2 \cdots [p-1]$, *where* $[p-1]$ *denotes the integer part of* $p-1$. *Then there exists* $\widetilde{C}_1(\Omega) > 0$ *such that for* $\lambda > \widetilde{C}_1 Q$, (1.1) *has at least two positive solutions, where* $Q := \inf_{r>0, s>0} \max\{\frac{r^{p-1}}{f(s)}, \frac{s^{p-1}}{g(r)}\}$.

6. Examples

Here we discuss various examples for Laplacian systems. In particular, we concentrate on the application of Theorem 1.1.

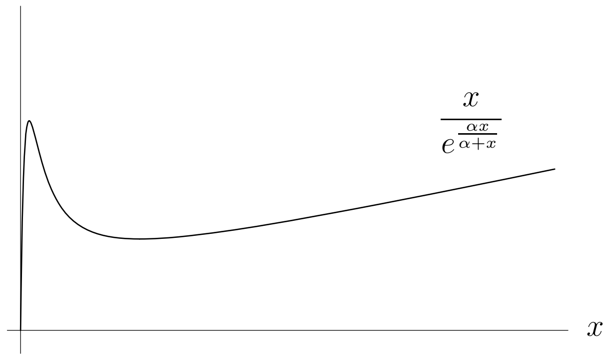
Example 1. Let $f(x) = e^{\frac{\alpha x}{\alpha + x}}$, $g(x) = e^x$. Clearly $f,\ g$ satisfies (H1) and (H2) as

$$\lim_{x\to\infty} \frac{f(Mg(x))}{x} = \lim_{x\to\infty} \frac{e^{\frac{\alpha M e^x}{\alpha + M e^x}}}{x} = 0.$$

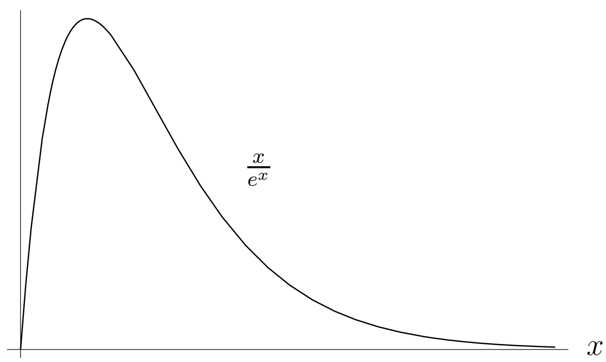
Choosing $a_1 = 1 = b_1$, $a_2 = \alpha = b_2 > 1$, we have $Q_1(1,1) := \min\big\{e^{-\frac{\alpha}{\alpha+1}}, e^{-1}\big\}$, $Q_2(\alpha, \alpha) := \max\big\{\alpha e^{-\frac{\alpha}{2}}, \alpha e^{-\alpha}\big\}$, thus

$$\begin{aligned}(Q_1/Q_2) &= \min\Big\{ \frac{e^{\alpha - \frac{\alpha}{\alpha+1}}}{\alpha},\ \frac{e^{\frac{\alpha}{2}-1}}{\alpha},\ \frac{e^{\alpha-1}}{\alpha},\ \frac{e^{\frac{\alpha}{2} - \frac{\alpha}{\alpha+1}}}{\alpha} \Big\} \\ &\geq \min\Big\{ \frac{e^{\alpha-1}}{\alpha},\ \frac{e^{\frac{\alpha}{2}-1}}{\alpha},\ \frac{e^{\alpha-1}}{\alpha},\ \frac{e^{\frac{\alpha}{2}-1}}{\alpha} \Big\} \geq \frac{e^{\frac{\alpha}{2}-1}}{\alpha}.\end{aligned}$$

For any Ω we can choose α so large that $Q_1/Q_2 > C(\Omega)$. Hence, Theorem 1.1 holds and there exist a range of λ for which there exists three positive solutions. See below the graph of $\frac{x}{f(x)}$ for $\alpha = 6$, and $\frac{x}{g(x)}$.



Graph of $\frac{x}{e^{\frac{\alpha x}{\alpha+x}}}$ for $\alpha = 6$



Graph of $\frac{x}{e^x}$

Example 2. Let $f(x) = e^{\frac{\alpha x}{\alpha+x}}$, $g(x) = (x+1)^p, p > 0$. Clearly $f,\ g$ satisfy (H1) and (H2) as

$$\lim_{x\to\infty} \frac{f(Mg(x))}{x} = 0.$$

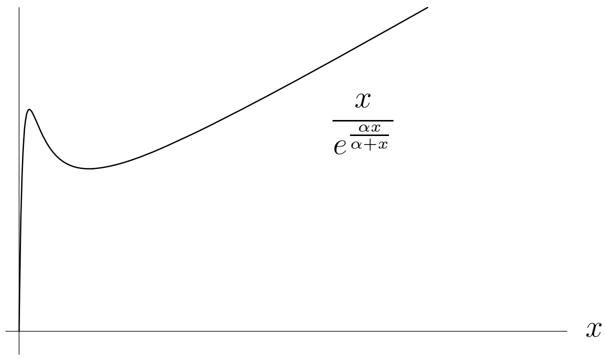
Choosing $a_1 = 1 = b_1$, $a_2 = \alpha^{\frac{2}{p}} - 1,\ b_2 = \alpha$, we have

$$Q_1(1,1) := \min\left\{e^{-\frac{\alpha}{\alpha+1}}, 2^{-p}\right\}, \quad Q_2(\alpha^{\frac{2}{p}} - 1, \alpha) := \max\left\{(\alpha^{\frac{2}{p}} - 1)e^{-\frac{\alpha}{2}}, \alpha^{-1}\right\},$$

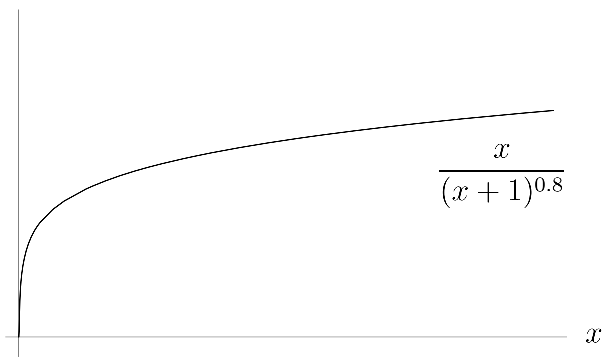
thus,

$$(Q_1/Q_2) = \min\left\{e^{-\frac{\alpha}{\alpha+1}}\alpha,\ \frac{e^{\frac{\alpha}{2}}}{2^p(\alpha^{\frac{2}{p}} - 1)},\ \frac{\alpha}{2^p},\ \frac{e^{\frac{\alpha}{2} - \frac{\alpha}{\alpha+1}}}{(\alpha^{\frac{2}{p}} - 1)}\right\}.$$

For any Ω we can choose α so large that $Q_1/Q_2 > C(\Omega)$. Hence, Theorem 1.1 holds and there exist a range of λ for which there exists three positive solutions. Note that for $p \leq 1$, $\frac{x}{g(x)}$ is nondecreasing. See below the graph of $\frac{x}{f(x)}$ for $\alpha = 5$ and $\frac{x}{g(x)}$ for $p = 0.8$.



Graph of $\frac{x}{e^{\frac{\alpha x}{\alpha+x}}}$ for $\alpha = 5$



Graph of $\frac{x}{(x+1)^{0.8}}$

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